A Portfolio Approach to

The Optimal Mix of Funded and Unfunded Pensions

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Abstract:

In this paper we address the optimal funding of pensions by means of portfolio choice approach. Considering the unfunded (Paygo) pension system as a "quasi-asset" with hedging and diversification properties, we derive the optimal portfolio mix of unfunded and Paygo systems within a Mean Variance and Bell linear exponential models. Our analysis involves both analytical computation and empirical estimations of optimal values using real long term data for equity, bonds and Paygo asset for several OECD countries and several time periods covering the time span 1897-2016. We find that in most cases a mix of both systems is desirable with a larger magnitude of Paygo system in the case of the Bell Framework as we capture attitudes towards asymmetry and tail risks that are typical to equity markets.

Keywords: portfolio choice, pension systems, social security, risk sharing

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1. Introduction

Our main objective is to analyze the optimal mix of funded and unfunded pension systems within a portfolio choice approach considering a mean variance and Bell (Linear exponential or Linex) preferences frameworks for a sample of eight OECD countries: the US, Canada, Australia, the UK, France, Germany, Sweden and Japan. When taking into consideration that the rates of return on both unfunded and funded pension systems are stochastic, the unfunded system also referred to as pay-as-you-go (Paygo) system can be interpreted as a "quasi-asset" displaying diversification and hedging properties.

Our paper belongs to the emerging strand of literature addressing the issue of optimal social security design in stochastic framework using a portfolio choice approach. This literature follows in the footsteps of Merton (1983) who initially addressed the portfolio and diversification effects of unfunded pension systems.

We depart from other papers in the literature by capturing attitudes towards asymmetry and tail risks that are representative of equity markets. We isolate asymmetry and tail risks from the more traditional notion of risk by taking into account the two parameters of Linex utility function. This function also satisfies the desirable "One-Switch" rule that should be characteristic of all rational decision making Bell (1988, 1995). Among the four families of One-Switch rule functions, it is the only one that also satisfies other desirable properties such as decreasing absolute risk aversion and non-satiety with respect to consumption. It combines a linear part and a negative exponential (or CARA) part.

The majority of pension systems in developed countries are unfunded or pay-as-you-go (paygo) systems. In such a scheme a contribution tax is levied on the labor income of current workers and is aimed to finance pensions of current retirees who are mainly former workers that have reached retirement age and are entitled to a pension.

The unfunded pension system is often compared to the funded system where current workers' contributions are rather dedicated to asset accumulation intended to pay future benefits.

The central argument for funded pension systems in the literature is based on returns comparison. The implicit rate of return of the unfunded system is equal to the growth rate of aggregate wage income which is lower than the real rate of interest (Aaron 1966, Abel et al., 1989; Feldstein, 1974, 1996). Investment in funded systems would *ceteris paribus* yield higher returns than in unfunded systems.

This rate of return argument is stronger in countries with lower population growth and ageing populations. The cost of unfunded pensions would unequally be borne by younger workers. One important ground of the literature on social security reforms lies on the fact that ageing populations degrade the financial sustainability of social security systems that are paygo financed.

On the other hand, papers that initially provided arguments for unfunded pension systems allowed for the possibility of dynamic inefficiency or capital over accumulation (Diamond 1965; Samuelson 1975). Savings may be higher than optimal, which may lead the rate of return on capital to be lower than wages income growth rate. Unfunded pensions would allow for redistributing consumption across generations and restore efficiency.

Both strands of papers, however, did not take into account variance and portfolio effects which were later addressed by Merton (1983) and Merton et al (1987).

Under uncertainty, a lower expected rate of return of unfunded pension system does not necessarily imply the unfunded system is dominated by a funded system. This is essentially the case if we assume that the rates of returns on both unfunded and funded systems are stochastic. The pay-as-you-go system can be then interpreted as a "quasi-asset".

The illuminating insight that a mix of funded and unfunded pensions systems may be optimal due to diversification was initiated by Merton (1983). Merton characterizes optimal tax-transfer programs within a theoretical general equilibrium model.

An unfunded system even with low implicit rate is desirable as it contributes to hedge other risks individuals may face during their lifetime. It allows one generation to trade in the human capital of the next generation. The paygo quasi-asset is not spanned by other available assets due to an imperfect correlation between the aggregate wage income growth rate and stock returns. Therefore, a mix between funded and unfunded systems is optimal as it allows for risk diversification and risk sharing.

Merton (1983) interpretation of unfunded pension as quasi-asset was further explored by Dutta et al. (2000), Matsen and Thogersen (2004), De Menil et al (2006) and Knell (2010).

This insightful body of literature addresses the issue of optimal social security design in stochastic framework using a portfolio choice approach. It departs from Merton general equilibrium setting and rather considers partial equilibrium portfolio maximization.

Dutta et al. (2000) offer a simplified formal portfolio choice approach using a static mean variance framework. They allow for risk diversification and show that the optimum is a mix of funded and unfunded systems. They complement the analysis by considering the desirable combination of bonds and equities in the funded pillar. Matsen and Thogersen (2004) derive analytical formulas for the optimal size of the paygo system and the optimal magnitude of the funded pillar in the stock market. Unlike Dutta et al. they don't consider a static but rather a dynamic setting with a two period overlapping generation model and analyze distinct risk concepts. De Menil et al (2006) also use an overlapping generation model and identify conditions under which a mix of funded and unfunded pension systems is optimal. Knell (2010) derives the optimal portfolio mix when people care about their consumption relative to a reference group.

In our paper we allow for explicit portfolio maximization and further explore the diversification properties of the Paygo asset and derive the optimal portfolio mix of pensions for different levels of risk aversion and different preferences modelling, we then conduct empirical estimations for various time periods using real data going back to 1897. We manage to capture attitudes towards asymmetry and tail risks that are important characteristics of equity markets. To our knowledge, this is the first time such long term analysis is conducted and the first time such notions of risk are isolated and directly taken into account when solving for the optimal mix of pension systems.

We first consider a mean variance theoretical framework and derive optimal portfolio allocation for eight countries; the US, Canada, Australia, the UK, France, Germany, Sweden and Japan. We then consider Linex preferences that display some desirable properties and also allow to account not only for the mean and variance components but also for some important stylized facts of asset returns such as asymmetry and heavy tails.

Our analysis involves not only analytical computation but also empirical estimations of the theoretical optimal values using real long term data for equity, bonds and Paygo asset. The results obtained are compared across countries, times periods, degrees of risk aversion and across utility functions and preferences.

We find that in most cases a mix of both systems is desirable with a larger magnitude of Paygo system in the case of the Linex Framework as we capture attitudes towards asymmetry and tail risks typical to equity markets. These results are robust to data sample and time periods.

The paper is structured as follows: Section 2 presents the optimal design of pension system in a Mean Variance then in a Linear Exponential Framework. In Section 3 we introduce our data and summary statistics for equity, bonds and Paygo for our sample of eight countries. In section 4 we provide empirical estimates of the optimal theoretical values. Finally, Section 5 concludes.

2. The model set up and optimal design of the pension system

2.1 The Mean Variance Framework

In this section, we introduce a mean-variance portfolio choice framework. Without loss of generality, we abstract from the notion of consumers' heterogeneity and consider a model where the expected utility of a representative agent is given by:

$$EU(C) = EC - \frac{\gamma}{2} Var(C) \quad (1)$$

where C is the consumption level of the representative individual.

A contribution tax τ to the unfunded system is levied on the wage income W; τ is a proportion of W where $0 < \tau < 1$. The remaining proportion of the wage denoted $\lambda = (1-\tau)W$, is saved and invested in the stocks available in the financial markets. The return on the pension system is: $1 + \lambda R_e + (1-\lambda) R_p$

Where R_e and R_p are respectively the equity return and the Paygo asset return. Both are stochastic variables.

With wage income normalized to 1, the level of consumption C can then be written as:

$$C = 1 + \lambda R_e + (1-\lambda) R_p \quad (2)$$

The mean and variance of consumption C are:

$$E(C) = 1 + \lambda \mu_e + (1-\lambda) \mu_p \quad (3)$$

$$Var(C) = \lambda^2 \sigma_e^2 + (1-\lambda)^2 \sigma_p^2 + 2\lambda(1-\lambda)\sigma_{ep} \quad (4)$$

Where μ_e and μ_p are respectively the mean of equity return and the mean of Paygo asset.

The optimal fraction of national wage income to be invested in the pension funded pillar λ^* is obtained by maximizing the expected utility with respect to λ:

$$\lambda^* = \frac{\mu_e - \mu_p + \gamma(\sigma_p^2 - \sigma_{ep})}{\gamma \sigma_{e-p}^2} \quad (5)$$

After solving for the optimal mix of funded and unfunded pension system, another important consideration is the optimal allocation of bonds and equity within the funded pillar. The level of consumption C can then be rewritten as:

$$C = 1 + \lambda a R_e + \lambda(1-a)R_b + (1-\lambda)R_p \tag{6}$$

Where R_b is the stochastic return on bonds, and a is the share of funded pension dedicated to equity. For a fixed λ the optimal fraction a^* is obtained by maximizing the expected utility with respect to a. The mean and variance of consumption C are:

$$E(C) = 1 + a\lambda\mu_e + (1-a)\lambda\mu_b + (1-\lambda)\mu_p \tag{7}$$

$$Var(C) = \lambda^2 a^2 \sigma_e^2 + \lambda^2(1-a)^2\sigma_b^2 + (1-\lambda)^2\sigma_p^2 + 2\lambda^2 a(1-a)\sigma_{eb} + 2\lambda(1-\lambda)\sigma_{ep} + 2\lambda(1-\lambda)(1-a)\sigma_{bp} \tag{8}$$

Given λ the optimal fraction a^* is obtained by maximizing the expected utility with respect to a.

$$a^* = \frac{\lambda(\mu_e - \mu_b) + \gamma\left(\lambda^2\sigma_b^2 - \lambda^2\sigma_{eb} + \lambda(1-\lambda)\sigma_{bp}\right)}{\gamma\lambda^2\sigma_{e-b}^2} \tag{9}$$

It is straightforward to notice that for a given fraction of funding λ, the optimal fraction a^* to be allocated to equity within the funded pillar increases in the mean of equity returns and decreases in the mean of bonds returns. The optimal fraction a^* is also decreasing in the covariance between bonds and equity returns. For a given fraction of funding λ the lower this covariance, the higher will be the optimal fraction to be allocated to equity instead of bonds.

2.2 Linear exponential preferences Framework

Whilst the mean variance framework à la Dutta et al (2000) remains a good benchmark; this type of preferences is known for displaying some non-desirable properties such as satiety and growing risk aversion.

In his illuminating papers, Bell (1988, 1995) argues that rational decision making should be characterized by a utility function that satisfies the one-switch rule. The linear exponential (Linex) utility function is one of the four families that satisfy this rule. In addition, among the four categories it is the only one that also satisfies other desirable properties such as decreasing absolute risk aversion. Unlike the quadratic utility function, it also satisfies non-satiety with respect to consumption. Most importantly, the Linex utility function will allow to isolate asymmetry and tail risks that are representative of equity markets (Capocci et al 2006; Cavenaile et al 2011).

The intuition behind the one-switch rule can be illustrated as follows: Let us consider two alternatives Y and Z where Y has the highest expectation but the worst possible outcome $E(Y) > E(Z)$. In spite of its low expected payoff, Z is more attractive at low levels of wealth as it allows risk averse decision makers to avoid the worst outcome. This alternative becomes however less and less appealing as the wealth increases. When the number n of such preference reversal is such that $n \leq 1$, then the corresponding utility function it said to satisfy the one-switch rule.

The Linex utility function satisfying the one-switch rule and displaying decreasing absolute risk aversion and non-satiety properties can be written as:

$$U(C) = C - bexp(-dC)$$

Where C is consumption and b and d are positive constants. It combines a linear part and a negative exponential (or CARA) part.

The appealing characteristics of this utility has been further studied by Bell and Fishburn (2001); Sandvik and Thornlund-Petersen (2010) and Denuit et al (2013); reinforcing the arguments in favor of Linex preferences.

The model set up and optimal design of the pension system

In this section, the representative individual preferences are represented by a Linex utility function. This is given by:

$$U(C) = C - bexp(-dC) \quad (10)$$

Where C is the consumption level of the representative individual and b and d are parameters accounting for attitude towards risk.

The level of consumption C is defined by equation (2) where λ is the proportion of the wage invested in stocks available in the financial markets:

$$C = 1 + \lambda R_e + (1 - \lambda)R_p$$

The expected utility to be maximized writes as:

$$EU(C) = E\big(1 + \lambda R_e + (1 - \lambda)\mathrm{R_p}\big) - bE\left[exp\left(-d\big(1 + \lambda R_e + (1 - \lambda)\mathrm{R_p}\big)\right)\right] \quad (11)$$

The optimal λ^* to be invested in the pension funded pillar must satisfy the following first-order condition:

$$\frac{dEU}{d\lambda^*} = \mu_e - \mu_p - bdE\left[exp\big(-d(1 + R_p)exp\left(-d\lambda\big(R_e - R_p\big)\right)\big(R_e - R_p\big)\right] = 0 \quad (12)$$

Property 1 *The solution λ^* is decreasing in b.*

This follows from the fact that the partial derivative of (12) with respect to b is negative. It is straightforward to see that this partial derivative writes as $-dE\left[exp\big(-d(1 + R_p\big)exp\left(-d\lambda\big(R_e - R_p\big)\right)\big(R_e - R_p\big)\right] < 0$

Property 2 *The solution λ^* is decreasing in d.*

This result follows from the fact that the partial derivative of (12) with respect to d is negative: $-bE\left[exp\big(-d(1 + R_p\big)exp\left(-d\lambda\big(R_e - R_p\big)\right)\big(R_e - R_p\big)\right] < 0$

Property 3 *An increase in the weight given to the Linex Linear component tempers risk aversion.*

It is also important to notice that if we were to give more weight in the utility function to the linear part as compared to the negative exponential part by multiplying the linear component by a positive parameter f such that $f > 1$; this would effect in tempering risk aversion.

In a Bell or Linex specification that would writes as: $U(C) = fC - bexp(-dC)$; the higher the parameter f, the higher will be the weight put on the risk neutral component and therefore the higher will be the tempering of risk aversion. An increase in parameter f will therefore result in higher fractions λ^* to be allocated to the funded pension system.

The following step is to compute the optimal allocation of bonds and equity within the funded pillar. The level of consumption C can then be rewritten as in equation (6):

$$C = 1 + \lambda a R_e + \lambda(1-a)R_b + (1-\lambda)R_p \quad (13)$$

Where R_b is the stochastic return on bonds; a is the share of equity and $(1-a)$ the share of bonds in the funded pillar.

For a fixed λ, the following expected utility is maximized with respect to a:

$$EU(C) = E\big(1 + \lambda a R_e + \lambda(1-a)R_b + (1-\lambda)R_p\big) - bE\left[exp\left(-d\big(1 + \lambda a R_e + \lambda(1-a)R_b + (1-\lambda)R_p\big)\right)\right] \quad (14)$$

The optimal fraction a^* of equity to be invested in the funded pillar must satisfy the following first-order condition:

$$\frac{dEU}{da^*} = \lambda\mu_e - \lambda\mu_b - bd\lambda E\big[exp\big(-d(1+(1-\lambda)R_p\big)exp\big(-d\lambda a(R_e - R_b)\big)(R_e - R_b)\big] = 0 \quad (15)$$

3. The data and summary statistics

In this section we provide country specific summary statistics for eight countries; the US, Canada, Australia, the UK, France, Germany, Sweden and Japan.

In view of the long run nature of pension systems, we consider 10 year holding periods and our data set covers the time span 1897-2016. Since the time period from 1897 to 2016 cover exceptional events such as years of depression and two world wars, we do consider two additional time periods respectively starting from 1947 and from 1977.

We therefore consider 3 time periods: 1897-2016, 1947-2016 and 1977-2016. Concerning our first period covering 1897-2016, due to the non-availability of data for early years for Canada, UK, and Japan the period considered for these three countries is shorter and covers 1917-2016. All figures for all countries and time periods are obtained from our own calculations using sets of data we constructed starting from raw data on equity indices and bonds prices collected from Global Financial Database.

The equity returns for each country are represented by returns on the most representative equity index for that country during the overall period. Bonds markets returns are approximated by the return on the 10-year government bonds.

Concerning the return on Paygo asset, since data on wage growth are not available for most countries over the time span from 1897 to 2016, the aggregate wage income is approximated by GDP growth. Following the same method of Dutta et al. (2000, 2002), Jermann (1999) in most his long run computations, Matsen and Thogersen (2004), Demesnil (2006) and Knell (2010). The GDP and inflation data are collected from Global Financial Database.

Table 1: Total Return on Equity and Paygo Asset 1897, 1917 - 2016

Country	Equity		Unfunded		$\sigma_{e,p}$	$\rho_{e,p}$
	Mean	St. dev.	Mean	St. dev.		
Australia	7.21%	15.36%	3.23%	4.35%	-1.35	-0.20
Canada	5.52%	9.54%	3.47%	6.54%	-2.03	-0.33
France	2.59%	20.56%	1.49%	10.50%	9.07	0.42
Germany	2.37%	47.26%	0.96%	36.80%	82.30	0.47
Japan	3.68%	21.29%	0.51%	28.58%	4.43	0.73
Sweden	5.53%	16.43%	2.90%	5.13%	-0.55	-0.06
UK	5.38%	12.37%	2.21%	3.76%	1.92	0.41
US	5.89%	13.90%	2.94%	5.00%	0.91	0.13

Period: 1897 – 2016 for Australia, France, Germany, Sweden and USA. Period: 1917 – 2016 for Canada, Japan and UK.

Table 2: Total Return on Equity and Paygo Asset 1947 - 2016

Country	Equity		Unfunded		$\sigma_{e,p}$	$\rho_{e,p}$
	Mean	St. dev.	Mean	St. dev.		
Australia	5.81%	15.48%	3.74%	4.66%	0.71	0.10
Canada	5.63%	10.11%	3.55%	5.77%	-0.05	-0.01
France	5.28%	18.36%	3.10%	5.56%	-1.27	-0.12
Germany	9.65%	33.20%	4.26%	12.25%	11.04	0.27
Japan	5.52%	15.09%	3.47%	10.58%	10.02	0.63
Sweden	7.82%	14.56%	3.13%	3.59%	-2.98	-0.57
UK	5.44%	14.79%	2.16%	3.34%	2.75	0.56
US	6.67%	12.71%	2.76%	3.77%	2.53	0.53

Period 1947 – 2016.

Table 3: Total Return on Equity and Paygo Asset 1977 - 2016

Country	Equity		Unfunded		$\sigma_{e,p}$	$\rho_{e,p}$
	Mean	St. dev.	Mean	St. dev.		
Australia	7.00%	16.47%	2.82%	2.28%	0.52	0.14
Canada	6.05%	6.49%	2.18%	2.62%	0.66	0.39
France	7.78%	19.04%	1.85%	3.15%	5.49	0.92
Germany	10.77%	33.94%	1.73%	2.36%	0.41	0.05
Japan	3.39%	16.90%	1.64%	5.67%	5.66	0.59
Sweden	10.61%	12.09%	2.39%	2.51%	-0.41	-0.13
UK	7.30%	15.12%	1.88%	3.90%	5.06	0.86
US	6.40%	7.54%	2.04%	2.21%	0.41	0.25

Period: 1977 – 2016.

Note: figures for all Tables 1, 2 and 3 are obtained from own calculations. The annualized mean on equity or the paygo asset is obtained by computing the sample mean of the [log(10-year holding period return)]/10. The annualized standard deviation is defined as the sample standard deviation of the [log (10-year holding period excess return)]/$\sqrt{10}$.

Tables 1, 2 and 3 report the empirical estimates for all periods for the means, standard deviations, covariances and correlation coefficients of the annualized 10-year holding-period for equity indices and Paygo asset returns for all eight countries.

The means and standard deviations of equity returns are replicated in the first and second columns. The third and fourth columns report the Paygo asset means and standard deviations. The last two columns of the table report covariances and correlation coefficients between real equity returns and real Paygo asset returns.

As expected, equity display higher returns but also higher standard deviations as compared to Paygo asset returns. There are however significant differences in the risk return profile across countries.

Table 1 covering the period 1897-2016 shows that across our sample of countries, Australia displays the highest return on equity of 7.21% with a relatively low standard deviation of

15.36%. The US and Canada have both relatively higher return on equity and low standard deviations. Canada displays the lowest correlation of equity and Paygo asset returns, we therefore expect the share of unfunded system to be higher for this country. With low correlations and low covariances, the hedging benefits of a Paygo system are higher.

Germany displays the lowest return on equity of 2.37% with the highest standard deviation of 47.26%; this must be due to the significant world war impact on the country risk return profile. The return on equity for Germany increases significantly as we consider recent time periods in Tables 2 and 3.

More generally, equity returns are higher across countries as we consider more recent time spans. And the returns on the Paygo asset are the highest for the time period from 1947 to 2016.

Table 4: Total Return on Bonds 1897, 1917 - 2016

Country	Mean	St. dev.	$\sigma_{b,e}$	$\sigma_{b,p}$	$\rho_{b,e}$	$\rho_{b,p}$
Australia	1.92%	3.99%	7.66	-1.92	0.40	-0.35
Canada	2.21%	2.70%	2.06	-3.51	0.25	-0.63
France	-2.14%	9.31%	37.06	13.55	0.61	0.44
Germany	0.30%	8.35%	63.21	86.36	0.51	0.89
Japan	0.51%	5.17%	14.97	24.69	0.43	0.53
Sweden	1.73%	2.64%	5.90	-0.61	0.43	-0.14
UK	2.30%	3.53%	10.93	0.16	0.79	0.04
US	1.17%	2.45%	4.63	-1.78	0.43	-0.46

Period: 1897 – 2016 for Australia, France, Germany, Sweden and USA. Period: 1917 – 2016 for Canada, Japan and UK.

Table 5: Total Return on Bonds 1947 - 2016

Country						
	Mean	St. dev.	$\sigma_{b,e}$	$\sigma_{b,p}$	$\rho_{b,e}$	$\rho_{b,p}$
Australia	1.89%	4.94%	9.08	-1.90	0.38	-0.26
Canada	1.90%	2.62%	0.02	-2.84	0.00	-0.60
France	0.60%	5.36%	7.46	-6.78	0.24	-0.72
Germany	3.39%	1.19%	-8.67	-2.02	-0.70	-0.44
Japan	0.92%	4.60%	-4.39	1.37	-0.20	0.09
Sweden	1.86%	2.92%	8.03	-1.64	0.60	-0.49
UK	2.01%	3.89%	14.79	0.75	0.81	0.18
US	1.48%	2.12%	0.51	-0.89	0.06	-0.35

Period: 1947 – 2016.

Table 6: Total Return on Bonds 1977 - 2016

Country						
	Mean	St. dev.	$\sigma_{b,e}$	$\sigma_{b,p}$	$\rho_{b,e}$	$\rho_{b,p}$
Australia	4.75%	3.22%	-5.97	0.62	-0.36	0.27
Canada	3.35%	2.18%	0.70	0.92	0.16	0.51
France	3.90%	2.03%	-3.64	0.02	-0.30	0.01
Germany	3.56%	0.75%	-6.29	0.10	-0.78	0.18
Japan	3.31%	1.59%	4.26	2.54	0.50	0.89
Sweden	3.69%	2.48%	-0.34	0.92	-0.04	0.46
UK	4.60%	1.86%	6.89	2.25	0.78	0.98
US	2.64%	1.74%	1.16	0.24	0.28	0.20

Period: 1977 – 2016.

Tables 4, 5 and 6 report long term summary statistics for bonds. As for previous tables, figures are obtained from own calculations. The first and second columns respectively report bonds real returns and standard deviations. The third column reports the covariance between bonds and equity and the fourth column the covariance between bonds and the Paygo asset. The correlation coefficients are replicated in the fifth and sixth columns.

These figures show that bonds have displayed lower real returns with significant standard deviations. The correlations between bonds and equity are also positive for all countries.

Table 4 shows that real returns on bonds are the lowest for France when we consider data from 1897 to 2016, with a negative value of -2.14%; France also displays the highest standard deviation across countries reaching 9.31%. During this same time period, the returns on bonds are the highest for the UK and Canada and also display lower standard deviations for both countries. The correlations between bonds and equity are positive for all countries and the correlations between bonds and Paygo are negative for the US, Canada, Sweden and Australia.

The returns on bonds increase as we consider more recent time periods in Tables 5 and 6 and are the highest from 1977 to 2016.

4. Empirical estimations of optimal funding

In this section we provide empirical estimations of the optimal mix of Paygo and funded systems as well as the optimal portfolio allocation for the eight countries in the case of Mean Variance then Linex preferences.

4.1. Optimal levels of funding in a Mean Variance Framework

Table 7: Risk Aversion and Optimal Funding 1897, 1917 - 2016

	$\gamma = 10$	$\gamma = 20$	$\gamma = 40$
Australia	100%	59%	24%
Canada	81%	23%	0%
France	26%	10%	2%
Germany	0%	0%	0%
Japan	0%	0%	0%
Sweden	96%	41%	14%
UK	100%	100%	65%
US	100%	66%	29%

Period: 1897 – 2016 for Australia, France, Germany, Sweden and USA. Period: 1917 – 2016 for Canada, Japan and UK.

Table 8: Risk Aversion and Optimal Funding 1947 - 2016

	$\gamma = 10$	$\gamma = 20$	$\gamma = 40$
Australia	78%	36%	15%
Canada	100%	51%	13%
France	44%	17%	3%
Germany	48%	22%	9%
Japan	100%	65%	29%
Sweden	100%	67%	26%
UK	100%	100%	56%
US	100%	100%	87%

Period: 1947 – 2016.

Table 9: Risk Aversion and Optimal Funding 1977 - 2016

	$\gamma = 10$	$\gamma = 20$	$\gamma = 40$
Australia	100%	74%	37%
Canada	100%	100%	100%
France	100%	100%	74%
Germany	79%	39%	20%
Japan	98%	55%	33%
Sweden	100%	100%	100%
UK	100%	100%	100%
US	100%	100%	100%

Period: 1977 – 2016.

Tables 7, 8 and 9 report optimal values obtained from equation (5) for different levels of risk aversion γ, we consider degrees of risk aversion of the same level as Dutta et al (2000). Depending on the country and the degree of risk aversion, the optimal share of funding takes all values from 0% to 100%. The optimal level of funding λ^* also depends on the covariances and the magnitude between the risk-return profile of equity and Paygo asset.

Table 7 shows that due to France risk return profile of Paygo and equity, it is optimal for this country to have relatively larger fractions of unfunded pension system. The correlations between Paygo and equity are negative for Canada and Sweden; this explains that there is more scope for unfunded pension system in these countries. With low correlations and low covariances, the hedging benefits of a Paygo system are higher.

Australia also displays considerably low correlation but also lower optimal fraction of Paygo system. This is due to the relatively high return and low variance profile of the equity market in this country. In the specific case of Germany where the equity market displays both low returns and significantly high standard deviations, there is no scope for funded pension system even with lower degrees of risk aversion.

As we consider more recent time spans in tables 7, 8 and 9 the optimal fraction of funded pillar becomes higher as a natural consequence of generally higher returns on bonds and equity.
It is also interesting to confront the optimal obtained fractions to the ones that are prevailing in reality for the various countries; for instance, the fractions of Paygo and funded systems are respectively of 95% - 5% for France, 63% - 37% for the UK and 64% - 36% for the US; see OECD Pensions Outlook 2016.

The optimal portfolio allocation in funded systems

Table 10: Portfolio allocation - Share of equity in the funded pillar 1897, 1917 - 2016

Country	Funding ratio		
	λ = 0.25	λ = 0.5	λ = 0.75
Australia	100%	100%	100%
Canada	100%	100%	100%
France	100%	100%	100%
Germany	100%	81%	38%
Japan	100%	100%	100%
Sweden	100%	100%	100%
UK	100%	100%	100%
US	100%	100%	100%

Period: 1897 – 2016 for Australia, France, Germany, Sweden and USA. Period: 1917 – 2016 for Canada, Japan and UK.

Table 11: Portfolio allocation - Share of equity in the funded pillar 1947 - 2016

Country	Funding ratio		
	λ = 0.25	λ = 0.5	λ = 0.75
Australia	100%	100%	100%
Canada	100%	100%	100%
France	100%	100%	100%
Germany	100%	100%	72%
Japan	100%	100%	100%
Sweden	100%	100%	100%
UK	100%	100%	100%
US	100%	100%	100%

Period: 1947 – 2016.

Table 12: Portfolio allocation - Share of equity in the funded pillar 1977 - 2016

Country	Funding ratio		
	λ = 0.25	λ = 0.5	λ = 0.75
Australia	100%	100%	94%
Canada	100%	100%	100%
France	100%	100%	100%
Germany	100%	100%	80%
Japan	41%	11%	1%
Sweden	100%	100%	100%
UK	100%	100%	100%
US	100%	100%	100%

Period: 1977 – 2016.

Table 10, 11 and 12 report optimal values a^* obtained from equation (9) for a given level of funding λ for different levels of risk aversion γ. Optimal values a^* are truncated at 100%

Depending on the level of funding and degree of risk aversion the optimal fraction of equity in the funded pillar ranges from 1% to 100%. For low degrees of risk aversion and low levels of funding there is no scope for bonds in the optimal portfolio. However, as the level λ of funding increases alongside with the degree of risk aversion γ, there is more room for bonds in the funded pillar. This is due to the fact that when the part of unfunded pension is low, bonds represent another low risk alternative and another risk hedge to be held and this in spite of the fact that bonds generally display lower returns than unfunded pension.

In the specific case of Germany, it is optimal to include bonds in the funded pillar even for relatively low γ and λ this is due the significantly high standard deviations and lower returns on equity for this country for the earliest time period, and to higher equity standard deviations and low bonds standard deviations during the second and third time periods.

In the case of Japan, it becomes very interesting to include bonds when we consider the period from1977 to 2016, this is due to the quiet low returns on equity during this time period.

In most cases however, portfolios in the funded pillar should be optimally invested in equity rather than a mix of bonds and equity. Bonds seem to be dominated by a combination of Paygo asset and equity.

4.2. Optimal levels of funding in a Linear Exponential preferences framework

In this section we provide an empirical estimation of the optimal mix of funded and unfunded pension systems in the case of the Linex utility function.
Table 13, 14 and 15 report optimal values obtained from equation (12) for different levels of risk parameter d. Parameters b and d of the Linex utility function can be read in terms of attitude towards risk. Coefficient d relates to asymmetry and tails of returns distribution while coefficient b accounts for a more traditional notion of risk aversion (Capocci et al 2006; Cavenaile et al 2011).

A high value of coefficient b reflects a general lower risk tolerance as a weight given to risk in the utility function. While the higher d the higher the weight attributed to the possibility of negative outcomes; a high value of d reflects a fear of extreme risks whilst a low value of d reflects the fear of frequent and small risks. The parameter d is to be interpreted as relative riskiness and b as the degree of tolerance to this riskiness.

The optimal fractions to be allocated to equity are obtained by introducing the empirical counterpart of equation (12), the arithmetic mean is used as an empirical estimate of the expectation. The First Order condition in the case of Linex preferences can therefore be written as:

$$\mu_e - \mu_p - bd\frac{1}{N-1}\sum_{i=1}^{N}\left[exp\left(-d(1+R_{pi}\right)exp\left(-d\lambda\left(R_{ei}-R_{pi}\right)\right)\left(R_{ei}-R_{pi}\right)\right] = 0 \qquad (16)$$

Table 13: Risk Parameter d and Optimal Funding 1897, 1917 - 2016

	d = 0.1	d = 0.2	d = 0.3	d = 0.4
Australia	3.20%	2.57%	2.07%	1.68%
Canada	1.64%	1.04%	1.04%	0.83%
France	0.89%	0.71%	0.57%	0.46%
Germany	1.12%	0.87%	0.65%	0.47%
Japan	2.52%	1.99%	1.55%	1.20%
Sweden	2.12%	1.70%	1.36%	1.10%
UK	2.53%	2.02%	1.61%	1.30%
US	2.37%	1.90%	1.53%	1.24%

Period: 1897 – 2016 for Australia, France, Germany, Sweden and USA. Period: 1917 – 2016 for Canada, Japan and UK.

Table 14: Risk Parameter d and Optimal Funding 1947 - 2016

	d = 0.1	d = 0.2	d = 0.3	d = 0.4
Australia	1.63%	1,28%	1,00%	0,79%
Canada	1,64%	1,29%	1,01%	0,79%
France	1,72%	1,35%	1,05%	0,82%
Germany	4,26%	3,35%	2,63%	2,07%
Japan	1,62%	1,27%	1,00%	0,78%
Sweden	3,70%	2,90%	2,27%	1,78%
UK	2,59%	2,03%	1,59%	1,25%
US	3,09%	2,43%	1,90%	1,49%

Period: 1947 – 2016.

Table 15: Risk Parameter d and Optimal Funding 1977 - 2016

	d = 0.1	d = 0.2	d = 0.3	d = 0.4
Australia	3.18%	2.37%	1.73%	1.23%
Canada	2.94%	2.19%	1.59%	1.13%
France	4.51%	3.36%	2.44%	1.73%
Germany	6.86%	5.11%	3.71%	2.62%
Japan	1.33%	0.99%	0.72%	0.51%
Sweden	6.24%	4.65%	3.38%	2.40%
UK	4.11%	3.06%	2.23%	1.58%
US	3.31%	2.47%	1.79%	1.27%

Period: 1977 – 2016.

The optimal fractions λ^* to be allocated to the funded pillar in the case of Bell Linex preferences are reported in tables 13, 14 and 15. As expected, for a given b the optimal fraction λ^* to be allocated to the funded pillar is decreasing in d for all countries and across all time periods. This is consistent with some stylized facts of equity returns among which asymmetry and heavy tails. The fraction to be allocated to the funded pillar tend to be lower for earlier time periods, this is due to higher risks during these periods. Similarly, for a fixed d the higher parameter b the lower would be the optimal fraction λ^* of national wage income to be allocated to the funded pillar. This is a consequence of equity generally bearing more risk than the Paygo asset.

Depending on the country and the attitude towards asymmetry and heavy tails, the optimal share of funding ranges from 0.46% to 6.86%.

It is important to highlight that obtained figures are lower than the ones obtained in the case of Mean Variance preferences and that this result is robust to times periods. This is due to the fact than when we factor in the attitude towards some stylized facts of equity returns among which asymmetry and heavy tails, the optimal fraction to be allocated to equity and therefore to the funded pension system tends to be smaller.

5. Concluding remarks

This paper has shed a new light on the optimal portfolio mix of funded and unfunded pension systems. It has provided an analytical portfolio framework within a mean variance and Linex settings. It has captured asymmetry and tails events typical to equity markets and provided empirical estimations for a sample of eight OCDE countries for various time periods going back to 1897.

Capturing the diversification and hedging properties of the Paygo system as a "quasi-asset", a Paygo even with lower returns can be desirable to the representative individual depending on equity/ Paygo risk return profiles and levels or risk aversion. For many economies a mix of both pension systems is optimal, and the fraction to be allocated to the Paygo system becomes larger as we take into account asymmetry and tails risks.

Another insightful route to the issue of optimal mix would be to account for the regret sentiment inherent to pension funding using a regret theoretical framework. We will leave this promising point for our future research.

References

Aaron, H.J.,1966. The Social insurance paradox. *Canadian Journal of Economics*, 32, 371-74.

Abel, A.B., Mankiw, N.G., Summers, L.H., Zeckhauser, R.J., 1989. Assessing dynamic efficiency: Theory and evidence. *Review of Economic Studies* 56, 1–20.

Bell, D., 1988. One-switch utility functions and a measure of risk. *Management Science* 34, 1416-1424.

Bell, D., 1995. Risk, return and utility. *Management Science* 41, 23-30.

Bell, D., Fishburn, P.C. (2001). Strong one-switch utility. Management Science 47, 601-604.

Capocci, D., Duquenne, F., Hübner G., 2006. Diversifying using hedge funds: a utility-based approach. Working paper, HEC-ULg Management School.

Cavenaile, L., Coën, A. Hubner G., 2011. The Impact of illiquidity and higher moments of hedge fund returns on their risk adjusted performance and diversification potential. *Journal of Alternative Investments*, 9-29.

Denuit, M., Eeckhoudt, L., Schlesinger, H., 2013. When Ross meets Bell: The linex utility function. *Journal of Mathematical Economics* 49, 177-182.

Diamond, P.A., 1965. National debt in a neoclassical growth model. *American Economic Review* 55, 1026–1050.

De Menil, G., Murtin, F., Sheshinski, E., 2006. Planning for the optimal mix of paygo tax and funded savings. *Journal of Pension Economics and Finance* 5 (1), 1-25.

Dutta, J., Kapur, S., Orszag, J. M., 2000. A portfolio approach to the optimal funding of pensions. *Economics Letters* 69 (2), 201-206.

Feldstein, M.S., 1974. Social Security, Induced Retirement, and Aggregate Capital Accumulation. *Journal of Political Economy*, 82, 905-26.

Feldstein, M.S., 1996. The Missing Piece in Policy Analysis: Social Security Reform, *American Economic Review*, 86, 1-14.

Jermann, U., 1999. Social security and intergenerational, intragenerational and international risk sharing: A comment. Carnegie-Rochester Conference Series in Public Policy 50 205 212.

Knell, M., 2010. The optimal mix between funded and unfunded pension systems when people care about relative consumption. *Economica* 77 (308), 710-733.

Matsen, E., Thogersen, O., 2004. Designing social security. A portfolio choice approach. *European Economic Review* 48, 883-904.

Merton, R. C., 1983. On the role of social security as a means for efficient risk sharing in an economy where human capital is not tradable. in: Bodie, Z., Shoven, J., Eds.: Financial Aspects of the United States Pension System, Chicago University Press.

Merton, R.C, Bodie, Z., 1987. A Marcus Pension plan integration as insurance against social security risk. Bodie, Z., Shoven, J., Wise, D., Eds.: Issues in Pension Economics, Chicago University Press.

Samuelson, P.A, 1975. Optimum social security in a life-cycle growth model. *International Economic Review* 16, 538–544.

Sandvik, B., Thorlund-Petersen, L., 2010. Sensitivity analysis of risk-tolerance. *Decision Analysis*, 313-321.