Estimating LGD with stochastic collateral

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Abstract

This article addresses to the appropriate modeling of loss given default (LGD) for the retail business sector. We assume small or mid-size loans that are assigned in a standardized way and collateralized by residential or commercial property. The focus on this specific type of loans entails two major advantages: Firstly, reduction of complexity is followed by easier-to-grasp methodology and increased handiness of results when comparing with other recent approaches in the field. Secondly, the focussing allows to take into account the characteristic properties of the housing market and its underlying uncertainty and so choose a tailor-made modeling for the collateral. The choice of an exponential Ornstein-Uhlenbeck diffusion as the stochastic process of the collateral combines the desirable features with the charm of analytical solvability which seems to be of advantage as regards to acceptance among practitioners. Further key improvements of this approach are the explicit consideration of loan ranking, the disentanglement of the time of default and the time of liquidation as well as the introduction of liquidation cost.

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1 Introduction and motivation

Loss Given Default (LGD) is one of the key measures when modeling and managing credit risk. It captures the percental loss the bank faces in case of a defaulting obligor. Since 2006, the Basle Committee on Banking Supervision (BCBS) allows banks to use their own rating approaches for the purpose of calculating the required equity for credit collateralization – i.e. a so called *Internal Rating Based Approach* (IRBA). This concept stipulates the idea of expected loss as a product of three factors:

$$E[L] = PD \cdot EAD \cdot LGD.$$

The first factor PD represents the *probability of default* (PD), the second one EAD denotes the amount of unredeemed outstanding debt at the moment the obligor defaults, the *exposure at default* (EAD). The third component yet is the *loss given default* (LGD) as percentage of nonrecoverable debt related to EAD. Banks using the advanced rating approach are allowed to estimate these parameters single-handedly by means of internally developed methods. If one assumes that the EAD component is predictable to a great extent by means of amortization schedules, the problem reduces to an accurate estimation of LGD and PD. For the bank, reliable estimates of each component are important in equal measure: Correctly estimated (and low) values for LGD and/or PD lead to lower expected loss and therefore to lower capital requirements and a reduction of risk capital.

Before proceeding it is worth concretizing the concept of expected loss as introduced in the equation above. Let us consider a portfolio with N debtors. For each of them $i, i = 1, ..., N$, let D_i be the digital random variable that indicates whether the very debtor defaults with possible realizations for D_i to be 1 in case of default and 0 if no default occurs, i.e. D_i is a digital random variable or indicator function. Consequently, the random loss L_i in absolute

terms is

$$L_i = D_i \cdot LGD_i \cdot EAD_i,$$

where LGD_i and EAD_i are the percentage loss and the outstanding debt in case of default of the ith obligor. Taking expectations we receive

$$
\begin{aligned}
E[L_i] &= E[D_i \cdot LGD_i \cdot EAD_i] \\
&= E[D_i \cdot E[LGD_i \cdot EAD_i | D_i]] \\
&= P(D_i = 1) \cdot E[LGD_i \cdot EAD_i | D_i = 1] \\
&\quad + P(D_i = 0) \cdot E[LGD_i \cdot EAD_i | D_i = 0] \\
&= P(D_i = 1) \cdot E[LGD_i \cdot EAD_i | D_i = 1],
\end{aligned}
$$

where for the last line we identified the case of no default with the absence of any loss. Slightly simplifying notation by PD_i for $P(D_i = 1)$ and assuming deterministic exposure at default EAD_i, we obtain

$$E[L_i] = PD_i \cdot E[LGD_i | D_i = 1] \cdot EAD_i.$$

When we compare this to the equation provided by the BCBS, we state that the notation used of the latter implicitly assumes that both the percentage loss and the exposure at default are known with certainty. Deviating from that we assume in this article uncertainty with respect to the LGD and identify the LGD in the BCBS sense with the expected loss given default $E[LGD_i | D_i = 1]$.

Even though the equations and the discussion above stress the importance of LGD for banks, the scientific debate seems to be biased towards an over-intense discussion about default probabilities. One reason for that may be the fact that the conceptual requirements of the BCBS with respect to the loss given default had not been specified that clearly until recently. Another reason could be that in a general framework the modeling of LGD cannot be reasonably done without simultaneously modeling the PD component. There

are a number of empirical articles that indicate a dependence of PD and LGD:
Altman/Resti/Sironi (2001), Altman/Brady/Resti/Sironi (2005), Caselli/
Gatti/Querci (2008) and Acharya/Bharath/Srinivasan (2007) use regres-
sion based models to show that the economic cycle of an industrial sector
or of the whole economy explains both probabilittes of defaults and histori-
cally realized values of loss given default in a significant way. Hu/Perraudin
(2006) use extreme value theory to prove a direct correlation of PD and LGD
in the US bond market. Bade/Rösch/Scheule (2011) investigate corporate
loans and identify a correlation between the default process (i.e. PD) and
the process of recovery values (i.e. LGD) by means of maximum likelihood
methods.

The existing literature concerning theoretical aspects of LGD, however,
restricts itself to a very general and hardly applicable view of the topic. The
theoretical models usually account for the possible dependence of PD and
LGD in one of the following ways: Frye (2000), Dev/Pykhtin (2002), Hille-
brand (2005), van Damme (2011) und Jacobs (2011) model the recovery
rate as one random variable and the assets of the obligor as a second one and
let both of them be driven by one latent factor. Jokivuolle/Peura (2003) and
Pykhtin (2003) choose correlated stochastic processes for the firm value on
the one hand and the value of the collateral on the other hand. The result-
ing formulas are highly complex but still vague for lack of concretion towards
a realistic and practice-oriented type of collateral. Consequently, trying to
catch 'all by one', these approaches end up at the lowest common denomina-
tor. Typically, this common denominator is found to be geometric Brownian
motion which then again can neither satisfy researchers nor practitioners.

As we acknowledge the impossibility to capture the heterogeneity of dif-
ferent LGD estimation problems within one general and still powerful model,
we enter the alternative path of specification: In this paper we focus on one

single but typically quite important portion of a bank's credit portfolio, the part of the retail business where loans are collateralized by residential or commercial property. We look at loans, that are conferred in standard way to an obligor, which typically is represented by a private individual or a small or mid-size company[1].

We now explain how this focus allows us to neglect the phenomenon of correlated PD and LGD by means of economic latent factors described above and thereby jettison part of the methodological over-complexity. This modeling is supported by some recent empirical results of Grunert/Weber (2009) and Grunert (2010): Both articles use default histories of small bank loans to conclude that there is no significant correlation between a bunch of economic indicators and the realized recovery rates. On the other hand, the assumption of independence for the retail business is not at odds with the other studies cited above that confirmed a general correlation of default probability and loss given default: The relationships detected in Altman/Resti/ Sironi (2001), Altman/Brady/Resti/Sironi (2005), Hu/Perraudin (2006) as well as Bade/Rösch/Scheule (2011) relate to defaults of publicly traded bonds. Acharya/Bharath/Srinivasan (2007) derive their insights from some data covering large-cap credit portfolios. Evidently, both types of financing are crucially different to the case of a classical mortgage-backed loan. Only the work of Caselli/Gatti/Querci (2008) address their analysis to small and mid-size bank loans of an Italian bank. The authors again receive some general evidence for a correlation between LGD and macroeconomic factors, yet they explicitly stress that this evidence disappears when choosing only the loans that are collateralized by residential property.

Summarizing, for the special case of a mortgage-backed private loan, the assumption of independence of the local housing market on the one hand and

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[1]Large-size engagements with international companies are excluded, as they are crucially different for being structured in a much more complex way.

the solvency of the single obligor on the other hand, seems to be feasible or at least not a major limitation. But what we earn is much more: We obtain an increased analytic manageability which should also increase acceptance among practitioners considerably. Meanwhile, the model reduction fans out a multitude of possibilities for an adequate modeling of the collateralizing asset. As we focus on residential property, we have a look on mathematical models of real estate markets (commercial and residential).

With respect to the descriptive and empirical level, there is early work of Case/Shiller (1989), Case/Shiller (1990) and Hosios/Pesando (1991). All three emphasize the incompleteness of real estate markets and elaborate the phenomenon of serial correlation to be a key ingredient of an appropriate mathematical model. Furthermore, seasonality seems to play an important role for the studies dealing with local housing prices in Chicago (Case/Shiller) and Toronto (Hosios/Pesando). Englund/Ioannides (1997) affirm these relationships when investigating international data sets. A number of articles tries to ascribe these stylized facts to search-theoretic (see Wheaton (1990), Krainer (2001), Piazzesi/Schneider (2009), Novy-Marx (2009) and Diaz/ Jerez (2013)) and/or behavioristic (see e.g. Hott (2011)) mechanisms.

Despite of these insights the early literature dealing with derivative pricing in the real estate sector are based on the assumption of complete markets and geometric Brownian motion as stochastic model (see Titman/Torous (1989), Buttimer/Kau/Slawson (1997) and Björk/Clapham (2002)). Later models keep geometric Brownian motion as driving diffusive process but introduce equilibrium models to account for market incompleteness (see Geltner/Fisher (2007) and Cao/Wei (2010)). Crawford/Fratantoni (2003) suggest ARIMA- and GARCH- models for a realistic mapping of house price indices. The recent work of Fabozzi/Shiller/Tunaru (2010) again stresses the need of a mathematical model that incorporates serial correlation and

provides new empirical evidence. The model the authors suggest ties in with the approaches of Lo/Wang (1995) and Jokivuolle (1998) who deal with other serially correlated assets. Finally, Fabozzi/Shiller/Tunaru (2012) point out that the property of serial correlation must be regarded as a central requirement when modeling any kind of real estate. The process these authors use and which also Perelló/Sircar/Masoliver (2008) use in a slightly different context with stochastic volatility, is called exponential Ornstein-Uhlenbeck process. Against the background described above, we also adopt this stochastic process for the purpose of modeling the value of the collateralizing residential property.

The basic idea of this article is to use a conceptual analogy of option pricing theory for LGD modeling. More precisely, we interpret the loss profile of a debtor at default as kind of a put option, where the underlying is identified by the vale of the collateral. Our model for LGD estimation shows a number of advantages with respect to practical use that to the best of our knowledge have not been worked out within the literature before: Firstly, we explicitly differentiate between the time of default and the time of liquidation. This separation makes allowance for the fact that the liquidation procedure is preceded by several steps of administrative and/or legal character, which leads to a delay between the time of default and the start of the liquidation (see also Gürtler/Hibbeln (2013)). Secondly, we introduce a cost factor that captures the liquidation efforts that may also affect the amount of loss. This approach acknowledges the requirement of the Basle committee, which provides workout costs to be included in the definition of LGD. Thirdly, our model easily captures the existence of loan-specific rank structures. We regard this to be necessary as also the retail business is often affected by situations where one collateral is used for the securitization of more than one loan. If one or several other creditors are in a superior or in the same rank, this has an immediate effect on the bank's risk position.

Finally, the analytical tractability of our model allows to compute sensitivities of the LGD formula for key parameters in closed form which could prove useful for practitioners when thinking of risk steering.

The remainder of the paper is structured as follows: In section 2 we present the overall modeling framework. Section 3 provides a closed-form solution for the expected loss given default, while section 4 analyzes the derived formula and investigates sensitivities with respect to the parameters of the model. Section 5 concludes.

2   The basic model

We first simplify notation and drop the index i for the remainder of the article, i.e. we write $E[LGD|D = 1]$ for the expected loss given the default of a representative obligor. As mentioned above we build our model based on analogies taken from option pricing theory. For that purpose we need at least one source of uncertainty. An obvious candidate is the value of the collateral which we denote by C. Its future development being uncertain we regard $C = C_t$ as a stochastic process.

We use the following notation: The variable t indicates calendar time. Initially, this can be identified with the starting date of the loan, later it may be any point in time where revaluation of the loss given default is assessed. The future date under consideration where there may or may not be a default is denoted by T_D (time of default). The instructions of the Basle Committee postulate a time horizon of one year concerning the potential loss, i.e. $T_D = 1$. We generalize slightly treating T_D as a variable. As motivated above, for the sake of sufficient practical relevance we further assume that liquidation takes place with some delay at the point in time T_L (time of liquidation) with $t < T_D \leq T_L$. The difference $T_L - T_D$ then is the length of the liquidation

period. It clearly depends on the efficiency of the bank's liquidation division, but may also be driven by strategic considerations. Using historical default data, average values can be determined empirically.

The part of the loan that is unredeemed until time t is termed E_t (Exposure). As we discussed earlier, we reduce complexity by stating that the amortization schedule determines that time-dependent variable to great extent and therefore assuming E_t to be deterministic through time typically of decreasing shape[2]. For example, an installment loan should exhibit a step function with steps of equal height. The outstanding unredeemed part of the loan at time of default, is called exposure at default EAD, i.e. $E_{T_D} = EAD$.

Taking into account that liquidation actions of collateralizing assets are costly for the bank, we introduce the cost parameter k. It represents the average cost of liquidation as a percentage value in relation to the value of the collateral.

Moreover, we equip the model with different rank structures (especially second-tier engagements). Generally, we allow for a senior position of other banks/debtors with a nominal N which is served first when liquidating the collateral[3]. Concerning liquidation, the second-tier rank of the bank represents an additional exposure. Only if the net present value of the liquidation revenues (i.e. discounted to the time of default and net of liquidation costs) exceeds the nominal amount N being senior, the liquidation actions will reduce the bank's loss. On the other hand the bank will only benefit from liquidation revenues until the outstanding loan (and liquidation cost) is cov-

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[2]Exposures of non-deterministic type may be modeled using credit conversion factors (CCF).

[3]For simplification reasons, positions of equal rank are treated as junior positions taking a conservative view. A generalization to more complex rank constellations is though straightforward.

ered, i.e. we have a lower limit of the bank's potential loss at the value of zero. Consequently, the loss profile is linearly decreasing with the value of the collateral for collateral (net present) values between N (last value of full loss) and $N + EAD$ (first value with full recovery).

Finally, as have been done by Jokivuolle/Peura (2003) we assume that beside liquidation additional recovery efforts are made undertaken by the credit-granting bank. If the liquidation revenues do not cover the amount of unredeemed loan at default (EAD), the bank will try to collect the outstanding amount in a different vein. The share of this kind of recovery as a fraction of unredeemed exposure (scarp value) net any recovery cost is termed by γ and represents the residual recovery rate. Again, we assume that this ratio can be derived from historical data and so be accurately estimated in advance. Consequently, we define this parameter to be non-random. The interest rate r represents the bank-internally used refinancing rate. It consists of the risk-free rate plus an additional bank-specific risk premium and is also assumed to be constant here.

With these assumptions, the loss profile LP at default reads as follows:

$$
LP_{T_D} = \begin{cases} EAD, & \text{if } 0 \leq e^{-r(T_L - T_D)}(1-k)C_{T_L} \leq N \\ EAD + N - e^{-r(T_L - T_D)}(1-k)C_{T_L}, \\ & \text{if } N < e^{-r(T_L - T_D)}(1-k)C_{T_L} < EAD + N \\ 0, & \text{if } EAD + N \leq e^{-r(T_L - T_D)}(1-k)C_{T_L}. \end{cases} \tag{2.1}
$$

It is easy to see that the structure of the loss profile looks like the payoff of an option. If we drop the nominal N in a first step, the option type is that of a long put option with strike EAD, i.e. the loss is in-the-money as soon as the underlying (the discounted collateral net cost) falls below the EAD[4]. If

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[4]Indeed, the liquidation payoff has the structure of a short put as one could interpret the liquidation payment at default as the duty to buy the collateral at the price of the EAD. As we are interested in loss profiles, the signs turn around.

we additionally take the nominal into account, we have an offsetting position for collateral values between zero and N, i.e. a short put with strike N. The combined position is that of a bear put spread.

The expected loss given default is given by the expected loss profile related to the total exposure and multiplied by $1 - \gamma$ which accounts for residual recovery:

$$E[LGD_{T_D}] = (1 - \gamma)\frac{1}{EAD}E[LP_{T_D}|\tau = T_D], \qquad (2.2)$$

where we have $E[LGD_{T_D}] = E[LGD|D = 1]$ and τ is the set of all possible times of default. For the special case where $N = 0$ (no senior positions) and $T_D = T_L$ (no liquidation delay) we obtain

$$
\begin{aligned}
E[LGD_{T_D}] &= (1 - \gamma)\frac{1}{EAD}E[\max(EAD - (1-k)C_{T_D}, 0)|\tau = T_D] \\
&= (1 - \gamma)E\Big[\frac{EAD^{NC}}{EAD}|\tau = T_D\Big].
\end{aligned}
\qquad (2.3)
$$

Here, the parameter EAD^{NC} represents the non-collateralized share of the unredeemed loan. For fully uncollateralized loans, we have $EAD^{NC} = EAD$, so that the LGD is only driven by the residual recovery process.

For the remainder of the article, we assume collateralized credit engagements. Treating the exposure at default as non-random, the only random factor influencing the loss profile and therefore the loss given default, is the value of the collateral. Focussing on residential or commercial property, we model the value of the collateral as an exponential Ornstein-Uhlenbeck-process (OU) as motivated above. Moreover, we assume that the market value of the collateral asset can be represented at any point in time $T > t$ as

$$C_T = C_t e^{Y_T}, \qquad (2.4)$$

where Y_T is an OU-process. As noted above, the exponential OU-process was proposed by Fabozzi/Shiller/Tunaru (2012) for the development of real estate property values and captures characteristic features of the housing market:

- Returns of real estate assets fluctuate over time, though they exhibit some mean-reverting trend to a long-term average.

- Returns are serially correlated over time.

Furthermore, the use of this process guarantees the collateral value to stay positive. The dynamics of the return process Y_t are described by the following stochastic differential equation:

$$dY_t = \Big(\frac{d\Psi_t}{dt} + \kappa(\Psi_t - Y_t)\Big)dt + \sigma dW_t. \qquad (2.5)$$

Ψ_t represents the average long-term level of the return process on property. This level is exogenous and depends on several features of real estate markets. It is time-dependent and may show different shapes: one could imagine e.g. a linear trend of the form $\Psi_t = \alpha + \beta t$ or a cyclical (seasonal) trend with $\Psi_t = \alpha_1 \cos(\beta_1 t) + \alpha_2 \sin(\beta_2 t)$. The parameter $\kappa > 0$ determines the speed at which mean-reversion takes place, the parameter $\sigma > 0$ indicates the volatility of the process, while by W_t we denote a standard Brownian motion with zero expectation and variance t. In the basic model, σ is assumed to be constant, extensions to a time-dependent volatility $\sigma = \sigma_t$ are though easy to accomplish. From a methodological point of view, the model is a generalization of the model of Vasicek (1977), which is the standard setup for modeling interest rates with constant long-term mean ($\Psi_t = c$).

At this point of the paper it is important to emphasize the following: We will make use of aspects of option pricing theory, which suggest themselves when looking at the loss profile and its option-like payoff. However, we still derive a traditional expectation under the physical measure. From a conceptual point of view, this has nothing to do with option pricing or a fair value approach for the credit portfolio or loss profile. Instead, we are asked to provide an estimate for the expected loss as one risk measure required by

the Basle Committee[5]. Consequently, the fact that the real estate market is incomplete – which is important when valuing real estate derivatives (see Fabozzi/Shiller/Tunaru (2012)) – does not play a major role here. Mathematically speaking, that means that there is no change of measure or risk premium here. Furthermore, the idea of discounting the future claim to today is not appropriate when thinking of the future loss at default time T_D that can be expected today.

It is straightforward to show that for all $t < T$ the solution of equation (2.5) is given by

$$Y_T = \Psi_T - (\Psi_t - Y_t)e^{-\kappa(T-t)} + \sigma \int_t^T e^{-\kappa(T-s)}dW_s. \qquad (2.6)$$

Obviously, Y_T is a Gaussian process with the following conditional moments:

$$E[Y_T \mid Y_t] = \Psi_T - (\Psi_t - Y_t)e^{-\kappa(T-t)} \qquad (2.7)$$

$$Var[Y_T \mid Y_t] = \frac{\sigma^2}{2\kappa}\Big(1 - e^{-2\kappa(T-t)}\Big). \qquad (2.8)$$

As one should expect, for the long-term limit we obtain

$$\lim_{T\to\infty} E[Y_T \mid Y_t] = \Psi_\infty$$

while the long-term variance is

$$\lim_{T\to\infty} Var[Y_T \mid Y_t] = \frac{\sigma^2}{2\kappa}.$$

We state that even though the expected value can increase without any bound depending on the shape of Ψ, the variance remains limited. Two more interesting features of the process have to do with the parameter κ. We have

$$\lim_{\kappa\to\infty} E[Y_T \mid Y_t] = \Psi_T$$

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[5]Though an isolated view on this measure may be misleading, in combination with other risk measures (e.g. value at risk) the information may nevertheless prove useful.

$$\lim_{\kappa \to \infty} Var[Y_T \mid Y_t] = 0$$

as well as

$$\lim_{\kappa \to 0} E[Y_T \mid Y_t] = \Psi_T - \Psi_t + Y_t$$

$$\lim_{\kappa \to 0} Var[Y_T \mid Y_t] = \sigma^2 (T - t).$$

For an infinite speed of mean-reversion, the return process becomes deterministic. For $\kappa = 0$ however, there is no feedback between the current level of return Y_t and the number Ψ_t, we have a classical Brownian motion plus a deterministic drift component.

We conclude this section by bringing the results from the return to the price level of the collateral. We receive

$$E[C_T \mid C_t] \quad = \quad C_t e^{\mu_Y + \frac{1}{2}\sigma_Y^2} \tag{2.9}$$

$$Var[C_T \mid C_t] \quad = \quad C_t^2 e^{2\mu_Y + \sigma_Y^2} (e^{\sigma_Y^2} - 1), \tag{2.10}$$

where $\mu_Y = \mu_{Y_T}$ and $\sigma_Y^2 = \sigma_{Y_T}^2$ are the moments of the return process Y_T given by equations (2.7) and (2.8). Additionally, we observe the probability that the collateral value C_T exceeds a critical value x is given by

$$P(C_T > x) = \Phi(d), \tag{2.11}$$

with $d = (\ln(C_t/x) + \mu_Y)/\sigma_Y$ and $\Phi(x)$ is the standard normal distribution at point x, i.e.

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} dz.$$

With $\lim_{x \to 0} d = \infty$, we obtain $P(C_T > 0) = 1$, i.e. the value of the collateral cannot become negative.

3 The expected loss given default

In this section we derive an analytical solution for the expected loss given default. With the model setup of the previous section and the related assumptions, the LGD is influenced by a couple of parameters and we can write

$$E[LGD_{T_D}] = E[LGD_{T_D}(EAD, N, C, k, T_D, T_L, \gamma, r, \Psi, \kappa, \sigma)].$$

We start by providing the main result concerning the expected loss given default:

Theorem 3.1 *As of time t, the expected loss given default for an obligor defaulting at time T_D is given by*

$$E[LGD_{T_D}] = (1 - \gamma)\Big[\Big(1 + \frac{N}{EAD}\Big)\Phi(-d) - \frac{N}{EAD}\Phi(-d^*)$$
$$-(1 - k)e^{-r(T_L - T_D)}\frac{C_t}{EAD}e^{\mu_Y + \frac{1}{2}\sigma_Y^2}\Big[\Phi(-(d + \sigma_Y)) - \Phi(-(d^* + \sigma_Y))\Big]\Big],$$
(3.1)

where

$$d = \frac{\ln\frac{C_t}{X} + \mu_Y}{\sigma_Y} \tag{3.2}$$

$$d^* = \frac{\ln\frac{C_t}{X^*} + \mu_Y}{\sigma_Y} \tag{3.3}$$

$$X = \frac{EAD + N}{1 - k}e^{r(T_L - T_D)} \tag{3.4}$$

$$X^* = \frac{N}{1 - k}e^{r(T_L - T_D)} \tag{3.5}$$

$$\mu_Y = \mu_{Y_{T_L}} = \Psi_{T_L} - (\Psi_t - Y_t)e^{-\kappa(T_L - t)} \tag{3.6}$$

$$\sigma_Y^2 = \sigma_{Y_{T_L}}^2 = \frac{\sigma^2}{2\kappa}\Big(1 - e^{-2\kappa(T_L - t)}\Big), \tag{3.7}$$

and $\Phi(\cdot)$ being the cumulative density function of the standard normal distribution.

For the proof, see Appendix A1.

Remark 3.2 *In the special case where $N = 0$, i.e. d.h. the bank is the only senior rank creditor, the formula for the LGD can be simplified as follows:*

$$E[LGD_{T_D}] = (1-\gamma)\Big[\Phi(-d) - (1-k)e^{-r(T_L-T_D)}\frac{C_t}{EAD}e^{\mu_Y+\frac{1}{2}\sigma_Y^2}\Phi(-(d+\sigma_Y))\Big],$$
(3.8)

where d, μ_Y and σ_Y are as above while $X = EAD/(1-k)e^{r(T_L-T_D)}$ is adjusted.

The formulae satisfy an economically interesting homogeneity condition.

Proposition 3.3 *For any $\pi > 0$ we have*

$$E[LGD_{T_D}(\pi EAD, \pi N, \pi C, k, T_D, T_L, \gamma, r, \Psi, \kappa, \sigma)]$$
$$= E[LGD_{T_D}(EAD, N, C, k, T_D, T_L, \gamma, r, \Psi, \kappa, \sigma)],$$
(3.9)

e.g., the expected loss given default is homogeneous of degree zero with respect to EAD, N and C.

The proof is an immediate consequence of equations (3.1)-(3.5). The economical statement of this mathematical finding is that for the LGD only the relationship between the three parameters matters, while absolute values have no influence. This ensures that all obligors are considered in the same way (especially, independent of the firm size).

We investigate the analytic structure of the derived formula a little deeper. For the purpose of a better understanding, we look at some examples based on concrete numbers. First, we analyze graphically, how the rank structure influences the expected LGD. We compare two cases of loans with even characteristics except that the first loan exhibits a senior position of a third party amounting to a nominal of $N = 50$, whereas in the second case this is not the case ($N = 0$). We let the long-term mean of real estate returns be constant at a level of $\Psi_t = 0.06$ and choose the other parameters as follows: $EAD =$
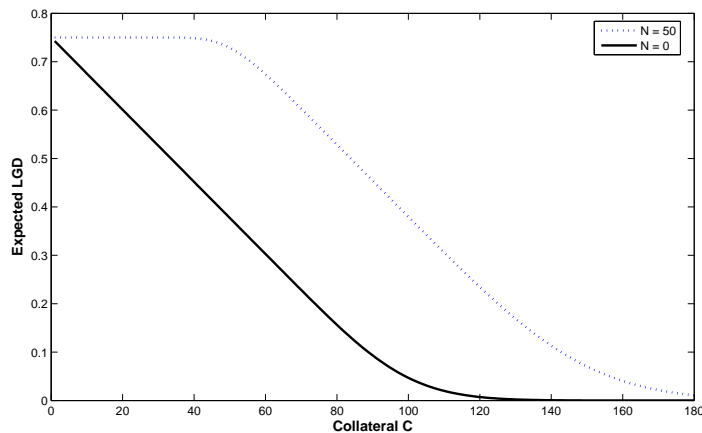
Figure 1: LGD estimates for two different tranches.

$100, T_D = 1, T_L = 1.5, k = 0.05, r = 0.05, Y = 0.04, \gamma = 0.25, \kappa = 2, \sigma = 0.3$. The expected LGD depending on today's value of the collateral is depicted in Figure 1 for both cases.

As expected, in both cases, the expected LGD is decreasing with increasing values of the collateral. The main difference, however, is that for the first scenario the existence of the third-party nominal ($N = 50$) shifts the LGD values considerably upwards. As long as the collateral only suffices to serve the senior position, there is no recovery from the liquidation. In our model, the residual recovery process characterized by the parameter $\gamma = 25\%$ limits the loss to a maximum of 75%.

We now relax the assumptions assuming a linear trend for the long-term mean of returns on residential property, i.e. $\Psi_t = \alpha + \beta t$. Again, we choose two different scenarios: one scenario where the housing market is prospering, the other scenario where it diminishes. The values for the scenarios are given in Table 1.

The other parameters that are equal for both cases are: $T_D = 1, EAD =$

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Scenarios for the real estate market		
Prospering market	$\alpha = 0.02$	$\beta = 0.08$
Diminishing market	$\alpha = 0.02$	$\beta = -0.08$

Table 1: Two different scenarios for the real estate market used in the next example.

$100, N = 50, k = 0.05, r = 0.05, Y = 0.04, \kappa = 1, \sigma = 0.3$ and $\gamma = 0.25$. We combine the two scenarios with two liquidation periods of different length: in the first case liquidation is supposed to take place six months after default ($T_L = 1.5$), in the second case 18 months after default ($T_L = 2.5$). The corresponding length of the liquidation period $T_L - T_D$ is 0.5 and 1.5 years, respectively. Figure 2 shows the expected LGD over the collateral value for these different speeds of liquidation assuming the positive (left graph of the figure) and negative (right graph) development of the housing market, respectively.

Again, due to the nominal N, there is no liquidation revenue for the bank as long as the collateral value lies below the claims of the senior parties. It is the residual recovery process that reduces the loss to a level of 75% ($(1 - \gamma) \cdot EAD$). More interestingly, the effect of the two scenarios becomes clear, not only in terms of the absolute level of the curves: For the assumption of a long-term prospering market, the delay of liquidation proceeds may be beneficial as higher revenues are earned from liquidation of a more valuable collateralizing asset. For the diminishing real estate market, the increased liquidation period is however disadvantageous in all respects.

Finally, we analyze the effect of different cost factors on the LGD value. We also keep today's value of the collateral flexible and focus on the scenario of an upturn in the market for residential property ($\alpha = 0.02, \beta = 0.08$). In our example, the cost factor may vary between 1% and 20%, the other
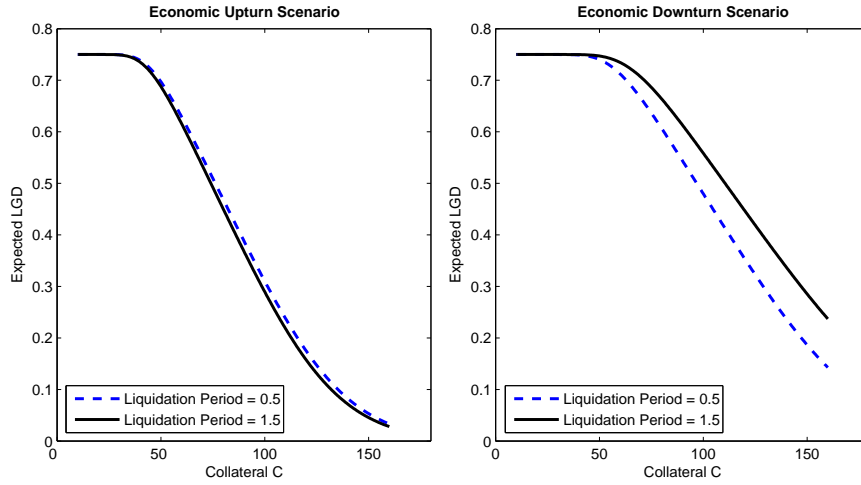
Figure 2: LGD estimates for two different market scenarios and two different liquidation periods.

parameters are $EAD = 100, N = 50, T_D = 1, T_L = 2, r = 0.05, Y = 0.04, \kappa = 2, \sigma = 0.3$ and $\gamma = 0.25$. We obtain a three-dimensional surface which is displayed in Figure 3.

One clearly observes that the cost factor k influences the expected LGD in a negative way which is in line with our intuition. The effect is most pronounced for collateral values close to the EAD. If the collateral value is too small to cover the nominal N of the senior parties, the cost factor is of minor relevance, the same holds true when the collateral value is far beyond the EAD.
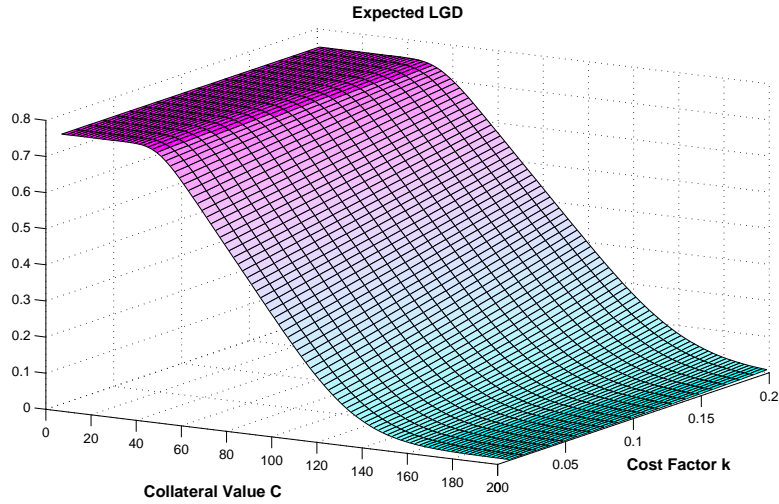
Figure 3: LGD estimates for different combinations of collateral values C and cost factors k.

4 Sensitivities

It is crucial for any bank to quantify, evaluate and control the default risk of their credit portfolio. As any information that serves this purpose should be valuable, it might prove useful to know about the influence of single parameters on the expected loss or the expected LGD, respectively. Clearly, the value of this information is evident when thinking of designing new credit contracts: The bank then may quantify how contract parameters as for example the value of the collateral influences the expected LGD. But also for existing engagements there is a multitude of applications, e.g. the knowledge about the influence of liquidation efficiency and cost factors on the LGD may lead to consequences on liquidation policy.

By means of our model we are able to quantify key sensitivities of the LGD formula with respect to its parameters in closed form. While one could

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investigate influence of any model parameter, we focus on an (in our opinion) appropriate choice of the most important ones.

The first sensitivity we derive is the one with respect to the value of the collateral C, which – in analogy to option pricing theory – we name *Delta*:

Proposition 4.1 *For the* Delta *of the expected LGD (Theorem 3.1) it holds:*

$$
\begin{aligned}
\frac{\partial E[LGD_{T_D}]}{\partial C} \;=\; & (1-\gamma)\Big[-\frac{n(d)}{\sigma_Y \cdot C} + \frac{N}{EAD \cdot \sigma_Y \cdot C}(n(d^*) - n(d)) \\
& -(1-k)e^{-r(T_L-T_D)}\frac{e^{\mu_Y+\frac{1}{2}\sigma_Y^2}}{EAD}\Big[\Phi(-(d+\sigma_Y)) - \Phi(-(d^*+\sigma_Y)) \\
& +\frac{1}{\sigma_Y}(n(d^*+\sigma_Y) - n(d+\sigma_Y))\Big]\Big],
\end{aligned}
\tag{4.1}
$$

where $\Phi(\cdot)$ are the cumulative density function and $n(\cdot)$ the density function of the standard normal distribution.

For the proof, see Appendix A2.

Remark 4.2 *For the special case $N = 0$ (no other senior positions), the formula for the* Delta *can be written as*

$$
\frac{\partial E[LGD_{T_D}]}{\partial C} = (1-\gamma)\Big[-\frac{n(d)}{\sigma_Y \cdot C} - (1-k)e^{-r(T_L-T_D)}\frac{e^{\mu_Y+\frac{1}{2}\sigma_Y^2}}{EAD}\Big[\Phi(-(d+\sigma_Y)) - \frac{n(d+\sigma_Y)}{\sigma_Y}\Big]\Big],
\tag{4.2}
$$

where we have $X = EAD/(1-k)e^{r(T_L-T_D)}$.

For a graphical illustration we use the same example as in the previous section (Figure 1) where we had the case with a nominal $N = 50$ as well as without any nominal $N = 0$. For ease of interpretation, we exclude the residual recovery actions here setting $\gamma = 0$. The other parameter values were $EAD = 100, T_D = 1, T_L = 1.5, k = 0.05, r = 0.05, Y = 0.04, \kappa = 2, \sigma = 0.3$ and $\Psi_t = 0.06$. Figure 4 depicts the *Delta* values for the two cases over the collateral value.
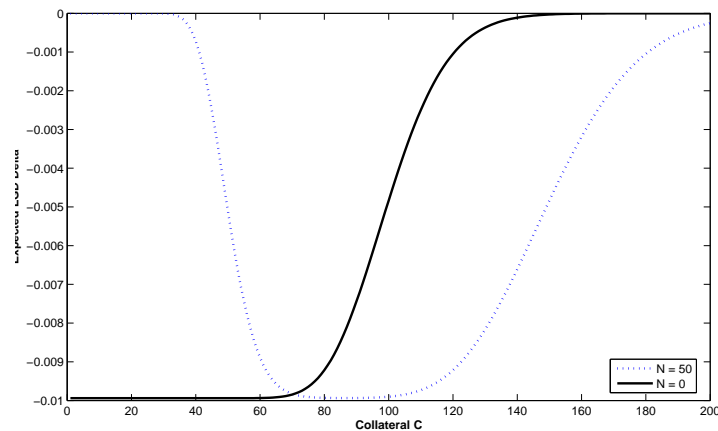
Figure 4: LGD Delta estimates for two different tranches.

Obviously, if the lending bank is in senior position ($N = 0$), the delta is at a constant level of minus one percentage point for very low collateral values. This value results from the fact that an increase of the collateral value by one unit also reduces the expected loss in total amounts by (almost) one unit. Divided by the EAD, which is 100 in this example, we receive the effect on the expected loss rate there to be -1%. With the analogy to option pricing stated above, recall that the payoff profile of LGD without nominal N resembles the position of a long put. For a collateral value close to zero, this option is far in-the-money. With increasing value of the collateral, the reduction on the expected loss is lower and finally tends to a level of zero for high collateral values. Here, the put option is far out-of the money, so that a further increase of the underlying value has nearly no effect on the value of the option.

If a third-party senior position exists with a nominal, there is an additional effect. For very low collateral values, the effect on the expected LGD is close to zero: Here, an increase of the collateral's value only helps the first-

tier creditors because the probability of any liquidation revenue that exceeds the nominal is rather low. Coming closer to the nominal, the potential loss reduction strengthens more and more. After exceeding the nominal value we have a shifted version of the first case described above: from the maximum effect of -1.00%, a further increase of the collateral's value will first result in weaker effects and finally fade out. Once more making use of option theory terminology, the LGD profile can be described as a combination of a short put with strike $N = 50$ and a long put with strike $N + EAD = 150$ (i.e. a bear put spread): For low collateral values, both options are far in-the-money and the effects (+1% for the short put and -1% for the long put) offset each other. Approaching the strike of the short put, the potential effect of the short put decreases from the value of +1% to zero while the long put is still far in-the money with a delta of -1% dominates, the sum of the two therefore evolves from 0 to -1%. With the collateral's value increasing further, the short put loses any influence and we have the shape of the long put's delta as in the example without any senior third-party.

The second sensitivity we investigate more deeply is the partial derivative of the expected loss given default with respect to the time of liquidation T_L:

Proposition 4.3 *The* Tau *of the expected LGD is given by:*

$$
\begin{aligned}
\frac{\partial E[LGD_{T_D}]}{\partial T_L} =\ & (1-\gamma)\Big[-n(d)\cdot[A-(\ln C^*-r(T_L-T_D)+\mu_Y)\cdot B] \\
& +\frac{N}{EAD}\Big[-n(d)\cdot[A-(\ln C^*-r(T_L-T_D)+\mu_Y)\cdot B] \\
& +n(d^*)\cdot[A-(\ln C^{**}-r(T_L-T_D)+\mu_Y)\cdot B]\Big] \\
& -(1-k)\frac{C}{EAD}e^{-r(T_L-T_D)+\mu_Y+\frac{1}{2}\sigma_Y^2}\Big[\Big(-r+\frac{\partial\Psi_{T_L}}{\partial T_L}+\kappa(\Psi_t-Y_t)e^{-\kappa(T_L-t)} \\
& +\frac{1}{2}\sigma^2 e^{-2\kappa(T_L-t)}\Big)\Big(\Phi(-(d+\sigma_Y))-\Phi(-(d^*+\sigma_Y))\Big) \\
& -n(d+\sigma_Y)\cdot[A-(\ln C^*-r(T_L-T_D)+\mu_Y)\cdot B+\frac{\partial\sigma_Y}{\partial T_L}] \\
& +n(d^*+\sigma_Y)\cdot[A-(\ln C^{**}-r(T_L-T_D)+\mu_Y)\cdot B+\frac{\partial\sigma_Y}{\partial T_L}]\Big]\Big], \quad (4.3)
\end{aligned}
$$

where

$$
A = \frac{-r+\frac{\partial\Psi_{T_L}}{\partial T_L}+\kappa(\Psi_t-Y_t)e^{-\kappa(T_L-t)}}{\sqrt{\frac{\sigma^2}{2\kappa}(1-e^{-2\kappa(T_L-t)})}}, \quad (4.4)
$$

$$
B = \frac{e^{-2\kappa(T_L-t)}}{2\sigma}\Big(\frac{2\kappa}{1-e^{-2\kappa(T_L-t)}}\Big)^{\frac{3}{2}}, \quad (4.5)
$$

$$
C^* = \ln\frac{C(1-k)}{N+EAD}, \qquad C^{**} = \ln\frac{C(1-k)}{N}, \quad (4.6)
$$

as well as

$$
\frac{\partial\sigma_Y}{\partial T_L} = \frac{\sigma}{2}\sqrt{2\kappa}\frac{e^{-2\kappa(T_L-t)}}{\sqrt{1-e^{-2\kappa(T_L-t)}}}, \quad (4.7)
$$

and where $\Phi(\cdot)$ are the cumulative density function and $n(\cdot)$ the density function of the standard normal distribution.

For the proof, see Appendix A3.

Figure 5 depicts the *Tau* values for the two scenarios over the collateral value. Note that we choose the nominal position N and the residual recovery rate γ to be zero in order to ease the interpretation. The other parameter

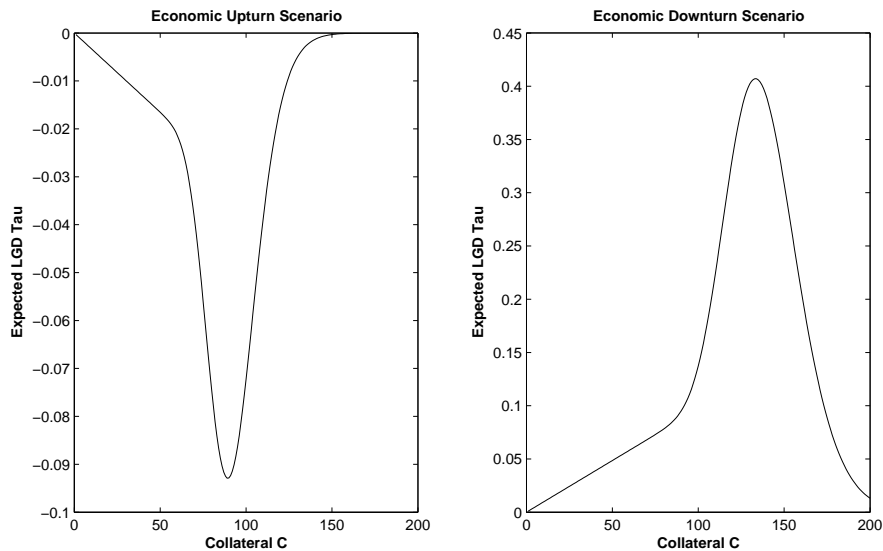values are $EAD = 100, T_D = 1, T_L = 2.5, k = 0.05, r = 0.05, Y = 0.04, \kappa = 2$ and $\sigma = 0.3$.

Figure 5: LGD Tau estimates for two different market scenarios.

As already suggested by Figure 2, the effect observable in this example heavily depends on the choice of the long-term mean Ψ_{T_L}, yielding a decrease in loss and hence a negative *Tau* for an upward-tending real estate market and vice versa. For both scenarios, the absolute value of the partial derivative takes a maximum value in the vicinity of the EAD. As goes along with intuition, for highly collateralized engagements ($C >> EAD$), the fluctuations of the collateral value induced by the prolongation of time play a dwindling role.

5 Conclusion

In this paper, we derived closed-form solutions for the loss given default (LGD) quotas. We therefor focussed on a selected subcase of LGD where loans of the retail sector are collateralized by residential or commercial real estate property. In our opinion, this specification is feasible and reasonable in more than one regards: first, it is one of the most important cases for the retail sector, second, it facilitates the disentanglement of PD and LGD which makes computations tractable and third, it allows for a case-specific and powerful modeling of the collateral value. To the best of our knowledge, the (careful) use of option pricing analogies with respect to methodology and interpretation of sensitivities has not been worked out before in that clearness. For sake of broad applicability, we incorporated typical problems of practitioners' workaday life like cost factors, residual recovery, liquidation periods and existing third-party nominal. Still being in a handy size, we hope that the formulae is attractive for practitioners with useful tools as sensitivities and intuition-feeding graphical illustrations. On the other hand, it may be a starting point for researchers to develop further sector-specific LGD formulae based on option-pricing arguments.

A Appendix

A.1

Proof of Theorem 3.1: First, we note that $E[LP_{T_D}|\tau = T_D]$ may be written as

$$
\begin{aligned}
E[LP_{T_D}|\tau = T_D] \ = \ & E[\max(E + N - e^{-r(T_L - T_D)}(1-k)C_{T_L}, 0)] \\
& - E[\max(N - e^{-r(T_L - T_D)}(1-k)C_{T_L}, 0)]. \quad \text{(A.1)}
\end{aligned}
$$

Furthermore, we know that the return process Y_{T_L} (conditioned on Y_t) is a normal variable with expectation μ_Y and variance σ_Y^2 given in Theorem 3.1.

Hence the density function of the collateral C_{T_L} equals

$$p(C_{T_L}|C_t) = \frac{1}{C_{T_L}\sqrt{2\pi\sigma_Y^2}} \exp\Big(-\frac{1}{2}\Big(\frac{\ln C_{T_L} - (\ln C_t + \mu_Y)}{\sigma_Y}\Big)^2\Big). \qquad \text{(A.2)}$$

The calculation of the expected loss profile as a sum of two expectations is done straightforwardly and is left as an exercise. The result is the formula stated in the Theorem. □

A.2

The proof of the formula for the delta is done fairly easy. We treat the parameter d and d^* as functions of C and get

$$\frac{\partial d(C)}{\partial C} = \frac{\partial d^*(C)}{\partial C} = \frac{1}{\sigma_Y \cdot C}. \qquad \text{(A.3)}$$

Applying the chain rule for derivatives we obtain

$$\frac{\partial \Phi(-d)}{\partial C} = -\frac{n(d)}{\sigma_Y \cdot C}, \qquad \text{(A.4)}$$

and

$$\frac{\partial \Phi(-d^*)}{\partial C} = -\frac{n(d^*)}{\sigma_Y \cdot C}. \qquad \text{(A.5)}$$

A straightforward application of the product rule establishes the formula. □

A.3

The derivation of the sensitivity formula with respect to the parameter T_L is slightly more cumbersome. Similarly to the delta case, we treat the variables d, d^*, μ_Y and σ_Y as function of T_L. We have

$$d(T_L) = \frac{\ln C^* - r(T_L - T_D) + \mu_Y(T_L)}{\sigma_Y(T_L)}, \qquad \text{(A.6)}$$

where $C^* = \ln(C(1-k)/(N+EAD))$ (analogously for d^*). Further,

$$\frac{\partial \mu(T_L)}{\partial T_L} = \frac{\partial \Psi_{T_L}}{\partial T_L} + \kappa(\Psi_t - Y_t)e^{-\kappa(T_L - t)}, \qquad \text{(A.7)}$$

and

$$\frac{\partial \sigma_Y(T_L)}{\partial T_L} = \frac{\sigma}{2}\sqrt{2\kappa}\frac{e^{-2\kappa(T_L-t)}}{\sqrt{1-e^{-2\kappa(T_L-t)}}}. \tag{A.8}$$

Application of the chain rule gives

$$\frac{\partial \Phi(-d)}{\partial C} = -n(d)\frac{\partial d(T_L)}{\partial T_L} \tag{A.9}$$

(analogously for d^*). The final expression follows after applying and combining the product and quotient rule for derivatives and aggregation. $\qquad\square$

References

V.V. Acharya, S.T. Bharath and A. Srinivasan, *Does industry-wide distress affect defaulted firms? Evidence from creditor recoveries*, Journal of Financial Economics, Vol. 85, 2007, pp. 787-821.

E.I. Altman, A. Resti and A. Sironi, *Analyzing and Explaining Default Recovery Rates*, ISDA Research Report, London, 2001.

E.I. Altman, B. Brady, A. Resti and A. Sironi, *The Link between Default and Recovery Rates: Theory, Empirical Evidence and Implications*, Journal of Business, Vol. 78 (6), 2005, pp. 2203-27.

B. Bade, D. Rösch and H. Scheule, *Default and Recovery Risk Dependencies in a Simple Credit Risk Model*, European Financial Management, Vol. 17 (1), 2011, pp. 120-144.

T. Björk and E. Clapham, *On the pricing of real estate index linked swaps*, Journal of Housing Economics, Vol. 11 (4), 2002, pp. 418-432.

R.J. Buttimer Jr, J.B. Kau, and V.C. Slawson Jr., *A model for pricing securities dependent upon a real estate index*, Journal of Housing Economics, Vol. 6 (1), 1997, pp. 16-30.

M. Cao and J. Wei, *Valuation of Housing Index Derivatives*, Journal of Futures Markets, Vol. 30 (7), 2010, pp. 660-688.

K.E. Case, R.J. Shiller, *The efficiency of the market for single-family homes*, The American Economic Review, Vol. 79 (1), 1989, pp. 125-137.

K.E. Case, R.J. Shiller, *Forecasting prices and excess returns in the housing market*, Real Estate Economics, Vol. 18 (3), 1990, pp. 253-273.

S. Caselli, S. Gatti and F. Querci, *The sensitivity of the loss given default rate to systematic risk: new empirical evidence on banl loans*, Journal of Financial Services Research, Vol. 34 (1), 2008, pp. 1-34.

P. Ciurlia and A. Gheno, *A model for pricing real estate derivatives with stochastic interest rates*, Mathematical and Computer Modelling, Vol. 50 (1), 2009, pp. 233-247.

G.W. Crawford and M.C. Fratantoni (2003), *Assessing the Forecasting Performance of Regime-Switching, ARIMA and GARCH Models of House Prices*, Real Estate Economics, Vol. 31 (2), 2003, pp. 223-243.

A. Dev and M. Pykhtin, *Analytical approach to credit risk modelling*, Risk, March 2002, pp. 26-32.

A. Díaz and B. Jerez, *House Prices, Sales, And Time On The Market: A Search-Theoretic Framework*, International Economic Review, Vol. 54 (3), 2013, pp. 837-872.

P. Englund and Y.M. Ioannides, *House price dynamics: an international empirical perspective*, Journal of Housing Economics, Vol. 6 (2), 1997, pp. 119-136.

F.J. Fabozzi, R.J. Shiller and R.S. Tunaru, *Property Derivatives for Managing European Real-Estate Risk*, European Financial Management, Vol. 16 (1), 2010, pp. 8-26.

F.J. Fabozzi, R.J. Shiller and R.S. Tunaru, *A Pricing Framework for Real Estate Derivatives*, European Financial Management, Vol. 18 (5), 2012, pp. 762-789.

J. Franks, A. de Servigny and S. Davydenko, *A comparative analysis of the recovery process and recovery rates for private companies in the U.K., France, and Germany*, Standard and poor's risk solutions, 2004.

J. Frye, *Collateral Dammage*, Risk, April 2000, pp. 91-94.

D. Geltner, J.D. Fisher, *Pricing and index considerations in commercial real estate derivatives*, The Journal of Portfolio Management, Vol. 33 (5), 2007, pp. 99-118.

J. Grunert, *Verwertungserlöse von Kreditsicherheiten: Eine empirische Analyse notleidender Unternehmenskredite*, Zeitschrift für Betriebswirtschaft, Vol. 80, 2010, pp. 1305-1323.

J. Grunert and M. Weber, *Recovery rates of commercial lending: Empirical evidence for German companies*, Journal of Banking & Finance, Vol. 33, 2009, pp. 505-513.

M. Gürtler and M. Hibbeln, *Improvements in loss given default forecasts for bank loans*, Journal of Banking & Finance, Vol. 37, 2013, pp. 2354-2366.

M. Hillebrand, *Modeling and estimating dependent loss given default*, Working Paper TU München, 2005.

A.J. Hosios, J.E. Pesando, *Measuring prices in resale housing markets in Canada: evidence and implications*, Journal of Housing Economics, Vol. 1 (4), 1991, pp. 303-317.

C. Hott, *Lending behavior and real estate prices*, Journal of Banking & Finance, Vol. 35 (9), 2011, pp. 2429-2442.

Y. Hu and W. Perraudin, *The Dependence of Recovery Rates and Defaults*, Risk Control Research Paper No. 6/1 , 2006.

M. Jacobs Jr., *Empirical Implementation of a 2-Factor Structural Model for Loss-Given-Default*, Cass-Capco Institute Paper Series on Risk, 2011.

E. Jokivuolle, *Pricing European options on autocorrelated indexes*, The Journal of Derivatives, Vol. 6 (2), 1998, pp. 39-52.

E. Jokivuolle and S. Peura, *Incorporating Collateral Value Uncertainty in Loss Given Default Estimates and Loan-to-value Ratios*, European Financial Management, Vol. 9 (3), 2003, pp. 299-314.

J. Krainer, *A theory of liquidity in residential real estate markets*, Journal of Urban Economics, Vol. 49 (1), 2001, pp. 32-53.

A.W. Lo, J. Wang, *Implementing option pricing models when asset returns are predictable*, The Journal of Finance, Vol. 50 (1), 1995, pp. 87-129.

J. Perelló, R. Sircar and J. Masoliver, *Option pricing under stochastic volatility: the exponential Ornstein-Uhlenbeck model*, Journal of Statistical Mechanics: Theory and Experiment, Vol. 6 , 2008, pp. 1-22.

R. Novy-Marx, *Hot and Cold Markets*, Real Estate Economics, Vol. 37 (1), 2009, pp. 1-22.

M. Piazzesi and M. Schneider, *Momentum traders in the housing market: survey evidence and a search model*, Working paper No.w14669, National Bureau of Economic Research, 2009.

M. Pykhtin, *Unexpected recovery risk*, Risk, August 2003, pp. 74-78.

S. Titman and W. Torous, *Valuing Commercial Mortgages: An Empirical Investigation of the Contingent-Claims Approach to Pricing Risky Debt*, The Journal of Finance, Vol. 44 (2), 1989, pp. 345-373.

G. Van Damme, *A generic framework for stochastic Loss-Given-Default*, Journal of Computational and Applied Mathematics, Vol. 235, 2011, pp. 2523-2550.

O. Vasicek, *An equilibrium characterization of the term structure*, Journal of financial economics, Vol.5 (2), pp. 177-188.

W.C. Wheaton, *Vacancy, search, and prices in a housing market matching model*, Journal of Political Economy, Vol. 98 (6), 1990, pp. 1270-1292.