Developing Real Option Game Models

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Abstract

By mixing concepts from both game theoretic analysis and real options theory, an investment decision in a competitive market can be seen as a "game" between firms, as firms implicitly take into account other firms' reactions to their own investment actions. We review several real option game models, suggesting which critical problems have been "solved" by considering game theory, and which significant problems have not been adequately addressed. We provide some insights on the plausible empirical applications, or shortfalls in applications to date, and suggest some promising avenues for future research.

keywords: Real Option Game, Games of Investment Timing, Pre-emption, War of Attrition.

1. Introduction

An investment decision in competitive markets is a "game" among firms, since in making investment decisions, firms implicitly take into account what they think will be the other firms' reactions to their own investment actions, and they know that their competitors think the same way. Consequently, as game theory aims to provide an abstract framework for modeling situations involving interdependent choices, and real options theory is appropriate for most investment decisions, a merger between these two theories appears to be a logic step.

The first paper in the real options literature to consider interactions between firms was Frank Smets (1993), who created a new branch of real option models taking into account the interactions between firms.

In the current literature, a "standard" real option game ("SROG") model is where the value of the investment is treated as a state variable that follows a known process[3]; time is considered infinite and continuous; the investment cost is sunk, indivisible and fixed[4]; firms are not financially constrained; the investment problem is studied in isolation as if it were the only asset on the firm's balance sheet[5] (i.e., the game is played on a single project); and the number of firms holding the option to invest is usually two[6] (duopoly). The focus of the analysis is the derivation of the firms' value functions and their respective investment thresholds, under the assumption that either firms are risk-neutral or the stochastic evolution of the variable(s) underlying the investment value is spanned by the current instantaneous returns from a portfolio of securities that can be traded continuously without transaction costs in a perfectly competitive capital market.

The two most common investment games are the "pre-emption game" and the "war of attrition game", both usually formulated as a "zero-sum game". In the pre-emption game, it is assumed that there is a first-mover advantage that gives firms an incentive to be the first to invest; in the attrition game, it is assumed that there is a second-mover advantage that gives firms an incentive to be the

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[3] Typically, geometric Brownian motion (gBm) and mean reverting processes, stochastic processes with jumps, birth and death processes, or combinations of these processes.

[4] There are papers, however, where this assumption is relaxed. See, for instance, Robert Pindyck (1993), where it is assumed that due to physical difficulties in completing a project, which can only be resolved as the project proceeds, and to uncertainty about the price of the project inputs, the investment costs are uncertain; see also Avinash Dixit and Robert Pindyck (1994), chapter 6, where, in a slightly different context, the same assumption is made.

[5] This is a weakness of the SROG models in the sense that the full dynamics of an industry is not analyzed. Fridick Baldursson (1998) and Joseph Williams (1993), who analyze the dynamics of oligopolistic industries, are exceptions to this rule.

[6] See Romain Bouis, Kuno Huisman and Peter Kort (2009) for an example of a real options model with three firms.

second to invest. Furthermore, the firm's advantage to invest first/second is, usually, assumed to be partial[7] (i.e., the investment of the leader (pre-emption) or the follower (war of attrition) does not completely eliminate the revenues of its opponent); the investment game is treated as a "one-shot game (i.e., firms are allowed to invest only once) where firms are allowed to invest (play) either sequentially or simultaneously, or both; cooperation between firms is not allowed; the market for the project, underlying the investment decision, is considered to be complete and frictionless; and firms are assumed to be ex-ante symmetric and ex-post either symmetric or asymmetric, and can only improve their profits by reducing the profits of rivals (zero-sum game).

In addition, in a SROG model[8], the way the firm's investment thresholds are defined, in the firm's strategy space, depends on the number of underlying variables used. Thus, in models that use just one underlying variable, the firm's investment threshold is defined by a point; in models that use two underlying variables, the firm's investment threshold is defined by a line; and in models that use three or more underlying variables, the firm's investment threshold is defined by a surface or other more complex space structures. However, regardless of the number of underlying variables used in the real options model, the principle underlying the use of the investment threshold(s), derived through the real options valuation technique, remains the same: "a firm should invest as soon as its investment threshold is crossed the first time".

Non-standard ROG ("NSROG") relax some of these assumptions and constraints. In Table 1, in the Appendix, we summarize the types and assumptions of several NSROG.

The three most basic elements that characterize a game are the players, their strategies and payoffs. Translating these to a ROG, the players are the firms that hold the option to invest (investment opportunity), the strategies are the choices "invest"/"defer" and the payoffs are the firms' value functions. Additionally, to be fully characterized, a game still needs to be specified in terms of what sort of knowledge (complete/incomplete) and information (perfect/imperfect, symmetric/asymmetric) the players have at each point in time (node of the game-tree) and regarding the history of the game; what type of game is being played (a "one-shot" game, a "zero-sum" game,

[7] Exceptions to this rule are Bart Lambrecht and William Perraudin (2003) and Pauli Murto and Jussi Keppo (2002) models, which are derived for a context of complete pre-emption.
[8] By ("ROG") we mean an investment game or activity where firms' payoffs are derived combining game theory concepts with the real options methodology.

a sequential/simultaneous[9] game, a cooperative/non-cooperative game); and whether mixed strategies are allowed[10].

Even though, at a first glance, the adaptability of game theory concepts to real option models seems obvious and straightforward, there are some differences between a “standard” ROG and a “standard” game like those which are illustrated in basic game theory textbooks.

One difference between a “standard” game and a SROG is the way the players’ payoffs are given. In “standard” games such as the “prisoners’ dilemma”, the “grab-the-dollar”, the “burning the bridge” or the “battle-of-the-sexes” games, the player’s payoffs are deterministic, while in SROGs they are given by sometimes complex mathematical functions that depend on one, or more, stochastic underlying variables. This fact changes radically the rules under which the game equilibrium is determined, because if the players’ payoffs depend on time, and time is continuous, the game is played in continuous-time. But, if the game is played in a continuous-time and players can move at any time, what does the strategy “move immediately after” mean? In the real options literature, the approach used to overcome this problem is based on Drew Fudenberg and Jean Tirole (1985), which develops a new formalism for modeling games of timing, permitting a continuous-time representation of the limit of discrete-time mixed-strategy equilibria[11].

In a further Section, we discuss in more detail some of the most important differences, from the point of view of the mathematical formulation of the model, between continuous-time ROG and discrete-time ROG, as well as some potential time-consistency and formal and structure-coherence problems which may arise in a continuous-time framework.

The main principle underlying game theory is that those involved in strategic decisions are affected not only by their own choices but also by the decisions of others. Game theory started with the work of John von Neumann in the 1920s, which culminated in his book with Oskar Morgenstern published in 1944. Von Neumann and Morgenstern studied “zero-sum” games where the interests of two players are strictly opposed. John Nash (1950, 1953) treated the more general and realistic case of a mixture of common interests and rivalry for any number of players. Others, notably Reinhard Selten and John Harsanyi (1988), studied even more complex games with sequences of moves and games with asymmetric information.

[9] In real option sequential games the players’ payoff depend on time and are usually called the “Leader” and the “Follower” value functions.
[10] The papers reviewed here are organized according to all these categories in table 2, by author contributions.
[11] Fudenberg and Tirole (1985) contributions to real option game models are discussed in section 2.

With the development of game theory, a formal analysis of competitive interactions became possible in economics and business strategy. Game theory provides a way to think about social interactions of individuals, by bringing them together and examining the equilibrium of the game in which these strategies interact, on the assumption that every person (economic agent) has his own aims and strategies. There are four main specifications for a game: the players, the actions available to them, the timing of these actions and the payoff structure of each possible outcome. The players are assumed to be rational (i.e., each player is aware of the rationality of the other players and acts accordingly) and their rationality is accepted as a common knowledge[12]. Once the structure of a game understood and the strategies of the players set, the solution of the game can be determined using Nash (1950, 1953), which uses novel mathematical techniques to prove the existence of equilibrium in a very general class of games.

Game-theoretic models can be divided into games with or without “perfect information” and with or without “complete information”. “Perfect information” means that the players know all previous decisions of all the players in each decision node; “complete information” means that the complete structure of the game, including all the actions of the players and the possible outcomes, is common knowledge[13]. Sometimes, it may be unclear to each firm where its rival is at each point in time and so the assumption of complete information may not be realistic[14]. In addition, games can also be classified according to whether cooperation among players is allowed or not. In the former case, the game is called a “cooperative game”, in the later, it is called a “non-cooperative game”. In “non-cooperative games” it is assumed that players cannot make a binding agreement. That is, each cooperative outcome must be sustained by Nash equilibrium strategies. On the other hand, in “cooperative games”, firms have no choice but to cooperate. Many real life investment situations exhibit both cooperative and non-cooperative features.

The Nash equilibrium is a concept commonly used in the real options literature. Translated to real option game models, when competing for the revenues from an investment, if firms reach a point where there is a set of strategies with the property that no firm can benefit by changing its strategy

[12] Note that, although game theory assumes rationality on the part of the players in a game, people may act in imperfectly rational ways. There are many unexplained phenomena assuming rationality. However, in business and economic decisions, this assumption may be a good start for gaining a better understanding of what is going on around us.

[13] The distinction between incomplete and imperfect information is somewhat semantic (see Tirole (1988), p. 174, for more details). For instance, in R&D investment games, firms may have “incomplete information” about the quality or success of each other’s research effort and “imperfect information” about how much their rivals have invested in R&D.

[14] It is quite common, for instance, that a firm, before an investment decision, is uncertain about the strategic implications of its action, such as whether it will make its rival back down or reciprocate, whether its rival will take it as a serious threat or not.

while its opponent keeps its strategies unchanged, then that set of strategies, and the corresponding firms' payoffs, constitute a Nash equilibrium. This notion captures a steady state of the play of a strategic game in which each firm holds the correct expectation about its rival's behavior and acts rationally. Although seldom used in the real options literature, the notion of a real option "mixed strategy Nash equilibrium" is designated to model a steady state investment game in which firms' choices are not deterministic but regulated by probabilistic rules. In this case we study a real option Bayesian Nash equilibrium, which, in its essence, is the Nash equilibrium of the Bayesian version of the real option game, i.e., the Nash equilibrium we obtain when we consider not only the strategic structure of the real option game but also the probability distributions over the firms' different (potential) characters or types. For instance, consider a *N*-firm real option game. A Bayesian version of this game would consist of: i) a finite set of potential types for each firm, ii) a finite set of perfect information games, each corresponding to one of the potential combinations of the firms' different types and, iii) a probability distribution over a firm's type, reflecting the beliefs of its opponents about its true type.

A game can be represented in a "normal-form" or in an "extensive-form". In the "normal-form representation", each player, simultaneously, chooses a strategy, and the combination of the strategies chosen by the players determines a payoff for each player. In the "extensive-form representation" we specify: (i) the players in the game, (ii) when each player has the move, (iii) what each player can do at each opportunity to move, (iv) what each player knows at each opportunity to move, and (v) the payoff received by each player for each combination of moves that can be chosen by players[15].

In our review we select an extensive number of papers, published or in progress, modeling investment decisions considering uncertainty and competition, developed over the last two decades. Our goal is to highlight many of the contributions to the literature on ROG, relate these results to the known empirical evidence, if any, and suggest new avenues for future research.

This paper is organized as follows. In section 2, we introduce basic aspects of the SROG models, discuss the mathematical formulation, principles and methodologies commonly used, such as the derivation of the firms' payoffs, and respective investment thresholds, and the determination of firms' dominant strategies and game equilibrium(a). In addition, we analyze, and contrast, the differences between discrete-time real option games and continuous-time real option games. In section 3, as a complement to our discussions, we briefly introduce real option-related literature,

[15] For a detailed description about game representation techniques see Robert Gibbons (1992), pp. 2-12, for the normal-form representation, and pp. 115-129, for the extensive-form representation.

namely, "continuous-time games of timing" and "deterministic" and "stochastic" investment models. Section 4 reviews two decades of academic research on "standard" and "non standard" ROG models. Tables 1 and 2, in the Appendix, classify these articles by game characteristics. Section 5 surveys the limited empirical research and suggests some testiable hypotheses. Section 6 concludes and suggests new avenues for research.

2. Real Option Game Framework

We first review standard monopoly real option models, and then provide the basic framework for standard strategic real option models.

2.1 Monopoly Market

The standard real option model for a monopoly market can be described as follows: there is a single firm with the possibility of investing *I* in a project that yields a flow of income X_t, where X_t follows a gBm process given by equation (1).

$$dX_t = \mu_X X_t dt + \sigma_X X_t dz \tag{1}$$

where, μ_X is the instantaneous conditional expected percentage change in X_t per unit of time (also known as the *drift*) and σ_X is the instantaneous conditional standard deviation per unit of time in X_t (also known as the *volatility*). Both of these variables are assumed to be constant over time and the condition $\mu_X < r$ holds, where *r* is the riskless interest rate, and *dz* is the increment of a standard Wiener process for the variable X_t. Given the assumptions above, using standard real options procedures the derivation of the firm's value function and investment threshold is straightforward (see Robert McDonald and Daniel Siegel, 1986).

The firm's value function is given by (for simplicity of notation we neglect the subscript t in the variable X):

$$F(X) = \begin{cases} AX^{\beta} & \text{if } X \le X^* \\ X - I & \text{if } X > X^* \end{cases} \tag{2}$$

with,

$$A = \left(\frac{\beta - 1}{\beta}\right)^{\beta-1} \frac{1}{\beta}\left(\frac{1}{I}\right)^{\beta-1} \tag{3}$$

I is the constant investment cost, and β is the positive root of the following quadratic function:

$$\frac{1}{2}\sigma^2\beta\left(\beta-1\right)+\left(r-\delta\right)\beta-r=0 \tag{4}$$

that is,

$$\beta=\frac{1}{2}-\frac{(r-\delta)}{\sigma^2}+\sqrt{\left[\frac{(r-\delta)}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2r}{\sigma^2}}\;>1 \tag{5}$$

with $\delta=r-\mu_X$.

The firm's optimal investment strategy consists in investing as soon as X_t first crosses X^*, where X^* is given by equation (6):

$$X^*=\frac{\beta}{\beta-1}I \tag{6}$$

Since $\beta>1$, the investment rule specifies that the firm should not invest before the value of the project has exceeded *I* by a certain amount.

This is the fundamental result from irreversible investment analysis under uncertainty. The essence of the investment timing strategy is to find a critical project value, X^*, at which the value from postponing the investment further equals the net present value of the project $X-I$. As soon as this value (investment threshold) is reached, the firm should invest. Since this is the solution for a monopoly market, the investment threshold, X^*, is sometimes referred to in the literature as the "non-strategic investment threshold", recognizing the fact that it is the firm's optimal threshold value on the assumption that its payoff is independent of other firms' actions[16].

2.2 Duopoly Market

In the real options literature there are models concerned with an exclusive (monopolistic) projects, in the sense that only one firm holds the opportunity to invest, and models concerned with non-exclusive projects, leading usually to sequential investments (leader/follower models). The former

[16] Note, however, that investments in large projects in monopoly markets can have an effect on the value of the monopolistic firm similar to the entrance of a new competitor. For instance, Jussi Keppo and Hao Lu (2003) derived a real options model for a monopolistic electricity market where due to the size of the new electricity plant, its operation will affect the market supply and the path of the electricity prices, and consequently, the value of the firm's currently active projects.

case, characterizes a game of one firm against nature, the later characterizes a standard ROG. Ideally, in ROG models the choice regarding leadership in the investment should be endogenous to the derivation of the firms' value functions and investment thresholds and the determination of the equilibrium(a) of the game. However, the mathematics for doing so are complex and, consequently, in the real options literature, so far, the approach that has been followed in this regard has been to assign, deterministically or by flipping a coin, the leader and the follower roles[17].

Consider an industry comprised of two identical firms, where each firm possesses an option to invest in the same (and unique) project that will produce a unit of output[18]. Furthermore, assume that the cost of the investment is I and irreversible and the cash flow stream from the investment is uncertain. In such context the payoff of each firm is affected by the actions (strategy) of its opponent. Then consider the extreme case where not only the project is unique but also as soon as one firm invests, it becomes worthless for the firm which has not invested, i.e., at time t when one firm triggers its investment, the investment opportunity is completely lost for the other firm. Consequently, due to the fear of losing the investment opportunity, each firm has a strong incentive to invest before its opponent as long as its payoff is positive. Hence, firms have an incentive to invest earlier than suggested by the monopoly solution (6).

Avinash Dixit and Robert Pindyck (1994), chapter 9, Kuno Huisman (2001), Dean Paxson and Helena Pinto (2005), among others, developed real option models for leader/follower competition settings. In these models, at a first moment of the investment game, only one firm invests and becomes the leader, achieving a (perhaps temporary) monopolist payoff; in a subsequent moment, a second firm is allowed to invest if that becomes optimal, and becomes the follower, with both firms thereafter sharing the payoff of a duopoly market. More specifically, assume that the firms' revenue flow is given by (7),

$$X(t)\left[D_{k_i,k_j}\right] \tag{7}$$

where $X(t)$ is the market revenue flow and D_{k_i,k_j} is a deterministic factor representing the proportion of the market revenue allocated to each firm for each investment scenario, with $i, j \in \{L, F\}$, where L means "leader" and F "follower", and $k \in \{0,1\}$, where 0 means that firm is not active and 1 means that firm is active.

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[17] Joseph Williams (1993) and Steven Grenadier (1996) are among the few exceptions to this rule.
[18] In this section we rely on Smets (1993).

Each firm contemplates two choices, whether it should be the first to exercise (becoming the leader) or the second to exercise (entering the market as a follower), having, for each of these strategies, an optimal time to act. The equilibrium set of exercise strategies is derived by letting the firms choose their roles, starting from the value of the follower and then working backwards in a dynamic programming fashion to determine the leader's value function. Denoting $F_F(X)$ as the value of the follower and assuming that firms are risk-neutral, $F_F(X)$ must solve the following equilibrium differential equation:

$$\frac{1}{2}\sigma_X{}^2 X^2 \frac{\partial^2 F_F(X)}{\partial X^2} + \mu_X X \frac{\partial F_F(X)}{\partial X} - rF_F(X) = 0 \tag{8}$$

The differential equation (8) must be solved subject to the boundary conditions (9) and (10), which ensure that the follower chooses the optimal exercise strategy:

$$F_F(X_F^*) = \frac{X_F^*\left[D_{1_F,1_L}\right]}{r-\mu_x} - I \tag{9}$$

$$F_F\,'(X_F^*) = \frac{D_{1_F,1_L}}{r-\mu_X} \tag{10}$$

where $D_{1_F,1_L}$ is the follower's market share when both firms are active, X_F^* is the follower's investment threshold, and I is the investment cost.

According to the real option theory, the optimal strategy for the follower is to exercise the first moment that $X_t > X_F^*$. The boundary condition (9) is the value-matching condition. It states that at the moment the follower's option is exercised its net payoff is $X_F^*[D_{1_F,1_L}]/(r-\mu_X)-I$ (the discounted expected present value of the duopoly cash flow in perpetuity). The boundary condition (10) is called the "smooth-pasting" or "high-contact" condition, and ensures that the exercise trigger is chosen to maximize the value of the option. Through this procedure we get closed-form solutions for the leader's and the follower's value functions, $F_F(X_t)$ and $F_L(X)$, respectively, and for the follower's investment threshold, X_F^*. These solutions are given below:

$$F_F(X) = \begin{cases} \left(\dfrac{I}{\beta - 1}\right)\left(\dfrac{X}{X_F^*}\right)^{\beta} & \text{if } X < X_F^* \\ \dfrac{X[D_{1_F,1_L}]}{r - \mu_X} - I & \text{if } X \geq X_F^* \end{cases} \tag{11}$$

$$X_F^* = \left(\frac{\beta}{\beta - 1}\right)\left(\frac{r - \mu_X}{D_{1_F,1_L}}\right) I \tag{1 2}$$

And,

$$F_L(X) = \begin{cases} \dfrac{X[D_{1_L,0_F}]}{r - \mu_X} + \dfrac{[D_{1_L,1_F} - D_{1_L,0_F}]}{D_{1_L,1_F}} \dfrac{\beta}{\beta - 1}\left(\dfrac{X}{X_F^*}\right)^{\beta} I - I & \text{if } X < X_F^* \\ \dfrac{X[D_{1_L,1_F}]}{r - \mu_X} & \text{if } X \geq X_F^* \end{cases} \tag{13}$$

where $D_{1_L,0_F}$ and $D_{1_L,1_F}$ are the leader's market shares when it is alone in the market and when it is active with the follower, respectively.

The expression for the leader's investment threshold, X_L^*, is derived by equalizing, for $X \geq X_F^*$, expressions (11) and (13), replacing variable X by X_L^* and solving the resulting equation for X_L^*.

Finally, when both firms invest simultaneously they will share the duopoly cash flow in perpetuity given by equation (14).

$$F_{L(F)}(X) = \frac{X\ [D_{1_{L(F)},1_{F(L)}}]}{r - \mu_X} - I \tag{14}$$

In the real options literature there are models for duopoly markets, such as Pauli Murto and Jussi Keppo (2002), where simultaneous investment is not allowed. In such cases, without any loss of insight, we can assume that "if the two firms want to invest simultaneously, then the one with the highest value, X, gets the project; if the project has the same value for both firms and both want to invest at the same time, the one who gets the project is chosen randomly using an even uniform distribution." With few exceptions, in the literature it is generally assumed that both players can observe all the parameters of the model (drift, volatility, etc) and the evolution of the random variable *dz* given in (1) [19].

2.2.1 Competition Setting

The Smets (1993) framework consists in the (deterministic) definition of a certain number of competition factors, each assigned to a particular investment scenario, all governed by an inequality. These competition factors, and the respective inequality, are the key elements in the determination of the firms' dominant strategy at each node of the game-tree and the resultant equilibrium of the game.

2.2.2 Dominant Strategies and Game Equilibrium

For a standard duopoly pre-emption game, the formulation of the game setting can be described as follows: there are two idle firms, each with two strategies available "invest"/"defer" which can lead to three different game scenarios: (i) both firms inactive; (ii) one firm, the leader, active and the other firm, the follower, inactive; (iii) both firms active, with the leader the first to invest. To each of these investment scenarios correspond different firms' payoffs, given by equation (17), conditioned by one (or several) competition factors governed by an inequality similar to the one below:

$$D_{1_i 0_j} > D_{1_i 1_j} > D_{0_i 0_j} \tag{15}$$

The competition factors are represented by $D_{k_i k_j}$, with $\mathrm{k} \in \{0,1\}$, where "zero" means inactive, "one" means active[20] and *i*, in this case, denotes the leader (L) and j denotes the follower (F).

Following the notation above we can redefine inequality (15) for each of the firms. For the leader it would be:

$$D_{1_L 0_F} > D_{1_L 1_F} > D_{0_L 0_F} \tag{16}$$

[19] Two exceptions are Jean-Paul Décamps, Mariotti, and Stéphane Villeneuve (2002), who studied a competitive investment problem where firms have imperfect information regarding those variables, and Ariane Reiss (1998) who derived a real option model for a patent race where the actions of the investors are formulated in a non-game theoretic framework.

[20] Note that this notation allows models with a wider range of investment scenarios. For instance, in Alcino Azevedo and Paxson (2009), $D_{k_i k_j}$ is defined with $\mathrm{k} \in \{0,1,2,12\}$, with "0" and "1" meaning the same as above, and "2" and "12" representing investment scenarios where firms are active but with, respectively, technology 2 alone and both technology 1 and technology 2 at the same time.

The economic interpretation for the relationship between the first two factors, $D_{1_L 0_F} > D_{1_L 1_F}$, is that the leader's revenue market share is higher when operating alone than when operating with the follower; the economic interpretation for the relationship between the second and the third factors, $D_{1_L 1_F} > D_{0_L 0_F}$, is that the leader's market share is higher when it operates with the follower than when it is idle.

After the definition of the competition factors, their economic meaning and the inequality that govern the relationship between the competition factors, we can determine at each node of the investment game-tree, the firms' dominant strategy, and study the equilibrium of the game. Note that, the example used above is a "zero-sum pre-emption game" with the two firms competing for a percentage of the market revenues where, for each investment scenario, the dominate share is deterministically assigned to the leader, and the follower is given a proportion of the total market revenues upon entry (see Andrianos Tsekrekos, 2003). These deterministic competition factors can take, however, more sophisticated forms and different meanings, but, essentially, the framework described above to derive the firms' payoffs, determine the dominant strategies at each node of the investment game-tree, and study the equilibrium of the game is the same.

Figure 1 illustrates the relationship between the leader's competition factors and the firms' investment thresholds.

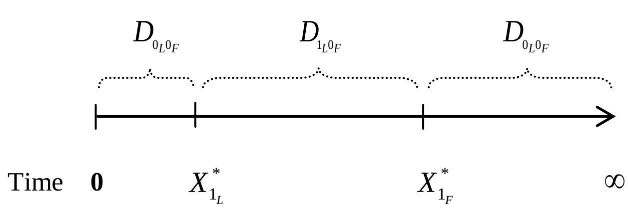

Figure 1 – Duopoly Pre-emption Game: Leader/Follower Investment Thresholds

2.2.3 The Firms' Payoffs

Using the general form for the representation of the firms' values as a function of *t*, with $t > 0$ at the beginning of the game, the firms' revenue flow is given by:

$$F_{i_k j_k} = X_t \left[D_{k_i k_j} \right] \tag{17}$$

where, X_t is the underlying variable (for instance, market revenues); $D_{k_i k_j}$ represents the competition factors, with $k \in \{0,1\}$, where "0" means that the firm to which is assigned this competition factor is inactive and "1" means that the firm is active, with *i, j* denoting the leader (L) or the follower (F).

The existence of a first mover's advantage (pre-emption game) is one assumption underlying the derivation of the SROG model, and so there is no need to make this assumption explicit in the inequality. However, in order to do so we just need to introduce a new pair of competition factors in inequality (16), $D_{1_L 1_F} > D_{1_F 1_L}$, and it would become $D_{1_L 0_F} > D_{1_L 1_F} > D_{1_F 1_L} > D_{0_L 0_F}$ with the second and third competition factors ensuring that the market revenue share of the leader, $D_{1_L 1_F}$, is greater than that of the follower, $D_{1_F 1_L}$, when both firms are active.

This framework also allows for the treatment of other types of investment games, such as a second mover's advantage context (war of attrition game). In addition, the first mover's advantage can be set as temporary or permanent. If permanent, we assume that inequality (16) holds forever, i.e., as soon as the follower enters the market, both firms share the market revenues in a static and pre-defined way, governed by the competition factors and the respective investment game inequality, with an advantage for the leader. If temporary, it is assumed that, at some stage of the game, with both firms active, a new market share arrangement will take place, reducing, or even eliminating, the leader's initial market share advantage. New entries or exits of existing players are not allowed.

The firms' value functions (payoffs) can incorporate one or several competition factors and, as mentioned earlier, a key parameter for the comparison of the firms' payoffs, at each node of the game-tree, is (are) the competition factor(s), which determines the payoff assigned to each firm and investment strategies available. The information underlying each competition factor/game inequality is then transposed to the firms' payoffs and allows the determination of the firms' dominant strategy at each node of the game-tree. When the leader is active and the follower is idle, the leader's payoff function is:

$$F_{L_1 F_0} = X_t \left[D_{1_L 0_F} \right] \tag{18}$$

Following similar procedures as those described above, the payoff functions for the leader and the follower when both firms are active are given, respectively, by:

$$F_{L_1F_1} = X_t \left[D_{1_L1_F} \right] \quad (19)$$

$$F_{F_1L_1} = X_t \left[D_{1_F1_L} \right] \quad (20)$$

Going back to inequality (16) we can see that $D_{1_L0_F} > D_{1_L1_F}$ and $D_{1_L1_F} > D_{1_F1_L}$, hence $F_{L_1F_0} > F_{L_1F_1}$ and $F_{L_1F_1} > F_{F_1L_1}$. Similar rationale is used to determine firms' dominant strategies at each node of the game-tree and the equilibrium of the game. Both firms are assumed to have common knowledge about inequality (16).

2.2.4 Two-Player Pre-emption Game

The pre-emption game is one of the most common games used in the real option literature, usually formulated as a two-player game where investment costs are sunk, firms' payoffs uncertain, time is assumed to be continuous and the horizon of the investment game infinite. Real options theory shows that when an investor has a monopoly over an investment opportunity, where the investment cost is sunk and the revenues are uncertain, there is an option value to wait which is an incentive to delay the investment opportunity more than the net present value methodology suggests. The more uncertain are the revenues, the more valuable is the option to wait. However, when competition is introduced into the investment problem, for a *ceteris paribus* analysis, the intuition is that the value of the option to wait erodes. The higher the competition among firms, the less valuable is the option to wait (defer) the investment.

In modeling duopoly pre-emption investment games using the combined real options and games framework one key element which is common to almost all ROG models is the use of the Fudenberg and Tirole (1985) principle of rent equalization. According to this principle, the erosion in the value of the option to defer the investment is caused by the fact that each firm fears being pre-empted in the market by its rival due to the existence of a first mover-advantage. Consequently, each firm knows that by investing a little earlier than its opponent, they will get a revenue advantage. When this advantage is sufficiently high, firms will try to pre-empt each other, leading them to invest earlier than would be the case otherwise.

Fudenberg and Tirole (1985) use the example of a new technology adoption game to illustrate the effect of pre-emption in games of timing, showing that the threat of pre-emption equalizes rents in a duopoly, thus the term "principle of rent equalization". Figure 2 illustrates how this principle works.

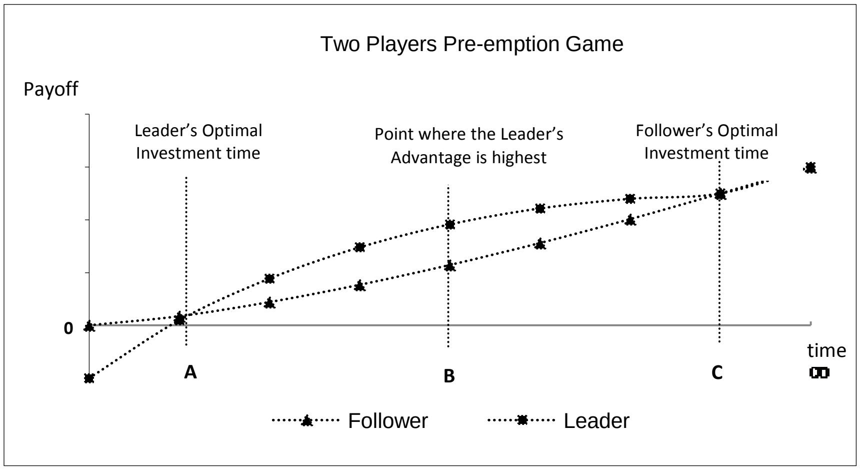

Figure 2 – Two-Player Pre-emption Game

In Figure 2, there are three different regions on the timeline: $[0, A)$, $[A, C)$ and $[C, \infty)$. In the interval $[0, A)$ the payoff of the follower is higher than that of the leader; in the interval $[A, C)$ the payoff of the leader is higher than that of the follower; and in the interval $[C, \infty)$ both players have the same payoff. In addition, we can see that point B is the point at which the leader's advantage reaches a maximum. In absence of the pre-emption effect, the optimal investment time for the leader would be point where the difference between the its payoff and the follower's payoff is highest (point B). However, in a context where there is a first-mover advantage, because firms are afraid of being pre-empted, the leader invests at point A, a point where the payoffs (rents) from being the leader and the follower are equalized.

Note that, in the interval $(A, B]$ there are an infinite number of timing strategies that would lead to a better payoff for the leader than the strategy to invest at time A. However, in a game where firms have perfect, complete and symmetric information about the game, both firms know that, in the interval $(A, B]$, if they invest an instant before the opponent they will get a payoff advantage, and this competition to pre-empt the rival leads both firms to target their investment at point A where each firm has 50 percent chance of being the leader. In these cases, the leader is chosen by flipping a coin. As soon as one firm achieves the leadership in the investment, for the follower, the optimal time to invest is point C. After the follower investment both firms will share the market revenues in a pre-assigned way, i.e., according to the information underlying inequality (16).

2.2.5 Discrete-time game *Versus* Continuous-time game

SROG are focused on symmetric, Markov, sub-game perfect equilibrium exercise strategies in which each firm's exercise strategy, conditional upon the other's exercise strategy, is value-maximizing. It is a Markov equilibrium in the sense that it is considered that the state of the decision process tomorrow is only affected by the state of the decision process today, and not by the other states before that; and it is a "subgame perfect equilibrium" because the players' strategies must constitute a Nash equilibrium in every subgame of the original game.

In continuous-time games with an infinite horizon, the time index *t*, is defined in the domain $t \in [0,\infty)$. Hence, given the relative values of the leader and the follower for a given current value of X_t, we are allowed to construct the equilibrium set of exercise strategies for each firm. SROG are usually formulated in continuous-time, so there is an obvious link between the literature on real option game models and the literature on continuous-time games of timing. Below we briefly introduce, discuss and illustrate the concept of continuous-time games and its relation with the SROG models, relying mainly on the works of Carolyn Pitchik (1981), David Kreps and Robert Wilson (1982a,b), Fudenberg and Tirole (1985), Partha Dasgupta and Eric Maskin (1986a,b), Leon Simon and Maxwell Stinchcombe (1989), Stinchcombe (1992), James Bergin (1992), Prajit Dutta and Aldo Rustichini (1995), and Rida Laraki, Eilon Solan, and Nicolas Vieille (2005).

As discussed earlier, for a sequential real options game in continuous-time, there is no definition for "the last period" and the "next period"[21]. This restricts the set of possible strategic game equilibria[22] and introduces potential time-consistency problems into real option game models. The formulation of firms' investment strategies in continuous-time is complex. Fudenberg and Tirole (1985) highlight the fact that there is a loss of information inherent in representing continuous-time equilibria as the limits of discrete-time mixed strategy equilibria. To correct this they extend the strategy space to specify not only the cumulative probability that player *i* has invested, but also the "intensity" with which each player invests at times "just after" the probability has jumped to one. An investor's strategy is defined as a "collection of simple strategies" satisfying an "inter-temporal consistency condition".

More specifically, a simple strategy for investor *i* in a game starting at a positive level θ of the state variable is a pair of real-value functions $\left(G_i(\theta), \varepsilon_i(\theta)\right) : (0,\infty)\times(0,\infty) \to [0,1]\times[0,1]$

[21] See Fudenberg and Tirole (1985), Simon and Stinchcombe (1989) and Bergin (1992) for detailed discussions on this problem.
[22] For instance, the follower's strategy "invest immediately after the leader" cannot be accommodated.

satisfying certain conditions ensuring that G_i is a cumulative distribution function, and that when $\varepsilon_i > 0$, $G_i = 1$ (i.e., if the intensity of atoms in the interval $[\theta, \theta + d\theta]$ is positive, the investor is sure to invest by θ). A collection of strategies for investor *i*, $\left(G_i^\theta(.), \varepsilon_i^\theta(.)\right)$, is the set of simple strategies that satisfy inter-temporal consistency conditions.

Although this formulation uses mixed strategies, the equilibrium outcomes are equivalent to those in which investors employ pure strategies. Consequently, the analysis will proceed as if each agent uses a pure Markovian strategy, i.e., a stopping rule specifying a critical value or "trigger point" for the exogenous variable θ at which the investor invests[23]. Fudenberg and Tirole (1985) employ a deterministic framework. Their methodology has been extended to a real option stochastic environment.

An investment game can be represented using one of the following techniques: i) a normal-form representation or ii) an extensive-form representation. The choice between these two types of representation depends on the type of investment game. Figures 3 and 4 illustrate a sequential investment game using a normal-form representation and an extensive-form representation, respectively.

		Firm j	
		Defer	Invest
Firm i	Defer	Repeat game	$[F_F(X_t), F_L(X_t)]$
	Invest	$[F_L(X_t), F_F(X_t)]$	$[F_S(X_t), F_S(X_t)]$

Figure 3 – Normal-Form Representation: Sequential Real Option Duopoly Game

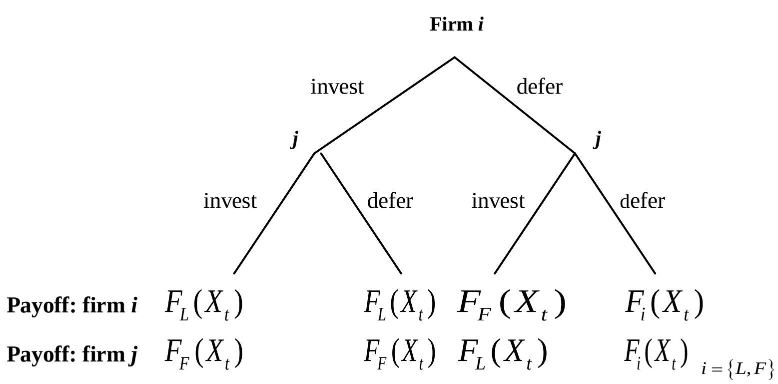

Figure 4 – Extensive-Form Representation: Sequential Real Option Duopoly Game

[23] Note, however, that this is for convenience only given that underlying the analysis is an extended space with mixed strategies (a good discussion about this issue can be found also in Robin Mason and Helen Weeds, 2001).

In Figure 3 the concept of "timing strategy", implicit in a sequential ROG, and the sequence of the players' moves is not as intuitive as in Figure 4, which explains the convenience of using the extensive-form representation to describe this type of game rather than the normal-form representation. In both of the representations above, however, the leader's and the follower's payoffs are represented by the same expressions $F_L(X_t)$ and $F_F(X_t)$, expressions (13) and (11), respectively. $F_S(X_t)$, in Figure 3, is the leader's and the follower's payoffs when both firms invest simultaneously, expression (14).

The subscript t in $F_L(X_t)$, $F_F(X_t)$ and $F_S(X_t)$, denotes the fact that X is not static but varies over time, meaning that as time changes so do the firms' payoffs. Consequently, in practice, for each firm, Figures 3 and 4 display different payoffs at each instant of the game. An intuitive view of the dynamic nature of the firms' payoffs, "timing strategy" and the Fudenberg and Tirole (1985) methodology of using the discrete-time framework as a *proxy* of the continuous-time approach is the elaborated representation of a duopoly ROG given in Figure 5.

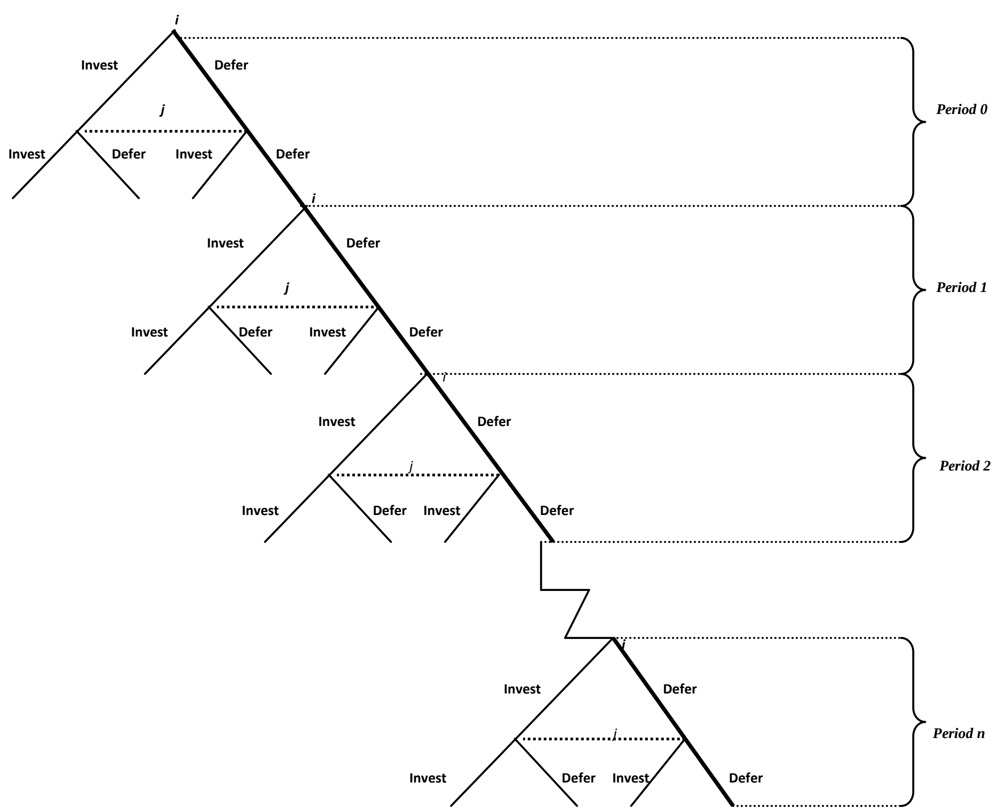

Figure 5 – Illustrative Extensive-Form: Continuous-Time Real Options Duopoly Game

An additional aspect that Figure 5 makes easier to see is the fact that in a duopoly sequential game where firms have two strategies available (invest/defer), although they can choose the strategy "invest" only once, they are allowed to choose the strategy "defer" an infinite number of times, since in a continuous-time framework, in between any two instants of the game where firms do not choose the strategy "invest", they have chosen, theoretically, an infinite number of times the strategy "defer"[24].

ROG models usually assume that time is infinite. This assumption is a mathematical convenience to derive the firms' payoffs and respective investment thresholds. However, it is not appropriate for many investment projects. From the point of view of the equilibrium of the game, there are differences between games where the option to invest matures at some particular point in time, and games where the option to invest can be held in perpetuity. However, this problem has passed "unnoticed" because the focus of our analysis has been directed not to the "timing strategy", chronologically speaking, but to the time at which the value of the investment (i.e., the underlying variable) reaches a threshold, regardless at which chronological point that occurs.

Using (13) and (11) we plot, in Figure 6, the leader's and the follower's payoff functions, respectively, whose shapes are standard (see Dixit and Pindyck, 1994).

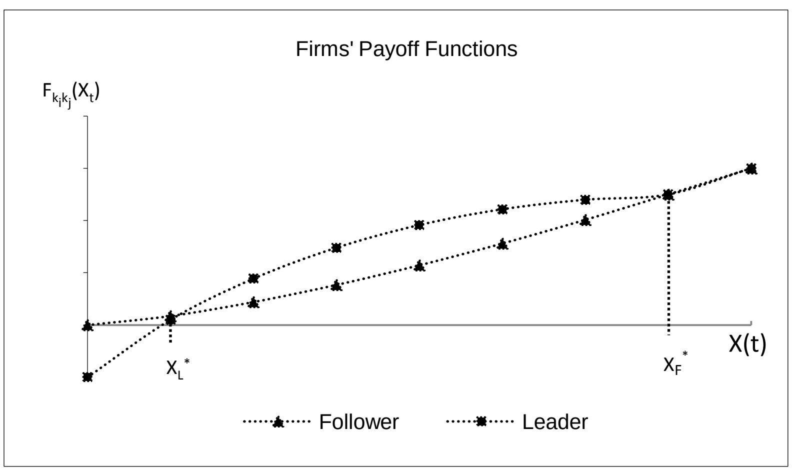

Figure 6 – Firms' Investment Thresholds for a Two-player Pre-emption Game

In Figure 6, there exists a unique point $X_L^* \in \left(0, X_F^*\right)$ with the following properties:

[24] Note that this does not happen, for instance, in the "The Prisoner's Dilemma" game because it is a "simultaneous-one-shot" game, where players can choose only once either "confess" or "defect".

$$
\begin{array}{ll}
F_L(X_t) < F_F(X_t) & \text{if } X_t < X_L^* \\
F_L(X_t) = F_F(X_t) & \text{if } X_t = X_L^* \\
F_L(X_t) > F_F(X_t) & \text{if } X_L^* < X_t < X_F^* \\
F_L(X_t) = F_F(X_t) & \text{if } X_t \geq X_F^*
\end{array}
$$

which demonstrates that there is a unique value at which the payoffs to both the leader and the follower are equal. At any point below X_L^* each firm prefers to be the follower; at X_L^* the benefits of a potentially temporary monopoly just equal the costs of paying the exercise price earlier; at any point above between X_L^* and X_F^* each firm prefers to be the leader; for $X_t \geq X_F^*$, the value of leading, following or simultaneous exercise are equal (.

Figure 6 shows the results for a scenario where after the follower investment both firms share a (permanent) symmetric market share (the initial leader's advantage is eliminated). Hence, both lines overlap after point X_F^*. However, the real option framework above allows (through inequality 16) any other market arrangement. For instance, if after the follower investment the leader market share is reduced but a certain (permanent) advantage is kept, so the leader's payoff function would be parallel to and above the follower's payoff function from point X_F^* onwards.

3. Real Options-Related Literature

3.1 Auction Theory

More recently, there are some works combining real option and auction theories, such as Jøril Maeland (2002, 2006, 2007) and Steven Anderson, Friedman, and Ryan Oprea (2010). Albert Moel and Peter Tufano (2000) also provide a good discussion about the potential advantages of combining both theories. By nature, auction models are "winner takes all" games. The models above are reviewed in detail in section 4.2.3 (see pp. 40-43)[25]. The assumptions underlying the Lambrech and Perraudin (2003) lead to a multi-firm equilibrium similar to that arising from models of first price auctions under incomplete information with a continuum of types (see pp. 627-629).

3.2 Continuous-time Games of Timing

There is a rich literature on continuous-time games of timing. As mentioned earlier, real option game models are usually formulated in continuous-time. To reduce complexity, one key assumption

[25] Moel and Tufano (2000) paper provides hints about the good prospects of combining both real options and auction theories, although the model presented and the analyses provided are based on auction theory only. Hence, their model is not reviewed here.

for modeling continuous-time games as the limit of discrete-time is to prevent firms from exiting and re-entering repeatedly. However, this assumption is not realistic for many investments[26].

Carolyn Pitchik (1981), following Guillermo Owen (1976), studies the necessary and sufficient conditions for the existence of a dominating equilibrium point in a "two-person non-zero sum game of timing" and the problem of pre-emption in a competitive race. David Kreps and Robert Wilson (1982a) propose a new criterion for equilibria of extensive-form games, in the spirit of Selten's perfectness criteria, and study the topological structure of the set of sequential equilibria. Kreps and Wilson (1982b) study the effect of "reputation" and "imperfect information" on the outcomes of a game, starting from the observation that in multistage games, players may seek early in the game to acquire a reputation for being "tough" or "benevolent" or something else. Pankaj Ghemawat and Barry Nalebuff (1985) apply game theory concepts to when and how a firm exits first from a declining industry where shrinking demand creates pressure for capacity to be reduced. Hendricks and Wilson (1985) investigate the relation between the equilibria of discrete and continuous-time formulations of the "war of attrition" game and show that there is no analogue in continuous-time for the variety of types of discrete-time equilibrium. Generally there is no one to one correspondence between the equilibria of the continuous-time with the limiting distributions of the equilibria of discrete-time games.

Dasgupta and Maskin (1986a,b) extend the previous literature by studying the existence of Nash equilibrium in games where an agent's payoffs functions are discontinuous. Fudenberg and David Levine (1986) provide necessary and sufficient conditions for equilibria of a game to arise as limit of ε-equilibria of games with smaller strategy spaces. Ken Hendricks and Charles Wilson (1987) provide a complete characterization of the equilibria for a class of pre-emption games, when time is continuous and information is complete, that allows for asymmetric payoffs and an arbitrary time horizon. Ken Hendricks, Andrew Weiss, and Charles Wilson (1988) present a general analysis of the "war of attrition" in continuous-time with complete information. Simon and Stinchcombe (1989) propose a new framework for continuous-time games that conforms as closely as possible to the conventional discrete-time framework, taking the view that continuous-time can be seen as "discrete-time" but with a grid that is infinitely fine[27]. Chi-Fu Huang and Lode Li (1990) prove the existence of a Nash equilibrium for a set of continuous-time stopping games when certain monotonicity conditions are satisfied.

[26] See John Weyant and Tao Yao (2005) for a good discussion on this issue.
[27] This is the approach that has been followed in the real options literature in continuous-time real option games.

Following Hendricks and Wilson (1985) and Simon and Stinchcombe (1989), Bergin (1992) tackles the problem of the difficulties involved in modeling continuous-time strategic behavior, since “time is not well ordered”, and develops a general repeated game model over an arbitrary time domain. Stinchcome (1992) defines the maximal set of strategies for continuous-time games, characterized by two conditions: (i) a strategy must identify an agent’s next move time, and (ii) agents’ only initiate finitely many points in time. Dutta and Rustichini (1993) study a general class of stopping games with pure strategy sub-game perfect equilibria and show that there always exists a natural class Markov-perfect equilibria. Bergin and Macleod (1993) develop a model of strategic behavior in continuous-time games of complete information, excluding conventional repeated games in discrete-time as a special case. Rune Stenbacka and Mihkel Tombak (1994) introduce experience effects into a duopoly game of timing the adoption of a new technology which exhibit exogenous technological progress, concluding that a higher level of technological uncertainty increases the extent of dispersion between the equilibrium timings of adoption and that the equilibrium timings are even more dispersed when the leader takes the follower’s reaction into account. Dutta and Rustichini (1995) study a class of two-player continuous-time stochastic games in which agents can make (costly) discrete or discontinuous changes in the variables that affect their payoffs and show that in these games there are Markov-perfect equilibria of the two-sided (s, S) rule type. Laraki et al. (2005) address the question of the existence of equilibrium in general timing games with complete information. These papers, along with many others, paved the progress towards more sophisticated methodologies to treat games in continuous-time, which are implicitly or explicitly used in “continuous-time real option game” models.

3.3 Other Investment Game Frameworks

There are also other branches of real options-related literature which although based on radically different theories, assumptions and mathematical formulations have been good source of insights to developing new real option game models. Many of these approaches have been converted into ROG models. Robert Lucas and Eduard Prescott (1971), David Mills (1988), John Leahy (1993) and Fridick Baldursson and Ioannis Karatzas (1997) derive models for a wide range of investment contexts. Jennifer Reinganum (1981a), Reinganum (1981b), Reinganum (1982), Richard Gilbert and David Newbery (1982), Reinganum (1983), Richard Gilbert and Richard Harris (1984), Richard Jensen (1992), Hendricks (1992) and Stenbacka and Tombak (1994) derive models for investments in new technologies. All these models consider strategic interactions among firms but using two different frameworks: (1) “deterministic”, where the variables that drive the value of the investment are assumed to be deterministic, (2) “non-option stochastic”, where the variables that drive the value of the investment are assumed to be stochastic but there are no (real) options involved, and (3) “auction framework”.

3.3.1 Deterministic

This branch of literature analyses timing games of entry and exit in a deterministic framework. Essentially, these are stopping games where the underlying process is simply time itself.

Reinganum (1981a) notes that the perfection of a new and superior technology confers neither private nor social benefit until that technology is adopted and employed by potential users. In an industry with substantial entry costs, perfection and adoption of an innovation are not necessarily coterminous. She studies the diffusion of new technologies considering an industry composed of two firms, each using current best-practice technology, assuming that the firms are operating at Nash equilibrium output levels, generating a market price (given demand) and profit allocation. When a cost-reducing innovation is announced, each firm must determine when (if ever) to adopt it, based in part upon the discounted cost of implementing the new technology and in part upon the behavior of the rival firm.

Reinganum (1981b) investigates the issues related to industrial research and development, in particular, situations in which two firms are rivals in developing a new process or device. She notes that in such cases there is, sometimes, a distinct advantage to being the first to produce a new product or implement a new technology, but since only the first to succeed realizes this advantage, each firm’s profits will depend upon the research efforts of its rival, which suggests a game-theoretic approach. In addition, she develops a theory of optimal resource allocation to R&D, under the assumption of uncertain technical advance and in presence of game-playing rivals, and finds that the Nash equilibrium and the socially optimal rates of investment do not coincide.

Reinganum (1982) addresses the problem of resource allocation to R&D in an n-firm industry using differential games. Following Reinganum (1981a, 1981b, 1982), Gilbert and Newbery (1982) enquire whether institutions such as the patent system create opportunities for the firms with monopoly power to maintain their monopoly power. They show that, under certain conditions, a firm with monopoly power has an incentive to maintain its monopoly power by patenting new technologies before potential competitors and that this activity can lead to patents that are neither used nor licensed to others (“sleeping patents”).

Reinganum (1983) applies two-person, nonzero-sum game theory to a problem in the economics of technology adoption, extending previous papers by considering differentiable mixed-strategy equilibria. Gilbert and Harris (1984) develop a theory of competition in markets with indivisible and irreversible investments, noting that in markets with increasing returns to investment scale, competition occurs over both the amount and timing of the new capital construction and that the consequences of competition depend on the strategies and information available to the competitors.

Jensen (1992) examines the welfare effects of adopting an innovation when there is uncertainty about whether it will succeed or fail, noting that the incentives of firms to adopt a new process need not coincide with maximum expected consumer surplus or social welfare if there is uncertainty before the process is adopted and if the only loss from failure is a fixed cost. Additionally, he finds that in some cases no firm will adopt an innovation likely to fail, although expected welfare is maximized if one adopts. There are cases where both firms will adopt an innovation likely to succeed, although expected welfare is maximized if only one firm adopts.

Hendricks (1992) studies the effects of uncertainty on the timing of adoption of a new technology in a duopoly. Firms are assumed to be uncertain about the innovation capabilities of their rivals and the profitability of the adoption, which creates a richer and, in some respects, more plausible theory of adoption where rents from delayed adoption are always realized and returns are not equalized across adoption times.

Mills (1988) examines timing and profits in investment-timing games where two or more firms compete to make an indivisible one-time investment, showing that the perfect-Nash equilibrium timing strategies eliminate rents only when it is costless for rivals to threaten pre-emption credibly.

Stenbacka and Tombak (1994) introduce the effect of experience into a duopoly game of timing of adoption of new technologies that exhibit exogenous technological progress. Their results show that a higher level of uncertainty increases the extent of dispersion between the equilibrium timings of adoption and that the equilibrium timings are even more dispersed when the leader takes the follower’s reaction into account.

3.3.2 Stochastic

There are several stochastic models which have not (yet) been converted into ROG models. Robert Lucas and Eduard Prescott (1971) assume that the actual and anticipated prices have the same probability distribution, or that price expectations are rational, and the social optimality of the equilibrium in a discrete-time Markov chain model is established and determines a time series behavior of investment, output, and prices for a competitive industry with stochastic demand. Starting from the real options insight about the effect of irreversibility on a firm’s investment decision, Leahy (1993) shows that the equilibrium entry time under free entry is the same as the optimal entry time of a myopic firm who ignores future entry by competitors, even considering the effect that entry may have on the mean and variance of the output price process. Following Leahy (1993), Baldursson and Karatzas (1997) establish the links between social optimum, equilibrium, and optimum of a myopic investor under a general stochastic demand process utilizing singular stochastic control theory. Their main focus is on a partial equilibrium model of a competitive industry. In Leahy (1993) the industry is composed of a continuum of infinitesimally small firms which incur irretrievable costs as they enter or exit. It is argued that each firm can be myopic as regards future investment in the industry and yet its decision will be optimal. The investment game is formulated in discrete-time and the model is applicable only to specific industries in which demand is linear in the sense that the methodology does not work for more general investment game specifications.

4. ROG Models

We classify ROG models as "standard" and "non-standard". Standard Real Option Game (SROG) models use game concepts/formulations which fit with the standard approach used within the real options literature, briefly described in sections 1 and 2. Their main contribution to the literature lies on the results and practical application found or on the projects' value underlying variables used or mathematical frameworks used, rather than on the novelty of the game concepts/formulation or assumptions used. Non-standard real option game (NSROG) models use game concepts/formulations which do not fit with the standard approach used in the real options literature. These models address the critical issues of (1) the determinates of leadership, (2) ex-ante and ex-post asymmetric firms, (3) games where the "winner-takes-all", or there is a "war of attrition", or cooperative repeated games, or market sharing is dynamic, (4) games of incomplete information, (5) oligopolies, and highly competitive industries, or duopolies where exit is feasible, (6) capacity choice strategies, (7) projects with several stochastic elements, and (8) consideration of several other innovative factors, not found in SROG.

Note that the citations below usually focus on only critical parts of each article, ignoring other, possibly important, aspects.

4.1 SROG Models

The literature combining the real options valuation technique with game theory concepts started with Smets (1993), which derived, for a duopoly market, a continuous-time model of strategic real option exercise under product market competition. His paper assumes that entry is irreversible, demand stochastic and simultaneous investment may arise only when the leadership role is exogenously pre-assigned.

Han Smit and L.A. Ankum (1993) combine the real options approach of investment timing with basic principles of game theory and industrial organization. Using simple standard game assumptions/formulations they illustrate the influence of competition on project value and investment timing. The “time” variable is assumed to be discrete.

Following Smets (1993), Dixit and Pindyck (1994) supply a basic ROG model for duopoly markets. Grenadier (2000a) provides a good summary of existing literature on game-theoretic option models. Grenadier (2000b) illustrates how intersection of real options and game theory provides powerful new insights into the behavior of economic agents under uncertainty, with examples from real estate development in an oligopoly and oil exploration investment decisions with symmetric information.

Lambrecht (1999) and Domingo Joaquin and Kirt Butler (1999) present models where competing firms have opportunities to invest in discrete investment projects and where the investment game is played on the timing of these investments. Huisman (2001) develops several innovative new technology adoptions game models for models for duopoly markets.

Tom Cottrell and Gordon Sick (2001) study first-mover advantage, starting from their belief that fear of losing first-mover advantages causes managers to ignore standard real options analysis completely and simply go ahead with any project that they think has a positive net present value. Their results show that by considering the merits of a delayed-entry follower strategy, value enhancing managers will want to be suitably cautious before ignoring real options analysis.

Lambrecht (2001) investigates the interaction between market entry, company foreclosure, and capital structure in a duopoly. Firms have complete information with respect to all model parameters, including their opponent’s and are restricted to a single entry/exit trigger strategy (one-shot entry/exit game). He extends the standard exit model by allowing financially distressed firms to renegotiate their debt contracts through a one-off debt exchange offer and found that firms with high bankruptcy costs or with prospects of profit improvement can get bigger reductions on their debt repayments.

Starting from the intuition that infrastructure investments generate other investment opportunities, and in doing so change the strategic position of the firm, Smit (2003) analyses the optional and strategic features of infrastructure investments. Mason and Weeds (2005) show that, in duopoly markets with positive externalities, greater uncertainty can raise the leader’s value more than the follower's. Hence, the leader may act sooner, but, as uncertainty increases a switch in this pattern of equilibrium investment is possible, which may hasten investment.

Han Smit and Lenos Trigeorgis (2004, 2006), are good reviews of SROG models, with several illustrations of model applications. Benoît Chevalier-Roignant and Lenos Trigeorgis (2010) supply good illustrations of the interception between real options and game theories, and practical examples about how to use both together to address investment decision for several industries and economic contexts.

SROG often make the assumption that leadership is determined by flipping a coin, an unlikely and unsatisfactory assumption.

4.2 NSROG Models

In this section we organize and review several NSROG models focusing mainly on the following game aspects: (i) degree of competition, (ii) asymmetric between firms (i.e., ex-ante and/or ex-post), (iii) dynamic *versus* static market sharing, (iv) cooperative games, (v) games with incomplete information (vi) multi-factor models, (vii) capacity choice, and (viii) other innovative parameters.

4.2.1 Degree of Competition

SROG assume there is a simple duopoly, leadership is determined or randomly chosen, and there is one follower. This is characteristic of only a few industries.

i) N-Rivals

Williams (1993) provides the first rigorous derivation of a Nash-equilibrium in a real options framework. He derives an equilibrium set of exercises strategies for real estate developers where equilibrium development is symmetric and simultaneous. In equilibrium all developers build at the maximum feasible rate whenever income rises above a critical value, and each developer conjectures correctly that each other developer currently builds at his optimal rate. The aggregate demand for the good or service and its supply of developed assets are proportional to power functions of the income. The optimal building rate depends on an exogenous factor which changes stochastically through time and affects the aggregate demand. Additionally, it is assumed that the number of identical owners of undeveloped assets is constant over time, and that the owners have an equal number of undeveloped assets. This model provides investment thresholds which, in equilibrium, all market players, simultaneously, should use to optimize their investment, regardless of the type of market (monopoly, oligopoly or perfect competition) in which they operate.

Grenadier (1996) develops an equilibrium game framework for strategic option exercise games for duopoly markets. He suggests a possible explanation for why some markets may experience building booms in the face of declining demand and property values. In contrast to Williams (1993), where equilibrium real estate development is symmetric and simultaneous, equilibrium real estate development may arise endogenously as either simultaneously or sequentially, depending on the initial conditions and the parameter values. If at the beginning of the game, $t = 0$, the variable underlying the value of the real estate development, $X(t)$, is below the trigger value determined for the leader entry time, $X(0) < X_L$, one developer will wait until the trigger X_L is reached, and the other will wait until the trigger X_F is reached. Therefore, developers will be indifferent between leading and following. If $X(0) \in \left[X_L, X_F\right)$, each will race to build immediately. The random winner of the race will then build, and the loser will wait until the trigger X_F is reached. If $X(0) \geq X_F$, an equilibrium will be characterized by simultaneous exercise. The firms' value functions for this game are similar to those presented in Figure 6 (p. 21) and therefore the same game equilibrium(a) analyses apply.

Ariene Reiss (1998) derives a real option model for when a firm should patent and adopt an innovation if the arrival time of competitors follows a Poisson process. The innovation value change over time is defined by the following differential equation:

$$dCF = \alpha CFdt + \sigma CFdz - CFdq \tag{20}$$

Where, *CF* is the stochastic net cash flow in perpetuity, α is the expected grow rate of the net cash flow, σ is the volatility of the net cash flow, and dq is the increment of a Poisson process and independent of dz:

$$dq = \begin{cases} 0 & \text{with probability } (1 - \lambda dt) \\ 1 & \text{with probability } \lambda dt \end{cases} \tag{21}$$

Reiss finds four different option exercise strategies and respective investment thresholds. The model applies to markets where there is competition, but does not specify the number of market participants. Instead, the intensity of rivalry is specified through a constant hazard rate λdt which can be regarded as a measure of intensity of rivalry, since the expected arrival time of competition decreases with an increasing hazard rate. The characterization of the investment game is, however, incomplete. For instance, if the innovation game is played in a context where firms are ex-ante symmetric and have complete, perfect and symmetric information, then simultaneous investment may occur. However, this outcome is not allowed. In addition, the market is not explicitly characterized, so the model may apply to several types of competition and market structures. If it is used for oligopoly or perfect competition markets with complete, perfect and symmetric information, all market participants would be guided in their investment decision by the same, and unique, investment scenario thresholds. Consequently, the option to invest in the innovation project would be simultaneously exercised by all market players and the value of the innovation project would decrease significantly for each player, a scenario not discussed in the paper. Finally, the model is derived for a pre-emption investment game with competition exogenously set. This later aspect is a weakness of the model, shared by most of other investment game models in the real options literature.

Naohiko Baba (2001) derives a leader/follower real options model to optimize a bank's entry decisions into a duopolistic loan market in an attempt to shed light on the prolonged slump in the Japanese loan market in the 1990s. He focuses on the differences resulting from the alternative assumptions regarding whether the roles of leader and follower are interchangeable or pre-determined, and shows that when the roles are pre-determined, as in the case of the Japanese main bank system, both leader and follower banks have a greater incentive to wait until the loan demand condition improves sufficiently than when the roles are interchangeable.

Grenadier (2002) provides a general tractable approach for deriving equilibrium investment strategies in a continuous-time Cournot-Nash equilibrium framework, with more than two competitors. Each firm faces a sequence of investment opportunities and must determine an exercise strategy for its path of investment. The cash flows from investment are determined by a continuous-time stochastic shock process as well as the investment strategies of all firms in the industry. A symmetric Nash equilibrium in exercise policies is determined such that each firm's equilibrium exercise strategy is optimal, conditional on its competitors following their equilibrium exercise strategies. The resulting equilibrium is quite simple and shows that the impact of competition on exercise strategies is substantial. More specifically, he shows that competition drastically erodes the value of the option to wait and leads to investment at very near the zero net present value threshold.

Martin Nielson (2002) extends the oligopolistic industry result described in Dixit and Pindyck (1994) for investments with positive externalities and scenarios where the monopolist has multiple investment opportunities. His results show that, with decreased profit flow, a monopolist always makes its first investment later than the leader among two competitive firms. It makes no difference for the first investment whether the monopolist has access to one or two investment projects. A monopolist will make its second investment earlier than the follower if the profit loss, due to increased competition is larger than that due to increased supply.

Pauli Murto, Näsäkkälä, and Jussi Keppo (2004) assume an oligopoly market for a homogeneous non-storable commodity, where the demand evolves stochastically. Firms make investments in order to adjust their production cost functions or production capacities, allowing for the timing of lumpy investment projects under uncertainty in a discrete-time state-space game. There are several large firms which move sequentially, ensuring a Markov-perfect Nash equilibrium. Once the equilibrium has been solved, Monte Carlo simulation is used to form probability distributions for the firms' cash flow patterns and completed investments, information which can be used to value the firms operations.

Martin Odening, Muβhoff, Hirschauer, and Alfons Balmann (2007) study investment decisions for markets where perfect competition holds. Firms are risk neutral, price takers and produce with the same "constant returns to scale" technology at a constant variable cost per unit, investments are irreversible and infinitely divisible with capital stock subjected to depreciation at a given rate. The demand shock follows a gBm diffusion process. Using simulations, they demonstrate that myopic planning may lead to non-optimal investment strategies. They quantify the degree of sub-optimality and propose measures to reduce the error.

Romain Bouis, Huisman, Peter Kort (2009) derive a real option game model for case where more than two identical firms are present. More specifically, they develop a model for a market with three firms and show that in case of three firms the investment timing of the first investor lies between the one and the two-firm case. In addition, they show that in equilibria where firms invest sequentially, the timing of the first investor in case of *n+2 firms* always lies between the timing of the *n* and *n+1 firm* case; increased competition can delay rather than hasten investment; and market entry occurs earlier when the number of anticipated market entrants is small and even. Their results are numerically extended to the *n-firm* case.

ii) Entry-Exit

In the traditional real options game framework, "ex post" losses are infrequent. Since a monopolistic invests at a substantial premium, the likelihood for large asset value reversals is remote. Hence, a SROG is not appropriate for explaining boom-and-bust markets such as real estate, where periodic overbuilding results in waves of high vacancy and foreclosure rates.

Laarni Bulan, Mayer, and Tsuriel Somerville (2002) use 1,214 individual real estate projects built in Vancouver, Canada, between 1979 and 1998, and conclude that competition erodes option value.

Murto (2004) examines a declining duopoly market where firms must choose when to exit from the market, considering a Markov-perfect equilibrium. He finds that with low degree of uncertainty there is a unique equilibrium, where one of the firms always exits before the other, and, when uncertainty is increased, another equilibrium with the reverse order of exit may appear, ruining the uniqueness. The occurrence of this event depends on the degree of asymmetry in the firm specific parameters.

Francisco Ruiz-Aliseda (2005) develops a ROG entry/exit model for a duopoly market where firms have to decide at each instant of time whether to be in or out of a market that expands up to a random date and dies thereafter. Firms are asymmetric only on the opportunity costs of usage of the assets they employ, so their investments are not equally recoverable. His results show that the destructive effect of the threat of pre-emption on option values is modified if the rival's commitment to remain active after investing is not credible.

Makoto Goto, Takashima, Tsujimura, and Takahiro Ohno (2008) provide a feasible exit for a follower only, when profitability or the market declines, when then the leader reverts to a monopoly position awaiting re-entry of the follower.

4.2.2 Asymmetry between Firms

The sources of asymmetries between firms are extensive, such as different abilities to learn (learning rates), different organization flexibility, or different liquidity constrains or benefits/losses from positive/negative externalities due to other firms' investments, or different sunk/operating costs or combinations of several of these asymmetries. In ROG, asymmetries between firms can exist "ex-ante" (i.e., only before the investment decision has been made), "ex-post" (i.e., only after the investment decisions has been made), or "ex-ante" and "ex-post" with the same nature or different natures and sizes. They can be introduced in the ROG model as a deterministic variable (i.e., exogenously set) or as an endogenous variable (i.e., guided by, for instance, probabilistic distribution function(s) endogenously considered in the model).

In this section we focus our analysis mainly on the nature of the asymmetry between firms, the way it is incorporated in ROG model and its effects on the model results, neglecting, therefore, other possible important aspects of the models.

i) Ex-post

Smets (1993) ROG framework is flexible regarding the ex-post asymmetry(ies) between firms. This is deterministically set through the competition inequality (15) described in section 2 (see page 13).

ii) Ex-ante

More difficult is, however, the derivation of real option models where firms are assumed to be ex-ante asymmetric, since this has implications in the mathematical derivation of firms' value functions and investment thresholds, and the game equilibrium. Therefore, in this section we neglect the ex-post asymmetry aspect in ROG models and focus our review mainly on the ROG models where firms are assumed to be ex-ante asymmetric. In table 2 we show further details about the models relevant parameters and assumptions.

Grenadier (1999) relaxes the real option models standard assumptions that timing of the exercise is simultaneous and uninformative and that agents are perfectly informed about the parameters of their opponents' real options. He assumes, instead, that agents are imperfectly and differentially informed and may impute the private information of others by observing their exercise (or lack of exercise) decisions, and concludes that in markets with both public and private information, the exercise of options must be determined as part of a strategic equilibrium. He presents a model of equilibrium option exercise policies and information revelation in markets with private signals. The model provides insights into the patterns of exercise in a variety of realistic economic settings.

Huisman (2001) develops detailed studies for dynamic duopoly markets where two ex-ante asymmetric firms compete for revenues underlying the adoption of new technologies. He identifies three types of investment equilibria: (i) sequential equilibrium, which occurs when cost asymmetry is high so that low-cost firm has a dominant competitive advantage; (ii) pre-emptive equilibrium, which occurs when cost asymmetry is not substantial; and (iii) simultaneous investment equilibrium, where both firms enter at the same threshold level.

Maeland (2002, 2006 and 2007) model firm are also assumed to be ex-ante asymmetric. These models are reviewed in section about models combining real options and auctions theories (see p. 27). The mathematics underlying the assumptions regarding the asymmetry between firms is briefly described in Table 1[28].

Yao-Wen Hsu and Bart Lambrecht (2003) incorporate asymmetric information into a model which examines the investment behavior of an incumbent and a potential entrant that are competing for a

[28] The mathematical framework and model assumptions used by Maeland to set the asymmetry between firms in these three articles is very similar. Hence, to save space, in Table 1 we describe Maeland (2002) model only.

patent with a stochastic payoff. They assume that the challenger has complete information about the incumbent, whereas the later does not know the precise value of its opponent's investment cost. Their results show that even a small probability of being pre-empted gives the informationally-disadvantaged firm an incentive to invest at the break-even point where it is indifferent between investing and being pre-empted.

Thomas Sparla (2004) examines exercise policies for closure options in a duopoly with uncertain (inverse) demand and strong and possible asymmetric strategic externalities. He shows that for asymmetric duopoly markets the level of the demand uncertainty may affect the number of prevailing equilibria. More specifically, he finds that duopolists disinvest later than a monopolist and earlier than myopic firms, and that increases in market price volatility make strategic externalities less important.

Smit and Trigeorgis (2004, chapter 7) use an integrated real options and game-theoretic framework for strategic R&D investments to analyze a duopoly two-stage games where the growth option value of R&D depends on endogenous competitive reactions. Firms choose output levels endogenously and may have different (asymmetric) production costs as a result of R&D, investment timing differences or learning. R&D investment decisions are made under asymmetric (imperfect) information, i.e., without being able to observe all the important strategic factors, such as the success of their rival's R&D efforts and their precise cost functions, and therefore, firms may have an incentive to provide partial or misleading information over the success of its R&D efforts.

Jean-Paul Décamps and Thomas Mariotti (2004) develop a duopoly model of investment in which firms have incomplete but symmetric information about the value of the investment project, but asymmetric information about their investment costs. The objective of the paper is to study the learning externality due to the increase in the signal's quality generated by the leader's investment. The investment project may be of low or high quality. If the project is of low quality, then players eventually learn this by observing failures that occur according to a Poisson process. By contrast, a high-quality project never fails. The paper addresses this game as a war of attrition game and, therefore, it is discussed in more detail in the "WOA" games review section (see p. 37-38).

Grzegorz Pawlina and Peter Kort (2006) focus on the impact of investment cost asymmetry on optimal real option exercise strategies for a duopoly market. Sources of potential investment cost asymmetries may be due to different liquidity constraints or organizational flexibility at implementing a new production technology. Firms differ ex-ante regarding the required sunk cost associated with the investment and have several. They show that for the context given there are three equilibrium strategies, as well as, the critical levels of cost asymmetry which delineate the equilibrium regions, as a function of the model underlying variables. Their results show that within a certain range of asymmetry level, a marginal increase in the investment cost of the firm with the cost disadvantage can enhance its own value and reduce its opponent's value, which is a somewhat "surprising" and counterintuitive result.

Jean Kong and Yue Kwon (2007) examine strategic investment pre-emptive games for a duopoly market with uncertain revenues and asymmetric firms in terms of investment costs and revenue flows. Compared to other models where firms are assumed to be asymmetric (ex-ante or ex-post), this model has a higher level of generality under the assumption of asymmetry in both cost and revenue, generating therefore a richer set of strategic equilibriums.

Takahiro Watanabe (2010) study a duopoly ex-ante asymmetric game. Both firms want to optimize their investment decision in a context where the profit flows has two uncertain parameters, one know only by the incumbent and the other shared by both firms. The incumbent is assumed to have a higher expected profit and therefore it invests earlier than the entrant. However, this earlier move reveals the, until then, incumbent private information, which accelerates the entrant investment. Knowing the signaling effect of its investment the incumbent may hides the private information strategically. Watanabe characterize the equilibrium conditions for the incumbent with such strategic information to invest.

Nalin Kulatilaka and Enrico Perotti (1998) provide a strategic rationale for growth options under uncertainty and imperfect competition. In a market with strategic competition, investment confers a greater capability to take advantage of future growth opportunities. This strategic advantage leads to the capture of a greater share of the market, either by dissuading entry or by inducing competitors to "make room" for the stronger competitor. When the strategic advantage is strong, increased uncertainty encourages investment in growth options; when the strategic effect is weak the reverse is true. An increase in systematic risk discourages the acquisition of growth options. These results contradict the view that volatility is a strong disincentive for investment. The authors analyze both the case where firms are ex-ante symmetric (i.e., firms are ex-ante identical) and the case where firms are ex-ante asymmetric. The former leads to simultaneous strategic entry by all market players, the latter leads to a pre-emption game where one firm enters the market first.

ROG Models with Asymmetric Firms	Type of Asymmetry	Model Assumptions/Parameters
1. Kulatilaka and Perotti (1998)	**<u>Ex-ante symmetric</u>:** firms spent *i* (inv. cost) and get a production **(ex-post) cost/unit advantage**. **<u>Monopoly (M)</u>:** P(Q) is the inverse demand function The market price is given as a function of total supply Q: $P(Q,\theta)=\theta-Q$, where θ is a random variable distributed on (0, oo) **<u>If no (N) initial investment</u>**: Firm will choose output level: $Q_M^N=1/2(\theta-K)$ with associated profits: $\pi_M^N=1/4(\theta-K)^2$ It will produce if: $\theta<\theta_M^*\equiv K$ **<u>In there is an initial (I) investment:</u>** Firm reduces the future unit cost to k, where k < K, due to the learning, logistic and product development improvements. $[K-k]$ is the firm's capacity cost advantage after investment. If θ < k the firm will not produce, else, it will choose an output: $Q_M^I=1/2(\theta-k)$ With associated profits: $\pi_M^I=1/4(\theta-k)^2$	**<u>Duopoly</u>:** Firm 1 chooses whether to make a strategic investment at time 0. Firm 2 may choose to enter the market at time 1, with a unit production cost of K. If both firms produce, the market outcome is Cournot competition. **<u>If firm 1 makes no (N) initial investment at time zero:</u>** Ex-post Firm 1 has no strategic advantage vis-à-vis the competitor. If both firms choose to produce they will face the same production cost K. As long as θ is equal or greater than K, the outcome is a "symmetric Cournot equilibrium". Each firm produces: $Q_1^N=(\theta-K)/3$ which yields a profit equal to: $\pi_1^N=1/9(\theta-K)^2$ If θ < K neither firms will produce, as the marginal cost revenue falls below cost. Hence, θ* = K can be interpreted as the symmetric Cournot entry point, below which no production takes place. **<u>If firm 1 investments (I) at time zero:</u>** Market interaction is affected which is acknowledged by firm 2 when making its output decision. Therefore, if both firms produce: Firm 1 will choose an output level: $Q_1^I=1/3(\theta+K-2k)$ With associated profits equal to: $\pi_1^I 1/9(\theta+K-2k)^2$ The Cournot entry point for the competitor is now equal to: $\theta^{**}\equiv 2K-k$
2. Joaquin and Butler (1999)	**<u>Ex-ante</u>:** cost-revenue asymmetry	To be completed later. Not access to yet…….
3. Grenadier (1999)	**<u>Ex-ante symmetric</u>:** agents may impute (asymmetrically) the private information of others by observing their exercise (or lack of exercise) decisions.	$\theta=\mu+S_1+S_2+...+S_n$ θ - information to be known μ - expected value of θ Signals S_i with $i\in\{1,...,n\}$ - independent, mean-zero random variables.
4. Huisman (2001)	**<u>Ex-ante symmetric (ch. 8)</u>:** impact of **investment cost asymmetry** on the optimal real option exercise strategy and the value of firms in a duopoly. **Ex-post symmetric (ch. 7, 9):** firms are ex-ante symmetric and ex-post asymmetric, if one firm takes the leadership in the investment.	The instantaneous profit of firm i, with $i\in\{1,2\}$ is given by: $\pi_{N_iN_j}(x)=xD_{N_iN_j}$, where $N_k=\begin{cases}0 \text{ if firm k has not invested}\\1 \text{ if firm k has invested}\end{cases}$ x - profit flows. $D_{N_iN_j}$ - deterministic contribution for the profit function. Competition inequality: $D_{10}>D_{00}; D_{10}>D_{11}; D_{00}>D_{01}$.

1. Murto and Keppo (2002)	**<u>Ex-ante symmetric</u>:** firms may have different valuations for the investment project and **<u>incomplete information</u>** about each other's project values, although the **<u>complete information</u>** investment scenario is also analyzed. As soon as one firm triggers the investment for the first time the value of the investment for the others jumps to zero (i.e., this is a WTA game).	Firm i's project value follows the following process: $dV_i = \alpha_i V_i dt + \sigma_i V_i dz_i - V_i dq$ where $dq = \begin{cases} 1 & \text{with prob. } \lambda dt \\ 0 & \text{with prob. } 1\text{-}\lambda dt \end{cases}$ λ is a constant hazard rate of losing the investment opportunity, i ε{1,…,n}, and the rest of the variables have the usual meanings. The competitors' project values mapping is defined by: $\Gamma_i : R^n_+ \longrightarrow \{0,1\}$ where "0" means "do not invest"; "1" means "invest" The game ends for firm *i* when $\Gamma_i \in (V_1,...,V_n) = 1$ for the first time. Firm *i* gets the payoff $[V_i - I_i]$, where I_i is the investment cost of firm *i*. Firms i's opponents get a zero payoff.
5. Maeland (2002) <u>Note</u>: Maeland (2007) uses a similar mathematical framework. To save space, herein we describe Maeland (2002) only.	**<u>Ex-ante symmetric</u>:** multi-agent game where each agent, *i*, has private information about his own costs of the investment but has no private information about the competitors' costs. Auctioneer does not observe the n agents' investment cost parameter, but it is common knowledge that the values are drawn from the same distribution.	Agent *i* has private information about his own inv. cost: K^i Competitors' costs are defined by a vector: $K^{-i} = (K^1,...,K^{i-1},K^{i+1},...,K^n)$ The inv. cost values are drawn from the same distribution function: $F(.)$
7. Hsu and Lambrecht (2003)	**<u>Ex-ante symmetric</u>:** In a patent race, the challenger has complete information about the incumbent, whereas the incumbent does not know the precise value of the opponent's investment cost.	π_{1i} - incumbent's profit without the new patent π_{2i} - incumbent's profit if succeeds in the new patent δ_i - incumbent's profit if the entrant acquires the new patent The following inequality holds: $\pi_{2i} > \pi_{1i} > \delta_i$
8. Sparla (2004)	**<u>Ex-ante symmetric</u>:** production capacity reduction model where, for instance, one firm has higher variable costs than the other, or faces different intensity of competition, or different uncertainty about the cash flows.	$\bar{q}$ - capacity level before capacity-reduction $\underline{q}$ - capacity level after capacity-reduction $R_1 = \frac{R(\bar{q},q) - R(\underline{q},\underline{q})}{R(\bar{q},\underline{q}) - R(\underline{q},\underline{q})}$; $R_2 = \frac{R(\bar{q},\bar{q}) - R(\underline{q},\bar{q})}{R(\bar{q},\underline{q}) - R(\underline{q},\underline{q})}$ The following inequality holds for second-mover advantages: $R_1 < 1 < R_2$
9. Smit and Trigeorgis (2004, ch. 7)	**<u>Ex-ante symmetric</u>:** reducing future production costs via making a strategic R&D investment in an innovative new production process, or alternatively, by investing earlier in production capacity. With learning, the marginal cost of firms *i* is assumed to decline exponentially with cumulative production at a learning rate γ . **The rate of learning** (i.e., how fast operational cost declines when cumulative production increases), and is likely to be (**asymmetric**) firm-specific.	Duopoly: firm *i*, with $i = A, B$ Cumulative production: $\sum Q_{1t} = \left(Q_{it} + \sum Q_{it-1}\right)$ Marginal cost of firm *i*: $c_i(Q_{it}) \equiv c_i^F + c_i^L e^{-\gamma \sum Q_{it}}$ Where, c_i^F - floor level of the marginal cost of firm i c_i^L - current level of the marginal cost of firm i
10. Décamps and Mariotti (2004)	**<u>Ex-ante symmetric</u>:** the return of the project is assumed to be the same for both players and independent of whom invests first. Players have incomplete but symmetric information about the project's value, but asymmetric information about their investment costs and, possibly, different opportunity	Player *i*, with $i = 1, 2$ Firm *i*'s sunk cost: θ^i Investment project can be of <u>low</u> or <u>high</u> quality. Prior probability that the project is of high quality is: $p_0 \in (0,1)$ Players can learn about the quality of the project through public signals, modeled as a Poisson process B with failure rate: $\lambda^B > 0$.

	costs of investment.	High-quality project never fails, and generates a profit: $d > 0$ per unit. Low-quality project may fail according to: $\lambda^L > 0$ Follower is observing a Poisson process F with intensity: $\lambda^F = \lambda^B + \lambda^L$, where: λ^F ,is the expected rate of failure for the follower's project; λ^B is the ex-ante firm's rate of failure based on the background public signal, and λ^L is the observed rate of failure for the leader's project.
11. Savva and Scholtes (2005) **Note**: To save space we herein we present firms' payoffs and conditions for the non-cooperative scenario only although a similar framework is also used in the papers to derive the payoffs for the cooperative scenario.	**Ex-ante symmetric:** two-firm two stages (discrete-time) cooperative/non-cooperative game. Focus on bilateral partnership deals (R&D, product commercialization projects). **Firms are ex-ante symmetric** (example: small biotech company *versus* large pharmaceutical company attempting to agree on a partnership regarding the commercialization of a nearly finished product) and share **asymmetric information** about the expected revenues from the partnership.	**Non-cooperative**: C_B - cash flow from the (drug) project, biotech company I_B - Investment cost, biotech company **Cooperative scenario**: C_{B+P} - cash flow from the (drug) project, shared by both firms I_{B+P} - Investment cost, shared by both firms **Conditions for cooperation**: $x_B \geq C_B - I_B$ $x_P \geq 0$ $x_B + x_p = C_{B+P} - I_{B+P}$ Where, x_B - revenue share from the deal, biotech company x_P - revenue share from the deal, large pharmaceutical company ▪ Biotech company's profit share, x_B , with the following inequality holding: $C_B - I_B \leq x_B \leq C_{B+P} - I_{B+P}$ ▪ Pharmaceutical company's profit share: $x_P = C_{B+P} - I_{B+P} - x_B$.
12. Pawlina and Kort (2006)	**Ex-ante symmetric:** impact of **investment cost asymmetry** on the optimal real option exercise strategy and the value of firms in a duopoly.	Instantaneous profit of firm i, with $i \in \{1,2\}$: $\pi_{N_iN_j}(x) = xD_{N_iN_j}$ where, $N_k = \begin{cases} 0 \text{ if firm k has not invested} \\ 1 \text{ if firm k has invested} \end{cases}$ $D_{N_iN_j}$ - deterministic contribution for the profit function. The following inequality holds: $D_{10} > D_{00}; D_{10} > D_{11}; D_{00} > D_{01}$
13. Maeland (2006)	**Ex-ante symmetric:** the owner of a project holds a real option to invest but needs specialized expertise. There are n firms with the expertise competing for the right to manage the project. Each firm chooses an unobservable effort that influences the probability of its investment cost level. When the effort is made each firm observes its own investment costs but not the competitors'. The model analyses the effect of agency conflicts and asymmetric information on firms' investment behavior.	Ex-ante each firm i makes a costly effort. A high effort increases the probability of being a "low cost" type. Ex-post, i.e., when an effort is made, firm i observes its cost, Ki, of making the investment. The vector of the reported investment cost is: $\hat{K}_i = \left[\hat{K}_1, \hat{K}_2, ..., \hat{K}_n\right]$ Each firm reports an investment cost, $\hat{K}_i \in \left[\underline{K}, \bar{K}\right]$ to the owner of the project.
14. Kong and Kwok (2007)	**Ex-ante symmetric: asymmetry** on both the investment **sunk cost** and the **revenue flows** of the two competing firms.	Instantaneous revenue flow for firm i at state j is: $\pi_t^{ij} = D_{ij}\theta_t \; i \in \{1,2\}$, where, θ - revenue flows. $D_{N_iN_j}$ - constant multiplier, $i \in \{1,2\}$ and $j \in \{m,d\}$ The following inequalities hold: If negative externality: $0 < D_{id} < D_{im}$, $i = 1,2$ If positive externality: $0 < D_{im} < D_{id}$, $i = 1,2$

Table 1 – ROG with Ex-ante and/or Ex-post Asymmetric Firms

4.2.3 Static *versus* Dynamic Market Sharing

Departing from the Tsekrekos (2003) assumption that the market share of the leader and the follower remains constant after the follower enters, several authors consider dynamic games, where (i) immediately (patent) or eventually (brand dominance) the winner-takes-all (“WTA”), or (ii) there is a war-of-attrition (“WOA”) so eventually one of the firms shrinks or disappears, or (iii) dominate shares are allocated dynamically over time among firms, or (iv) there are repeated cooperative games, where market shares are maintained through collusion.

i) Static

Weeds (2002) provides a real option game model to study R&D investments in a WTA patent system with irreversible investment cost and uncertain revenues. The technological success of the project is probabilistic and the economic value of the patent to be won evolves stochastically over time. Economic uncertainty gives rise to option values and a tendency for delay. The WTA nature of the patent system generates a first-mover advantage that counteracts the incentive to delay. She finds that, comparing with the optimal cooperative investment pattern, investment is more delayed when firms act non-cooperatively as each holds back from investing in the fear of starting a patent race.

Jøril Maeland (2002, 2006) combine real options theory with auction theory to optimize investment decisions for a WTA investment game with two or more firms sharing asymmetric information.

Maeland (2002) models an investment decision where there is a project whose owner holds the option to implement it and organizes an action where privately informed agents can participate. Firms have asymmetric information about the cost of the investment, i.e., each investor has private information about its own costs but no private information about the cost of the competitors. The investment strategy is formulated as an optimal stopping problem and is delegated to the winner of the auction. She shows that asymmetric information causes “an additional wedge between affecting the critical price of the implementation, with the inverse hazard rate being a key component”. Maeland (2006), develops an investment model for contexts where there are *n* firms technically capable of managing a project which is being auctioned, in a context where agency conflicts and information asymmetries hold. Each firm chooses an effort, not observed by its competitors, which affects the probability of its investment cost level. Modeled as a “winner takes all” game, and by mixing both auction concepts and the real options theory she concludes that the winner of the contract is the firm that reports the lowest investment cost, the private information problem increases the critical price of investment compared to the case of no inefficiency and that the effect of moral hazard in the investment trigger is ambiguous.

Maeland (2007) analyzes an investment decision for context similar to that described in Maeland (2002). However, in this case, she shows that the auction participants' private information increases the project owner's cost of exercising the option, leading perhaps to under-investment, and that the investment strategy is independent of the number of privately informed agents participating in the auction.

Anderson, et al. (2010), study a complete pre-emption investment games, theoretically and empirically, based on auction and real options theory. They characterize the symmetric Bayesian-Nash equilibrium of the pre-emption game with an arbitrary number of firms and model investment opportunities available to *n+1* investors whose value, *V*, is publicly observed and evolves according to a gBm process with known parameters where each investor has a privately known avoidable cost of investing and when the first mover pre-empts it obtains the entire value *V* (i.e., the winner takes all the revenues). In addition, firms are uncertain about their rival's costs (i.e., they are ex-ante symmetric but have incomplete information about the game). This model extends that of Lambrecht and Perraudin (2003) to more than two players and is explicitly rooted in auction theory as well as in real options theory.

Murto and Keppo (2002) study also investment firms' behaviors for WTA games. However, they address the investment problem under the assumption that firms have incomplete information and therefore, hence this paper is described in detail in the review section about ROG models with incomplete information (see pp. 41-44).

Jaco Thijssen, Huisman, and Peter Kort (2002) study pre-emption (i.e., first-mover advantage) games and WOA (i.e., second-mover advantage) games extending the strategy spaces and equilibrium concepts as introduced in Fudenberg and Tirole (1985). Marcel Boyer, Gravel, Mariotti, and Moreaux (2001) had made a similar attempt, but their adaptation is less suitable to modeling WOA games. The attempts to extend the firms' strategic space and equilibrium concepts tried to overcome a weakness underlying real options models such as those of Grenadier (1996) and Weeds (2002), who assume that, at the pre-emption point, only one firm can succeed in investing, an unsatisfactory assumption given that firms are assumed to be ex-ante symmetric, and, therefore, there is no *a priori* ground for assuming that firms are not allowed to invest simultaneously even if it is optimal for both to do so.

Cottrell and Sick (2002) discuss the follower advantages, providing practical examples of successful delay in the context of a real option on innovation, such as the ability to learn more about a technology before irreversibly committing scarce resources, the advantage of observing market reaction to product design and features, and the avoidance of sunk investment in obsolete technology.

Jean-Paul Décamps and Thomas Mariotti (2004) develop a duopoly model of investment in which each player learns about the quality of a common value project by observing some public background information, and possibly the experience of a rival. Investment costs are assumed to be private information and the background signal takes the form of a Poisson process conditional on the quality of the project being low. Their results show that the resulting WOA game has a unique symmetric equilibrium which depends on initial public beliefs. They determine the impact of changes in the cost and signal distributions on investment timing, and how equilibrium is affected when a first-mover advantage is introduced. Firms have incomplete but symmetric information about the value of the investment project, but asymmetric information about their investment costs. In addition, firms' payoffs incorporate both a common and a private value component. The return of the project is assumed to be the same for both players, and independent of whom invests first. There are two sources of public information. A background signal provides free information about the value of the project, independently of firms' investment decisions. Once a leader has made an investment, an additional signal is generated that may be used by the follower to optimally adjust its investment decision. Their aim is to study the learning externality due to the increase in the signal quality generated by the leader's investment. By delaying investment, each firm tries to convince the rival that their own cost is high, and thus that the rival should invest first. The difference with a "pure" common real option game model is that each player does not care about the information of the rival *per se*, but only in so far as it measures the likelihood of investing second and hence of benefitting from a better signal. The common belief process is Markovian, in the sense that the equilibrium trigger of each firm depends on the initial belief about the value of the project. The more optimistic firms are (ex-ante) about the quality of the project, the higher is the game-equilibrium trigger.

ii) Dynamic

Lorenzo Garlappi (2001) develops a large discrete-time nonzero-sum stochastic game for two all-equity financed single project firms competing in the development of a project that requires N phases to be completed. At each date before completion, the two firms must decide, simultaneously, whether to keep working on the project in the attempt to reach the next hurdle, or to wait. In making their decisions on whether to undertake a phase of the investment or not, the firms consider: (i) their position in the investment race, i.e., the number of stages completed; (ii) a signal, δ, in the form of potential cash flows generated by the completed project, modeled as a geometric random walk; (iii) a random variable $n(t)$ that represents the number of phases completed by firm A at time t; and (iv) an analogous variable, $m(t)$, for the opponent of firm A, where, δ, $n(t)$ and $m(t)$ are common knowledge, that is at each stage of the game both firms know the potential cash flows generated by the completed project and the number of phases completed by the opponent. For mathematical tractability, simultaneous success in the investment race is not allowed. In addition, Garlappi (2001) studies the effect of cooperation and pre-emption on the value of the investment race and the risk premium, i.e., the discount rate to be used in evaluating future uncertain cash flows.

Paulo Pereira and Artur Rodrigues (2010) assume that when the follower(s) enter, the market may expand due to network or commonality effects, so that over stages it is conceivable that everyone wins as the number of firms expands.

4.2.4 Cooperative Games

Mason and Weeds (2001) demonstrate that strategic interactions can have important consequences for irreversible and uncertain investments. Pre-emption significantly decreases investment option values. Relative to the cooperative outcome, externalities introduce inefficiencies in the investment decisions, and pre-emption and externalities combined can actually hasten, rather than delay, investment. The model is derived for a duopoly market with or without cooperation. The innovative of this model is that it does not impose exogenously an asymmetry between firms, but, instead allows the first-mover to be determined endogenously. The authors derive two versions of the model. In the first version, the roles of the leader and the follower are pre-assigned exogenously; in the second version, the roles are determined endogenously[29].

Marcel Boyer, Lasserre, Mariotti, Michel Moreaux (2001) develop the conventional literature on strategic investment with a deterministic formulation and perfect foresight by firms, most notably Gilbert and Harris (1984), Fudenberg and Tirole (1985), and Mills (1988). They focus on a duopoly in a homogeneous growing product market with incremental indivisible capacity investments, paying also attention to the role of uncertainty and the speed of market development on investment strategies and competition. Firms are assumed to have access to the same technology and time is continuous. Their results show that collusion is likely when the industry is made up of two active firms of equal size, and the market is volatile and develops quickly.

[29] That is, the leader invests according to the Fudenberg and Tirole (1985) principle of rent equalization, described in section 2.

Nicos Savva and Stefan Scholtes (2005) examine partnerships bilateral deals under uncertainty but with downstream flexibility. Their analysis is focused on the effect of options on the synergy underlying the deal, distinguishing between cooperative options, which are exercised jointly and in the interest of maximizing the total deal value, and non-cooperative options, which are exercised unilaterally in the interest of one partner's payoff. In this partnership game model, although not explicitly stated, firms are ex-ante asymmetric and share incomplete and imperfect information about the true intentions of each other regarding the deal. The investment is analyzed as a two-firm two-stage (discrete-time) game with both firms holding cooperative and non-cooperative options regarding the partnership deal. They provide results for the cooperative and the non-cooperative game equilibrium, and reach discussions about optimal investment behaviors for such economic contexts.

Weyant and Yao (2005) derive a model for investments in R&D projects for contexts where there is competition, and market and technical uncertainty. Firms make R&D investment decisions on an ongoing basis before the success of the project, and these repeated strategic interactions may facilitate self-enforcement tacit collusion. Hence, the authors study the possibility of defining a collusion (cooperative) equilibrium based on the use of a trigger strategy with information time lag. They find that when there is a long time lag, a pre-emptive (non-cooperative) equilibrium emerges in which the option values of delay are reduced by competition; but when the information time lag is sufficiently short, a collusion (cooperative) equilibrium emerges in which investment is delayed more than for a monopoly.

Armada, Manuel, Lawrence Kryzanowski, and Paulo Pereira (2009) introduce the concept of "hidden rivals" so that proportions of the market share can be taken not only by positioned firms but also by hidden (not yet disclosed) competitors. They derive value functions and investment thresholds for two firms facing hidden competitors, for scenarios where firms are ex-ante symmetric and ex-post asymmetric with a permanent market share for the leader, or ex-ante asymmetric in terms of investment costs and ex-post symmetric or asymmetric. The formulation used to treat competition is standard. The scenario where the two positioned firms agree to invest together (cooperate) reducing the join investment cost and eliminating the fear of pre-emption from each other's actions is also considered.

4.2.5 Incomplete Information

Similarly to the case of asymmetries between firms, real option games models with incomplete information study situations where firms have incomplete information about the game variables, for instance, their opponents' ability to learn, or benefits/losses from positive/negative externalities due to others' actions, or investment or operating cost (or cost functions). In this type of models there is a structural element of uncertainty (incomplete information) besides the standard diffusion component of the value process. In the RO literature, models of "incomplete information", usually, make the assumption that it holds ex-ante and ex-post, and forever. In this section focus we focus our analysis mainly on the effect of the assumption of firms' incomplete information about the ROG neglecting, therefore, other possible important aspects of the models.

Spiros Martzoukos and Eleftherios Zacharias (2001) develop a real options duopoly game model to study the optimization of R&D value enhancement in the presence of spillover effects, for contexts where firms have the option to enhance value by doing R&D and/or acquiring more information about the project. Due to information spillovers, firms act strategically by optimizing their behavior conditional on the actions of their counterpart, and are assumed to have incomplete information about the investment game. Each firm must decide: (i) how much of its investment effort should be shared by its rival (i.e., the level of coordination) and (ii) how much to spend on direct actions to enhance the project's value (i.e., R&D, advertisement, etc) given the spillover effects. The solution for the firms' optimal strategic and tactical R&D decision-making is found as the solution of these two-stage game, and the results show that this decision is heavily dependent on the effectiveness of R&D investments, their cost, and the degree of coordination that is optimal for the two firms, whose optimality varies over time.

Jean-Paul Décamps, Mariotti, and Stéphane Villeneuve (2002) investigate the impact of incomplete information on firms' investment strategies. They study the optimal time to invest in an indivisible project whose value, while still perfectly observable, is driven by a parameter that is unknown to the decision maker ex-ante, that is, there is a structural element of uncertainty besides the standard diffusion component of the value process. They argue that this captures in a simple way a variety of empirically relevant investment situations. For instance, a firm might ignore the exact growth characteristics of a market where it contemplates investing. An owner who considers selling an asset might ignore how the willingness to pay of potential buyers will evolve in the future. By observing the evolution of the asset value, the decision maker can update beliefs about the uncertain drift of the value process. However, this information is noisy, since it does not allow one to distinguish perfectly between the relative contributions of the drift and diffusion components to the instantaneous variations of the project value. Consequently, they use filtering and Martingale techniques to show that the optimal investment region is characterized by a continuous and non-decreasing boundary in the value state space. The decision maker always benefits from being uncertain about the drift of the value process, always preferring the option to invest in a project with an unknown drift to that of investing in a project with constant drift equal to the prior expectation of the drift in the first option. Consequently, one might expect the value of claims on structurally uncertain assets (for instance, in an emerging sector in which future growth prospects are uncertain) to be higher than that of claims on assets in more traditional sectors with otherwise identical risk characteristics. This is a real options game model for monopoly markets[30], i.e., an investment game with just one player playing against nature. However, its rich mathematical formation, originality of the economic context to which it applies, and insights for the development of real options for competition game settings justify its inclusion in this review.

Murto and Keppo (2002) develop a game-theoretic model to study the competition for a single investment opportunity under uncertainty, where firms do not know the rival valuations for the project. The investment game is modeled as a WTA game, i.e., as soon as one firm triggers the investment for the first time the value of the investment for the others jumps to zero. They provide results for both scenarios: (i) the scenario where firms know the value of each other's projects (complete information) and (ii) the scenario where they do not know the value of each other's projects (incomplete information). It is shown that the structure of the information about the rival's valuation for the project has an important effect on the equilibrium of the game. More specifically, their results show that if there are at least two firms with the same valuation for the project (complete information), then the competition completely eliminates all profits and when one of the firms invests in the project it is indifferent between investing and not investing. If, on the other hand, one of the firms has some advantage (due to incomplete information of its rivals regarding the value of its project) over the others (for instance, the investment cost is lower or the value of the project is higher for this firm than for the others), then, in equilibrium, that firm gets a positive payoff.

Bart Lambrecht and William Perraudin (2003) derive a full dynamic model of investment under uncertainty for first-mover advantage contexts, assuming firms have incomplete information about each other. Firms observe their own investment cost, but know only that the cost of rivals is an independent draw from a distribution which has a continuous differentiable density with strictly positive support on an open interval. Their approach leads to a Bayesian Nash equilibrium where each firm invests strategically. The inclusion of incomplete information yields quite rich implications for the equity return distributions of companies holding real options subject to possible pre-emption. The model predicts that returns on such equities will contain jumps and that the

[30] Hence, it is not included in Table 1.

volatility associated with those jumps will be negatively correlated across competing firms, unlike more standard volatility attributable to news on the general prospects of the industry.

Li, Yuanshun and Gordon Sick (2010) examine empirically the equilibrium of firms investment decision for a context where firms' output price and production volume are uncertain, firms may choose to invest cooperatively or competitively, and there are economies of scale (i.e., network effects). In their game setting, interacting firms play a real option bargaining and exercise game under incomplete information. Their results show that the probability of cooperation is positively affected by the network effect and negatively affected by the real option exercise price[31].

Models like those of Maeland (2002, 2006, 2007), Hsu and Lambrecht (2003), Décamps and Mariotti (2004), a part from the assumption about the incomplete information also assume that firms are ex-ante asymmetric of about some parameters of the model. Hence, they are also reviewed in a previous section where we address the NSROG with "asymmetric information" (see pp. 33-39).

ROG Models with Incomplete Information	Incomplete Information Type	Key Model Parameters
1. Martzoukos and Zacharias (2001)	**Ex-ante**: duopoly market, the **information is incomplete** in the sense that the controls' outcome is random. These controls outcome mechanism is classified according to 2 types: (i) "pure learning control actions", with the sole purpose of information acquisition that reduce uncertainty, and (ii) "impact control actions" with direct value enhancement (such as cost reduction) purpose.	**Assumptions**: Two firms have an investment opportunity and the possibility of enhancing its value through: (i) direct actions such as R&D that improves product attributes or reduces costs, advertisement, etc, or (ii) indirectly through information acquisition such as exploratory drilling, market research, etc. Due to spillovers, each firm's action affects the other firm and firms can act strategically taking advantage of the positive spillovers, or taking pre-emptive actions to avoid negative spillovers. In equilibrium the degree of coordination can be high or low and the implementation of strategy by each firm can either be implicit or explicit (i.e., by forming a research joint venture). Ex-ante firms know the probability distribution of the outcome (denoted as "random controls"). Advertisements, process improvement, product attribute enhancement, etc. are actions that result directly in adding value. "Pure learning actions" are intended to improve the information about the project's underlying variables (i.e., potential sales price/quantity, etc.). The project's value follows the following stochastic process: $\frac{dS}{S} = \mu dt + \sigma dz^R + \sum_{i=1}^{N} k_i dq$ where, dZ^R is an increment of a standard Wiener process in the real probability measure, dq is a jump counter for managerial activation of action *i*, i.e., a control (not random) variable.

[31] This is an empirical work so it is not described in Table 2.

2. Murto and Keppo (2002)	**<u>Ex-ante</u>**: asymmetric (different valuations for the investment project) and **<u>incomplete information</u>** about each other's project values (although the **<u>complete information</u>** scenario is also analyzed). As soon as one firm triggers the investment for the first time, the value of the investment for the others jumps to zero (this is a WTA game).	Firm $i \in \{1,...,n\}$ Competitors' project values mapping: $\Gamma_i : R_+^n \longrightarrow \{0,1\}$ "0" means "do not invest"; "1" means "invest" Project values evolve according to gBm. If firm $i,\ \Gamma_i \in (V_1,...,V_n) = 1$ The game ends when firm when $i,\ \Gamma_i \in (V_1,...,V_n) = 1$ for the first time. If so, firm i gets payoff *Vi-Ii* (with *Ii* being the inv. cost) and its opponents get zero. Firm's project value follows the following process: $dV_i = \alpha_i V_i dt + \sigma_i V_i dz_i - V_i dq$ where, $dq = \begin{cases} 1 & \text{with prob. } \lambda dt \\ 0 & \text{with prob. } 1\text{-}\lambda dt \end{cases}$ and, λ is a constant hazard rate of losing the inv. opportunity and the rest of the gBm process differential equations have the usual meanings.
3. Décamps, et al. (2002) <u>Note</u>: this is a monopoly game of one firm against nature. The framework used is not the same as that used in ROG.	**<u>Ex-ante</u>**: monopoly investment decision in an indivisible project whose value is perfectly observable but driven by a parameter that is unknown to the decision maker ex-ante.	**<u>Assumption/framework</u>**: Infinitely lived decision maker, Risky project, sunk cost, irreversible investment, time is continuous, The value of the project follows a Brownian motion process: $dV_t = \mu dt + \sigma dW_t$ where the standard Wiener process is independent of μ. **<u>Information Structure</u>**: A key assumption is that the decision maker does not know ex-ante the true value of μ. Ex-post, the decision maker perfectly observes the value process V but neither the drift nor the evolution of W. The only information available to the decision maker is generated by the value process. The decision problem is to find a stopping time $\tau^* \in \tau^V$ such that: $\sup_{\tau \in \tau^V} \mathrm{E}\left[e^{-r\tau}(V_\tau - I) \right] = \mathrm{E}\left[e^{-r\tau^*}(V_{\tau^*} - I) \right]$
4. Maeland (2002) <u>Note</u>: Maeland (2006, 2007) use a similar mathematical framework. Hence, to save space, these articles are not described here.	**<u>Ex-ante:</u>** multi-agent game where each agent, *i*, has private information about his own costs of the investment but has no private information about the competitors' costs. Auctioneer does not observe the n agents' inv. cost parameter, but it is common knowledge that the values are drawn from the same distribution.	Agent *i* has private information about his own inv. cost: K^i Competitors' costs are defined by a vector: $K^{-i} = (K^1,...,K^{i-1},K^{i+1},...,K^n)$ The inv. cost values are drawn from the same distribution function: $F(.)$

5. Lambrecht and Perraudin (2003) Note: The assumptions underlying this model mean that the multiple-firm equilibrium is similar to those that arise in models of "first price auctions" under incomplete information with a continuum of types.	**Ex-ante:** Duopoly/multi-firm markets. This paper introduces **incomplete information** and **pre-emption** into an equilibrium model of firms that have the opportunity to enter into a new market. The model focuses mainly on duopoly markets, although a multi-firm equilibrium is also derived, and shows that the optimal investment trigger may lie anywhere between the zero-NPV trigger (the so called Marshallian trigger) and the firm's optimal monopolistic, non-competitive trigger (referred as the non-strategic option trigger), depending An application illustrative example is the "spatial investment strategy" followed by the US discount retailer Wal-Mart, described in the well-known Harvard Business School case study by Ghemawat (1986).	**Multi-firm Equilibrium:** Assume that there are 2 firms, labeled $i = 1, 2$ which can invest in the income inflow $X(t)$ described by the following equation: $dx = \mu X dt + \sigma X dB$, where B is a standard Brownian motion and μ and σ are the drift and volatility of the variable *X*, respectively. **Duopoly market with a threat of pre-emption**: Suppose a firm i can invest at a cost, Ki, in the income stream, X(t), describe above. However, another firm j may invest first, in which case firm i loses any further opportunity to invest. To introduce incomplete information, it is assumed that firm i conjectures that firm j invests when X(t) first crosses some level $\bar{X}_j$, and that $\bar{X}_j$ is an independent draw from a distribution $F_j(\bar{X}_j)$, where $F_j(\bar{X}_j)$ has a continuously differentiable density $F_j^{'}(\bar{X})$ with positive support on an interval, $[\bar{x}_L, \bar{x}_U]$. **Incomplete Information**: is introduced by supposing that the *i*th firm observes its own cost, Ki, but knows only that Kj, $j \neq i$, is an independent draw from a distribution G(k). G(k) has a continuously differentiable density, G'(k), with strictly positive support on an open interval (k_L, K_U).
6. Hsu and Lambrecht (2003) Note: The framework used in this paper to take into account incomplete information is similar to that used in Lambrecht and Perraudin (2003)	**Ex-ante:** in a patent race, the challenger has complete information about the incumbent but the incumbent has incomplete information about its opponent's investment cost (i.e., it does not know the precise value of the opponent's investment cost). See pp. 25-28 for a comparative analysis between models with symmetric/asymmetric and complete/incomplete information.	**Assumptions:** Before entry occurs, the incumbent produces only one product that has a patent of infinite duration. Entry into the monopolized market can be gained only by patenting a substitute for the incumbent's present product. The costs of acquiring the new patent are *Ki* for the incumbent and *ke* for the entrant. *Ki* is publicly known; Ke known only by the entrant, the incumbent knows that it is drawn from a probability distribution G(Ke) that has a continuous probability density function G'(Ke) and a positive support [KL,KU], i.e., this framework is similar to that of Lambrecht and Perraudin (2003). **Formulation**: As soon as the patent is acquired, the second product will be launched without any further cost. Depending on whether and by whom the second patent has been acquired, the market structure will be (i) a monopoly with only one product, (ii) a monopoly with two product, or (iii) a duopoly with two products. There are no capacity constrains, no production costs and, therefore, no returns to scale. Before the second product is launched the incumbent makes a profit of: $\pi_{1i} x_t$, where, x_t is a stochastic variable representing demand shocks and follows a standard gBm process given by: $dx_t = x_t \mu dt + x_t \sigma dW_t$ π_{1i} is a strictly positive constant and the following inequality holds: $\pi_{2i} > \pi_{1i} > \delta_i$, where, π_{1i} - incumbent's profit without the new patent π_{2i} - incumbent's profit if succeeds in the new patent. δ_i - incumbent's profit if the entrant acquires the new patent.

Table 2 – ROG with Incomplete Information

4.2.6 Multi-Factor Models

Typically the process underlying most real options is gBm for a single factor, but several authors have introduced multi-factor projects, with values affected by uncertain price, quantity, operating cost, and with sometimes stochastic investment costs.

Huisman (2001) supplies several real options models, for several different economic contexts, applied to new technology investments, namely models for monopoly and duopoly markets, with constant and non-constant investment costs, with one, two or multiple new technology(ies) available, with and without technological uncertainty.

Paxson and Pinto (2003) derive, for a duopoly market, firms' value functions assuming that the leader's market share evolves according to an immigration (birth) and death process. Paxson and Pinto (2005) provide a real options model for a duopoly market using two stochastic underlying variables and show that the degree of correlation between the two variables results in different value functions and investment thresholds, especially for the follower. They also consider the case where firms invest simultaneously in a non pre-emption game. Paxson and Arun Melmane (2009) assume that market share for search engines evolves deterministically, but is subject to synergy shocks from complementary activities.

Huisman and Kort (2004) assume a new technology adoption game similar in many respects to that studied previously, except that arrival of a second and more efficient technology is assumed to follow a Poisson process.

Azevedo and Paxson (2009) develop a real options model for a duopoly market to optimize investment decisions on new technologies whose functions are complementary (adding a second technology improves the efficiency of the first technology). They arrive at analytical and quasi-analytical solutions for the leader and the follower value functions and their respective investment thresholds. At the beginning of the investment game, firms have two technologies available, whose functions are complements. There is an option to adopt both technologies at the same time or at different times, in a context where the evolution of the gains that can be made through the adoption of the technology(ies) and the cost of the technologies are uncertain. Their results contradict the conventional wisdom which says that "when a production process requires two extremely complementary inputs, a firm should upgrade (or replace) them simultaneously". They find that when uncertainty about revenues and the price of the two technologies is considered it might be optimal for the leader and the follower to adopt the two technologies asynchronously, first, the technology whose price is decreasing at a lower rate and then the technology whose price is decreasing more rapidly.

4.2.7 Capacity Choice

Most of the real option game models reviewed above ignore the operating decisions that may arise once the investment is completed.

Filipe Aguerrevere (2003) studies strategic investment behavior in a real options framework that includes more realistic features of investment projects such as capacity choice. He considers the effects of competitive interactions on investment decisions and the dynamics of the price of a storable commodity, in a model of incremental investment with time to build and operating flexibility, extending the classic capacity choice models of Pindyck (1988), and Hua He and Robert Pindyck (1992). A firm must decide how much capacity to build initially and when to expand it later, and has the option not to use any incremental unit of capacity if demand falls. Aguerrevere shows that with time to build, more uncertainty may encourage the firm to hold more capacity, and that firms' optimal capacity may be larger under uncertainty than under certainty. This result contrasts with that from models of incremental investment which assume no "construction lags" and where it has been shown that there is a negative effect of uncertainty on capacity choice. Other works close to this approach are those of Baldursson (1998) and Grenadier (2002). However, Baldursson (1998) assumes that investment is instantaneous and installed capacity is fully utilized, and the example analyzed indicates that qualitatively the price process will be the same in oligopoly and perfect competition.

Grenadier (2002) develops an approach to solving for investment equilibrium that is applicable to a more general specification of demand. Neither assumes flexibility in the use of the installed capacity, so the resulting output price behavior is the same for different numbers of firms in the industry.

Murray Carlson, Dockner, Fisher, and Ron Giammarino (2006) study the relationships between industry and individual firm risk that reflect the strategic interplay of option exercise by imperfectly competitive firms, characterizing the industries as adolescent, juvenile and mature, and examining the risk dynamics of heterogeneous duopolistic firms that strategically manage options to expand and contract capacity.

Jianjun Wu (2006) explores the problem of firms' incentives to expand capacity using a ROG model, where two *ex-ante* identical firms can choose capacity and investment timing regarding the entry into a new industry. Demand grows until an unknown maturity date and declines thereafter. Firms are allowed to entry and exit when it is optimal to do so.

Aguerrevere (2009) studies the effects of competitive interactions among firms on asset returns in a real options framework. More specifically, he investigates how competition in the product market affects the link between firms' real investment decisions and their return dynamics in competitive industries. The assets in place and firms' growth options have different sensitivities to market uncertainty, and the behavior of the market participants influences the relative importance of these components of the value of the firms. At any time t firms play a static Cournot game where each chooses its output level to maximize its profits, and the optimal investment decision is an endogenous Nash equilibrium solution in investment strategies, where "production capacity" is the strategic variable. Aguerrevere shows that when firms have the ability to vary their capacity utilization in response to a shock in demand, output price volatility is increasing in the number of firms in the industry; firms in competitive industries are riskier when demand is low because operating leverage makes assets in place riskier than growth options; without production costs increased competition always reduces risk, and suggests that an unconditional competition premium must be associated with high production costs, while low production costs lead to an unconditional concentration premium.

4.2.8 Other Innovative Parameters

Some interesting innovative parameters include (i) correlated firm profitabilities, (ii) synergies in joint activities, and (iii) games between lenders and borrowers in restructuring debt.

Mark Shackleton, Tsekrekos, and Rafal Wojakowski (2004) analyze for a duopoly market the entry decision of the competing firms when rivals earn different but correlated uncertain profitabilities, allowing each firm's decision to be subject to a firm-specific stochastic variable. Their results show that, in the presence of entry costs, decision thresholds exhibit hysteresis. The range of hysteresis decreases as the correlation between firms increases. They determine an explicit measure for the expected time of each firm being active in the market and the probability of both rivals entering within a finite time. An illustration is supplied using the rivalry case of Airbus's launch of the A380 super carrier and Boeing's optimal response to that strategic move.

Alexandre Ziegler (2004) uses game theory to study leverage and bankruptcy, following Hayne Leland (1994), arguing that the payoff values of borrowers-lenders are strategic real options. There are several extensions of this approach over the past five years, including foreclosures and debt renegotiation strategies.

Sundaresan, Suresh and Neng Wang (2007) develop a framework to model the role of financial architecture on ex-ante growth option exercising decisions and firm value when debt offers tax benefits. They show that stronger equity holder's bargaining power lowers debt capacity, reduces firm value, and delays growth option exercising.

5. Empirical Research and Testable Hypotheses

5.1 Basis for First-Mover Advantage

Assuming some reasonable definition for what constitutes a first-mover advantage, for empirical work on oligopolies, there still remains the problem of distinguishing among later entrants. According to Marvin Lieberman and David Montgomery (1988), such entrants can be classified by (i) their numerical order of sequence of entry, (ii) elapsed time since entry of the pioneer, or (iii) general categories such as early follower, late follower, etc, although these categories may not be comparable across markets. Afterwards, we need to define measures of the first-mover advantage. Given that profit maximization is the primary objective of shareholders in modern theories of firm, economic profit or economic profit-related variables are the appropriate measure. However, disaggregate profit data are seldom available. Market share and rates of firm survival can also be used as surrogate measures, since both have been shown to be correlated with profits (although the correlation is not always high and causality is often ambiguous)[32].

The magnitude and duration of the first-mover advantage may depend on the point in time that the market is observed. For instance, a firm (first-mover) protected by a patent can earn substantial profits during the patent protection, but its profits can fall substantially once the patent expires, making its first-move less profitable than later moves if analyzed for all the life of the underlying project. Empirical answer to the questions of what conditions constitute a first-mover advantage and over what timeframe (ephemeral *versus* long-lived), and how does this vary by economic mechanisms and by industry, would be helpful to calibrate and test empirically real options models.

The concept of first-mover advantage helps to provide a unifying real options framework to analyze investments in competitive contexts. However, the mechanisms that benefit the first-mover advantage[33] may be counterbalanced by various disadvantages in the sense that late movers can benefit from, for instance, a "free-ride" on the pioneer's investment and a resolution of the market,

[32] Lieberman and Montgomery (1988), p. 51, provide a good discussion on this topic.

[33] In the literature of real options and economics, several possible justifications have been stated as the reason(s) for one firm taking the leadership in the investment, such as: (i) the technological leadership, namely through advantages derived from the "learning curve" and success in patent races; (ii) the pre-emption of assets, where by pre-empting rivals in the acquisition of scarce assets the first-mover gains advantage by controlling assets that already exist; and (iii) the buyer switching costs, where late entrants are required to invest extra resources to attract customers away from the first-mover firm.

technical and technological uncertainty, as Cottrell and Sick (2001) illustrate with realistic examples. In the real option literature, a first/second-mover advantage is defined as the ability of pioneering/follower firm to earn a positive economic profit in excess of the follower/leader. However, a given firm cannot simply choose whether or not to pioneer. Pioneering opportunities may arise endogenously, although it is not yet clear under what conditions do first/second-mover advantages arise and by what specific economic mechanisms. There are few answers for when it is in a firm's interest to pursue first-mover opportunities, as opposed to allowing rivals to make the pioneering investment. In the ROGs literature these conceptual issues are resolved through the (unsatisfactory) assumption that, for some hidden reason, one firm will invest first, and, if both firms might invest at the same time, one of them will become the leader by flipping a coin (even though, most models make the contradicting assumption that firms are initially symmetric). ROG models need to be more precise in elucidating first/second-mover economic mechanisms to avoid being too general, deterministic and definitionally elusive.

According to the theoretical models surveyed in Lieberman and Montgomery (1988), first-mover advantages arise from the existence of some initial asymmetry among firms. Without this asymmetry, it is argued, first-mover advantage does not arise. This assumption collides, however, with the assumption of ex-ante asymmetry between firms used in SROG models.

5.2 Measuring Competition

A problem in most industries is that competition cannot be measured directly, as costs and often also price data of single products are usually unavailable. Hence, indirect measures are needed. Most studies of competition rely on one of the two standard competition measures that capture the classic determinants of competition: the number and relative sizes of firms. Usually, the presence of more firms is associated with more competition. The simplest type of measure counts the number of competing firms. It is an easy measure, but it does not capture the relative sizes of firms, which can play an important role in competition. Another common measure of competition is the Herfindahl-Hirschman index (HHI). The HHI for a market is the sum of the squared market shares of all the firms competing in the market, where higher HHI means less competition. Recent articles have contributed to the introduction of new industry-related measures of competition, such as Pinelopi Goldberg and Michael Knetter (1999), who develop an approach to measure competition in export markets; Laurence Baker (2000) who presents a new technique to measure competition in the health care market; and more recently Michiel Leuvensteijn, et al. G(2007), who, based on the HHI, suggest a new measure of competition, the Boone indicator, for EU banking industry. Some of these techniques can be used to determine empirically the competition factors of ROG models and, therefore, to reduce the often arbitrary assumptions of investment leadership.

5.3 Testable Hypotheses

There are many testable hypotheses arising from the SROG and NSROG literature, although limited empirical testing or calibration of theoretical parameters to date In the Appendix, Table 2, last column, are some of the most common applications of SROG and NSROG: "R&D investments" (at a firm-level, patent race strategies, design of incentives for individual or group of researchers, allocation of funds among competing projects and setting of investment strategies, and, at a country level, setting of innovation policies, tax incentives and direct subsidies); "new technology adoptions" (timing the adoption of new technologies in contexts where there are one or several technologies available, with or without technological uncertainty, in competitive markets with first or second mover advantages); "production capacity choices" (when to expand/reduce capacity); and "real estate investments" (optimization of project design and location first-mover advantage) are among the most popular cases.

With the increasing sophistication of the information technologies and marketing monitoring techniques, frequent and public monitoring is today sometimes feasible, for instance in the cases of public marketing of innovations, FDA applications and patent applications. However, brand loyalty and differential pricing for the first-mover is not always transparent or measurable (for instance, for e-banking and internet service providers, only some companies provide frequent information on new accounts, churn and usage, and seldom provide detailed frequent information on profits (see Paxson, 2003, pp. 318-320). Progress on this monitoring area would allow the calibration of SROG and NSROG models and the empirical test of hypotheses.

6. Successes and Shortfalls

The ROG models reviewed above address modern questions in investment decisions, provide new solutions to investment problems, and contribute to a better understanding of the complex nature of firms' investment behavior in markets where uncertainty and competition hold. SROG advances real option models beyond monopolies by considering investments by rival firms, which alter market share, product profitability or market size. Standard determination of leadership by artificial assumptions and simple pre-emption has been improved in some NSROG articles assuming ex-ante asymmetric advantages, or wars of attrition.

The number of techniques and theories that have been combined with the real options framework/theory is high. Regarding the modeling of the ROG underlying variables the processes used range from the common gBm process to gBm with jumps, birth and death processes and mean-reverting processes; concerning the techniques used to combine the real options framework with those process, to optimize the investment decision/determine the equilibrium of the investment game range from the common game theory and the optimal control theories to the filtering and the auction theories. The scale of researchers working around the ROG topic and the accumulation of ROG models and respective results has been growing since the early nineties and are now substantial. Therefore, this is perhaps a good time to reflect on what has been done, what our successes and shortfalls were and which new research avenues are we able, or it would be wise, to tackle. This paper is an essential tool for carrying out such endeavor.

We anticipate that new and more sophisticated NSROG models will be developed in the near future, considering a broader range of competition settings, markets and economic contexts and, possibly, making better use of the powerful game theory mathematical techniques. Due to the high degree of innovation and intensity of competition, technological industries and more specifically R&D investment and the optimization of new technology adoptions timing and the launch of new products in the market will continue to be natural areas of application of real options game models.
There are few empirical works in the literature, and little empirical testing or calibration of theoretical parameters. The practical use of NSROG depends on such quantitative improvements.

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Appendix 1 - Game Theory Aspects underlying the most Relevant Literature on Real Option Games

	TableA (Papers)	Formalism		Game Information						Type of Game													Firms			Leadership		Industry or Application(s) (Suggested/used in the article)
		Discrete time	Continuous time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Time Horizon		Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exogenous	Endogenous	
																	Finite	Infinite										
1	Aguerrevere (2003)		x	x		x		x		x	x	x	x		x			x		x	x		x	x	x	x		Manufacturing
2	Aguerrevere (2009)		x	x		x		x		x	x		x		x			x		x	x				x	x		Manufacturing
2	Armada, et al. (2009)		x	x	x	x		x			x	x			x			x	x	x	x	x		x		x		Not Specifyed
4	Anderson, et al. (2010)		x		x	x		x			x	x	x	x	x			x		x	x		x	x	x		x	Mining & Retailing
5	Azevedo and Paxson (2009)		x	x		x		x			x	x			x			x		x	x			x		x		New Technology Adoptions
5	Baba (2001)		x	x		x		x			x	x			x			x		x		x		x		x	x	Banking
7	Bouis, et al. (2009)		x	x		x		x		x	x	x			x			x		x	x				x	x		Not Specifyed
8	Boyer, et al. (2001)		x			x		x			x	x			x			x	x	x	x			x			x	Manufacturing
9	Carlson, et al. (2006)		x	x		x		x			x	x			x			x		x		x		x		x		Manufacturing
10	Cottrell and Sick (2001)		x	x		x		x			x	x			x			x		x	x			x		Empirical research: with illustrations from several industries		Patent Race R&D Hardware Software
11	Cottrell and Sick (2002)		x	x		x		x			x	x			x			x		x	x			x		x		Innovation Investment projects
12	Décamps and Mariotti (2004)		x		x	x		x	x		x	x			x			x		x		x		x			x	Not Specifyed
13	Dixit and Pindyck (1994)		x	x		x		x			x	x			x			x		x	x		x	x	x	x		Not specifyed (Textbook)
14	Garlappi (2001)	x		x		x		x			x		x	x	x			x		x	x			x			x	Patent Races R&D ventures
15	Goto, et al. (2008)		x	x		x		x			x		x		x			x		x	x			x		x		Manufacturing
16	Grenadier (1996)		x	x		x		x		x	x	x			x			x		x	x			x		x		Real Estate

	Table A (cont.)	Formalism		Game Information						Type of Game													Firms			Leadership		Industry or Application(s)
	(Papers)	Discrete time	Continuous Time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Time Horizon		Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	> 2	Exosgenous	Endogenous	(Suggested/used in the article)
																	Finite	Infinite										
17	Grenadier (1999)		x	x			x		x	x	x	x	x		x			x		x		x		x	x	x	x	**Real Estate Oil well drilling Pharmaceutical**
18	Grenadier (2000a)		x	x		x		x		x	x	x	x		x			x		x	x	x		x	x	x	x	**Not Specifyed (Textbook)**
19	Grenadier (2000b)		x	x		x		x			x				x			x		x	x			x		x	x	**Real Estate Oil well drilling Pharmaceutical**
20	Grenadier (2002)		x	x		x		x		x	x	x	x		x			x		x	x				x		x	**Real Estate**
21	Huisman (2001)		x	x		x		x		x	x	x			x			x		x	x Ch 7, 9	x Ch 8		x		x		**New Technology Adoptions (Textbook)**
22	Huisman and Kort (2004)		x	x		x		x			x	x			x			x		x	x			x		x		**New Technology Adoptions**
23	Hsu and Lambrecht (2003)		x		x		x		x		x	x			x			x		x		x		x		x		**Patent Race**
24	Kong and Kwork (2007)		x	x		x		x		x	x	x			x			x		x		x		x		x		**Not Specifyed**
25	Kulatilaka and Perotti (1998)	x		x		x		x		x	x	x			x			x		x	x	x	x	x		x		**Standard** Investment Projects
26	Joaquim and Butler (1999)		x	x		x		x			x	x			x			x		x	x			x		x		**Not Specifyed** (published in a textbook)
27	Lambrecht (1999)		x	x		x		x			x	x			x			x		x	x			x	x		x	**Patent Race** (published in a textbook)
28	Lambrecht and Perraudin (2001)		x	x		x		x			x	x			x			x		x	x		x	x	x	x		**Baking**
29	Lambrecht and Perraudin (2003)		x		x	x		x	x		x	x			x			x		x	x		x	x	x		x	**Retailing**
30	Maeland (2006)		x		x		x		x		x		x	x				x		x		x			x	x		**Project Auction**
31	Maeland (2007)		x		x		x		x		x		x	x				x		x		x			x	x		**Project Auction**
32	Martzoukos and Zacharias (2001)		x		x		x	x		x	x		x		x			x		x	x			x			x	**Oil industry R&D**

	Table A (cont.) (Papers)	Formalism		Game Information						Type of Game													Firms			Leadership		Industry or Application(s)
		Discrete time	Continuous Time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Time Horizon: Finite	Time Horizon: Infinite	Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exosgenous	Endogenous	(Suggested/used in the article)
33	Mason and Weeds (2001)		x	x		x		x		x	x	x			x			x	x	x	x			x		x		**New Technology Adoptions Satellite Systems Internet**
34	Mason and Weeds (2005)		x	x		x		x		x	x	x			x			x		x	x	x		x		x		**Not Specifyed**
35	Murto and Keppo (2002)		x	x	x	x		x	x	x		x		x				x		x	x	x	x	x	x	x		**Tele communication**
36	Murto (2004)		x	x		x		x		x	x	x	x		x			x		x	x	x		x		x		**Not Specifyed**
37	Murto, et al. (2004)	x		x		x		x		x		x			x			x		x	x				x	x		**Non-storable Commodity**
38	Nielson (2002)		x	x		x		x		x	x	x			x			x		x				x	x	x		**Software hardware**
39	Odening, et al. (2007)		x	x		x		x		x	x	x			x			x		x	x				x		x	**Agriculture**
40	Pawlina and Kort (2006)		x	x		x		x		x	x	x			x			x		x		x		x		x		**Not Specifyed**
41	Paxson and Pinto (2005)		x	x		x		x		x	x	x			x			x		x	x			x		x		**Not Specifyed**
42	Paxson and Pinto (2003)		x	x		x		x		x	x	x			x			x		x	x			x		x		**Tele communication**
43	Paxson and Melmane (2009)		x	x		x		x			x	x			x			x		x	x			x		x		**Internet**
44	Pereira and Rodrigues (2010)		x	x		x		x			x	x				x		x		x	x				x	x		**Not Specifyed**
45	Reiss (1998)		x	x		x			x	x	x	x			x			x		x		x	x		x	x		**Patent Race**
46	Ruiz-Aliseda (2005)		x	x		x		x			x		x		x			x		x		x		x		x		**Not Specifyed**
47	Savva and Scholtes (2005)	x		x	x		x		x	x	x	x			x	x		x	x	x		x		x		x		**Biotech and Pharmaceutical**
48	Shackleton, et al. (2004)		x	x		x		x			x	x			x			x		x	x	x		x			x	**Aircraft Industry**

	Table A (cont.)	Formalism		Game Information						Type of Game													Firms			Leadership		Industry or Application(s)
																	Time Horizon											
	(Papers)	Discrete time	Continuous Time	Complete	Incomplete	Perfect	Imperfect	Symmetric	Asymmetric	Simultaneous	Sequential	One -shot	Large	Winner Takes All	Zero-sum	Nonzero sum	Finite	Infinite	Cooperative	Non cooperative	Ex-ante Symmetric	Ex-ante Asymmetric	1	2	>2	Exosgenous	Endogenous	(Suggested/used in the article)
49	Smets (1993)		x	x		x		x			x	x			x			x		x	x			x		x		Not Specifyed
50	Smit and Ankum (1993)	x		x		x		x		x	x	x			x			x		x	x			x		x		R&D Investment
51	Smit (2003)	x		x		x		x		x	x	x			x			x		x	x			x		x		Public Infrastructure R&D
52	Smit and Trigeorgis (2004)	x chapter 4, 5, 6, 7	x	x	x chapter 9	x	x chapter 7, 9	x	x chapter 7, 9		x	x		x chapter 8	x			x	x chapter 8	x	x		x	x		x		Electronics R&D Oil (Textbook)
53	Sparla (2004)		x	x		x		x		x	x	x			x			x		x	x	x		x		x		Manufacturing
54	Thijssen (2004)		x	x		x		x	x	x	x	x			x			x	x	x	x			x	x	x	x	R&D (Textbook)
55	Thijssen, et al. (2002)		x	x		x		x		x	x	x			x			x		x	x			x		x	x	Not Specifyed
56	Tsekrekos (2003)		x	x		x		x		x	x	x			x			x		x	x			x		x		Not Specifyed
57	Watanabe (2010)		x		x	x			x		x	x			x			x		x		x		x		x		Not Specifyed
58	Weeds (2002)		x	x		x		x			x	x		x				x	x	x	x			x		x		Patent Race
59	Weyant and Yao (2005)		x	x		x		x		x	x		x		x			x		x	x			x			x	R&D
60	Williams (1993)		x	x		x		x		x		x			x			x		x	x				x		x	Real Estate
61	Wu (2006)		x	x		x		x		x	x		x		x			x		x	x			x		x		Manufacturing

“Non-Standard” Real Option Game Models (Game Formulation)			
Game Information		**Type of Game**	
Incomplete	**Imperfect**	**Large Game**	**Winner-Takes-All**
1. Anderson, et al. (2010) 2. Armada, et al. (2009) 3. Anderson, et al. (2010) 4. Décamps et al. (2002) 5. Décamps and Mariotti (2004) 6. Hsu and Lambrecht (2003) 7. Lambrecht and Perraudin (2003) 8. Maeland (2006) 9. Maeland (2007) 10. Martzoukos and Zacharias (2001) 11. Murto and Keppo (2002) 12. Savva and Scholtes (2005) 13. Smit and Trigeorgis (2004, ch 9)	1. Grenadier (1999) 2. Hsu and Lambrecht (2003) 3. Maeland (2006) 4. Maeland (2007) 5. Martzoukos and Zacharias (2001) 6. Savva and Scholtes (2005) 7. Smit and Trigeorgis (2004, ch. 7, 9)	1. Aguerrevere (2003) 2. Aguerrevere (2009) 3. Anderson, et al. (2010) 4. Garlappi (2001) 5. Goto, et al. (2008) 6. Grenadier (1999) 7. Grenadier (2000a) 8. Grenadier (2002) 9. Maeland ((2006) 10. Maeland (2007) 11. Martzoukos and Zacharias (2001) 12. Murto (2004) 13. Ruiz-Aliseda (2005) 14. Weyant and Yao (2005) 15. Wu (2006)	1. Anderson, et al. (2010) 2. Garlappi (2001) 3. Maeland (2006) 4. Maeland (2007) 5. Murto and Keppo (2002) 6. Smit and Trigeorgis (2004, ch 8) 7. Weeds (2002)
Type of Game		**N-Firms**	**Leadership**
Cooperative	**Firms: Ex-ante Asymmetric**	**N > 2**	**Endogenous**
1. Anderson, et al. (2010) 2. Armada, et al. (2009) 3. Boyer et al. (2001) 4. Mason and Weeds (2001) 5. Savva and Scholtes (2005) 6. Smit and Trigeorgis (2004, ch 8) 7. Thijssen (2004) 8. Weeds (2002)	1. Armada, et al. (2009) 2. Baba (2001) 3. Carlson, et al. (2006) 4. Décamps and Mariotti (2004) 5. Grenadier (1999) 6. Grenadier (2000a) 7. Huisman (2001, ch. 8) 8. Hsu and Lambrecht (2003) 9. Kong and Kwork (2007) 10. Kulatilaka and Perotti (1998) 11. Maeland (2006) 12. Maeland (2007) 13. Mason and Weeds (2005) 14. Murto and Keppo (2002) 15. Murto (2004) 16. Pawlina and Kort (2006) 17. Reiss (1998) 18. Ruiz-Aliseda (2004) 19. Savva and Scholtes (2005) 20. Shackleton, et al. (2004) 21. Sparla (2004) 22. Watanabe (2010)	1. Aguerrevere (2003) 2. Aguerrevere (2009) 3. Anderson, et al. (2010) 4. Bouis, et al. (2009) 5. Dixit and Pindyck (1994), ch. 8, 9 6. Grenadier (1999) 7. Grenadier (2000a) 8. Grenadier (2002) 9. Lambrecht (1999) 10. Lambrecht and Perraudin (2001) 11. Lambrecht and Perraudin (2003) 12. Maeland (2006) 13. Maeland (2007) 14. Murto and Keppo (2002) 15. Murto, et al. (2004) 16. Nielson (2002) 17. Odening, et al. (2007) 18. Pereira and Rodrigues (2010) 19. Reiss (1998) 20. Thijssen (2004) 21. Williams (1993)	1. Andeson, et al. (2010) 2. Baba (2001) 3. Boyer, et al. (2001) 4. Boyer, et al. (2004) 5. Décamp and Mariotti (2004) 6. Garlappi (2001) 7. Grenadier (1999) 8. Grenadier (2000a) 9. Grenadier (2000b) 10. Grenadier (2002) 11. Lambrecht (1999) 12. Lambrecht and Perraudin (2003) 13. Martzoukos and Zacharias (2001) 14. Odening, et al. (2007) 15. Shackleton, et al. (2004) 16. Thijssen (2004) 17. Thijssen, et al. (2002) 18. Weyant and Yao (2005) 19. Williams (1993)

Table B – Game Formulation: Non-Standard Real Option Game Models

Benchmarking ROG Formulations/Results		
Game Information		**Type of Game**
Complete *vs* **Incomplete**	**Symmetric** *vs* **Asymmetric**	**Simultaneous** *vs* **Sequential**
1. Armada et al. (2009) 2. Murto and Keppo (2002) 3. Savva and Scholtes (2005) 4. Smit and Trigeorgis (2004)	1. Décamps and Mariotti (2004) 2. Lambracht and Perraudin (2003) 3. Murto and Keppo (2002) 4. Smit and Trigeorgis (2004)	1. Aguerrevere (2003) 2. Aguerrevere (2009) 3. Bouis, et al. (2009) 4. Grenadier (1996) 5. Grenadier (1999) 6. Grenadier (2000a) 7. Grenadier (2002) 8. Huisman (2001, ch.7, 8, 9) 9. Kong and Kwork (2007) 10. Kulatilaka and Perotti (1998) 11. Martzoukos and Zacharias (2001) 10. Mason and Weeds (2001) 11. Mason and Weeds (2005) 12. Murto (2004) 13. Nielson (2002) 14. Odening, et al. (2007) 15. Pawlina and Kort (2006) 16. Paxson and Pinto (2005) 17. Paxson and Pinto (2003) 18. Reiss (1998) 19. Savva and Scholtes (2005) 20. Smit and Ankun (1993) 21. Smit (2003) 22. Sparla (2004) 23. Thijssen, et al. (2004) 24. Thijssen, et al. (2002) 25. Tsekrekos (2003) 26. Weyant and Yao (2005) 27. Wu (2006)

Type of Game				**Firms**
One-shot vs **Large**	**Zero-sum** *vs* **Non-zero sum**	**Cooperative** *vs* **Non-Cooperative**	**Ex-ante symmetric** *vs* **Ex-ante asymmetric**	**Monopoly** vs **2-firms** or **2-firms** *vs* **N** (n>2)
1. Aguerrevere (2003) 2. Anderson, et al. (2010) 3. Grenadier (1999) 4. Grenadier (2000a) 5. Grenadier (2002) 6. Murto (2004)	1. Savva and Scholtes (2005)	1. Armada et al. (2009) 2. Boyer et al. (2001) 3. Mason and Weeds (2001) 4. Savva and Scholtes (2005) 5. Smit and Trigeorgis (2004) 6. Thijssen (2004) 7. Weeds (2002)	1. Armada et al. (2009) 2. Grenadier (2000a) 3. Huisman (2001) 4. Kulatilaka and Perotti (1998) 5. Mason and Weeds (2005) 6. Murto and Keppo (2002) 7. Murto (2004) 8. Shackleton, et al. (2004) 9. Sparla (2004)	1. Aguerrevere (2003) 2. Anderson, et al. (2010) 3. Dixit and Pindyck (1994) 4. Grenadier (1999) 5. Grenadier (2000a) 6. Kulatilaka and Perotti (1998) 7. Lambrecht (1999) 8. Lambrecht and Perraudin (2003) 9. Murto and Keppo (2002) 10. Nielson (2002) 11. Reiss (1998) 12. Smit and Trigeorgies (2004) 13. Thijssen (2004)

Table C – Benchmarking ROG Formulations/Results

Appendix 2 – Complementary Information

a) Complete/Incomplete information game: An investment game with complete information means that knowledge about other firms or players is available to all participants, i.e., each player knows its own payoffs (or payoff functions) and strategies available and the payoffs and strategies available to the other players.

b) Perfect/Imperfect information game: Complete and perfect information are not identical. Complete information refers to a state of knowledge about the structure of the game and objective functions of the players, while not necessarily having knowledge of actions. The distinction between incomplete and imperfect information is somewhat semantic. For instance, in R&D investment games, firms may have "incomplete" information about the quality or success of each other's research effort and "imperfect" information about how much their rivals have invested in R&D, or the building of an office building, the drilling of an exploratory oil well, and the commitment of a pharmaceutical company toward the research of a new drug all convey private information to other market participants.

c) Symmetric/Asymmetric information game: Symmetric information means that all players participating in a game share the same information about the game, i.e., there are no players with more or less, better or worst information than other players.

d) Ex-ante Symmetric/Asymmetric game: "Ex-ante symmetric firms" are firms which, before the game starts, are identical in all parameters of the investment game model.

e) One-shot/Large game: For simplicity, in this review we define "one-shot" ROG as a game where firms have only one option to "invest", and the game ends as soon as this option is exercised; and a "large" ROG as a game where players have more than two options to "invest" (for instance, R&D multi-stage investment, see Garlappi, 2001). In the literature of game theory, however, a larger game is a game with many players, each with one or several strategies available, or a game with one player with a larger number of strategies available, or a game with one player with only one strategy available but which can be exercised a large number of times.

f) Zero-sum/winner-takes-all game: A zero-sum game is a game where the player(s)' gain/loss is exactly balanced by the loss/gain of the other participant(s) in the game. In a winner-takes-all game, there is no payoff for the loser(s), i.e., in a WTA first-mover advantage (leader/follower) ROG, there is no payoff for the follower.

g) Monopoly game against nature: Investment decisions in monopolistic markets can be modeled as a game of one player against nature. The description of these types of games are not, however, the object of this literature review, hence we only include articles about monopolistic investment games when the insights underlying their respective models are good illustrations of the concept/framework we aim to introduce/describe.

h) Sequential/Simultaneous game: A sequential game is a game with at least two players where one, or several, player(s) move first, initiating a necessarily sequential game. A simultaneous game is a game where at least two players invest at the same time.

i) Cooperative/Non-cooperative game: In non-cooperative games it is assumed that players cannot make a binding agreement, i.e., each cooperative outcome must be sustained by Nash equilibrium strategies. In cooperative games, firms have no choice but to cooperate.

j) Endogenous/Exogenous leadership: In standard leader/follower ROG models, the leadership in the investment is exogenously set and simultaneous moves are not allowed. In case simultaneous investment occurs, the leadership in the investment is chosen by flipping a coin. In investment games with more than two firms (oligopoly and perfect competition markets), the determination of the sequence by which firms invest is more challenging. Other branches of literature have more general frameworks like that of Leahy (1993). However, those frameworks have the disadvantage of determining the optimal investment behavior of all market players without specifying what they should do in case one, or several, players move first initiating a necessarily sequential investment game. Essentially, this latter framework advises firms about the adjustments they should perform over time assuming that all of them will react to market shocks, necessarily, at the same time. Both frameworks carry some practical inconsistencies.

In this review we define “endogenous leadership” as the leadership which takes place in contexts where firms randomize their actions, otherwise, we define the leadership is “exogenous” (i.e., it is deterministic and based on pure and exogenously set strategies). For some models the classification endogenous/exogenous is not straightforward. For instance, Smets (2003) framework is defined here as leading to “exogenous” leadership (driven by the competition inequality). Note, however, that although his framework is not based on pure strategies (N-tuple of strategies) firms’ investment thresholds depend on stochastic (random) underlying variable(s). Therefore, our approach was to define the leadership as exogenous if the firms’ strategies are “pure strategies” (deterministic), and endogenous leadership if firms’ actions are randomized, leading for instance to Bayesian Nash-equilibrium as in Anderson, et al. 2010. There are also models where the value of the firm’s payoff from investment (real option exercise) is endogenous as it depends on the exercise strategies of all option holders. In these cases, the optimal exercise strategies cannot be derived in isolation but must be calculated as part of a game-theoretic equilibrium.

l) **Mixed Strategies:** In a mixed strategy there is an assignment of probability to each player’s (pure) strategy, which permits firms to randomly select the strategies available. The introduction of mixed strategies into continuous-time games adds complexity into the framework. The prevailing method in the real options literature to avoid the existence of mixed strategies is to rule out simultaneous option exercise (as in Smets, 1993, Dixit and Pindyck, 1994, ch. 9, Grenadier 1996 among others).