Fixed odds bookmaking with stochastic betting demands

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Abstract

This paper provides a model of bookmaking in the market for bets in a British horse race. The bookmaker faces the risk of unbalanced liability exposures. Even random shocks in the noisy betting demands are costly to the bookmaker since his book could become less balanced. In our model, the bookmaker sets appropriate odds to influence the betting flow to mitigate the risk. The stylized fact of the favorite-longshot bias arises from the model under some specific assumptions. Our model offers insights into the complexity of managing a series of state contingent exposures such as options.

JEL Classification: G10, G13

Keywords: Horse betting; Bookmaking; State-contingent claims

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1 Introduction

The market for bets in the British horse race provides a good opportunity to understand the pricing of state contingent claims.[1] Shin (1992, p. 426-427) points out:

> ...In its simplest formulation, the market for bets in an n-horse race corresponds to a market for contingent claims with n states of the world, where the ith state corresponds to the outcome that the ith horse wins the race. Moreover, the basic securities (Arrow-Debreu securities) which pay a dollar if a particular state obtains and nothing otherwise, have their prices determined by the betting odds. Since odds are offered on each horse, all basic securities are traded...

Since a horse betting market is an especially simple financial market, in which the complexity of the pricing problem is reduced, it provides a clear view of pricing issues which are more complicated elsewhere.

In a horse race, the role of the market maker is taken by the bookmaker, and the traders are played by the potential bettors (Shin (1992)). In a series of papers, Shin (1991, 1992 and 1993) analyzes the price setting strategy when a bookmaker faces asymmetric information. Both systematic and anecdotal evidence suggests the prevalence of insider trading in the market for bets (Crafts (1985), also in Shin (1993)). Shin's analysis also addresses the well-known stylized fact that the percentage mark-ups in the prices over the true probabilities is not uniform. In general, prices exhibit the *favorite-longshot bias* in which, the normalized prices on the favorites of the race understate the winning chances of these horses, while the normalized prices on the longshots exaggerate their winning chances.

In this paper we employ the setting of a horse race betting market to understand the market making strategy in a market of state contingent claims such as options.[2] A bookmaker and an option market maker share many similarities in managing their state contingent exposures. In a horse race, there is one and only one horse that wins

[1]The horse race betting system in this paper is the British one where a bookmaker sets odds for difference horses prior to the start of the race. It is not the system in the North America where odds are determined by the parimutuel method in which prices are proportional to amounts wagered.

[2]The market making in securities market has been analyzed by Glosten and Milgrom (1985), Kyle (1985), and in more general economic context, Gould and Verrecchia (1985), Grinblatt and Ross (1985), and Laffont and Maskin (1990).

the race. Similarly, if an option market maker deals with butterfly spreads instead of a single option, there is one and only one option of the spreads that pays off at expiry. Furthermore, both a bookmaker and an option market maker have a fixed expiry of state contingent claims. For a bookmaker, it occurs when the race finishes and the winner is declared. For an option market maker, it occurs when option contracts expire.

More importantly, in a horse betting market, a bookmaker sells liabilities over all horses and tries to avoid large liability exposure should any horse wins. Similarly, an option market maker wants to have a balanced portfolio over all option exposures and to avoid large losses should the value of underlying assets (or more precisely, the implied volatilities of the options) moves against him. Both a bookmaker and an option market maker use odds (prices) to influence betting (order) flow to maintain a balanced portfolio of their state contingent claims. Given these similarities, we attempt to extend the market making literature by taking up this natural approach to modeling market making state contingent claims.

Our chief concern in this paper is to address the problem of managing a portfolio of state contingent exposures. We model a representative bookmaker who faces the risk of an unbalanced book. A book is liable to become unbalanced because betting demands are noisy and the bookmaker may not know correct odds to quote. Furthermore, whenever his book becomes unbalanced, the bookmaker wants to re-balance it so that the problem of having high liability exposures can be alleviated. Even random shocks from noisy traders are costly to the bookmaker since his book could become less balanced.

The bookmaker in this model revises his odds to mitigate the risk. He influences the public betting flow by raising the normalized prices for horses with high initial liabilities and lowering the normalized prices for horses with low initial liabilities. Interestingly, the normalized prices exhibit the favorite-longshot bias in our model under some specific assumptions.

Letting the bookmaker to set several rounds of odds before the race starts gives us a better view of the bookmaker's strategy to manage liability exposures over time. The bookmaker continues to set odds to influence the noisy betting demand. His book generally gets more balanced over time and the normalized prices approach the competitive profit margin for providing his service.

Our model is related to the inventory models of the market making literature in which market makers change their prices in order to elicit unexpected imbalance of the buy and sell orders, in the direction of restoring their inventories to a preferred position.[3] The model offers insights into the complexity of managing a series of state contingent exposures such as options for a single expiry date.

This paper is organized as follows. For ease of exposition, in Section 2, we analyze the case of deterministic betting demands, i.e., there is no noise in betting demands. Section 3 considers the first round of odds setting in a model with noisy demands. In Section 4 the odds in the subsequent rounds are analyzed. Section 5 concludes the paper.

2 A simplified model: deterministic betting demands

2.1 Assumptions

A bookmaker faces many uncertainties. He is not sure about which horse will win the race, nor his wealth contingent each horse i winning. In this paper, we focus on how a bookmaker manages his wealth over different horses but do not specifically model how a bookmaker learns the true winning probabilities of these horses. For ease of exposition, we first look at the case where the public betting demands are deterministic functions of the odds. This is relaxed in the following section.

We consider an N-horse race in which each horse is indexed as $i = 1, \cdots, N$. Denote the bookmaker's subjective probability that the ith horse wins the race as p_i and the odds that the bookmaker quotes as q_i, $0 \leq p_i \leq 1$ and $0 \leq q_i \leq 1$ for all i. There is a one to one correspondence between the quoted odds and the prices of bets. For example, odds of k to l correspond to the price of $l/(k+l)$.[4]

We assume that the bookmaker instantaneously knows the total money he has already received, denoted as M, and his existing liabilities, denoted as L_i, on horse i. Denote W_i as the bookmaker's current wealth on horse i, we have:

$$W_i = M - L_i \tag{1}$$

When horse i wins, the bookmaker's wealth on this horse is the difference between the money he has already collected and his liability on this horse. Note the opposite signs before W_i and L_i: positive liability reduces bookmaker's wealth.

[3] See, for example, Garman (1976), Ho and Stoll (1981, 1983), Stoll (1978), and Amihud and Mendelson (1980).

[4] Specifically, one bets l to win k, that is, one pays $l/(k+l)$ to receive \$1 if his bet wins.

Assume the market betting demand function for horse i, denoted as Q_i, is given by:

$$Q_i = c\frac{p_i}{q_i} + b \tag{2}$$

where b and c are some positive constants. Clearly, the higher the odds q_i, the lower the demand Q_i. Note that the net market demand (Q_i) is equivalent to the newly created liability on horse i in this round of betting.

Denote R_i as the bookmaker's revenue on horse i, we have $R_i = Q_i q_i$. The bookmaker's total revenue is the sum of his revenues over all horses. Using equation (2), we have:

$$\begin{aligned}\sum_{i=1}^{N} R_i &= \sum_{i=1}^{N} Q_i q_i \\ &= \sum_{i=1}^{N} (c\frac{p_i}{q_i} + b) q_i \\ &= \sum_{i=1}^{N} (c p_i + b q_i) \\ &= c + ab \end{aligned} \tag{3}$$

where $\sum_{i=1}^{N} p_i = 1$ and $\sum_{i=1}^{N} q_i = a > 1$. Note that the bookmaker's wealth on each horse i (W_i in equation (1)) and his total wealth ($\sum_{i=1}^{N} W_i$) depend on his existing liabilities (L_i) and newly created liability (Q_i) in each round of betting.

By construction, the total revenue is constant and independent of q_i (equation (3)). Different odds affects the bookmaker's current liabilities on different horses (equation (2)) but does not affect his total revenue. Assuming a constant total revenue allows us to focus on the bookmaker's liability management problem. This assumption however is not unrealistic. A bookmaker's total revenue is relatively less volatile than his liability exposures over different horses. One can think of this assumption from the prospect of bettors rather than the bookmaker: bettors put the same amount of money every round of betting.

The sum of the odds on all horses ($\sum_{i=1}^{N} q_i$) requires some explanation. In a betting market, *Dutch books* refer to the portfolios that guarantee a payoff of one but whose price is less than one. To avoid the existence of Dutch books, the summation of all odds ($\sum_{i=1}^{N} q_i$) normally exceeds one and the difference between the sum and one is often called the over-roundness of the book. The over-roundness of the book gives the bookmaker positive profit margin for providing the service. We assume $\sum_{i=1}^{N} q_i = a$ where $a > 1$.[5] The value of a is restricted by the competition among bookmakers. Our analysis is a partial equilibrium one in which the competition results in the profit margin of $(a-1)$ for every bookmaker for providing the service. One could also think that regulators set this constraint to limit the bias of the odds in favor of the bookmaker.

Assuming the bookmaker has a negative exponential utility function, his expected utility of wealth over all horses is given by

$$E[U] = -\frac{1}{\lambda} \sum_{i=1}^{N} p_i e^{-\lambda W_i} \tag{4}$$

where λ is the bookmaker's risk aversion parameter.

The bookmaker's maximization problem is given by:

$$\max_{q_1, \cdots, q_N} E[U] \tag{5}$$
$$s.t. \sum_{i=1}^{N} q_i = a$$

The bookmaker sets odds $(q_1, \ldots, q_N)$ for different horses to maximize his expected utility of wealth, subject to the constraint on the roundness of his book.

The bookmaker's maximization problem can be solved using the Lagrange method. But the analytical solution is difficult to obtain, partly because the bookmaker's subjective probabilities $(p_1, \ldots, p_N)$ enter into the maximization problem, and because his utility function has an exponential form. We solve the problem numerically for a small number of horses in the next section.

2.2 A numerical solution

In this section, we solve the bookmaker's maximization problem (5) numerically. We first make assumptions of the parameter values in the model. Let's consider a 6-horse race, so $N = 6$. Let the bookmaker's total revenue be 100. Consistent with empirical evidence, the roundness of the book (a) is assumed to be 1.15, i.e., a 15% profit margin.[6]

[5] Shin (1992, 1993) also requires the similar assumption in his models. He interprets a as the bids posted by different bookmakers and each one submits his bids for monopoly rights to the betting market. The bookmaker who sets the lowest bids wins.

[6] Kuypers (2000) reports that, in football fixed odds betting the over-roundness of the book is remarkably constant at around 11.5% for all the major bookmakers. The average over-roundness in the

Table 1: **Parameter values used in the numerical solution** This table reports the parameter values used in the numerical solution. N is the number of horses. $\sum_{i=1}^{N} R_i$ is the total revenue. a is the roundness of the book. c is the constant part of the total revenue. λ is the bookmaker's risk aversion parameter. $p_{1,\ldots,6}$ is the bookmaker's subjective probability of each horse winning the race.

N	$\sum_{i=1}^{N} R_i$	a	c	λ	p_1	p_2	p_3	p_4	p_5	p_6
6	100	1.15	50	0.1	0.30	0.25	0.20	0.15	0.08	0.02

Let c, the constant part of the bookmaker's total revenue in equation (3), be 50. The value of b equals $b = \frac{100-c}{a} = 43.48$. Let the bookmaker's risk aversion parameter be 0.1. We also need to specify the bookmaker's subjective probabilities (p_i) for all horses. The sum of all these probabilities needs to be 1. Table 1 summarizes our assumptions of parameter values.

We are interested in how the bookmaker re-balances his book. We assume the bookmaker inherits different initial liabilities over different horses and he only has one chance to re-balance his book. In Section 4 we will discuss the more interesting case where the bookmaker can set several rounds of odds to re-balance his book when betting demands are stochastic. Since this simple one-shot model offers useful intuition, we first analyze this model and use the results as the basis for comparison with more complicated formulations.

We consider two cases of the bookmaker's initial liabilities. In the first case, the bookmaker inherits a flat book in which he has the same initial liabilities over all horses. In the second case, he inherits different liabilities over different horses. Table 2 reports the odds (q_i) and the normalized prices (q_i/p_i) for two initial liability (L_i) distributions.

Table 2-Panel A shows that when the initial liabilities are the same, the odds (q_i) are simply the products of the bookmaker's subjective probability (p_i) and the roundness of his book (a). The normalized prices (q_i/p_i) are constant across all horses and equal to 1.15. Intuitively, since the bookmaker already has a balanced book, his best strategy is to keep the same liability distribution and simply set odds subject to the constraint of

sample of 3382 games is 11.5% with a standard deviation of only 0.34. In horse races, the over-roundness varies considerably among bookmaker's notional implied profit margin (average 25.63%) and internet betting exchanges (with a maximum of 5%) (Smith, Paton and Vaughan Williams (2006)).

Table 2: **Bookmaker's odds: deterministic demands** This table reports the bookmaker's odds for different initial liabilities when the individual demands are deterministic. For horse i, p_i is the bookmaker's subjective probability of horse i winning the race; L_i is the initial liability that the bookmaker's inherits; q_i is the bookmaker's odds set for this round of betting; q_i/p_i is the normalized price. Panel A reports the results when the bookmaker's initial liabilities are the same. In Panel B the initial liabilities are highly imbalanced.

		Panel A: Same Initial Liability			Panel B: Different Initial Liabilities		
Horse i	p_i	L_i	q_i	q_i/p_i	L_i	q_i	q_i/p_i
1	0.30	0	0.35	1.15	-20	0.24	0.81
2	0.25	0	0.29	1.15	0	0.26	1.05
3	0.20	0	0.23	1.15	30	0.36	1.79
4	0.15	0	0.17	1.15	10	0.18	1.23
5	0.08	0	0.09	1.15	0	0.08	1.05
6	0.02	0	0.02	1.15	-10	0.02	0.92

the roundness of his book.

Table 2-Panel B reports the odds when the bookmaker inherits an unbalanced book. The initial liabilities range from a positive 30 (horse 3) to a negative -20 (horse 1).[7] The normalized prices show how the bookmaker re-balances his book. The bookmaker raises the prices for horses with larger initial liabilities (horse 3 and 4) and lowers the prices for horses with smaller initial liabilities (horse 1 and 6). Since bets contribute to new liabilities on different horses, he effectively encourages more bets on horses 1 and 6 and less bets on horses 3 and 4. In this way, the bookmaker achieves a more balanced book in which big liability exposures are reduced.

[7]Since a positive liability reduces his wealth (equation (1)), the bookmaker tries to reduce his positive liability exposure.

3 Stochastic betting demands: first round of odds setting

In this section, we assume that the bookmaker has stochastic betting demands for different horses (equation (2)) but still has a constant overall revenue. That is, the bookmaker knows his total revenue from all horses but is not sure about the exact amount from each horse.

We write the revenue from horse i, $\tilde{R}_i$, as:

$$\tilde{R}_i = cp_i + bq_i + \sigma_i \varepsilon_i \tag{6}$$

where σ_i is the standard deviation of the noise and ε_i is normally distributed random noise with zero mean and unit variance. In the numerical solution, we will examine two different σ structures. The betting demand for each horse is given by:

$$\tilde{Q}_i = \frac{\tilde{R}_i}{q_i} = c\frac{p_i}{q_i} + b + \frac{\sigma_i}{q_i}\varepsilon_i \tag{7}$$

Given the assumption of a constant overall revenue ($\sum_{i=1}^{N} R_i = c+ab$), we must have $\sum_{i=1}^{N} \sigma_i \varepsilon_i = 0$ in equation (6). For any two different horses i and j, we note that the random noises ε_i and ε_j are not quite independently distributed.

The bookmaker's wealth on each horse ($\tilde{W}_i$) is given by:

$$\tilde{W}_i = M - L_i - \tilde{Q}_i \tag{8}$$

where M is the total money he has already collected, L_i is the existing liability and $\tilde{Q}_i$ is the newly created liability. Note here $\tilde{W}_i$ is not deterministic but normally distributed. Let the mean of W_i be M_i and the variance be V_i. The bookmaker's expected utility is given by:

$$\begin{aligned} E[\tilde{U}] &= E[-\frac{1}{\lambda}\sum_{i=1}^{N} p_i e^{-\lambda \tilde{W}_i}] \\ &= -\frac{1}{\lambda}\sum_{i=1}^{N} p_i E[e^{-\lambda \tilde{W}_i}] \\ &= -\frac{1}{\lambda}\sum_{i=1}^{N} p_i e^{-\lambda(M_i - \frac{1}{2}\lambda V_i)} \end{aligned}$$

since $e^{-\lambda \tilde{W}_i}$ is lognormally distributed. Computationally it is as easy to optimize this as it was for the deterministic case in the previous section.

The bookmaker's maximization problem is given by:

$$\max_{q_1,\cdots,q_N} E[\tilde{U}] \tag{9}$$
$$s.t. \sum_{i=1}^{N} q_i = a$$

We now proceed to solve this maximization problem for two different σ structures.

3.1 Constant σ

Here we assume σ_i's of equation (6) are the same for all horses and equal to 1.[8] Table 3-Panel A(B) reports the bookmaker's odds when the bookmaker inherits the same (different) initial liabilities.

Table 3-Panel A shows that normalized prices (q_i/p_i) are no longer the same as in the case of deterministic market demands (Table 2-Panel A). In particular, the normalized prices exhibit the *favorite-longshot bias.* That is, the bookmaker reduces the normalized prices for horses with high winning chance (horse 1 and 2) and increases the normalized prices for horses with low winning chance (horse 5 and 6). The favorite-longshot bias arises from this model because the constant σ has a disproportional effect on the newly created liability for different horses. The effect is much stronger for the longshots (e.g., horse 6) than for the favorites (e.g., horse 1) since the noise is more significant in affecting the newly created liability of the longshots. Specifically, the effect of constant σ's on newly created liabilities is given by σ_i/q_i (equation (7)). Since the longshots have smaller q_i's, they have larger (σ_i/q_i)'s. The longshots hence get penalized and the normalize prices (q_i/p_i) increase.

Table 3-Panel B reports the bookmaker's odds when he inherits different initial liabilities. Comparing to Table 2-Panel B, the normalized prices for horses 1, 2, 3 and 4 are reduced and for horses 5 and 6 are increased. These changes partly reflect the favorite-longshot bias that we have just discussed. The high normalized prices for horses 3 and 4 also suggest the bookmaker manages his liabilities by raising prices for horses with high initial liabilities to discourage betting flows.

[8]If we have constant σ's, they must be quite small to avoid the possibility of negative revenue for some horses.

Table 3: **Bookmaker's odds: stochastic demands with constant** σ This table reports the bookmaker's odds for different initial liabilities when market betting demands are stochastic. We assume here the standard deviation σ_i of the individual revenue R_i is constant. For horse i, p_i is the bookmaker's subjective probability of horse i winning the race; L_i is the initial liability that the bookmaker's inherits; q_i is the bookmaker's odds set for this round of betting; q_i/p_i is the normalized prices. Panel A reports the results when the bookmaker's initial liabilities are the same. In Panel B the initial liabilities are highly imbalanced.

		Panel A: Same Initial Liability			Panel B: Different Initial Liabilities		
Horse i	p_i	L_i	q_i	q_i/p_i	L_i	q_i	q_i/p_i
1	0.30	0	0.33	1.11	-20	0.23	0.79
2	0.25	0	0.28	1.11	0	0.26	1.03
3	0.20	0	0.22	1.12	30	0.34	1.71
4	0.15	0	0.17	1.14	10	0.18	1.22
5	0.08	0	0.10	1.23	0	0.09	1.15
6	0.02	0	0.04	2.22	-10	0.04	1.93

3.2 Proportional σ

Now we assume that σ_i is proportional to the bookmaker's subjective probability p_i. Let $\sigma_i = m \times p_i$ and $m = 6$. Table 4-Panel A(B) reports the bookmaker's odds when the bookmaker inherits the same (different) initial liabilities.

Table 4-Panel A shows that the bookmaker's odds are the same as in the case of deterministic betting demands (Table 2-Panel A). All normalized prices are the same and equal to the roundness of the book (1.15). Intuitively, when σ_i is proportional to p_i, the standard deviation of the newly created liability $\tilde{Q}_i$ is proportional to (p_i/q_i) (equation (7)) and is constant if the ratio of (p_i/q_i) is constant. Since the ratio of (p_i/q_i) is constant here, the noise does not affect the bookmaker's decision and his odds are the same as in the case of deterministic betting demands. The normalized prices satisfy the bookmaker's constraint on the roundness of his book.

Table 4-Panel B shows the odds when the bookmaker inherits different initial liabil-

Table 4: **Bookmaker's odds: stochastic demands with proportional** σ This table reports the bookmaker's odds for different initial liabilities when market betting demands are stochastic. We assume here the standard deviation σ_i of individual revenue R_i is proportional to the bookmaker's subjective probability p_i of horse i winning the race. For horse i, p_i is the bookmaker's subjective probability of horse i winning the race; L_i is the initial liability that the bookmaker's inherits; q_i is the bookmaker's odds set for this round of betting; q_i/p_i is the normalized prices. Panel A reports the results when the bookmaker's initial liabilities are the same. In Panel B the initial liabilities are highly imbalanced.

		Panel A: Same Initial Liability			Panel B: Different Initial Liabilities		
Horse i	p_i	L_i	q_i	q_i/p_i	L_i	q_i	q_i/p_i
1	0.30	0	0.35	1.15	-20	0.25	0.82
2	0.25	0	0.29	1.15	0	0.26	1.06
3	0.20	0	0.23	1.15	30	0.35	1.75
4	0.15	0	0.17	1.15	10	0.18	1.23
5	0.08	0	0.09	1.15	0	0.08	1.06
6	0.02	0	0.02	1.15	-10	0.02	0.93

ities. Since now σ_i is proportional to p_i, the horses with high p_i/q_i (or low q_i/p_i, the normalized prices) get penalized. Comparing to case of deterministic betting demands (Table 2-Panel B), previous bargains (with low normalized prices (q_i/p_i)) get penalized and their normalized prices increase (horses 1, 2, 5 and 6). Horses previously with high normalized prices (horses 3 and 4) see price improvements with decreasing prices. Horses with high initial liabilities (horses 3 and 4) still have relatively high prices, suggesting the bookmaker manages his book by setting odds to influence the betting flow.

4 Several rounds of odds setting

A bookmaker typically sets several rounds of odds before a race starts. In each subsequent round, given his initial liability positions, the bookmaker sets his odds. The betting flow in respond to his odds is his newly created liability on each horse. This liability adds to his existing liability and this becomes his initial liability for the next round. On one hand, more opportunities to re-balance his book give the bookmaker more flexibility in setting odds. On the other hand, the betting flow in respond to current odds could make the book even less balanced. Under this more realistic formulation we can examine how the bookmaker manages his liabilities exposures over time. We will show how quickly he eliminates his liability exposures and how the normalized prices of his odds evolve over time.

The correct maximization for multiple rounds of odds setting would be that the bookmaker maximizes his expected utility of terminal wealth. In this section we examine a numerically approximate case where the bookmaker is myopic: he maximizes his expected utility in each round of betting.

This section is divided into two parts. The first subsection deals with the odds in subsequent rounds in terms of expected demand functions. The second part introduces noise in calculating the odds. In each subsection, we proceed under our assumptions of the two σ structures, i.e., the constant σ's and the proportional σ's.

4.1 Expected demands

Here we calculate the odds in the subsequent rounds in terms of expected betting demands. We do so by taking the expectation of the betting demand function (7). Since noise has a zero mean, the random noise itself does not affect expected wealth (equation (8)), but the variance of noise affects the odds through the bookmaker's maximization problem (9).

Table 5 reports five rounds ($L0$ to $L4$) of odds setting under our assumption of two σ structures. In each case, we let the bookmaker start with the same unbalanced book. We also report the normalized prices (q_i/p_i) for every horse in each round. Figure 1 shows the distributions of the book and the normalized prices. Numbers 1-5 indicate the first round to the fifth round. Note that the bookmaking in this model is quite profitable: he obtains positive wealth (negative liabilities) over every horses after two rounds of betting.[9]

[9] There is however no clear relationship between the normalized prices and the liabilities for different horses, possibly because that the risk component (V_i) of the bookmaker's utility function has a stronger effect than the endowment component (M_i) in his maximization problem.

Figure 1: **Distributions of books and normalized prices: expected demand functions** These figures show the distributions of liability positions and normalized prices over 5 rounds of bets setting for 6 horses. Panel A, B (C, D) reports the distributions of liabilities and normalized prices when σ's are constant (proportional to p_i's). Numbers 1-5 indicate the 1st to 5th round of odds setting.

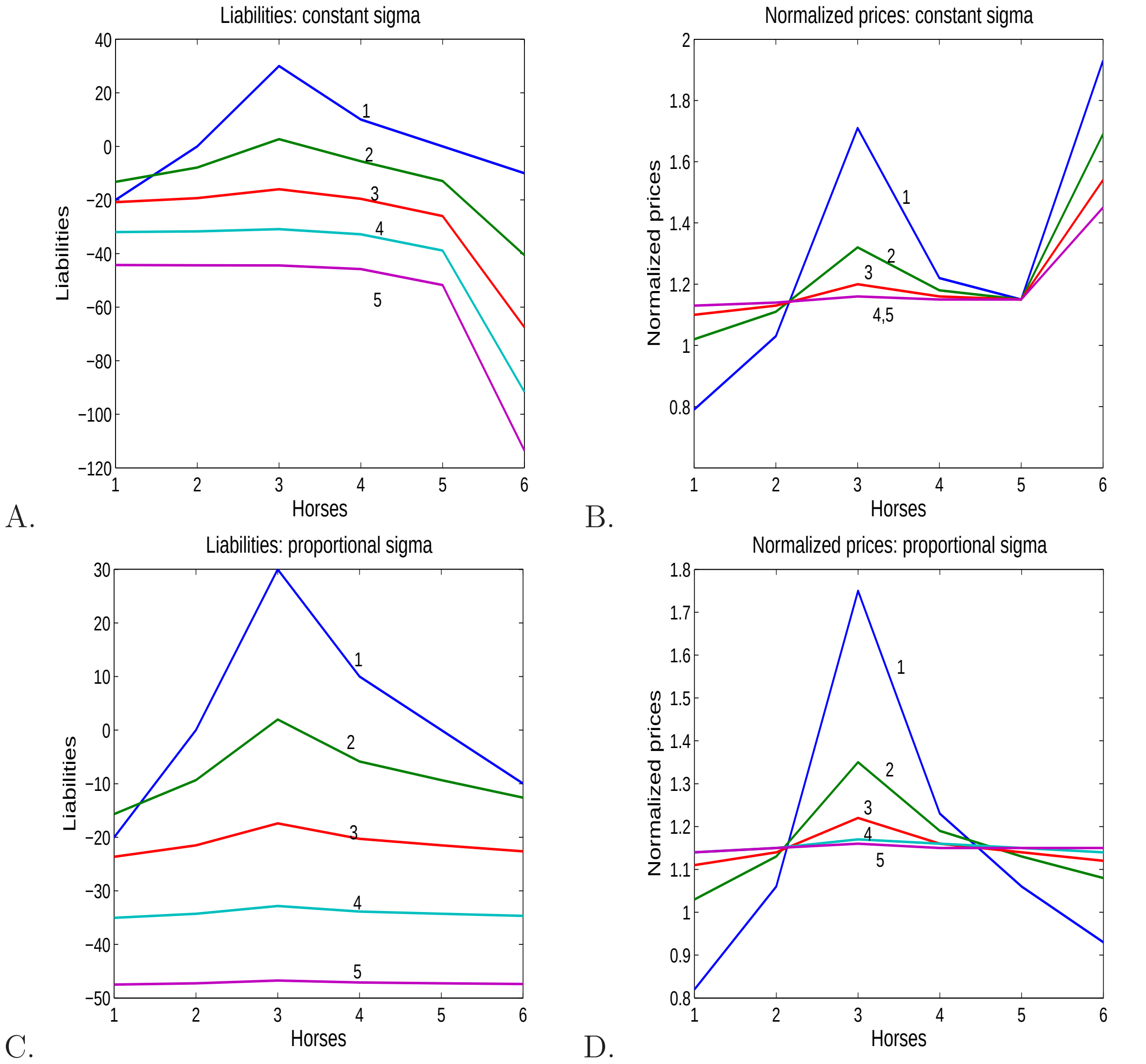

Table 5: **Several rounds of odds setting: expected demands** This table reports the odds for five rounds of betting. L indicates the bookmaker's liability positions over 6 horses. For horse i, $L0_i$ is the initial liability and $L1_i - L4_i$ are subsequent liabilities; q_i/p_i is the normalized price. SD is the standard deviation of respective liabilities position. Panel A reports the odds when σ's are constant. Panel B reports the odds when σ_i is proportional to p_i.

Panel A: Odds with constant σ

Horse	p_i	$L0_i$	q_i/p_i	$L1_i$	q_i/p_i	$L2_i$	q_i/p_i	$L3_i$	q_i/p_i	$L4_i$	q_i/p_i
1	0.30	-20.00	0.79	-13.24	1.02	-20.80	1.10	-31.98	1.13	-44.29	1.13
2	0.25	0.00	1.03	-7.88	1.11	-19.33	1.13	-31.70	1.14	-44.38	1.14
3	0.20	30.00	1.71	2.68	1.32	-15.99	1.20	-30.89	1.16	-44.44	1.16
4	0.15	10.00	1.22	-5.57	1.18	-19.57	1.16	-32.80	1.15	-45.79	1.15
5	0.08	0.00	1.15	-12.95	1.15	-25.99	1.15	-38.90	1.15	-51.77	1.15
6	0.02	-10.00	1.93	-40.60	1.69	-67.53	1.54	-91.63	1.45	-113.55	1.45
SD		17.22		14.76		19.54		20.10		27.67	

Panel B: Odds with proportional σ

Horse	p_i	$L0_i$	q_i/p_i	$L1_i$	q_i/p_i	$L2_i$	q_i/p_i	$L3_i$	q_i/p_i	$L4_i$	q_i/p_i
1	0.30	-20.00	0.82	-15.68	1.03	-23.65	1.11	-35.03	1.14	-47.50	1.14
2	0.25	0.00	1.06	-9.35	1.13	-21.51	1.14	-34.29	1.15	-47.25	1.15
3	0.20	30.00	1.75	1.99	1.35	-17.41	1.22	-32.84	1.17	-46.74	1.16
4	0.15	10.00	1.23	-5.87	1.19	-20.29	1.16	-33.86	1.16	-47.10	1.15
5	0.08	0.00	1.06	-9.35	1.13	-21.51	1.14	-34.29	1.15	-47.25	1.15
6	0.02	-10.00	0.93	-12.60	1.08	-22.62	1.12	-34.67	1.14	-47.38	1.15
SD		17.22		6.11		2.16		0.76		0.27	

4.1.1 Constant σ

Table 5-Panel A reports the odds when σ's all equal to 1. Recall that constant σ's introduce the favorite-longshot bias. The normalized prices (q_i/p_i) are generally higher for the longshot (horse 6) and lower for the favorites (horse 1). After five rounds of betting, the normalized prices are close to the roundness of the book (1.15) for all horses except horse 6.

As the bookmaker continues to re-balance his book over time, his book generally becomes less volatile. The standard deviation of his liability positions decreases from 17.22 of the initial book to 14.76 in the second round. After that, the standard deviation gradually increases slightly. Close observation suggests that the increase is due to the strong negative liability of horse 6 (the standard deviation of the liabilities on horses 1-5 is 3.21 in the fifth round), which comes from the large variance of the newly created liability on horse 6 over time.[10]

4.1.2 Proportional σ

Table 5-Panel B reports the bookmaker's odds when σ_i is proportional to his subjective probability p_i. Recall that proportional σ's of stochastic betting demands penalize the previous bargains by raising their prices but do not introduce the favorite-longshot bias. Panel B shows that over several rounds, the bargains in the previous rounds always get penalized in the next round. Consequently, all normalized prices are *pulled* towards the roundness of the book (1.15). Furthermore, the absence of the favorite-longshot bias helps the bookmaker balance his book more quickly compared to the case of constant σ's. The standard deviation decreases from 17.22 of the initial book to 0.27 in the fifth round.

4.2 Noisy demands

Now we calculate the odds using the stochastic betting demand function (7) in which the noise ε_i affects the bookmaker's newly created liabilities $\tilde{Q}_i$. Given our assumption of a constant total revenue, $\sum_{i=1}^{N} \sigma_i \varepsilon_i$ of equation (6) needs to be zero. Hence we cannot simply use random numbers to simulate the noise. In the Appendix, we discuss our simulation method of noise $\sigma_i \varepsilon_i$ that satisfies $\sum_{i=1}^{N} \sigma_i \varepsilon_i = 0$. Table 6 reports the odds with noisy demands under our assumption of two σ structures. Figure 2 illustrates the distributions of the book and the normalized prices. Numbers 1-5 indicate the first round to the fifth round of odds setting. Similar to the case of expected demands, the bookmaker obtains positive wealth (negative liabilities) over all horses just after two rounds of odds setting.

4.2.1 Constant σ

Table 6-Panel A reports the odds with stochastic betting demands with constant σ's. Comparing to the case of the expected demands, the noise in the demands makes the

[10]Recall that the standard deviation of the newly created liability in equation (7) is σ_i / q_i. Since horse 6 has the smallest odds (q_6), given the constant σ's, it has the biggest standard deviation.

Figure 2: **Distributions of books and normalized prices: noisy demand functions** This figure shows the distributions of liability positions and normalized prices over 5 rounds of bets setting for 6 horses. Panel A, B (C, D) reports the distributions of liabilities and normalized prices when σ's are constant (proportional to p_i's). Numbers 1-5 indicate the 1st to 5th round of odds setting.

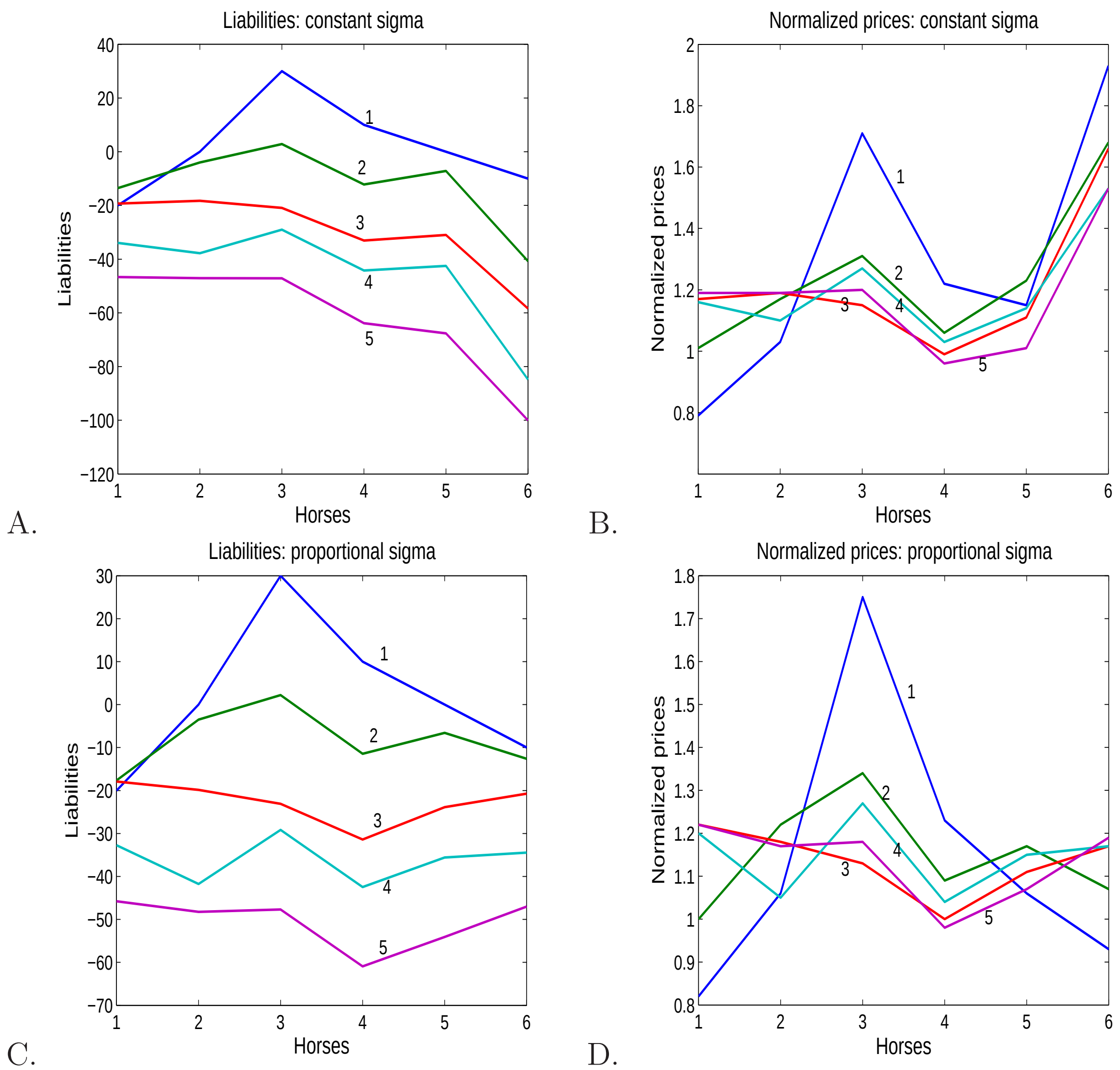

Table 6: **Several rounds of odds setting: noisy demands** This table reports the odds for five rounds of betting. L indicates the bookmaker's liability positions over 6 horses. For horse i, $L0_i$ is the initial liability and $L1_i - L4_i$ are subsequent liabilities; q_i/p_i is the normalized price. SD is the standard deviation of respective liabilities position. Panel A reports the odds that the bookmaker sets when σ's are constant. Panel B reports the odds when σ_i is proportional to p_i.

Panel A: Odds with constant σ

Horse	p_i	$L0_i$	q_i/p_i	$L1_i$	q_i/p_i	$L2_i$	q_i/p_i	$L3_i$	q_i/p_i	$L4_i$	q_i/p_i
1	0.30	-20.00	0.79	-13.55	1.01	-19.30	1.17	-33.96	1.16	-44.69	1.19
2	0.25	0.00	1.03	-3.99	1.17	-18.25	1.19	-37.78	1.10	-47.10	1.19
3	0.20	30.00	1.71	2.86	1.31	-20.93	1.15	-29.01	1.27	-47.16	1.20
4	0.15	10.00	1.22	-12.19	1.06	-33.05	0.99	-44.24	1.03	-63.85	0.96
5	0.08	0.00	1.15	-7.16	1.23	-30.99	1.11	-42.53	1.14	-67.62	1.01
6	0.02	-10.00	1.93	-40.74	1.68	-58.41	1.66	-84.82	1.53	-100.03	1.53
SD		17.22		15.07		15.18		20.10		20.78	

Panel B: Odds with proportional σ

Horse	p_i	$L0_i$	q_i/p_i	$L1_i$	q_i/p_i	$L2_i$	q_i/p_i	$L3_i$	q_i/p_i	$L4_i$	q_i/p_i
1	0.30	-20.00	0.82	-17.66	1.00	-17.90	1.22	-32.72	1.20	-45.78	1.22
2	0.25	0.00	1.06	-3.51	1.22	-19.87	1.18	-41.77	1.05	-48.23	1.17
3	0.20	30.00	1.75	2.20	1.34	-23.11	1.13	-29.16	1.27	-47.70	1.18
4	0.15	10.00	1.23	-14.45	1.09	-31.42	1.00	-42.47	1.04	-60.92	0.98
5	0.08	0.00	1.06	-6.57	1.17	-23.87	1.11	-35.59	1.15	-54.08	1.07
6	0.02	-10.00	0.93	-12.63	1.07	-20.72	1.17	-34.44	1.17	-47.02	1.19
SD		17.22		7.11		4.74		5.20		5.81	

overall book more volatile. In the last round, the standard deviation of the liabilities over horse 1-5 is 10.36, compared to 3.21 in the case of the expected demands. The strong negative liability on horse 6 increases the overall standard deviations of the book over the last several rounds.

Noisy demands also make the normalized prices more volatile (Figure 2-Panel B vs. Figure 1-Panel B). The favorite-longshot bias still exists in the case of constant σ's. Normalized prices are higher for the longshot (horse 6) and lower for the favorite (horse 1).

With noisy betting demands, the bookmaker has a more difficult job to balance his book. When the betting demands are deterministic, the bookmaker needs to balance his book by setting appropriate odds to influence the betting flow. With noisy betting demands, even the random shocks in betting demands are costly to the bookmaker as these random noise makes the book less balanced.

4.2.2 Proportional σ

Table 6-Panel B reports the odds with proportional σ's. As we have discussed, proportional σ's penalize the previous bargains by raising their prices. The normalized prices are generally pulled towards the roundness of the book (1.15).

Without the favorite-longshot bias, the liability positions are less volatile compared to the case of constant σ's. The standard deviations are reduced to around 5 to 6 after five rounds of betting as compared to around 20 in the case of constant σ's. Noisy betting demands make both the book and normalized prices more volatile compared to the case of expected betting demands (Figure 1-Panel C, D vs. Figure 2-Panel C, D).

5 Conclusion

Wagering markets are particularly simple financial markets in which many important economic issues have been analyzed.[11] A horse race betting market is one form of wagering markets. Shin (1991, 1992, 1993) studies the market for bets in a British horse race to better understand the pricing of state contingent claims with asymmetric information. In this paper, we take this natural approach to study the market making of state contingent claims such as options. We do so by analyzing how a bookmaker in a British horse race manages his liability exposures over different horses.

A bookmaker needs to maintain a balanced book in which high liability exposures are avoided. The bookmaker also does not know the correct odds to quote. The noisy betting demands complicate the signal in his observation of betting flows and make his book less balanced. Even random shocks from noisy bettors are costly to the bookmaker since his book could become less balanced.

In our model, the bookmaker revises his odds to mitigate the risk. He influences the public betting flow by raising the normalized prices for horses with high initial liabilities

[11] Investors' risk preference and market information efficiency are examples of these issues. See Sauer (1998) for a recent survey of the economics of wagering markets.

and lowering the normalized prices for horses with low initial liabilities. The bettors find less attractive prices for horses with larger initial liabilities and hence place smaller bets on these horses. Through this new round of betting, the bookmaker achieves a more balanced book. Allowing the bookmaker to set several rounds of odds before the race starts gives a better view of the bookmaker's odds setting strategy and its impact on the public betting flow over time.

One interesting result of our model is that the favorite-longshot bias arises under some specific assumptions. Shin (1991, 1992, 1993) argues that the favorite-longshot bias comes from asymmetric information in the betting market. Our model hence gives another prospective of this stylized fact of the British horse race market.

Possible extensions of our model include incorporating the bookmaker's learning of the true winning probabilities of different horses from his noisy betting flow observation. With learning, a bookmaker could also behave strategically when he has more opportunities to set odds before a race starts. It would then improve our understanding of the complicated problem of market making state contingent claims.

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A Simulation of random shocks in section 4.2

In this section, we show how to simulate the shocks in the revenue function (7) that satisfies $\sum_{i=1}^{N} \sigma_i \varepsilon_i = 0$.

We write $\tilde{v}_i = \sigma_i \varepsilon_i$. We wish to simulate $\tilde{v}_i, ..., \tilde{v}_N$ with the standard deviations $\sigma_i, ..., \sigma_N$ such that $\sum_{i=1}^{N} \tilde{v}_i = 0$.

We simulate:

$$\tilde{v}_i = \sqrt{r_i}\epsilon_i - r_i \sum_{j=1}^{N} \sqrt{r_j}\epsilon_j \tag{10}$$

where r_i is a constant and ϵ_i is the normally distributed noise with mean zero and variance V. Equation (10) gives $\sum_{i=1}^{N} \tilde{v}_i = 0$ as long as $\sum_{i=1}^{N} r_i = 1$. The variance of $\tilde{v}_i$ is given by:

$$var(\tilde{v}_i) = (r_i - r_i^2 \sum_{j=1}^{N} r_j)V = r_i(1 - r_i)V$$

We require $r_i - r_i^2 = \frac{\sigma_i^2}{V}$ for $i = 1, ..., N$ and $\sum_{i=1}^{N} r_i = 1$.

Given the assumption of σ_i's, we can solve for V and r_i. The random shocks in equation (7) can be simulated by using equation (10).