Black's simple discounting rule: A simple implementation✧

Claudio Loderer[a]

Universität Bern

John B. Long [b]

University of Rochester

Lukas Roth[c]

Pennsylvania State University

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[a] Universität Bern, Institut für Finanzmanagement, Engehaldenstrasse 4, 3012 Bern, Switzerland, email: claudio.loderer@ifm.unibe.ch, phone: +41 31 631 3775, fax: +41 31 631 8421. [b] University of Rochester, Simon School, Carol Simon Hall 3-208, Rochester, NY 14627, USA, email: long@simon.rochester.edu, phone: +1 585 275 3358. [c] Pennsylvania State University, Smeal College of Business, 367 Business Building, University Park, PA 16802, USA, email: lukas.roth@psu.edu, phone: +1 814 863 2390, fax: +1 814 865 3362

Abstract

We propose a simple implementation of Black's (1988) elegant discounting rule. The rule overcomes thorny problems that traditional valuation approaches struggle with, namely identifying the market portfolio, measuring project risk, and assessing the market risk premium. The implementation we propose consists of four steps: Finding a stock index or a benchmark stock that correlates with the project's cash flows; (b) estimating the percentiles of the distribution for which the stock return in question equals the risk-free rate; (c) obtaining information from managers to assess the cash flows that correspond to those particular percentiles in the cash flow distribution (i.e., the conditional cash flows); and (d) discounting those conditional cash flows at the risk-free rate.

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JEL classification: G12, G31

1. Introduction

Project value is usually computed by discounting the project's expected net cash flows with an appropriate cost of capital. The capital asset pricing model (CAPM) is typically used to measure that cost. Some of the problems in doing so include identifying the market portfolio, measuring risk, and computing risk premiums. Black (1988) has proposed a valuation rule that avoids those problems and can be used under all circumstances in which one can use the CAPM. The rule is elegant and simple, but it requires knowledge of the project's future conditional net cash flows—conditional on the relevant stock return being equal to the risk-free rate. Estimating conditional net cash flows, however, is not straightforward, which has probably dissuaded textbooks from recommending the rule and discouraged practitioners from adopting it—in spite of the fact that estimating mean net cash flows, as required under the traditional valuation approaches, is in many ways an equally daunting task. The purpose of this paper is to illustrate a simple way to estimate a project's conditional net cash flows.

The rule was originally derived by Black (1988) and later analyzed by Long (2000). Discussions are in, among others, Brennan (1995), Myers (1996), and Laitenberger and Löffler (2002). The elegance and simplicity with which Black's rule takes us around the problem of risk-adjustment comes, as we said, at the cost of having to assess conditional cash flows. The rule, however, moves the focus of the analyst away from the assessment of discount factors and puts it squarely on the more challenging, and arguably more relevant problem of gauging the project's cash flows. We claim that the problem of estimating conditional cash flows can be solved relatively easily. Black's rule would therefore seem to be a simpler tool to compute project value than traditional valuation approaches are.

The rule and our implementation also apply in cases in which the CAPM (or the APT) does not necessarily hold. What we need is linearity between project NCFs and the return on a benchmark security or index, and efficient markets with respect to the information set of relevance (see also Myers, 1996). The rule and our implementation are appropriate also for the case in which the correct asset pricing model is a multifactor model, such as Fama and French's (1993) Three Factor Model.

The rest of the paper is structured as follows. Section 2 uses an example to illustrate the problems associated with standard project-valuation approaches. Section 3 derives and discusses Black's rule. Section 4 shows how Black's rule can be implemented. Section 5 provides descriptive statistics about what we call risk-free percentiles, the key variable in the implementation we are proposing. The last section draws conclusions.

2. Project valuation under a standard CAPM approach

The following example takes us through the valuation of a simple project and illustrates the problems faced when assessing the appropriate cost of capital.

Example 1

Your friend owns a restaurant in downtown New York. He would like to spend a sabbatical year in Switzerland and is therefore willing to give you the opportunity to lease her restaurant during that time. The asking price is 400,000. To decide whether this opportunity is financially attractive, you have to estimate the average net cash flow (NCF) the restaurant will generate and discount that amount back to the present at a risk-adjusted discount rate. Let us review how you would normally proceed.

Assuming for simplicity that all cash flows accrue at year-end, you first have to estimate the average NCF. You could do so by projecting three alternative scenarios with respect to the overall state of the market:

	Overall market conditions		
	Pessimistic Scenario	Ordinary Scenario	Optimistic Scenario
Probability	25%	50%	25%
NCF	200,000	400,000	800,000

Using this information, you can compute:

Average NCF = 0.25×200,000 + 0.50×400,000 + 0.25×800,000 = 450,000.

The average NCF therefore equals 450,000. Next, you have to discount this amount back to the present. To do so, you need a risk-adjusted discount rate. If you want to apply the CAPM, you look for a risk-free rate, a beta, and a market risk premium. Estimating the risk-free rate is relatively simple, you can get fairly good proxies from the financial section of your newspaper. Suppose that rate is 2 percent.

The problem is obtaining an estimate of the other two variables. Both require knowledge of the “market” portfolio of risky assets. Analysts usually get around this obstacle by picking a stock index as a proxy. You can do the same. You could use the S&P 500. Having defined your “market” portfolio, you need a beta for your project. Since the restaurant in question is probably not traded, you have to find traded firms in the food and lodging industry. Suppose you do so and measure a beta of 1.2. To obtain an estimate of the market risk premium, you calculate the historical average difference between the return on the S&P 500 and the contemporaneous risk free rate. Let’s assume that premium is 5 percent.

All these variables imply a risk-adjusted return of 8 percent (= 2 percent + 1.2×5 percent). This rate refers to the cost of equity of comparable traded firms. To compute the overall cost of capital for your project, you have to correct for financial leverage. Suppose you do that and find a discount rate of 7 percent.

You now have all the information you need to compute the project’s value (in thousands):

$$\text{NPV (project)} = -400 + \frac{450}{1.07} = -400 + 421 = 21\,.$$

Based on this point estimate, the investment therefore appears to be marginally beneficial.

The example shows that, if we want to implement the traditional DCF rule, we have to solve the following problems:

- Forecasting the project's future net cash flows, assessing their probabilities, and calculating their mean values;
- Identifying the market portfolio of risky assets;
- Measuring the market risk premium;
- Finding the project's beta;
- Estimating the project's risk-adjusted discount rate (if the term structure of interest rates is not flat and the project extends over a number of years, we may need more than one discount rate);
- Discounting the forecasted net cash flows with the appropriate risk-adjusted discount rate(s).

Some of these problems are not easy to solve and require substantial guesswork. This is especially the case when it comes to assessing the appropriate risk-adjusted discount rate(s). To get around these problems, Black (1988) proposes an elegant, alternative valuation procedure. What follows provides an intuition for that procedure and discusses its implementation.

3. Black's discounting rule

Suppose, for simplicity, that our investment project generates only one net cash flow (NCF) at the end of the year (or at the end of a number of years). Also, suppose this NCF can be written as:

$$N\tilde{C}F = \alpha + \beta \times \tilde{R}_M + \tilde{\varepsilon}\,, \qquad (1)$$

where the tilde indicates a random variable, $\tilde{R}_M$ is the arithmetic rate of return[1] on the market portfolio (or a benchmark return such as the return on an industry portfolio or on some other security) during the year, α is a constant, β is the project's cash-flow beta (a risk measure), and

[1] In other words, the wealth-relative minus one.

$\tilde{\varepsilon}$ is an error variable with zero mean that is uncorrelated with $\tilde{R}_M$. The only important assumption is linearity between benchmark return and cash flow. The error term measures the project's firm-specific or idiosyncratic risk, i.e., possible disturbances in the net cash flow that are unrelated to market-wide events. These disturbances could occur if a secretary made a mistake when typing a contract, accidents in the company's plant, unforeseen delays in the delivery of raw materials, or luck in discovering design faults or in hiring employees.

Equation (1) tells us that the project's NCF is linearly related to the return on the market portfolio of risky assets. That is, if the project's beta is positive and we ignore the error term, higher market returns lead to higher net cash flows. Moreover, projects with higher cash flow betas react more strongly to changes in market returns—they are riskier.

To compute the value of the random NCF in equation (1), it helps to first rearrange equation (1) by writing:

$$N\tilde{C}F = \alpha + \beta \times (\tilde{R}_M - R_F) + \beta \times R_F + \tilde{\varepsilon}, \qquad (2)$$

where R_F is the risk-free rate. Written this way, the net cash flow is the sum of two random [$\beta \times (\tilde{R}_M - R_F)$ and $\tilde{\varepsilon}$] and two non-random [α and $\beta \times R_F$] amounts of money. Its value is therefore the sum of the values of those four terms. Since the two *non-random* quantities are known, we find their value by discounting them at the risk-free rate, namely by calculating

$\dfrac{\alpha}{1+R_F}$ and $\dfrac{\beta \times R_F}{1+R_F}$, respectively.

As for the value of the two *random* amounts of money, it is zero. To show that, we reason as follows. Recognize first that $\tilde{\varepsilon}$ represents pure risk, since it has an average value of zero. Taking that risk is like flipping a coin. What would you pay for a gamble that does not pay off anything on average? What would you pay, for instance, if someone tossed a coin and paid you 100 if heads came up but required you to pay 100 if tails came up? If you think about it, one would probably have to pay you to accept such a gamble. Its risk, however, is fully diversifiable since the gamble is uncorrelated with the market return—the gamble has no systematic risk.

Consequently, since the gamble does not pay anything on average while at the same time it does not impose any costs on you, it is worth nothing—the present value of $\tilde{\varepsilon}$ is zero.

As for the present value of the random amount $\beta \times (\tilde{R}_M - R_F)$, it is zero as well. The reason is that you can costlessly construct a replicating portfolio that yields that payoff. To see that, write out this expression as $\beta \times \tilde{R}_M - \beta \times R_F$, and realize that you can replicate that amount of money by simply borrowing the sum β at the risk-free rate and by investing it in the market portfolio.[2] In principle, since you have not invested any of your own funds, you should not expect to make any money with this strategy—otherwise, you would have found a money machine. After one year, the market portfolio will pay an amount $\beta \times (1 + \tilde{R}_M)$ and you will owe the bank an amount $\beta \times (1 + R_F)$. The difference is exactly the sum $\beta \times \tilde{R}_M - \beta \times R_F$ we are trying to value. For instance, if β equals 500, you can borrow 500 from the bank and invest that amount in the market portfolio. After one year, you will receive the amount $500 \times (\tilde{R}_M - R_F)$. Since this investment does not cost anything, it cannot be worth anything—otherwise you could become a millionaire by selling such financial claims to people who are silly enough to pay a positive price for it. Consequently, the value of the project's NCF equals:

$$\text{Current value of } N\tilde{C}F = \frac{\alpha}{1 + R_F} + \frac{\beta \times R_F}{1 + R_F} = \frac{\alpha + \beta \times R_F}{1 + R_F}. \tag{3}$$

The quantity $\alpha + \beta \times R_F$ in this expression equals the average value of the net cash flow when the market return equals the risk-free rate—i.e., it equals the average NCF conditional on that event. To see that, go back to equation (2), set $\tilde{R}_M = R_F$, and take expectations. The result is:

$$E\left[N\tilde{C}F \middle| \tilde{R}_M = R_F \right] = E\left[\alpha + \beta \times (\tilde{R}_M - R_F) + \beta \times R_F + \tilde{\varepsilon} \middle| \tilde{R}_M = R_F \right] = \alpha + \beta \times R_F, \tag{4}$$

where we use the fact that the mean of the error term, $\tilde{\varepsilon}$, is zero.

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[2] In reality, the bank will ask for security to cover your liability in case the market return is negative.

The expression $E\left[N\tilde{C}F \middle| \tilde{R}_M = R_F\right]$ is the so-called *conditional* expectation of the net cash flow— $E(N\tilde{C}F) = \alpha + \beta \times E(\tilde{R}_M)$ would be its *unconditional* expectation. Combining equations (3) and (4), we can express the present value of the project's net cash flow as:

$$\text{Current value of } N\tilde{C}F = \frac{\alpha + \beta \times R_F}{1 + R_F} = \frac{E\left[N\tilde{C}F \middle| \tilde{R}_M = R_F\right]}{1 + R_F}. \tag{5}$$

Equation (5) tells us that, to find the current value of a risky net cash flow, all we have to do is discount its conditional expected value at the risk-free rate. That means, we have to measure what the NCF would be on average in the event that the market return equals the risk-free rate (i.e., *conditional* on that event), and discount that number at the risk-free rate. "On average" means that we ignore the random term $\tilde{\varepsilon}$. By implication, the market or benchmark return can follow almost any distribution. All we require is that project cash flow and return be linearly related. The reason is that the replicating portfolio M is a linear combination of its component securities. We would be unable to use that combination to replicate a cash flow that was a nonlinear function of the return portfolio. All we can do is replicate linear functions of that return. We should also mention that the conditional NCF is the certainty equivalent of the NCF in question. This is Black's discounting rule. Let us illustrate its use with a simple numerical example.

Example 2

To compute the value of the project in the preceding restaurant example, we need an estimate of its mean conditional cash flow. Since the CAPM and Black's rule assign the same current value to the project's future cash flow, we can write:

$$\text{Current value of the future cash flow } = \frac{E(N\tilde{C}F)}{1+k} = \frac{E\left[N\tilde{C}F \middle| \tilde{R}_M = R_F\right]}{1 + R_F},$$

where k is the risk-adjusted discount rate under the CAPM. This expression implies a conditional mean cash flow of:

$$\text{Conditional mean cash flow } = E\left[N\tilde{C}F \middle| \tilde{R}_M = R_F\right] = E\left[N\tilde{C}F\right]\frac{1+R_F}{1+k} = 450 \times \frac{1.02}{1.07} = 429.$$

Hence, project value under Black's rule is:

$$\text{NPV (project)} = -400 + \frac{429}{1.02} = -400 + 421 = 21.$$

Note that the information needed to compute project value under the CAPM, namely information about k and $E(\tilde{NCF})$, also enables us to compute the project's conditional mean cash flow and, via the risk-free rate, the project's value as well. Hence, the implementation of Black's rule does not require more information than the implementation of valuation models such as the CAPM does. On the contrary, we will show further down that we can assess the project's conditional mean cash flow also without having to compute the project's cost of capital k. Thus, we do not need to know the market portfolio, the project's beta, and the market risk premium. The implementation of Black's rule is simpler.

Equation (5) applies also in the more common case in which projects extend over more than one period. In that case, we compute project value by valuing its conditional net cash flows separately according to equation (5). The only assumption we make is that project NCFs and multiperiod market returns are linearly related.

Black's discounting rule looks simpler to implement than the traditional DCF rules. If we know the conditional NCF, we can ignore the market risk premium and we don't need to know the project's beta. As we said, we don't even have to tell what the *market* portfolio is, since the rule applies also in the case of other benchmark portfolios or securities. These are considerable simplifications. Moreover, the rule holds in all situations in which the traditional valuation models such as the CAPM and the APT hold. The simple discounting rule does not work, however, when the net cash flows are a non-linear function of the market returns—but neither do the traditional valuation models. Options, for instance, are non-linear functions of the market return, since they have positive payoffs above the exercise price and zero payoffs below it.[3]

4. Implementing Black's discounting rule

The problem in applying Black's rule is the estimation of conditional net cash flows. As we said, these cash flows are those we observe on average when the return on the market portfolio equals the risk-free rate. Yet it is not clear how we can easily obtain meaningful estimates of those cash flows. One possible solution is to ask managers to tell us what the future net cash flows will be if the market return equals the risk-free rate, on average. Unfortunately, this approach does not seem to be very promising because it is unlikely that managers are

[3] See the discussion in Black (1988), p. 9–10, and Long (2000), p. 10–11.

consciously aware of that relation. We have to find an *indirect* way to elicit the information we want from them.

Figure 1 illustrates what we are after. The histogram on the left shows market (or any benchmark) returns as we would observe them if we used the CRSP Value Weighted Index as a proxy and they were generated under a normal distribution with the historical parameters estimated for the years 1926–2005—namely a mean of 9.54 percent and a standard deviation of 19.51 percent. In grey, we show the frequency of observations smaller than or equal to an assumed risk-free rate of 3.63 percent (the historical average annual return on 30-day T-bills). The diagram on the right-hand side of the figure uses these market returns and equation (1) to generate the net cash flows we would expect on a project with an assumed α of 100 and a cash flow beta of 800 (unlike return betas, cash flow betas have values that depend on project size: larger projects tend to have larger cash flow betas). The computation ignores the idiosyncratic risk component—i.e., the $\tilde{\varepsilon}$ term in equation (1). The grey area in the histogram on the right-hand side of the figure is defined by the interval of net cash flows produced by market returns smaller than or equal to the risk-free rate. The conditional net cash flow forecast we are interested in is the upper limit of that interval.

A possible heuristic procedure to generate these conditional forecasts is therefore to find the percentile of the distribution the market return falls into when it equals the risk-free rate—we are looking for the cumulative density at that point. Because of the monotone increasing relation between net cash flows and market returns, the associated net cash flow will fall into the same percentile of its own distribution (in other words, the grey areas in the two diagrams of Figure 1 are equal). For example, if the market return equals the risk-free rate at the 20th percentile of its distribution, then the implied conditional net cash flow will also correspond to the 20th percentile of its respective distribution. And once we know the percentile of the net cash flow distribution we are interested in, we can use managers' cash flow information to identify the conditional NCF we are looking for, namely the NCF that defines that 20th percentile. Let us refer to that percentile as the *risk-free* percentile.

We are assuming that the project's cash flow beta is positive. If that beta is negative, meaning that higher market returns induce more negative cash flows, the appropriate conditional forecast is the cash flow at the percentile equal to one minus the risk-free percentile.

Our implicit assumption is that, in providing NCF information, managers are intuitively able to abstract from the impact that firm-specific events can have on the cash flows of their projects. In other words, we assume that, in forecasting the possible future project NCFs, they are able to focus on the economy-wide (or industry-wide, if we use an industry index as a benchmark) causes of variation in those cash flows, such as the overall state of the economy, and ignore accidental, firm-specific events. We claim that paying no attention to firm-specific occurrences is a tendency that is quite natural. It would be very difficult for managers to forecast net cash flows based on speculations concerning the occurrence of fortuitous events such as secretarial mistakes or accidents in the company's plants—and besides, the set of possible occurrences is unlimited.

Our implementation of Black's rule involves the following four steps: (a) Finding a stock index or a benchmark stock that correlates with the project's cash flows; (b) estimating the percentiles of the distribution for which the stock return in question equals the risk-free rate; (c) obtaining information from managers to assess the cash flows that correspond to those particular percentiles in the cash flow distribution (i.e., the conditional cash flows); and (d) discounting those conditional cash flows at the risk-free rate. So far, we have assumed for simplicity that the stock index we need in our first step is the market portfolio. In what follows, we maintain that assumption and use the CRSP Value Weighted Index as a proxy. Remember, however, that whereas in the implementation of the CAPM we have to look for market portfolio proxies, we do not have to do so here. All we need is an index that correlates with the project cash flows. The next three sections describe steps (b) to (d) in the implementation of Black's rule.

4.1 Estimating the percentile of the distribution where $R_M = R_F$

The following table reports the historical distribution characteristics of the continuously compounded annual stock return on the CRSP Value Weighted Index. In the period of 1942–2005, the average return was 11.39 percent and the standard deviation was 15.58 percent. The table also shows the annual yields-to-maturity on Treasury securities with maturities between 1 and 5 years during the same time period.[4] We use those yields as proxies for the risk-free rate for different maturities.

	CRSP Value Weighted Index	Treasury yields				
	1942–2005	1 year	2 years	3 years	4 years	5 years
Average	11.39%	5.13%	5.24%	5.32%	5.39%	5.47%
Standard deviation	15.58%					

Suppose we have an investment project whose net cash flows are linearly related to the market return. Assuming the distribution of market returns remains the same over time, we can use the numbers in the table to assess our risk-free percentiles. Since investment projects can last several years, we assume that equation (1) holds with market returns measured over a different number of years, corresponding to the time horizon of the project's net cash flows. Consequently, the net cash flow two years ahead will be related to the market return over the next two years, the net cash flow three years ahead to the market return over the next three years, etc.

The table below uses the historical data to calculate the market return's average and standard deviation as well as the relevant risk-free rate for time horizons of one to five years. For example, the average market return over a three-year horizon is 34.17 percent, the market return's standard deviation is 26.99 percent,[5] and the risk-free rate is 15.96 percent.[6] On the basis of these values for a three-year horizon, the table calculates that the percentile of the distribution for which the market return equals the risk-free rate of 15.96 percent is 24.99 percent.[7] The table shows that the risk-free percentile falls from 34.39 percent for a time horizon of one year to 19.78 percent for a horizon of five. The reason for the decline is that the mean return increases faster with the investment horizon than the return dispersion does—the mean

[4] Since we don't have Treasury yields for 3- and 4-year maturities, we compute them as linear interpolations of the available 2- and 5-year yields.

[5] Given continuous compounding, and safe for rounding errors, the cumulative average return equals three times the annual average (34.17% = 3×11.39%). The associated standard deviation equals the square root of three times the annual standard deviation (26.99% = √3×15.58%).

[6] Given continuous compounding, that average yield equals three times the annualized three-year yield, namely 15.96% (= 3×5.32%).

[7] That percentile is computed by first setting the cumulative three-year stock return equal to the cumulative three-year risk-free rate and then standardizing the result with the cumulative average three-year stock return and its standard deviation. The standard normal variable in question equals (15.96–34.17)/26.99 = –0.6748 and the associated normal distribution is 24.99%.

excess return increases with T whereas the standard deviation of the excess return increases with the square root of T.

Year of NCF	Cumulative average R_M	Standard deviation of R_M	Cumulative risk-free rate	Percentile for which R_M equals or is smaller than the risk-free rate
1	11.39%	15.58%	5.13%	34.39%
2	22.78%	22.03%	10.48%	28.83%
3	34.17%	26.99%	15.96%	24.99%
4	45.56%	31.16%	21.56%	22.06%
5	56.95%	34.84%	27.35%	19.78%

4.2 Estimating the distribution of future net cash flows

The second step in our valuation approach is estimating the distribution of future net cash flows.[8] Most managers do not know that distribution in much detail. They know aspects of it, however. And, under normality, all we need is two points of that distribution. For example, they might have an idea about the mean of that distribution and an estimate of the probability that the cash flows will fall under a certain value.[9] Alternatively, they might be able to state mean values for various scenarios, such as a pessimistic and an optimistic one (actually, these are truncated means of the overall net-cash-flow distribution). And at the same time, they might have a rough idea of the probability with which the cash flows will fall under the average value under the pessimistic scenario, or exceed the average value under the optimistic one. As illustrated in the following example, we can use that information to pinpoint the full distribution of the future cash flows of an investment project.

[8] This section relies on input by a team of students (Heinz Brägger, Thomas Himmel, Roman Käser, Andreas Nauer, and Jürg Rippl) in the Rochester-Bern Executive M.B.A. Program who implemented Black’s rule for an actual investment project.
[9] Similar information is needed in other contexts to state value-at-risk or cash-flow-at-risk measures.

Example 3

Suppose we want to use Black's rule and are therefore interested in assessing the future distribution of net cash flows from a particular investment project during the second year of its life. The project manager provides us with the following data.

State of the market	Forecasted average NCF	Additional information
Pessimistic	300,000	Probability of observing lower values: 10 percent
Optimistic	1,500,000	Probability of observing higher values: 5 percent

Remembering the properties of normal distributions, we can use this information to find the unconditional mean (μ) and standard deviation (σ) of the distribution of the project's net cash flows. Given the properties of normal distributions, once we know those two parameters, we will have uniquely defined the distribution of future net cash flows. To obtain those estimates, we have to standardize the values we are given and realize that cumulative probabilities remain unaffected by the standardization. We can therefore write:

$$\Phi\left(\frac{300,000-\mu}{\sigma}\right)=0.1 \text{ and } \Phi\left(\frac{1,500,000-\mu}{\sigma}\right)=0.95.$$

The symbol $\Phi(\cdot)$ denotes cumulative probabilities. We can then invert these expressions and find the values of a standard normal variable which yield the probabilities in question. These values are:

$$\frac{300,000-\mu}{\sigma}=-1.282 \text{ and } \frac{1,500,000-\mu}{\sigma}=1.645.$$

Solving the first equation for the unconditional mean net cash flow, we obtain: $\mu=300,000+1.282\times\sigma$. We can then insert this expression in the second equation and find the standard deviation of net cash flows: $\sigma=409,976$. And putting this value back into the expression for the unconditional mean, we find: $\mu=300,000+1.282\times 409,976=825,589$. Hence, what managers tell us implies a normal NCF distribution with mean 825,589 and standard deviation 409,976.

4.3 Estimating the conditional cash flows

The third step in our valuation approach involves quantifying the conditional cash flows of our project. Given the information gathered in the two preceding sections, we can do so fairly easily. Let us illustrate this by extending the preceding example.

Example 3 (continued)

In step one above, we have established that the risk-free percentile for the NCF in year two is 28.83. That means the future conditional net cash flow we are looking for, NCF_C, is the one that defines the 28.83th percentile of the NCF distribution. The tables for standard normal variables can help us quantify that cash flow. To use them, however, we first have to standardize our net cash flow by deducting its mean estimate and dividing the result by the estimated standard deviation. The conditional net cash flow we are looking for has to meet the following condition:

$$\Phi\left(\frac{NCF_C-\mu}{\sigma}\right)=\Phi\left(\frac{NCF_C-825,589}{409,976}\right)=0.2883.$$

The tables for standard normal variables tell us that a cumulative probability of 0.2883 is associated with a z-value of –0.5582. We can therefore write:

$$\frac{\text{NCF}_\text{C} - 825{,}589}{409{,}976} = -0.5582,$$

and solve this expression for $\text{NCF}_\text{C} = -0.5582 \times 409{,}976 + 825{,}589 = 596{,}740$. The conditional net cash flow we are seeking is therefore 596,740.

4.4 Discounting the conditional cash flows at the risk-free rate

Computing the value of conditional cash flows simply requires discounting them at the risk-free rate. In our preceding example, the conditional cash flow in year two is 596,740. Since the two-year continuously compounded risk-free rate is 10.48 percent, the current value of the cash flow is:

Example 3 (continued)

$$\text{Current value} = \frac{596{,}740}{e^{0.1048}} = \frac{596{,}740}{1.1105} = 537{,}361.$$

4.5 A comprehensive example

Having discussed the individual steps to implement Black's rule, we are now ready to illustrate this valuation approach with a comprehensive example.

Example 4

A producer of plastic products is thinking of replacing one of its extrusion machines. The new machine costs 1.2 million. It requires less energy, and it is faster and more reliable than the current one. Upkeep and maintenance costs are about the same as for the old machine. The relevant horizon is five years. To decide, the producer has asked your opinion. You would like to base your advice on Black's rule.

Suppose you use the CRSP Value Weighted Index as a proxy for the market and the annual yields-to-maturity on Treasury securities with maturities between 1 and 5 years as measures of the risk-free rate. On the basis of these assumptions, we know from the preceding discussions that the relevant risk-free percentiles are as follows.

Year of NCF	Cumulative average R_M	Standard deviation of R_M	Cumulative risk-free rate	Risk-free percentile
1	11.39%	15.58%	5.13%	34.39%
2	22.78%	22.03%	10.48%	28.83%
3	34.17%	26.99%	15.96%	24.99%
4	45.56%	31.16%	21.56%	22.06%
5	56.95%	34.84%	27.35%	19.78%

To assess the distribution of future net cash flows, the project manager has given you the following data (all NCF are expressed in thousands).

Year of NCF	Average NCF State of market: Pessimistic	Probability of lower NCF	Average NCF State of market: Normal	Probability of lower NCF
1	200	10%	500	50%
2	300	10%	700	50%
3	300	10%	700	50%
4	200	10%	500	50%
5	100	10%	200	50%

Note that the manager is able to state the unconditional average future NCFs. Hence, you only need to estimate the standard deviation of the future NCFs to identify their distributions. You can do so by retracing what we did in step two of our procedure. And once you have those estimates, you can calculate the conditional NCFs you are searching for by following what we did in step three. The following table summarizes the resulting calculations (NCFs expressed in thousands).

Year of NCF	Estimated mean of future NCF distribution	Estimated standard deviation of future NCF distribution	Risk-free percentile	Estimated conditional NCF	Risk-free rate
1	500	234.09	34.39%	405.93	5.25%
2	700	312.12	28.83%	525.73	5.30%
3	700	312.12	24.99%	489.38	5.45%
4	500	234.09	22.06%	319.71	5.50%
5	200	78.03	19.78%	133.71	5.60%

Using the annualized risk-free rate listed in the table for each maturity, you can therefore compute:

$$\text{NPV} = -1,200 + \frac{405.93}{1.0525} + \frac{525.73}{1.0530^2} + \frac{489.38}{1.0545^3} + \frac{319.71}{1.0550^4} + \frac{133.71}{1.0560^5} = 437.07.$$

The value of the project is 437.07 thousand. Based on this point estimate, buying the machine appears to be financially attractive.

4.6 What financial analysts need to know

An important key to our implementation of Black's rule is the assessment of a project's future unconditional mean cash flows. With that information, we showed how one can estimate the project's conditional mean cash flows. Yet to assess the unconditional mean cash flows, managers have to be able to disregard firm-specific events such as secretarial mistakes [i.e., the disturbance factor ε in equation (1)] and to focus on the economy-wide sources of variation of project cash flows. We simply assumed they have that ability without much explanation. There are, however, arguments that help us make our case. The first is that idiosyncratic events cancel each other out over time. Hence, managers with long enough working experience should have learned to focus on economy-wide developments almost automatically. The second argument is that many managers are probably able to distinguish company-specific events from economy-wide changes. Risk management helps managers make that important distinction, since the two classes of events have different risk characteristics. Adverse firm-specific events, in particular, can be prevented by establishing appropriate internal guidelines and codes of conduct. To discourage the CEO from getting hurt while parachuting in his free time, for example, the firm can add a clause in the CEO's contract that forbids dangerous pastimes. In contrast, there is little an importer of Japanese high-tech equipment can do to prevent market-wide events such as a hike in the value of the yen. We claim that, if managers are able to distinguish between firm-specific and economy-wide developments, they also know that firm-specific events are unsystematic and therefore irrelevant in forecasting future project cash flows.

Of course, even if managers have not learned to disregard the idiosyncratic sources of cash-flow variation, the project analyst can always help them do so with the proper instructions.

It is important to recognize, however, that the problem of ignoring firm-specific considerations when having to forecast a project's future mean cash flows is not limited to our implementation of Black's rule. It confronts every user of the traditional discounted-cash-flow methods. In fact, one could argue that the problem is simpler under Black's rule. Black's rule does not require a market portfolio but rather a security or a tracking portfolio correlated with the project's cash flows. That security could be an industry index. If so, the rule to distinguish between systematic and idiosyncratic developments is that whatever is not company-specific is systematic. That simple rule fails, however, when systematic means market-wide. There are, for example, industry-specific disturbances that are not market-wide—the demise of the buggy-whip industry is an example. We have to ignore these disturbances if we want to implement the CAPM. Yet we do not face this tricky identification problem under Black's rule.

5. Empirical characteristics of risk-free percentiles

In order to implement Black's rule, we have to compute risk-free percentiles. In the preceding examples, we have used U.S. data from the years 1942–2005 to do so. Yet the use of data in the comparatively far past makes sense only if the percentiles in question are reasonably stationary over time. This section examines that issue for different investment horizons. We also want to know how these risk-free percentiles compare across capital markets—in integrated markets, we would expect similar percentiles.

5.1 Risk-free percentiles for one-year investment horizons: Historical estimates for the U.S.

Data for our computations are from the monthly CRSP files. The CRSP Value-Weighted Index is our proxy for the market portfolio and the 30-day T-bill rate our proxy for the risk-free rate. For simplicity, we work with excess returns (ER), defined as the difference between the annualized monthly market returns and the contemporaneous risk-free interest rates. The risk-free percentile for a given investment horizon is therefore the cumulative probability of an excess return equal to or smaller than zero over that horizon.

Table 1 reports distribution characteristics of risk-free percentiles for one-year horizons using annualized continuously compounded monthly returns during the years of 1926–2005. Panel A divides the overall sample period in decades. Column (1) and (2) give excess return averages and variances. For the full period, for example, the average excess return is 5.9 percent and the variance is 3.6 percent. Column (4) uses these parameter estimates to generate risk-free percentiles under the normal. These estimates go from 16.0 percent (1946–1955) to 56.6 percent (1966–1975), although most of the observations are between 30 percent and 48 percent. For the full 1925–2005 period, the estimate is 37.7 percent. To assess whether risk-free percentiles are stationary, we use a Wilcoxon rank-sum (Mann-Whitney) test to compare the distribution of monthly excess returns in each decade with that of the full 1926–2005 period (exclusive of the decade under consideration). The z-statistics of this test are reported in column (3). They suggest that the distributions of the monthly excess returns for each individual decade do not differ significantly from the distribution for the full period at customary levels of significance. Hence, the distribution of excess returns does not seem to change significantly over time, which means that the risk-free percentiles are stationary. A reasonable estimate of the risk-free percentiles over a time horizon of one month is therefore the 37.7 percent figure obtained for the overall 1926–2005 period. The exception is the 1966–1975 decade, which is significantly different from the overall distribution with confidence 0.95. Under the null hypothesis of stationarity, however, observing a significant difference for one decade out of eight is not very surprising.

Our computations assume normality. Column (5) of the table, however, rejects that assumption for most of the decades analyzed on the basis of a Shapiro-Wilk test. We therefore compute a parameter-free estimate of the risk-free percentiles by using the actual distributions of excess returns and assessing the percentile for which the excess return equals zero (the so-called exact method). Column (6) shows these percentiles. The historical estimates for the various decades lie much closer together than those obtained under a normal distribution. All except one estimate are between 34 and 46 percent. The exception is the estimate for the decade of 1966–1975, which equals 50.8 percent. For the full period, the risk-free percentile is 40.1 percent. That is not much different from the 37.7 percent obtained under the normal assumption. Hence, normality does not seem to be a bad approximation. Figure 2 confirms this conclusion graphically by plotting the historical histogram of the annualized excess returns in 1926–2005 and its normal approximation based on the estimated historical average of 5.9 percent and variance of 3.6 percent. As typical of security return distributions, the historical histogram is leptokurtic. Still, the normal approximation seems to be acceptable.

Panel B shows risk-free percentiles for successive 5-year time intervals. Our conclusions remain essentially the same. The percentiles given by the exact method appear to be stationary over time (the only exception is the 1971–1975 period) and, in most cases, not very different from the ones obtained under a normal.

Figure 3 illustrates the numbers in Panel B. It displays the exact 95 percent-confidence intervals for the risk-free percentile for each five-year period in the sample period (gray line). The exact risk-free percentile of 40.0 percent estimated for one-year horizons for the full 1926–2005 period is drawn as a black line. For no five-year period can we reject the hypothesis that the exact risk-free percentile is 40.0 percent. The only exception, as we have seen, is the 1971–1975 period.

5.2 Risk-free percentiles for different maturities

Having shown that the risk-free percentile for one-year investment horizons can be assumed to be stationary, we estimate risk-free percentiles for longer investment horizons. Table 2 displays the results for maturities up to ten years. For our computations, we assume normality and use the excess-return average and variance estimates reported in Table 1 for the full sample period of 1925–2005 (5.9 percent and 3.6 percent, respectively). For an investment horizon of T years, the average excess return is T×5.9 percent and the variance T×3.6 percent. For a four-year horizon, for example, the average excess return is 23.6 percent (=5.9 percent×4) and the return variance 7.8 percent (=3.6 percent×4). This implies a risk-free percentile of 24.2 percent. The risk-free percentiles go from 37.7 percent for a one-year horizon down to 16.1 percent for a ten-year horizon. Given the assumption that risk is progressively resolved over time, a longer investment horizon means a more substantial cumulative risk and therefore a smaller risk-free percentile—and smaller conditional cash flows.

5.3 Risk-free percentiles across international capital markets

We have investigated risk-free percentiles for different investment horizons based on U.S. data for the years 1926–2005. Next, we would like to know how these percentiles compare across capital markets internationally. The countries of interest are the U.K., Japan, and Switzerland. Table 3 computes historical one-year percentiles (exact percentiles) using data from Datastream for different five-year periods in 1971–2005. If capital markets were reasonably integrated, we should observe similar numbers. The evidence in columns (1), (3), and (5) supports that conjecture. For the overall period, all percentiles lie between 42 and 44 percent. That is not much different from the one-year risk-free percentile of 42 percent observed for the U.S. during the same years (Panel B of Table 1).

6. Conclusions

Black's rule gets around a number of estimation problems that face the analyst trying to implement traditional DCF valuation approaches. Among other things, he does not have to identify the market portfolio; all he needs to do is find a security or an index that is correlated with the project's cash flows. Moreover, he does not have to measure the project's risk. And he does not have to assess the market risk premium.

Black's (1988) rule can be used under all circumstances in which one can use the standard valuation approaches, including the CAPM and the APT. Moreover, it can be used in situations under which the CAPM and the APT cannot be used. The rule looks fairly simple, but it requires knowledge of the project's conditional cash flows (conditional on zero excess returns). Estimating those conditional cash flows, however, is not straightforward, which has probably dissuaded textbooks from recommending the rule and discouraged practitioners from adopting it. The purpose of this paper is to illustrate a simple way to estimate a project's conditional cash flows.

Our approach involves four steps: (a) Finding a stock index or a benchmark stock that correlates with the project's cash flows; (b) estimating the percentiles of the distribution for which the stock return in question equals the risk-free rate; (c) obtaining information from managers to assess the cash flows that correspond to those particular percentiles in the cash flow distribution (i.e., the conditional cash flows); and (d) discounting those conditional cash flows at the risk-free rate. The most difficult task would seem to be step (c), which means that the focus of the valuation effort is where it actually belongs, namely the project's cash flows. It is the estimation of the relevant cash flows that represents the greatest challenge to analysts in their project valuation exercises.

The normality assumption simplifies task (c) substantially without making Black's valuation approach more restrictive than the traditional valuation approaches—they all make the same assumption. Still, we could drop the assumption and work with alternative distributions, something that would not be possible under the traditional valuation approaches. For example, the project's cash flows might equal a constant plus one plus the return on a benchmark security for the period between now and the time of the cash flow (Black, 1988). If that return were normally distributed, then the cash flow would be lognormally distributed. In principle, our implementation would apply in that case as well. The easiest situation would be if managers could provide estimates of the mean and the standard deviation of the cash flows directly.

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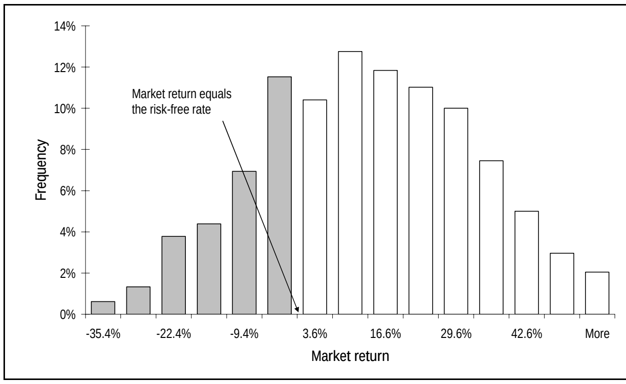
Figure 1

Market returns and associated mean conditional NCFs

The diagrams use the continuously compounded return on CRSP's Value Weighted Index as a proxy for the market return. That return is assumed to be normally distributed with parameter values equal to those observed during 1926–2005, namely an annual average of 9.54 percent and a standard deviation of 19.51 percent. The continuously compounded risk-free rate is 3.63 percent. The investment project is assumed to have a NCF with an α-value of 100 and a cash flow beta of 800.



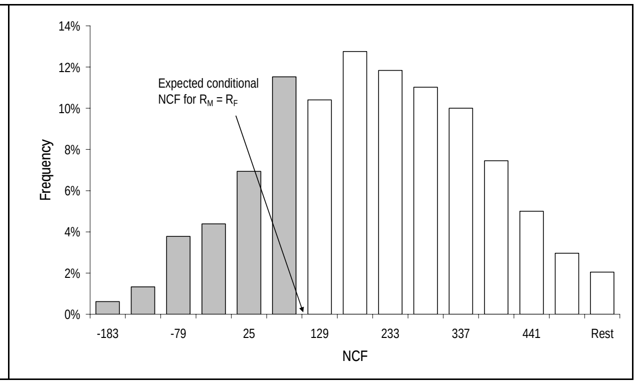


Table 1

Risk-free percentiles for one-year investment horizons in the U.S.

The table estimates risk-free percentiles for one-year investment horizons on the basis of monthly return U.S. data from the years 1926–2005. We use continuously compounded, annualized monthly returns. The CRSP Value Weighted Index is used as a proxy for the market index and yields on 90-day T-bills proxy for the risk-free return. Panel A shows estimates for individual decades in the overall period, and Panel B does so for 5-year intervals. The individual columns report: (1) The average excess returns measured as the difference between the return on the CRSP Value Weighted Index and the contemporaneous risk-free rate; (2) The excess return variances; (3) The z-statistics of a Wilcoxon rank-rum (Mann-Whitney) test of the hypothesis that the sub-periods come from the same distribution as the overall 1925–2005 period (excluding the sub-period in question); (4) The risk-free percentiles, defined as the probabilities that the excess returns are smaller than or equal to zero. Excess returns are assumed to be normally distributed with parameter values equal the average excess return and the variance of the excess return measured for the decade (five-year period) in question; (5) The z-statistics of a Shapiro-Wilk test of normality; (6) The exact percentiles, defined as the proportions of observed excess returns that are smaller than or equal to zero in any given period; (7) The binomial exact 95 percent-confidence intervals; (8) The number of monthly returns in each decade (five-year period). *, **, *** denote significance at the 10 percent, 5 percent, and 1 percent level, respectively (two-sided tests).

Panel A: Risk-free percentiles for individual decades in 1926–2005

Period	Average excess return	Variance of excess return	Wilcoxon rank-sum test (z-statistics)	Percentile of excess return ≤ 0, assuming normality	Shapiro-Wilk tests for normality (z-statistics)	Exact percentile of excess return ≤ 0	Binomial exact 95%-confidence interval	Number of obs.
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1926–1935	2.066%	10.948%	–0.021	47.5%	4.002***	42.5%	[33.5%, 51.8%]	120
1936–1945	8.422%	4.682%	–1.001	34.9%	5.006***	35.8%	[27.3%, 45.1%]	120
1946–1955	12.774%	1.644%	–1.242	16.0%	1.425*	37.5%	[28.8%, 46.8%]	120
1956–1965	8.024%	1.386%	–0.137	24.8%	3.123***	34.2%	[25.8%, 43.4%]	120
1966–1975	–2.760%	2.744%	2.210**	56.6%	0.709	50.8%**	[41.6%, 60.1%]	120
1976–1985	5.728%	2.250%	0.707	35.1%	0.930	45.8%	[36.7%, 55.2%]	120
1986–1995	7.626%	2.313%	–0.292	30.8%	5.645***	35.0%	[26.5%, 44.2%]	120
1996–2005	5.405%	2.667%	–0.223	37.0%	3.360***	39.2%	[30.4%, 48.5%]	120
1926–2005	5.911%	3.569%		37.7%	9.801***	40.1%	[37.0%, 43.3%]	960

Panel B: Risk-free percentiles for individual 5-year periods in 1926–2005

Period	Average excess return	Variance of excess return	Wilcoxon rank-sum test (z-statistics)	Percentile of excess return ≤ 0, assuming normality	Shapiro-Wilk tests for normality (z-statistics)	Exact percentile of excess return ≤ 0	Binomial exact 95%-confidence interval	Number of obs.
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1926–1930	0.839%	4.678%	–0.155	48.5%	3.455***	38.3%	[26.1%, 51.8%]	60
1931–1935	3.292%	17.400%	0.126	46.9%	1.248	46.7%	[33.7%, 60.0%]	60
1936–1940	0.785%	7.642%	0.032	48.9%	2.987***	40.0%	[27.6%, 53.5%]	60
1941–1945	16.059%	1.703%	–1.400	10.9%	1.916**	31.7%	[20.3%, 45.0%]	60
1946–1950	7.995%	2.084%	–0.362	29.0%	1.488*	40.0%	[27.6%, 53.5%]	60
1951–1955	17.554%	1.193%	–1.335	5.4%	–0.401	35.0%	[23.1%, 48.4%]	60
1956–1960	7.008%	1.395%	–0.018	27.6%	2.636***	35.0%	[23.1%, 48.4%]	60
1961–1965	9.040%	1.399%	–0.169	22.2%	2.513***	33.3%	[21.7%, 46.7%]	60
1966–1970	–1.700%	2.347%	1.206	54.4%	0.083	46.7%	[33.7%, 60.0%]	60
1971–1975	–3.819%	3.189%	1.814*	58.5%	0.812	55.0%**	[41.6%, 67.9%]	60
1976–1980	8.253%	2.465%	–0.254	30.0%	2.062***	41.7%	[29.1%, 55.1%]	60
1981–1985	3.203%	2.063%	1.219	41.2%	0.537	50.0%	[36.8%, 63.2%]	60
1986–1990	3.727%	3.698%	–0.120	42.3%	4.217***	40.0%	[27.6%, 53.5%]	60
1991–1995	11.526%	0.940%	–0.280	11.7%	1.519*	30.0%	[18.8%, 43.2%]	60
1996–2000	10.386%	2.903%	–1.094	27.1%	3.312***	36.7%	[24.6%, 50.1%]	60
2001–2005	0.425%	2.435%	0.790	48.9%	1.595*	41.7%	[29.1%, 55.1%]	60
1926–2005	5.911%	3.569%		37.7%	9.801***	40.1%	[37.0%, 43.3%]	960

Figure 2

Histogram of annualized monthly excess returns (U.S. data, 1926–2005)

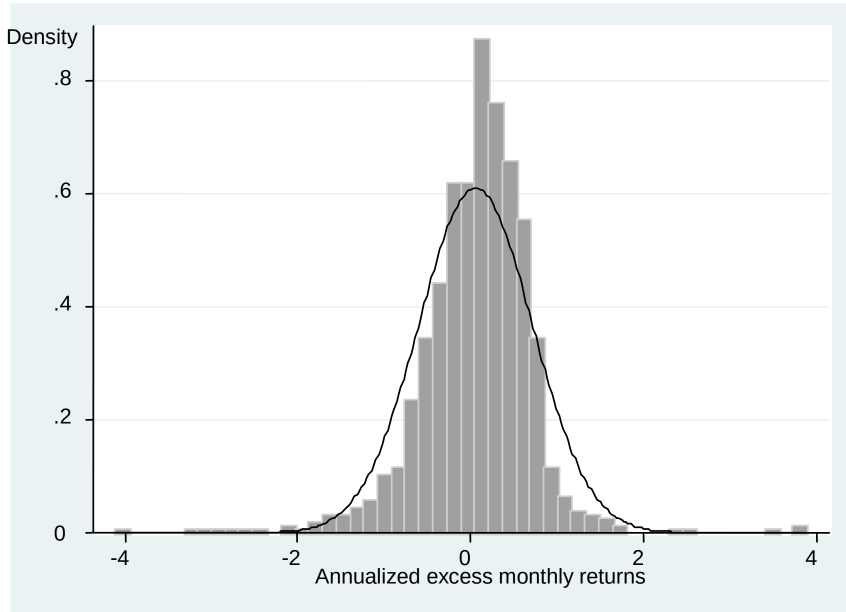


Figure 3

Risk-free percentiles for one-year investment horizons in the U.S.

The figure displays risk-free percentiles for one-year investment horizons using U.S. data from successive five-year periods during the years of 1926–2005. The numbers are from Panel B in Table 1. The black line shows the exact risk-free percentiles. The gray line is the confidence interval of the risk-free percentiles for the various five-year intervals.

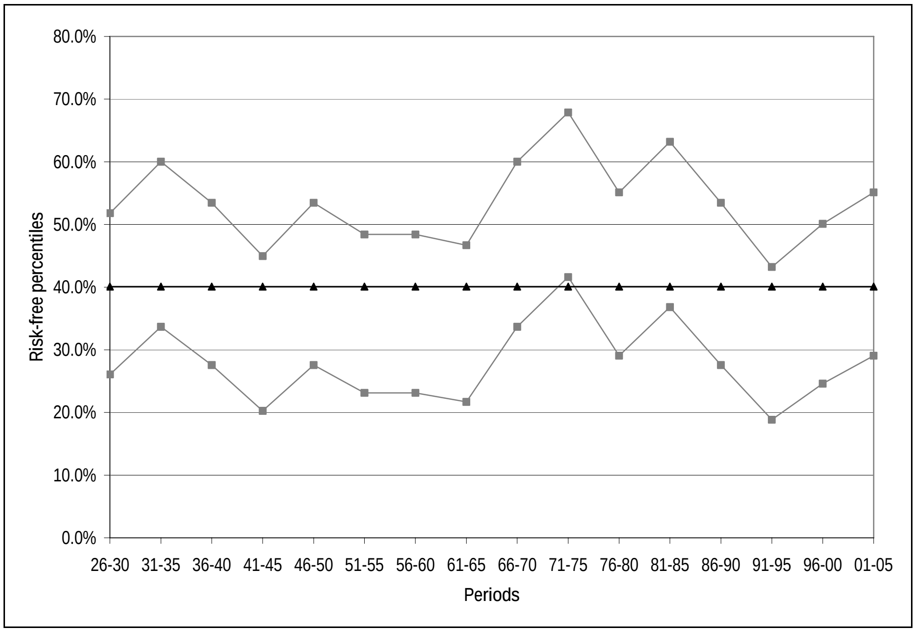


Table 2

Risk-free percentiles for different cash-flow maturities in the U.S.

This table reports risk-free percentiles for different cash-flow maturities. The CRSP Value Weighted Index is used as a proxy for the market index, and the yield on 90-day T-bills is chosen as a proxy for the risk-free interest rate. The data are from the years 1926–2005. The various columns show: (1) The investment horizon, i.e., the maturity of the cash flows of the hypothetical project; (2) The estimated mean cumulative excess returns for the different investment horizons (= 5.911 percent × investment horizon; 5.911 percent is from Table 1); (3) The variance of the excess returns (= 3.569 percent × investment horizon; 3.569 percent is from Table 1); (4) The risk-free percentiles assuming normality.

Investment horizon in years	Estimated mean cumulative excess returns	Estimated variance of the excess returns	Risk-free percentiles, assuming normality
(1)	(2)	(3)	(4)
1	5.9%	3.6%	37.7%
2	11.8%	7.1%	32.9%
3	17.7%	10.7%	29.4%
4	23.6%	14.3%	26.6%
5	29.6%	17.8%	24.2%
10	59.1%	35.7%	16.1%

Table 3

Risk-free percentiles for one-year investment horizons across international capital markets

The table reports risk-free percentiles for one-year investment horizons using data from foreign capital markets during the years of 1971–2005. The markets examined are the United Kingdom (U.K.), Japan, and Switzerland. Proxies for the market index are the MSCI indices for the countries in question. The risk-free rates are those reported by the individual countries' central bank. The columns (1), (3), and (5) report exact risk-free percentiles based on actual observations; The columns (2), (4), and (6) show the binomial 95 percent-confidence intervals for those percentiles; and (7) The number of monthly returns in each five-year period. *, **, *** denote significance at the 10 percent, 5 percent, and 1 percent level, respectively.

Period	U.K.		Japan		Switzerland		
	Exact risk-free percentiles	Exact binomial 95%-confidence intervals	Exact risk-free percentiles	Exact binomial 95%-confidence intervals	Exact risk-free percentiles	Exact binomial 95%-confidence intervals	Number of obs
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1971–1975	50.0%	[36.8%, 63.2%]	38.3%	[26.1%, 51.8%]	43.3%	[30.6%, 56.8%]	60
1976–1980	45.0%	[32.1%, 58.4%]	45.0%	[32.1%, 58.4%]	45.0%	[32.1%, 58.4%]	60
1981–1985	30.0%*	[18.8%, 43.2%]	33.3%	[21.7%, 46.7%]	40.0%	[27.6%, 53.5%]	60
1986–1990	43.3%	[30.6%, 56.8%]	40.0%	[27.6%, 53.5%]	55.0%*	[41.6%, 67.9%]	60
1991–1995	40.0%	[27.6%, 53.5%]	56.7%*	[43.2%, 69.4%]	35.0%	[23.1%, 48.4%]	60
1996–2000	43.3%	[30.6%, 56.8%]	51.7%	[38.4%, 64.8%]	30.0%*	[18.8%, 43.2%]	60
2001–2005	45.0%	[32.1%, 58.4%]	43.3%	[30.6%, 56.8%]	43.3%	[30.6%, 56.8%]	60
1971–2005	42.8%	[38.1%, 47.6%]	44.4%	[39.7%, 49.3%]	42.4%	[37.7%, 47.2%]	960