The Influences of Foreclosure Factors on the Value, Yield, Duration and Convexity of a Risky Mortgage

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Abstract

This paper presents a theoretical model that considers the influence of the foreclosure factors (including the foreclosure lag, the foreclosure costs and the ratio of the auction price to the unpaid balance of mortgage—hereafter denoted as APUPB ratio) on the analysis of the formulas for mortgage value, yield, duration and convexity. Furthermore, we used data obtained by FHA to perform the numerical analyses. The results indicate that the average default period of a mortgage and the average foreclosure lag are 2.4749 years and 1.9 years, respectively. For each percentage increase in APUPB ratio, the foreclosure lag decreases by about 4.57 days. The lender's opportunity cost is greater than the foreclosure settlement costs, and the total foreclosure costs are 25% of the house auction price. Moreover, the foreclosure lag, foreclosure costs and the discounted auction price (i.e., 1- APUPB ratio) are negatively correlated with the mortgage value, duration and convexity, but positively correlated with the mortgage yield. Also, the results reveal that a change in the default probability has the greatest influence on mortgage convexity and the next greatest on mortgage yield. Our model and results should provide portfolio managers and lenders a useful tool to more appropriately measure the yield, duration and convexity of a mortgage and thus help them to more effectively hedge their mortgage holdings.

Keywords: Yield, Duration, Convexity, Foreclosure Lag, Foreclosure Costs

1. Introduction

After the outbreak of the U.S. housing market crisis in 2007-2008, many financially suffering households could not afford to repay their mortgages, thus inducing foreclosures. Because the foreclosure process usually takes a long time, lenders bear the foreclosure settlement costs and the cost of the lost mortgage interest (opportunity cost) during this period. Previous studies have demonstrated that long foreclosure lags and high foreclosure costs indeed significantly influence the mortgage value, yield, duration and the lender’s loss given default (hereafter defined as LGD) (Clauretie 1990; Quigley and Van Order, 1991; Riddiough and Wyatt, 1994; Qi and Yang, 2009; Clauretie and Daneshvary, 2011; Park and Band, 2014). Thus, the goal of this study was to use a theoretical model that simultaneously considers the foreclosure factors (the settlement period of foreclosure—hereafter defined as foreclosure lag—foreclosure costs and the ratio of the auction price to the unpaid balance—hereafter denoted as APUPB ratio) to analyze their influence on the closed-form formulas of the value, yield, duration and convexity of a risky mortgage that we derived.

Every foreclosure for lenders may incur a large LGD caused by the uncertain recovery amount and uncertain recovery time, which are viewed as the lender’s recovery risk. The lender incurs an LGD because the amount recovered from the foreclosure sale of the collateral is less than the unpaid balance and foreclosure costs during the foreclosure lag. Several scholars have demonstrated that the estimated losses on a foreclosure range from 30% to 60% of the outstanding loan balance (Capone, 1996; Clauretie and Herzog, 1990; Ciochetti, 1997). The foreclosure lag, foreclosure costs and the discount of the auction price of the foreclosed property are the three main factors that affect the variability of the LGD.

A number of studies have found that the foreclosure lag has a large range.

Capozza and Thomson (2006) estimated that it can be 6.5 years for a seriously delinquent subprime loan, but Ambrose and Capone (1996) found an average duration of only 7.5 months for seriously delinquent FHA loans. Cordell et al. (2013) estimated that in judicial foreclosure states (statutory foreclosure states) in the US, the foreclosure lag is only 25 (16) months for the pre-foreclosure- crisis period and increases to 44 (22) months for the post-foreclosure-crisis period.

When foreclosing the collateral of a defaulted mortgage, the lender bears the foreclosure costs that come from the opportunity cost and the foreclosure settlement costs (accrued interest, legal fees, administrative costs, property taxes and maintenance costs). Cutts and Green (2005) estimated foreclosures to cost an average of $58,792. Wilson (1995) found that a 5-month delay in judicial states raised time-dependent costs by 5% of the loan balance. Cordell et al. (2013) showed the estimated foreclosure cost[1] to be 11% for the pre-foreclosure-crisis period and 9% for the post-foreclosure-crisis period. Qi and Yang (2009) and Park and Band (2014) demonstrated that the LGD in distressed housing markets is significantly higher than under normal housing conditions. Qi and Yang (2009) empirically demonstrated that the LGD is positively related to the duration of the foreclosure process. This implies that the longer the foreclosure lag and the higher the foreclosure costs, the greater the LGD incurred by the lenders. Tsai, Chiang and Miller (2016) empirically demonstrated that the opportunity cost per monetary unit of the outstanding mortgage balance is nearly twice the mortgage rate. This implies that for a defaulted mortgage the foreclosure costs significantly influence the lender's recovery amount from the foreclosed property.

Several researchers have found that mortgage defaults and the resulting property

[1] These calculated foreclosure costs are presented as a percentage of the unpaid balance (UPB).

foreclosures generate a discount of the auction price of the foreclosed property of 7% to 24%, depending on its location within the US (e.g., Shilling et al., 1990; Forgey et al., 1994; Hardin and Wolverton, 1996; Springer, 1996; Carroll et al., 1997; Pennington-Cross, 2006; Clauretie and Daneshvary, 2009).

The above studies imply that the high variability of the foreclosure lag, foreclosure costs and the discount ratio of the auction price induce a large change in the lender’s LGD. Since the foreclosure factors significantly influence the LGD, researchers have considered these foreclosure factors when investigating mortgages. Curley and Guttentag (1974) developed a procedure for calculating the yield on foreclosed FHA loans, taking account of the special costs to the lender associated with the foreclosure. They also found that the "net foreclosure gain or loss"[2] can be explained largely by the duration of the foreclosed loan from acquisition to settlement, the foreclosure lag with FHA and the lender's legal expenses. Ambrose et al. (1997) incorporated foreclosure lag in their model for a reasonably priced mortgage. Several studies have revealed that the borrower’s default behavior and the foreclosure procedure have a significant influence on mortgage yield (Ambrose and Capone, 1998). Harrison and Seiler (2015) found that the rate quotes posted by lenders are directly related to measures of foreclosure process risk, including the foreclosure lag. Obviously, the foreclosure factors (foreclosure lag, foreclosure costs and discount of the auction price of the foreclosed property) significantly influence mortgage value, yield and duration.

Accurately pricing a mortgage, determining its appropriate yield and accurately measuring its duration and convexity are important for both portfolio managers and

[2] Curley and Guttentag (1974) term the difference between the yield on a foreclosed loan and the expected yield as the "net foreclosure gain or loss".

market participants. Because the factors associated with foreclosure affect the uncertain recovery amount and uncertain recovery time, they should also affect the value, yield, duration and convexity of a mortgage. When valuing mortgage yield, duration and convexity, researchers often include the termination risk and interest rate risk in their models (Ott, 1986; Kau et al., 1993; Yang et al., 1998; Tsai et al., 2009) but in previous studies the foreclosure factors have never been simultaneous and fully included in the models. In view of that, this study was intended to derive the valuation formulas of the mortgage model, taking into account the factors associated with recovery risk (i.e., foreclosure lag, foreclosure costs and APUPB ratio).

The issues related with mortgage value, yield and duration have been analyzed using three approaches: the contingent-claim approach (Childs et al., 1997; Yang et al., 1998; Ambrose and Buttimer, 2000; Azevedo-Pereira et al., 2003), the intensity-form approach (Schwartz and Torous, 1989, 1993; Quigley and Van Order, 1990; Gong and Gyourko, 1998; Lambrecht et al., 2003; Ambrose and Sanders, 2003; Liao et al., 2008; Tsai et al., 2009; Tsai and Chiang, 2012) and the empirical analysis (Ott, 1986; Berger and Udell, 1990; Quigley and Van Order, 1990, 1995; Derosa et al., 1993; Haensly et al., 1993; Chiang et al., 2002; Lambrecht et al., 2003). The intensity-form approach evaluates the probabilities of prepayment and default based on hazard rate information. The termination probability is inserted into the model to determine the value, yield and duration of a risky mortgage. Because it is easy to derive the formulas for the value, yield, duration and convexity of a mortgage using the intensity-form approach, we used this method to model a risky mortgage contract.

It is important for portfolio managers and market participators to fully understand how the foreclosure factors influence mortgage value, yield, duration and convexity. However, only a few studies have completely addressed such issues. To gain an understanding of how mortgage yield, duration and convexity affect changes in the various foreclosure factors, we not only derived their formulas for the purpose of conducting sensitivity analyses, but we also performed numerical analyses using FHA data. Because the foreclosure is caused by the default event, we also describe three scenarios with different levels of default probabilities for showing the changes for mortgage value, yield, duration and convexity.

The remainder of this article is organized as follows. The next section presents the valuation framework. It includes the model design and explains how to derive the implicit formulas for value, yield, duration and convexity of a mortgage. In the third section, we report sensitivity analyses examining how the foreclosure parameters influence mortgage value, yield, duration and convexity. In the fourth section, we use numerical examples to illustrate the application of our model. The conclusion is the final section.

2. The valuation model

This section describes a model to valuate a fixed-rate mortgage (FRM), the basic building block of the mortgage market. We assume that the FRM is a fully amortized mortgage that has an initial mortgage principal M_0, a fixed coupon rate c and time to maturity of T years. This implies a continuous payout Y equal to:

$$Y = M(0) \times \frac{c}{1-e^{-cT}}. \tag{1}$$

The principal outstanding at time t, $M(t)$, is obtained by

$$M(t) = M(0) \times \frac{1-e^{-c(T-t)}}{1-e^{-cT}}. \tag{2}$$

Because the borrower usually has the option to prepay or default the mortgage contract, we denote the probability of survival, the probability of prepayment and the probability of default at time t as $S(t)$, $P^P(t)$ and $P^D(t)$, respectively. Since we concentrate on the effect of the factors regarding foreclosure (such as the foreclosure lag, foreclosure costs and the discount ratio of the auction price) on the mortgage value and its yield, duration and convexity, to avoid complicated specifications that would blur the point, we simply specify $S(t)$, $P^P(t)$ and $P^D(t)$ as deterministic variables in the valuation model. The value of the mortgage at time 0 is therefore (Liao et al., 2008):

$$V(0) = Y\int_0^T S(s)E[e^{-\int_0^s r(u)du}]ds + \int_0^T M(s)P^P(s)E[e^{-\int_0^s r(u)du}]ds$$

$$+\int_0^T M(s)P^D(s)E[(1-l(s))\,e^{-\int_0^s r(u)du}]ds\,, \quad (3)$$

where $V(0)$ is the mortgage value at time 0, $E[\cdot]$ is an expected operator risk-neutral measure, and $r(t)$ is the short interest rate.

To derive the solution of Equation (3), we need to specify the short interest rate process $r(t)$ and the loss rate process $l(t)$. To begin, we specify the interest rate as the extended Vasicek model, a single-factor model with deterministic volatility that can match an arbitrary initial forward-rate curve through the specification of the long-term short interest rate $\bar{r}(t)$ (Vasicek, 1977; Heath et al., 1992). Under the risk-neutral measure, the term-structure evolution is described by the dynamics of the short interest rate:

$$dr(t) = a(\bar{r}(t) - r(t))dt + \sigma_r dZ_r(t)\,, \quad (4)$$

where a is the speed of adjustment (a positive constant), σ_r is the volatility of the short interest rate (a positive constant), $\bar{r}(t)$ is the long-term short interest rate (a deterministic function of t), and $Z_r(t)$ is a standard Brownian motion under the risk-neutral measure.

In Equation (4), the short interest rate follows a mean-reverting process under a risk-neutral measure. As shown in Heath et al. (1992), to match an arbitrary initial forward-rate curve, one can set the following:

$$\bar{r}(u)=f(t,u)+\frac{1}{a}\left(\frac{\partial f(t,u)}{\partial u}+\frac{\sigma_r^2(1-e^{-2a(u-t)})}{2a}\right), \tag{5}$$

where $f(t,u)$ is the instantaneous forward rate. Combining Equations (4) and (5), the evolution of the short interest rate can be rewritten as

$$r(u)=f(t,u)+\frac{\sigma_r^2(e^{-a(u-t)}-1)^2}{2a^2}+\int_t^u \sigma_r e^{-a(u-v)}dZ_r(v). \tag{6}$$

Accordingly, we can define a zero coupon bond with \$1 face value and maturity date s. Its value at the time 0 can be described as the following:

$$B(0,s)=E[e^{-\int_0^s r(u)du}]=e^{-\mu_{X_0}(0,s)+\frac{1}{2}\sigma_{X_0}^2(0,s)}, \tag{7}$$

where

$B(0,s)$ is the value of zero coupon bond at the time 0;

$\mu_{X_0}(0,s)=f(0,s)s+\frac{\sigma_r^2}{2a^2}(s-\frac{2}{a}(1-e^{-as})+\frac{1}{2a}(1-e^{-2as}))$; and

$\sigma_{X_0}^2(0,s)=\frac{\sigma_r^2}{a^2}(s-\frac{2}{a}(1-e^{-as})+\frac{1}{2a}(1-e^{-2as}))$.

Based on Equation (7), one can obtain a closed-form formula for the first and second terms on the right-hand side of Equation (3):

$$Y\int_0^T S(s)E[e^{-\int_0^s r(u)du}]ds=Y\int_0^T S(s)B(0,s)ds, \text{ and} \tag{8}$$

$$\int_0^T M(s)P^P(s)E[e^{-\int_0^s r(u)du}]ds = \int_0^T M(s)P^P(s)B(0,s)ds\,. \tag{9}$$

To investigate the influence of the foreclosure factors on mortgages, we model the loss rate to include these factors. We assume that the lender chooses to force the property to be offered at a foreclosure auction when the mortgage borrower defaults and has no desire or ability to keep the property. If the property does not sell at foreclosure, the lender may take back the property so that it becomes real estate owned (REO) (Chan et al., 2014).[3] We specify in the model the APUPB ratio of the foreclosed collateral as κ. The APUPB ratio can also be viewed as the gross recovery rate for a defaulted mortgage, because it does not take off the foreclosure costs.

Note that κ may be greater than one, since there are defaulters with positive equity. Generally, defaulters are divided into two types: negative equity defaulters and positive equity defaulters (Foote et al., 2008; Pennington-Cross, 2010). Negative equity defaulters decide to default because the collateral value is less than the outstanding balance on the mortgage; thus, it may make financial sense to default. On the contrary, positive equity defaulters decide to default because unexpected trigger events (e.g., job loss, divorce, significant change in health status) make it difficult for them to continue making timely payments on their current mortgages. Under this situation, the value of the collateral exceeds the outstanding balance. Thus, κ may be larger than one.

Here the foreclosure lag is denoted as ξ. We assume ξ is a function of the APUPB ratio. A simple foreclosure lag function is described as

$$\xi = \xi_0 + \xi_1 \kappa\,, \tag{10}$$

[3] Because of our focus on the influence of the foreclosure lag on mortgages, we are not concerned here with REO.

where ξ_0 is a baseline foreclosure lag that is not influenced by the κ, ξ_1 is the marginal rate for the foreclosure lag corresponding to κ; specifically, it represents the unit change in ξ given a one-unit increase in κ.

Generally, the relationship between ξ and κ is ambiguous. When foreclosing the collateral of a defaulted mortgage, the lender faces the following four problems. First, because a longer foreclosure procedure increases the exposure of the financial institution to the financially distressed asset holding, the lender faces increased pressure from the financial supervisors. In view of that, the lender is motivated to sell the property quickly to meet the regulatory requirements related to the risk associated with the asset holding. Second, the uncertain foreclosure lag makes it difficult to forecast the recovery timing. Previous empirical studies have demonstrated that the estimated foreclosure lag ranges from 7.5 months to 6.5 years, depending on the mortgage type and property location (Ambrose and Capone, 1996; Capozza and Thomson, 2006; Pennington-Cross, 2010; Tsai et al., 2016).

Third, the unpredictability of the auction price of a foreclosed property makes it difficult to manage the risk of the LGD. In empirical studies, the estimated foreclosure discount has ranged wildly from 4% to 33%, depending on the mortgage type and location (Shilling et al., 1990; Forgey et al., 1994; Hardin and Wolverton, 1996; Springer, 1996; Clauretie and Daneshvary, 2009; Rogers, 2010; Campbell et al., 2011). Fourth, the foreclosure costs increase the lender's LGD and make it difficult to predict the recovery amount. Russell (1937) claimed that lenders incur foreclosure costs of approximately 5% of the property value. Posner and Zingales (2009) estimated the lender's incurred foreclosure costs to be 30% of the housing price. Park and Bang (2014) empirically demonstrated that each month in auction leads to an added LGD of 0.7%. This implies that the higher foreclosure costs result from the longer foreclosure lag; this increases the lender’s LGD.

Given the above four problems, the lender is motivated to sell the foreclosed property quickly to avoid carrying costs and to reduce pressures from the financial supervisors; thus, the house is most likely to be sold at a greater discount. However, a lower APUPB ratio may prompt the lender to sell the property quickly to avoid higher foreclosure costs, but a too low APUPB ratio could mean that the net recovery amount obtained from the foreclosed property will be less than the unpaid mortgage balance. By contrast, a higher APUPB ratio could increase the recovered amount but may discourage the buyer from participating in the foreclosure, thereby probably lengthening the foreclosure lag and subsequently increasing the foreclosure costs. Thus, for the lender, in theory there is a problematic trade-off between the APUPB ratio and the foreclosure lag.

Most empirical research has demonstrated that the lender’s motivation for a quick sale may lead to a significant discount from the market price. Forgey et al. (1994) argued that lenders who desire to reduce the period of time they hold a property will accept a lower price to reduce the marketing time. Springer (1996) found that foreclosed homes sold more quickly, with a 4 to 6 percent price discount. Empirical evidence from Clauretie and Daneshvary (2011) demonstrates that the greater the discount the shorter the marketing time, and for each percentage increase in price the marketing time increased by about 11 days. Based on these results, one can infer that the shorter foreclosure lag, the lower the APUPB ratio.

However, a number of empirical studies also have found a negative relationship between sale price and market time. This negative relationship has been attributed to a "stigma" effect of prolonged marketing time (Jud et al., 1996). Huang and Palmquist (2001) provided a very plausible explanation for this result: as the marketing time increases, sellers adjust their reservation price, resulting in a negative relationship between them. Accordingly, in Equation (10), if an increase in the APUPB ratio lengthens the foreclosure lag, we have $\xi_1 > 0$. On the contrary, if $\xi_1 < 0$, there is a "stigma" effect in the relation between the foreclosure lag and APUPB ratio.

The APUPB ratio represents the recovered funds per monetary unit of the outstanding balance of the defaulted mortgage before taking off the foreclosure costs. Thus, it also can be viewed as the gross recovery rate. The present value of the APUPB ratio at time s, ψ^{APUPB}, can be calculated as:

$$\psi^{\text{APUPB}} = \kappa\, e^{-\int_s^{s+\xi} r(u)du}. \tag{11}$$

Previous studies have demonstrated that the foreclosure settlement costs and the lender's opportunity cost during the foreclosure lag significantly behind the increase in the lender's LGD. Thus, we divide the total foreclosure costs into two components: the lender's opportunity cost and the foreclosure settlement costs. The lender's opportunity cost (OC) is defined as the maximum interest the lender can earn from the outstanding balance of the mortgage loan during the foreclosure lag. According to Tsai et al. (2016), we have

$$\psi^{\text{OC}} = e^{r_C \xi}\, e^{-\int_s^{s+\xi} r(u)du}, \tag{12}$$

where ψ^{OC} denotes the present value of the OC at time s during the entire foreclosure lag, namely, the lender's opportunity cost per monetary unit of the outstanding mortgage balance.

For measuring the total foreclosure settlement costs for a defaulted loan during a foreclosure lag, we assume the foreclosure settlement cost rate ($\phi(v)$) for each time period to be a fixed percentage of the outstanding principal of the defaulted mortgage. Thus the present value of the total foreclosure settlement costs at time s during the foreclosure lag can be obtained as follows:

$$\psi^{\mathrm{FC}} = \int_0^{\xi} \phi(v) e^{-\int_s^{s+v} r(u)du} dv , \tag{13}$$

where ψ^{FC} represents the foreclosure settlement costs per monetary unit of the outstanding mortgage balance. Moreover, for simplicity of expression, we assume $\phi(v) = \phi$ is a constant variable. The present value of the total foreclosure costs at time s is $\psi^{\mathrm{OC}} + \psi^{\mathrm{FC}}$.

According to the previous specifications, the loss rate, which depends on the factors associated with foreclosure, can be specified as follows:

$$l(s) = \max[0, 1 - (\psi^{\mathrm{APUPB}} - \psi^{\mathrm{OC}} - \psi^{\mathrm{FC}})]. \tag{14}$$

Substituting Equation (14) into Equation (3), one can obtain the value of the mortgage considering the foreclosure factors (APUPB ratio, foreclosure lag and foreclosure costs). Because the lenders mainly focus on the positive value of LGD, we assume $l(u) > 0$, which means that there is a loss rate when default occurs. Thus, we have

$$1 - l(u) = (\kappa - e^{r_C \xi})\ e^{-\int_s^{s+\xi} r(u)du} - \phi \int_0^{\xi} e^{-\int_s^{s+v} r(u)du} dv . \tag{15}$$

To solve for the expected value in the third term in Equation (3), we derive $E[(1 - l(s))\, e^{-\int_0^{s} r(u)du}]$ as the following:

$$E[(1 - l(s))\, e^{-\int_0^{s} r(u)du}]$$

$$= E[\,(\kappa - e^{r_C \xi})e^{-\int_0^{s+\xi} r(u)du}\,] - \int_0^{\xi} \phi E[e^{-\int_0^{s+v} r(u)du} dv]$$

$$= (\kappa - e^{r_C(\xi_0+\xi_1\kappa)})B(0, s+\xi_0+\xi_1\kappa) - \int_0^{\xi_0+\xi_1\kappa} \phi B(0, s+v)dv\,. \qquad (16)$$

Finally, substituting the results of Equations (8), (9) and (16) into Equation (3), we obtain the mortgage value, along with the interest risk, prepayment risk, default risk and recovery risk, under the risk-neutral measure:

$$V(0) = Y\int_0^T S(s)B(0,s)ds + \int_0^T M(s)P^P(s)B(0,s)ds$$

$$+\int_0^T M(s)P^D(s)(\kappa - e^{r_C(\xi_0+\xi_1\kappa)})B(0, s+\xi_0+\xi_1\kappa)ds$$

$$-\int_0^T \phi M(s)P^D(s)\left(\int_0^{\xi_0+\xi_1\kappa} B(0, s+v)dv\right)ds\,. \qquad (17)$$

In Equation (17), the first term represents the expected value of a mortgage that does not terminate until maturity. The second term represents the expected value of a mortgage that has been prepaid before maturity. The sum of the third and the fourth terms represents the expected value of a mortgage that has been defaulted before maturity, taking account of the foreclosure factors.

3. Sensitivity analyses

To determine the influence of the parameters related with foreclosure on the mortgage value, and yield, duration and convexity of a mortgage, we report sensitivity analyses in Subsections 3.1 and 3.2.

3.1 Analyzing the influence of the foreclosure parameters on the mortgage value

In our model, the mortgage value is jointly determined by the survival probability $S(s)$, the prepayment probability $P^P(s)$, the default probability $P^D(s)$, the values of the initial yield curve $f(t,s)$, the term-structure evolution parameters a and σ_r and the parameters regarding foreclosure (ϕ, ξ_0, ξ_1 and κ). Here we focus mainly on the influence of the foreclosure factors on the mortgage value. The sensitivity analyses of interest rate, prepayment probability and default probability are relegated to Appendix A.

According to the Leibniz integral rule, we have

$$\frac{d\int_0^{\xi} B(0,s+v)dv}{d\xi} = B(0,s+\xi)\,. \tag{18}$$

Thus, we display the partial derivative of the mortgage value with respect to the foreclosure factors as follows:

$$\frac{\partial V(0)}{\partial \xi_0} = \int_0^T M(s)P^D(s)\kappa \frac{B(0,\xi)}{\partial \xi} ds - \int_0^T M(s)P^D(s)\phi(B(0,s+\xi))ds < 0\,; \tag{19}$$

$$\frac{\partial V(0)}{\partial \xi_1} = \int_0^T M(s)P^D(s)\kappa^2 \frac{B(0,\xi)}{\partial \xi} ds - \int_0^T M(s)P^D(s)\phi\kappa B(0,s+\xi)ds < 0\,; \tag{20}$$

$$\frac{\partial V(0)}{\partial \phi} = -\int_0^T M(s)P^D(s)(\int_0^{\xi_0+\xi_1\kappa} B(0,s+v)dv\,)ds < 0\,; \tag{21}$$

$$\begin{aligned}\frac{\partial V(0)}{\partial \kappa} &= \int_0^T M(s)P^D(s)B(0,\xi)ds + \int_0^T M(s)P^D(s)\kappa \frac{\partial B(0,\xi)}{\partial \xi}(\xi_1)ds \\ &\quad - \int_0^T M(s)P^D(s)\phi B(0,s+\xi)(\xi_1)ds\,.\end{aligned} \tag{22}$$

where $\frac{\partial B(0,\xi)}{\partial \xi} = -B(0,\xi)f(0,\xi) < 0$.[4]

Equations (19)-(21) show that there are negative relationships of the mortgage value and the parameters related with the parameters of the function of foreclosure lag (ξ_0 and ξ_1) and foreclosure settlement cost (ϕ). They reveal that an increase in the

[4] The derivation of the formula of $\frac{\partial B(0,\xi)}{\partial \xi}$ is shown in Appendix A.

values for the parameters of the function of foreclosure lag and foreclosure settlement costs decreases the mortgage value. Furthermore, the relationship between the mortgage value and the APUPB ratio depends on the sign of ξ_1. If $\xi_1 > 0$, $\frac{\partial V(0)}{\partial \kappa}$ cannot be reliably judged as positive or negative. The impact of the APUPB ratio on the mortgage value depends on whether the sum of the first and second terms is large or less than the value of the third term on the right-hand side of Equation (22). The main reason is that although an increase in the APUPB ratio (i.e., the gross recovery rate) can increase the mortgage value, it also can lead to an increase in the foreclosure lag and foreclosure costs, thereby decreasing the mortgage value. If $\xi_1 < 0$, we have $\frac{\partial V(0)}{\partial \kappa} > 0$. This means that if there is a "stigma" effect (i.e., $\xi_1 < 0$), an increase in the APUPB ratio may increase the mortgage value.

3.2 Analyzing the influence of the foreclosure parameters on the mortgage yield, duration and convexity

Sensitivity analyses of the impact of the foreclosure parameters (ξ_0, ξ_1, κ and ϕ) on the yield, duration and convexity of a mortgage can be easily performed using our valuation formula. We first derive the explicit formulas of yield, duration and convexity, and then we determine the effect of the foreclosure parameters on them by deriving their partial derivatives with respect to these parameters.

To start, with R defined as the risk-adjusted yield of a risky mortgage required by the mortgage holder at time 0, the mortgage value at time 0 can be expressed as

$$V(0) = Y\int_0^T \exp(-Rs)ds = Y(\frac{1-e^{-RT}}{R}). \tag{23}$$

As shown in Equations (1) and (2), we have $M(t) = Y\frac{1-e^{-c(T-t)}}{c}$. Thus Equation (17) can be rewriting as

$$V(0)=Y\int_0^T S(s)B(0,s)+c^{-1}(1-e^{-c(T-s)})P^P(s)B(0,s)$$

$$+c^{-1}(1-e^{-c(T-s)})P^D(s)(\kappa-e^{c(\xi_0+\xi_1\kappa)})B(0,s+\xi_0+\xi_1\kappa)$$

$$-c^{-1}(1-e^{-c(T-s)})P^D(s)\phi(\int_0^{\xi_0+\xi_1\kappa} B(0,s+v)dv))ds\,. \tag{24}$$

According to Equations (23) and (24), we can express the yield of the mortgage as the following function:

$$R=f(S(\cdot),P^P(\cdot),P^D(\cdot),B(0,.),c,T,\xi_0,\xi_1,\kappa,\phi)\,. \tag{25}$$

Then, the formulas for the duration and convexity of a mortgage can be obtained as follows (see Appendix B):

$$D=-\frac{1}{V(0)}\frac{\partial V(0)}{\partial R}=T+R^{-1}-T(1-e^{-RT})^{-1}\text{, and} \tag{26}$$

$$C=\frac{1}{V(0)}\frac{\partial^2 V(0)}{\partial R^2}=2R^{-2}+(2R^{-1}T+T^2)e^{-RT}(1-e^{-RT})^{-1}. \tag{27}$$

To conduct the sensitivity analyses on yield, duration and convexity of a mortgage, for simplicity of expression, we let φ denote the model parameters (ξ_0, ξ_1, κ and ϕ). The partial derivatives of the yield, duration and convexity of variable φ can be obtained as follows (see Appendix B):

$$\frac{\partial R}{\partial \varphi}=-A_1\frac{\partial V(0)}{\partial \varphi}, \tag{28}$$

$$\frac{\partial D}{\partial \varphi}=-A_2\frac{\partial R}{\partial \varphi}=A_1A_2\frac{\partial V(0)}{\partial \varphi}\text{, and} \tag{29}$$

$$\frac{\partial C}{\partial \varphi}=-A_3\frac{\partial R}{\partial \varphi}=A_1A_3\frac{\partial V(0)}{\partial \varphi}. \tag{30}$$

where $A_1 = \dfrac{1}{V(0)D} > 0$,

$A_2 = R^{-2} + T^2 e^{-RT} > 0$, and

$A_3 = 4R^{-3} + e^{-RT}(1 - e^{-RT})^{-1}$

$$\times (2R^{-2}T + (2R^{-1}T^2 + T^3)\ (2R^{-2}T + (2R^{-1}T^2 + T^3)e^{-RT}(1 - e^{-RT})^{-1}) > 0 .$$

According to Equations (28)-(30), we can use the results of the sensitivity analyses of the foreclosure parameters on the mortgage value to infer the impacts of the foreclosure parameters on the mortgage yield, duration and convexity. Because A_1, A_2 and A_3 are all greater than zero, the influence of the foreclosure parameters on the mortgage yield is inversely related to that on the mortgage value, but their influence on the mortgage duration and convexity are positively related to that on the mortgage value. For example, according to Equation (21), an increase in the foreclosure cost (ϕ) decreases the mortgage value. However, according to Equations (28)-(30), it increases the mortgage yield but decreases the mortgage duration and convexity. The rest can be deduced by analogy.

The results show that all the foreclosure factors except κ are positively correlated with the mortgage yield, but they are negatively correlated with the mortgage duration and convexity. This reveals that when the foreclosure factors increase, the expected present value of the recovered amount received from the foreclosed loan decreases. Therefore, a larger loss risk induces the lender to require a higher yield to compensate for the losses. Moreover, our results imply that a higher recovery risk leads to a shorter mortgage duration and less convexity. This inference is similar to the arguments in Chance (1990), Derosa et al. (1993) and Tsai et al. (2009).

4. Numerical analyses

Here we use numerical examples to illustrate the application of our model. We use data from the FHA databank to estimate the parameters of the foreclosure lag and foreclosure costs. Subsection 4.1 describes the data for the foreclosure sale of collateral for defaulted mortgages. In Subsection 4.2, we show the estimates of the parameters of the foreclosure factors and the analyses of foreclosure factors on the mortgage value, yield, duration and convexity

4.1 Data descriptions

The FHA databank contains the insured mortgage contracts for 13,153,880 U.S. loans. We selected a mortgage contract with 30-year maturity for our example. The sample period starts on 1 Jan. 1998 and ends on 30 Dec. 2011. We define the foreclosure lag as beginning on the date the borrower stops monthly payments and ending on the date the house is sold. The foreclosure settlement cost is defined as the sum of the maintenance expenses, the repair fees, the settlement charges and the tax expenses.[5] After data cleansing, we collected 387,757 samples for the numerical analyses.[6]

Table 1 presents the descriptive statistics (mean, standard deviation, median, maximum and minimum values) for the mortgages, the initial loans, the auction prices of the foreclosed properties, the foreclosure lags and the various foreclosure costs. The average auction price of the foreclosed properties ($103,430) is more than the average

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[5] The maintenance expenses are calculated using accounting rule 15-1502 (06/2008). The repair fees are calculated using the same rule. Settlement charges are required by HUD. The tax expenses are calculated using accounting rule 15-1503 (06/2008).

[6] We delete the data when the defaulted mortgage has the following characteristics: 1. missing data; 2. the auction price less than 500 dollars; and 3. negative values for the maintenance expenses, repair fees, and tax expenses.

initial loan amount ($102,870). The mean default period for the mortgage contracts is 2.4749 years after the mortgage origination date. The average foreclosure lag is approximately 1.9 years, but for a seriously delinquent subprime loan it is approximately 12 years.

[Insert Table 1 here]

The foreclosure settlement costs come mainly from the maintenance expenses and the settlement charges. The sum of these two costs is 86.53% of the total settlement foreclosure costs, which are approximately 11.43% of the house auction price (11,817.89/103,430). The opportunity cost is defined as the maximum accrued interest the lender can earn from the unpaid balance (UPB) of the defaulted mortgage during the foreclosure lag. The mean, standard deviation, minimum and maximum of lender's opportunity cost are $14,074, $10,010, $507.86 and $434,140, respectively. On average, the lender's opportunity cost is approximately 13.6% of the house auction price (14,074/103,430). The lender's opportunity cost (13.6%) is greater than the foreclosure settlement costs (11.4%). Although previous studies did not include the lender's opportunity cost in the foreclosure costs, our results reveal that the opportunity cost for defaulted mortgages should not be ignored. After the total foreclosure costs are taken off, the net recovery amount ($77,538.11)[7] is less than the outstanding balance ($100,050). The total foreclosure costs are approximately 25% of the house auction price. This is similar to the findings in Posner and Zingales (2009), in which the lender's incurred foreclosure costs are estimated as 30% of the housing price.

The above results allow us to calculate the following four ratios: the ratio of the auction price to the initial housing price, the APUPB ratio (κ), the ratio of the

[7] This amount is calculated as 103,430 -11,817.89 -14,074 = 77,538.11.

opportunity cost to the UPB and the ratio of the foreclosure settlement cost to the UPB. From the mean of the ratio of the auction price of the collateral to the initial housing price (96.74%), we can see that the average auction price of the foreclosed house is 3.26% less than its average initial value. That the mean APUPB ratio (104%) is greater than 100% implies that it is positive equity defaulters in our sample. Moreover, the average ratio of opportunity cost (14.406%) is more than the average ratio of foreclosure settlement costs (12.493%). If the foreclosure costs include only the foreclosure settlement costs, our result (12.493%) is similar to the findings in Cordell et al. (2013), in which the foreclosure costs were estimated as 11% of the unpaid balance. When the foreclosure costs include the foreclosure settlement costs and the opportunity cost, the mean of the total foreclosure costs is more than 26% of the UPB. This indicates that the opportunity cost is a very important part of the lender's total foreclosure costs. On average, the lender's gross recovery rate is 104%, but the net recovery rate is 78% (104% - 26%); that is, the average LGD is 22% for lenders.

4.2 Numerical results for the influences of foreclosure factors on the mortgage value, yield, duration and convexity

Because we focus on analyses of the effects of the foreclosure factors, we let the prepayment and default probabilities be deterministic values. Data pertaining to the prepayment and default probabilities were obtained from the Department of Housing & Urban Development's 2010 FHA annual actuarial report.[8] These data were sampled yearly from 1998 to 2010. The prepayment and default probabilities, obtained by calculating their means for each contract year, are shown in Table 2.

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[8] The title of the report is "Actuarial Review of the Federal Housing Administration Mutual Mortgage Insurance Fund (Excluding HECMs) for Fiscal Year 2010." HUD's website is http://portal.hud.gov/hudportal/HUD.

[Insert Table 2 here]

For the numerical example it is necessary to estimate the values of ξ_0 and ξ_1, as shown in Equation (10). These estimates are presented in Table 3, where it is shown that the estimate of the baseline foreclosure lag (ξ_0) is 1.9169, which is significant at the 1% level. This means that the expected baseline foreclosure lag, which is the foreclosure lag not influenced by the APUPB ratio, is 1.9169 years. The estimated marginal rate for the foreclosure lag influenced by the APUPB ratio (ξ_1) is -0.0125 (years) which is also significant at the 1% level. In other words, there is a negative relationship between the foreclosure lag and the APUPB ratio. A higher APUPB ratio decreases the foreclosure lag. This means that our case exemplifies the "stigma" effect. The result also shows that for each percentage increase in APUPB ratio, the foreclosure lag decreases by about 4.57 (0.0125 × 365) days. Obviously, our results imply that the loss rate (1 - APUPB ratio) is positively related with the foreclosure lag. This result is consistent with the findings in Qi and Yang (2009) and Tsai et al. (2016), in which the LGD was positively related to the length of the foreclosure process. Tsai et al. (2016) explained this phenomenon as possibly caused by a greater discount (e.g., lower auction price) for the collateral because the building is old or in a bad location, resulting in a longer foreclosure lag.

< Insert Table 3 here >

For the numerical analyses, we let the parameters for a fixed-rate mortgage be $M_0 = \$100$ million, r_C = 8%, and $T = 30$ years. For simplicity, we let the interest rate be a constant value, r = 4%. Thus, we can obtain $B(0,T) = \exp(-0.04T)$. According to the estimated results in Tables 1 and 3, the basic parameters for calculating the mortgage value, yield and duration are κ = 104%, ξ_0 = 1.9169, ξ_1 = -0.0125, η = 14.406% and ϕ = 12.493%. The numerical results for the mortgage value, yield, duration and convexity are shown in Table 4.

< Insert Table 4 here >

Because the foreclosure results mainly come from the default, we provide three scenarios to show how sensitive the value, yield, duration and convexity of the mortgage are to the changes in the default probability. In Scenario 1, we use the prepayment and default probabilities shown in Table 4 to obtain the mortgage value, yield, duration and convexity. The default probabilities in Scenarios 2 and 3 are assumed to be 5 times and 10 times the default probabilities in Scenario 1, respectively.

In Scenario 1, we have $\psi^{\mathrm{SPTOR}} = 0.9638$, $\psi^{\mathrm{OC}} = 0.1525$ and $\psi^{\mathrm{FC}} = 0.2290$. Accordingly, the lender's expected net recovery rate for a defaulted mortgage is 58.23% of the outstanding balance. The lender's LGD (38.15%) in our example is within the estimated losses within a foreclosure range from 30% to 60% of the outstanding loan balances in previous studies (Capone, 1996; Clauretie and Herzog, 1990; Ciochetti, 1997). Next, using the formulas shown in Equations (24)-(27), we obtain the mortgage value, yield, duration and convexity, respectively. The results in Table 4 reveal that as the default probability increases, the mortgage value, duration and convexity decrease, but the yield increases. These results conform to economic intuition. As a mortgage becomes riskier, lenders require a higher yield to compensate for the higher default risk. Moreover, our results imply that a higher risk of default leads to a shorter mortgage duration and less convexity. This finding confirms the inference from the sensitivity analyses in Subsection 3.2 and is similar to the arguments in Chance (1990), Derosa et al. (1993) and Tsai et al. (2009).

Compared the results in Scenarios 2 (Scenarios 3) with that in Scenarios 1, we find the mortgage value decreases 7.05% (13.61%), the yield increases 12.65% (25.86%), the duration decreases 3.89% (7.81%) and the convexity decreases 21.43%

(37.49%). These results reveal that the change in the default probability has the greatest influence on the mortgage value, yield, duration, and convexity. Nevertheless, the effect of the default risk on the mortgage convexity is greater than its effect on the mortgage duration. This finding conforms to the arguments in Tsai et al. (2009).

The numerical analyses of the influence of the parameters related to the foreclosure factors on mortgage value, yield, duration and convexity are shown in Figures 1-4. These figures show the analyses for the foreclosure factors, namely, the baseline foreclosure lag (ξ_0), the marginal rate for the foreclosure lag corresponding with the APUPB ratio (ξ_1), the APUPB ratio (κ) and the foreclosure settlement cost rate (ϕ). According to Figures 1 and 2, the influence of ξ_0 on the mortgage value, yield, duration and convexity are similar to the influence of ξ_1 on them. Accordingly, the parameters related to the foreclosure lag (ξ_0 and ξ_1) are negatively correlated with the mortgage value, duration and convexity. Only the mortgage yield is positively correlated with these two parameters.

Figures 3a, 3b, 3c and 3d show the numerical analyses for the influence of the APUPB ratio on the mortgage value, yield, duration and convexity, respectively. The results indicate that an increase in the APUPB ratio (the gross recovery rate) increases the mortgage value, duration and convexity, but it decreases the mortgage yield. These results imply that a decrease in the recovery risk reduces the mortgage yield, lengthens the mortgage duration and increases the convexity. Finally, the analyses for the influence of the foreclosure periodic settlement cost rate (ϕ) on the mortgage value, yield, duration and convexity are shown in Figure 4. The results show that an increase in ϕ decreases the mortgage value, duration and convexity, but it increase the mortgage yield.

The numerical results shown in Figures 1-4 confirm our inferences from the sensitivity analyses in Section 3. That is, the impact of the foreclosure factors on the mortgage yield is opposite to that on the mortgage value, but the impact of these factors on the mortgage duration and convexity are similar to that on the mortgage value.

< Insert Figures 1-4 here >

5. Conclusion

Measuring the values, yields, durations and convexities of mortgages is quite complicated, due to uncertainty in their changes that is caused by the interest rate risk, prepayment risk, default risk and recovery risk. When investigating mortgage values, yields, durations and convexities, most studies have focused on the impact of the interest rate risk, prepayment risk and default risk, but they seldom addressed the impact of the recovery risk resulting from the foreclosure. A number of previous studies have empirically demonstrated that the great variability of the foreclosure lag, the foreclosure costs and the discount ratio of the auction price induce a large change in the lender's LGD. Thus, for this study we decided to derive the explicit mortgage valuation formulas based on a model that includes the foreclosure factors. Further, we demonstrated how sensitive the value, yield, duration and convexity of a mortgage are to changes in these foreclosure factors by performing numerical and sensitive analyses. To the best of our knowledge, no previous studies have used a completely theoretical model to analyze this interesting issue.

We used data obtained from the FHA databank to conduct numerical analyses that illustrate the application of our model. The empirical results tell us that the average default point is 2.4749 years after the mortgage origination time and that the average foreclosure lag is approximately 1.9 years. The foreclosure settlement costs are approximately 11.43% (12.493%) of the house auction price (unpaid balance) and the lender's opportunity cost is approximately 13.6% (14.406%) of the house auction price. These results indicate that the opportunity cost is a very important part of the lender's total foreclosure cost, and thus it should not be ignored when calculating the latter. The total foreclosure cost is approximately 25% of the house auction price and 26% of the outstanding balance. Thus, after taking off the total foreclosure cost, the net recovery amount could be less than the outstanding balance of the defaulted mortgage.

Also, the empirical results reveal that the auction price of the foreclosed house represents a 3.26% discount of the initial value of the foreclosed property; on average, the lender's gross recovery rate (APUPB ratio) is 104%, but the net recovery rate is 78%; in other words, the lender incurs a 22% LGD. Since the APUPB ratio is larger than 100%, it is positive equity defaulter in our example.

Our estimates of the foreclosure lag parameters show that the expected baseline foreclosure lag, that is, the foreclosure lag not influenced by the APUPB ratio, is 1.9169 years. In addition, there is a negative relationship between the foreclosure lag and the APUPB ratio: for each percentage increase in the APUPB ratio, the foreclosure lag decreases by about 4.57 days. This phenomenon may result from a lower auction price for the collateral, perhaps due the building being old or in a bad location, resulting in a longer foreclosure lag.

The numerical results show that the factors related to the foreclosure lag, foreclosure costs and the discounted auction price (1 - APUPB ratio) are negatively correlated with the mortgage value, duration and convexity, but are positively correlated with the mortgage yield. These results imply that a higher recovery risk induces lenders to require a higher yield to compensate for their losses, and shortens the mortgage duration and convexity. These assertions are consistent with Chance (1990), Derosa et al. (1993) and Tsai et al. (2009). The numerical results also reveal that a change in the default probability has its greatest influence on the mortgage yield and convexity. Nevertheless, the effect of the default risk on the mortgage convexity is greater than its effect on the mortgage duration.

Since our formulas for the value, yield, duration and convexity of the mortgage reflect the impact of recovery risk more sensitively than the traditional formula does, they are more appropriate for assessing management risk. Therefore, our model and results should provide portfolio managers and lenders a useful tool to more appropriately measure the yield, duration and convexity of a mortgage; thus, using our formulas can improve the efficiency of their immunization strategies.

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Appendix A

This Appendix supports the sensitivity analyses of the effects of the interest rate, prepayment and default hazard rate parameters on the value of a mortgage. Let ϖ be the parameters in the interest rate model ($\varpi = f(0,s)$, a, σ_r s); we then have the partial derivative of ϖ on $V(0)$ as follows:

$$\frac{\partial V(0)}{\partial \varpi} = Y\int_0^T S(s)\frac{\partial B(0,s)}{\partial \varpi}ds + \int_0^T M(s)P^P(s)\frac{\partial B(0,s)}{\partial \varpi}ds$$
$$+\int_0^T M(s)P^D(s)(\kappa - e^{c(\xi_0+\xi_1\kappa)})\frac{\partial B(0,s+\xi_0+\xi_1\kappa)}{\partial \varpi}ds$$
$$-\int_0^T \phi M(s)P^D(s)(\int_0^{\xi_0+\xi_1\kappa}\frac{\partial B(0,s+v)}{\partial \varpi}dv)ds,$$

where $\frac{\partial B(0,s)}{\partial \varpi} = B(0,s)(-\frac{\partial \mu_{X_0}(0,s)}{\partial \varpi} + \frac{1}{2}\frac{\partial \sigma^2_{X_0}(0,s)}{\partial \varpi})$.

From Equation (7) we have

$$\frac{\partial \mu_{X_0}(0,s)}{\partial f(0,s)} = s,$$
$$\frac{\partial \mu_{X_0}(0,s)}{\partial a} = \sigma_r^2(-\frac{s}{a^3} + \frac{3(1-e^{-as})}{a^4} - \frac{e^{-as}s}{a^3} - \frac{3(1-e^{-2as})}{4a^4} + \frac{e^{-2as}s}{2a^3}),$$
$$\frac{\partial \sigma^2_{X_0}(0,s)}{\partial a} = \sigma_r^2(-\frac{2s}{a^3} + \frac{6(1-e^{-as})}{a^4} - \frac{2e^{-as}s}{a^3} - \frac{3(1-e^{-2as})}{2a^4} + \frac{e^{-2as}s}{a^3}),$$
$$\frac{\partial \mu_{X_0}(0,s)}{\partial \sigma_r} = \frac{\sigma_r}{a^2}(s - \frac{2}{a}(1-e^{-as}) + \frac{1}{2a}(1-e^{-2as})),$$
$$\frac{\partial \sigma^2_{X_0}(0,s)}{\partial \sigma_r} = \frac{2\sigma_r}{a^2}[s - \frac{2}{a}(1-e^{-as}) + \frac{1}{2a}(1-e^{-2as})],$$
$$\frac{\partial \mu_{X_0}(0,s)}{\partial s} = f(0,s) + \frac{\sigma_r^2}{2a^2}(1-2e^{-as}+e^{-2as}), \text{ and} \tag{A1}$$
$$\frac{\partial \sigma^2_{X_0}(0,s)}{\partial s} = \frac{\sigma_r^2}{a^2}(1-2e^{-as}+e^{-2as}). \tag{A2}$$

Let the survival function be defined as (Lancaster, 1992):

$$S(t) = e^{-\int_0^t(\theta(u)+\pi(u))du}, \tag{A3}$$

where $\theta(t)$ and $\pi(t)$ denote the hazard rates of the terminated mortgage, prepaid and defaulted at time t, respectively. We therefore have

$P^P(t) = \theta(t)S(t)$ and $P^D(t) = \pi(t)S(t)$.

For the prepayment and default hazard rate, we have

$$\frac{\partial V(0)}{\partial \gamma(s)} = Y\int_0^T \frac{\partial S(s)}{\partial \gamma(s)} B(0,s)ds + \int_0^T M(s)\frac{\partial P^P(s)}{\partial \gamma(s)} B(0,s)ds$$

$$+\int_0^T M(s)\frac{\partial P^D(s)}{\partial \gamma(s)}(\kappa - e^{r_C(\xi_0+\xi_1\kappa)})B(0,s+\xi_0+\xi_1\kappa)ds$$

$$-\int_0^T M(s)\frac{\partial P^D(s)}{\partial \gamma(s)}\phi(\int_0^{\xi_0+\xi_1\kappa} B(0,s+v)dv)ds,$$

where $\gamma(s) = \theta(s)$ and $\pi(s)$. According to Equation (A3), we have

$$\frac{\partial S(s)}{\partial \theta(s)} = -S(s); \quad \frac{\partial P^P(s)}{\partial \theta(s)} = S(s) - \theta(s)S(s); \quad \frac{\partial P^D(s)}{\partial \theta(s)} = -\pi(s)S(s);$$

$$\frac{\partial S(s)}{\partial \pi(s)} = -S(s); \quad \frac{\partial P^P(s)}{\partial \pi(s)} = -\theta(s)S(s); \text{ and } \frac{\partial P^D(s)}{\partial \pi(s)} = S(s) - \pi(s)S(s).$$

The following shows the derivation of the formula for $\frac{\partial B(0,\xi)}{\partial \xi}$ in Equations (20)-(24). According to Equation (7), we have

$$\frac{\partial B(0,\xi)}{\partial \xi} = B(0,\xi)(-\frac{\partial \mu_{X_0}(0,\xi)}{\partial \xi} + \frac{1}{2}\frac{\partial \sigma^2_{X_0}(0,\xi)}{\partial \xi}).$$

Using the formulas in Equations (A1) and (A2), we have

$$\frac{\partial \mu_{X_0}(0,\xi)}{\partial \xi} = f(0,\xi) + \frac{\sigma_r^2}{2a^2}(1-2e^{-a\xi}+e^{-2a\xi}), \text{ and}$$

$$\frac{\partial \sigma^2_{X_0}(0,\xi)}{\partial \xi} = \frac{\sigma_r^2}{a^2}(1-2e^{-a\xi}+e^{-2a\xi}).$$

Thus we have

$$\frac{\partial B(0,\xi)}{\partial \xi} = B(0,\xi)(-f(0,\xi)) < 0.$$

Appendix B

This appendix shows the formulas in Equations (28)-(30). According to Equation (23), we have

$$V(0) = YR^{-1}(1-e^{-RT}) . \tag{B1}$$

Thus, the differential of $V(0)$ by φ, where φ represents the model's parameters, namely ξ_0, ξ_1, ϕ, κ_0 and κ, can be expressed as follows:

$$\begin{aligned}
\frac{\partial V(0)}{\partial \varphi} &= \frac{\partial V(0)}{\partial R}\frac{\partial R}{\partial \varphi} \\
&= Y((\frac{\partial R^{-1}}{\partial R}(1-e^{-RT}) + R^{-1}\frac{\partial(1-e^{-RT})}{\partial R})\frac{\partial R}{\partial \varphi}) \\
&= -V(0)(-\frac{1}{V(0)}\frac{\partial V(0)}{\partial R})\frac{\partial R}{\partial \varphi} \\
&= -V(0)D\frac{\partial R}{\partial \varphi} .
\end{aligned}$$

In other words, we have

$$\frac{\partial R}{\partial \varphi} = -\frac{1}{V(0)D}\frac{\partial V(0)}{\partial \varphi} .$$

This is Equation (28).

The duration of mortgage is defined as follows:

$$D = -\frac{1}{V(0)}\frac{\partial V(0)}{\partial R} . \tag{B2}$$

According to Equation (B1), we have:

$$\begin{aligned}
\frac{\partial V(0)}{\partial R} &= Y(\frac{\partial R^{-1}}{\partial R}(1-e^{-RT}) + R^{-1}\frac{\partial(1-e^{-RT})}{\partial R}) \\
&= -R^{-1}Y\frac{(1-e^{-RT})}{R} + R^{-1}Ye^{-RT}T \\
&= -R^{-1}Y\frac{(1-e^{-RT})}{R} - T(Y\frac{-e^{-RT}+1-1}{R}) \\
&= -R^{-1}V(0) - TV(0) + \frac{TY}{R} .
\end{aligned} \tag{B3}$$

Substituting Equation (B3) into Equation (B2) and using Equation (B1), we have

$$D = \frac{1}{R} + T - \frac{TY}{V(0)R} = T + R^{-1} - T(1-e^{-RT})^{-1} .$$

The differential of D by φ can be expressed as follows:

$$\frac{\partial D}{\partial \varphi} = -R^{-2}\frac{\partial R}{\partial \varphi} - T^2e^{-RT}\frac{\partial R}{\partial \varphi} = -(R^{-2} + T^2e^{-RT})\frac{\partial R}{\partial \varphi} .$$

This is Equation (29).

Convexity is defined as follows:

$$C = \frac{1}{V(0)} \frac{\partial^2 V(0)}{\partial R^2} . \tag{B4}$$

According to Equation (B3), we have

$$\frac{\partial V(0)}{\partial R} = -R^{-1} Y \frac{(1-e^{-RT})}{R} + R^{-1} Y e^{-RT} T .$$

Thus the second differential of $V(0)$ by R can be expressed as follows:

$$\begin{aligned} \frac{\partial^2 V(0)}{\partial R^2} &= 2R^{-3}(1-e^{-RT})Y - R^{-2}(e^{-RT}T)Y - R^{-2}Ye^{-RT}T - R^{-1}Ye^{-RT}TT \\ &= 2R^{-3}(1-e^{-RT})Y - (R^{-1}T + R^{-1}T + T^2)R^{-1}Ye^{-RT} \\ &= 2R^{-2}V(0) - (2R^{-1}T + T^2)(V(0) - \frac{Y}{R}) . \end{aligned} \tag{B5}$$

Substituting Equation (B4) into Equation (B5), we have

$$\begin{aligned} C &= 2R^{-2} - (2R^{-1}T + T^2)(1 - \frac{Y}{RV(0)}) \\ &= 2R^{-2} - (2R^{-1}T + T^2)(1 - \frac{1}{1-e^{-RT}}) \\ &= 2R^{-2} + (2R^{-1}T + T^2)e^{-RT}(1-e^{-RT})^{-1} . \end{aligned}$$

The differential of C by φ can be expressed as follows:

$$\begin{aligned} \frac{\partial C}{\partial \varphi} = -\frac{\partial R}{\partial \varphi}[4R^{-3} + e^{-RT}(1-e^{-RT})^{-1} \\ \times (2R^{-2}T + (2R^{-1}T^2 + T^3)\ (2R^{-2}T + (2R^{-1}T^2 + T^3)e^{-RT}(1-e^{-RT})^{-1})] . \end{aligned}$$

This is Equation (30).

Figure 1: Sensitivity analyses of the influence of ξ_0 on mortgage value, yield, duration and convexity

Figure 1a: The relation of mortgage value and ξ_0

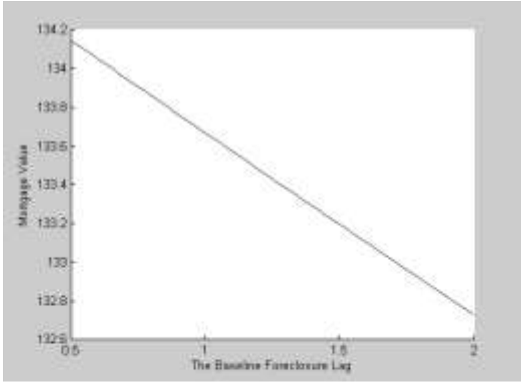


Figure 1c: The relation of mortgage duration and ξ_0

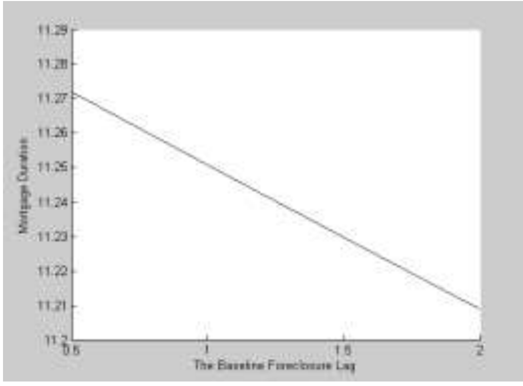


Figure 1b: The relation of mortgage yield and ξ_0

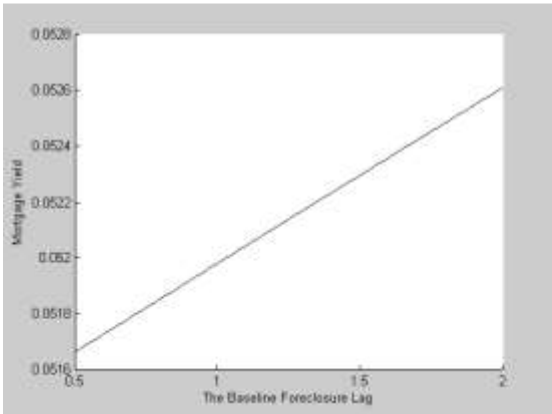
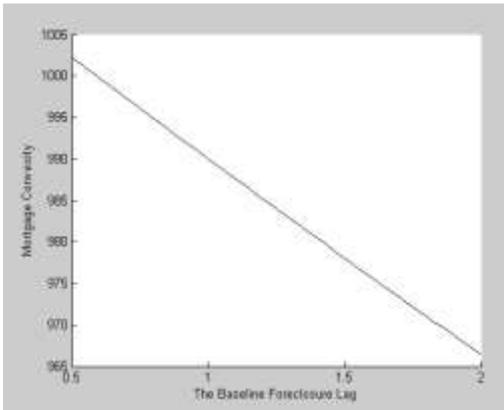


Figure 1d: The relation of mortgage convexity and ξ_0



Note: The x-axis represents ξ_0. The y-axis represents mortgage value, yield, duration and convexity in Figures 1a, 1b, 1c and 1d, respectively. ξ_0 ranges from 0.5 to 2.

Figure 2: Sensitivity analyses of the influence of ξ_1 on mortgage value, yield, duration and convexity

Figure 2a: The relation of mortgage value and ξ_1

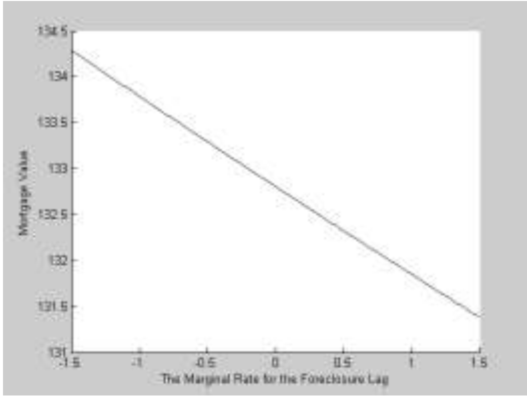


Figure 2c: The relation of mortgage duration and ξ_1

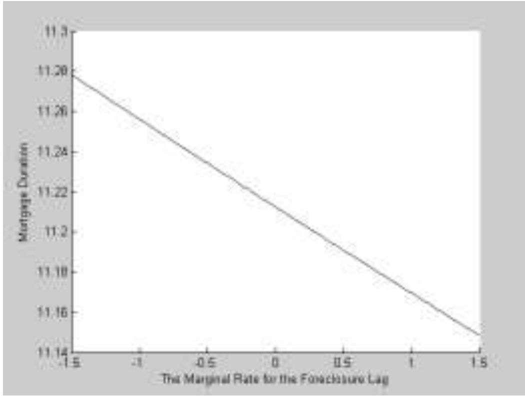


Figure 2b: The relation of mortgage yield and ξ_1

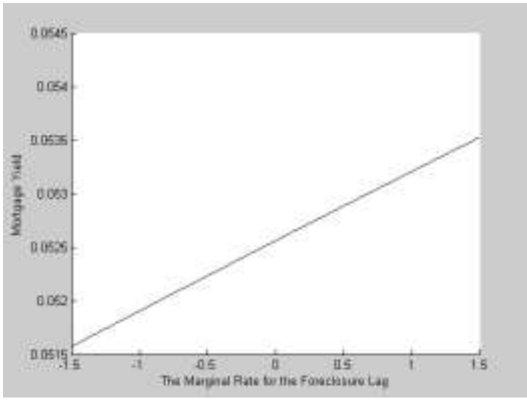
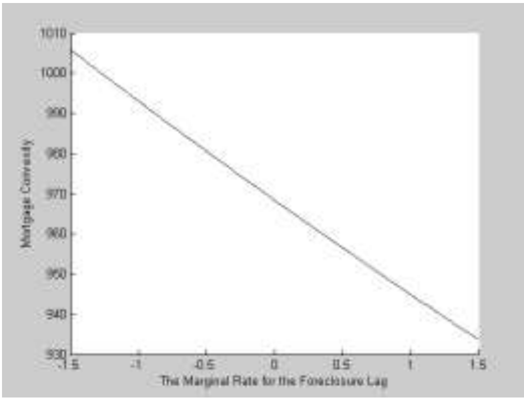


Figure 2d: The relation of mortgage convexity and ξ_1



Note: The x-axis represents ξ_1. The y-axis represents mortgage value, yield, duration and convexity in Figures 2a, 2b, 2c and 2d, respectively. ξ_1 ranges from -1.5 to 1.5.

Figure 3: Sensitivity analyses of the influence of κ on mortgage value, yield, duration and convexity

Figure 3a: The relation of mortgage value and κ

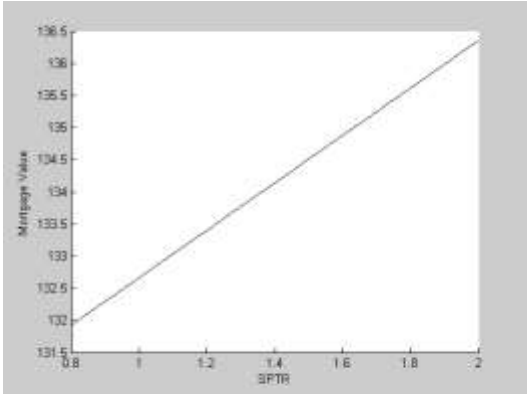


Figure 3c: The relation of mortgage duration and κ

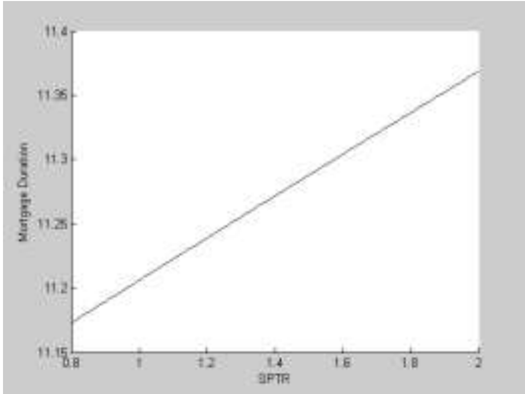


Figure 3b: The relation of mortgage yield and κ

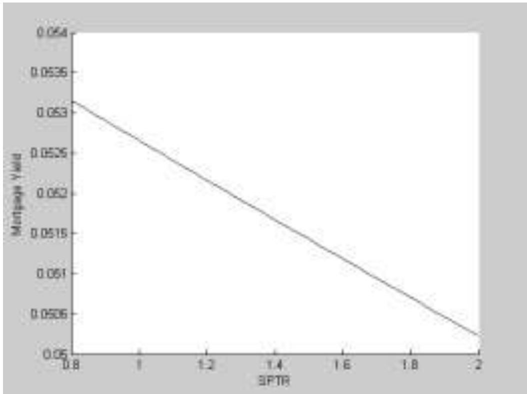
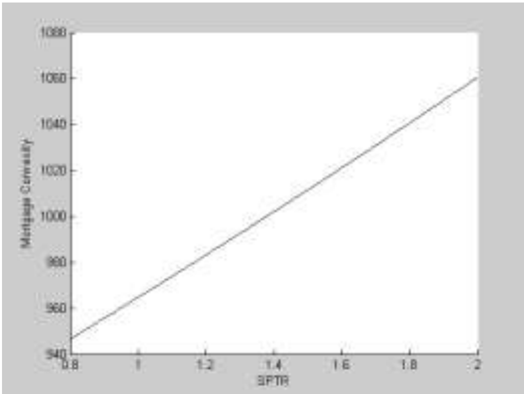


Figure 3d: The relation of mortgage convexity and κ



Note: The x-axis represents κ (APUPB ratio). The y-axis represents mortgage value, yield, duration and convexity in Figures 3a, 3b, 3c and 3d, respectively. κ ranges from 0.8 to 2.

Figure 4: Sensitivity analyses of the influence of ϕ on mortgage value, yield, duration and convexity

Figure 4a: The relation of mortgage value and

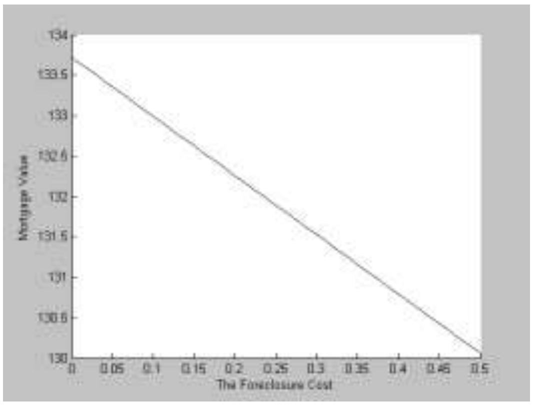


Figure 4c: The relation of mortgage duration and

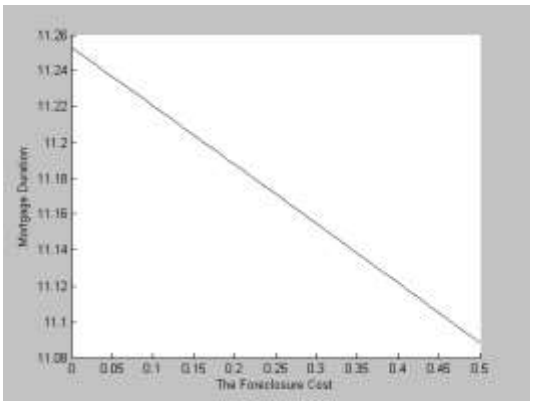


Figure 4b: The relation of mortgage yield and

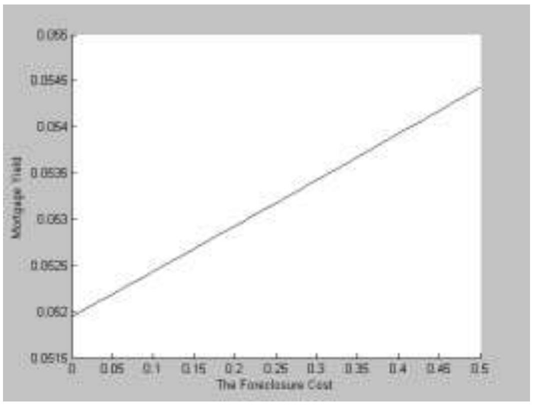
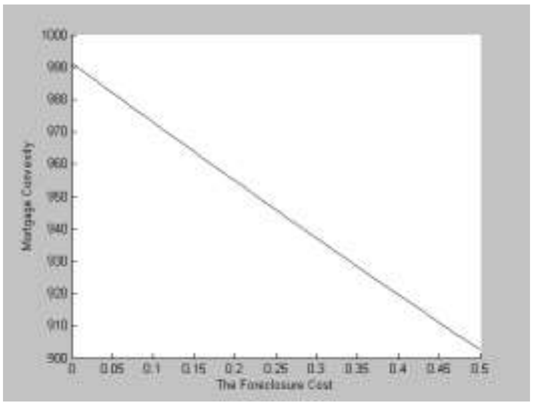


Figure 4d: The relation of mortgage convexity and



Note: The x-axis represents ϕ. The y-axis shows mortgage value, yield, duration and convexity in Figures 4a, 4b, 4c and 4d, respectively. ϕ ranges from 0 to 0.5.

Table 1: Summary statistics for mortgage and foreclosure factors

	Variables	Mean	Std	Min	Max
Mortgage (US dollar)	Mortgage rate	6.919%	1.0533%	1%	10.7%
	Initial loan	102,870	44,555	6,400	521,810
	Unpaid balance (UPB)	100,050	43,805	6,031.9	509,360
	Auction Price	103,430	48,583	503	1,000,000
Period (year)	Default period	2.4749	1.9094	0.076712	16.011
	Foreclosure lag (ξ)	1.9038	0.77798	0.42192	11.882
Foreclosure cost (US dollar)	Maintain expenses	5,562.3	3,697.1	19.44	298,030
	Settlement charges	4,663.4	3,264.3	0.01	114,260
	Repaired fees	314.09	1,184	0.01	142,680
	Tax expenses	1,278.1	1,422.5	0.01	79,884
	Opportunity cost	14,074	10,010	507.86	434,140
Ratio	Ratio of auction price on initial housing price	96.74%	25.69%	0.29983%	2,124.7%
	APUPB ratio (κ)	104%	27.771%	0.36732%	2,630.4%
	Ratio of opportunity cost on UPB	14.406%	7.5928%	0.90544%	168.17%
	Ratio of foreclosure settlement cost on UPB	12.493%	5.6468%	0.60125%	254.98%

Note: We used data from the FHA databank to estimate the foreclosure lag and foreclosure costs. The sample period starts on 1 Jan. 1998 and ends on 30 Dec. 2011. After data cleansing, we collected 387,757 samples for the numerical analyses. The total foreclosure costs are defined as the sum of the foreclosure settlement costs (maintenance expenses, repair fees, settlement charges, tax expenses) and the opportunity cost. The foreclosure lag is defined as beginning on the date the borrower stops monthly payments and ending on the date the house is sold. "Unpaid balance" (UPB) is the unpaid loan balance of the defaulted mortgage at the default time point. Interest during the foreclosure lag period represents the accrued interest, calculated based on the outstanding balance at the default time point, the mortgage rate and the foreclosure lag. The ratio of the auction price to the initial housing price is the auction price of the collateral divided by the initial housing price. The rest can be deduced by analogy. For each mortgage contract, we use its LTV ratio and its initial loan amount to calculate the initial housing price. "Mean", "Std", "Min" and "Max" stand for the mean, standard deviation, minimum and maximum, respectively.

Table 2: Summary of termination probabilities for each calendar year

Calendar Years	Prepayment probability	Default probability	Calendar Years	Prepayment probability	Default probability
1	0.0320%	0.0003%	16	8.1300%	0.6703%
2	0.4177%	0.0157%	17	8.2563%	0.3297%
3	1.0717%	0.1350%	18	8.9200%	0.3720%
4	1.1310%	0.3183%	19	9.0653%	0.5067%
5	1.1780%	0.3837%	20	9.4363%	0.6993%
6	1.4560%	0.3997%	21	9.3647%	0.7533%
7	2.4090%	0.3897%	22	9.1173%	0.8880%
8	3.4140%	0.3687%	23	8.9137%	0.7470%
9	5.1270%	0.6230%	24	9.0187%	0.5990%
10	6.0757%	0.8437%	25	9.5010%	0.8857%
11	6.5340%	0.9633%	26	9.7353%	1.5660%
12	6.1087%	0.5283%	27	10.1860%	1.6377%
13	6.9410%	0.5823%	28	13.8810%	1.4737%
14	7.3770%	0.7140%	29	14.3930%	2.5890%
15	7.6050%	0.6393%	30	13.0900%	1.6900%

Note: The data regarding the prepayment and default probabilities were obtained from the Department of Housing & Urban Development's 2010 FHA annual actuarial report. These data were sampled yearly from 1998 to 2010. This table shows the prepayment and default probabilities that are obtained by calculating their means for each calendar year the mortgage is issued.

Table 3: Estimates of foreclosure lag parameters

Parameters	ξ_0	ξ_1
Estimates	1.9169***	-0.0125***
P-value	0.0000	0.0054

Note: The estimates are obtained from Equation (13). "***" means significant at the 1% level.

Table 4: Numerical results for mortgage value, yield, duration and convexity

	Value	Yield	Duration	Convexity
Scenario 1	132.82	5.255	11.213	968.65
Scenario 2	123.45	5.92	10.777	761
Scenario 3	114.74	6.6138	10.337	605.51

Note: The basic parameters used for calculating the mortgage value, yield, duration and convexity are: $M_0 = 100$, $T = 30$, $r = 4\%$, $r_C = 8\%$, $\kappa = 104\%$, $\xi_0 = 1.9169$, $\xi_1 = -0.0125$, $\eta = 14.406\%$ and $\phi = 12.493\%$. The mortgage value, yield, duration and convexity are calculated from Equations (24), (25), (26) and (27), respectively. In Scenario 1, we use the prepayment and default probabilities shown in Table 3 to calculate the mortgage value, yield, duration and convexity. The default probabilities in Scenarios 2 and 3 are assumed to be 5 times greater and 10 times greater, respectively, than the default probability in Scenario 1.