Macroprudential Policy, Difference in Beliefs and Growth: What is the Role of Risk Premia? A Model of Macroprudential Policy *

Daren Wei†

*I am grateful for helpful comments from Harjoat Bhamra, Alexander Michaelides, David Miles and Tarun Ramadorai. All remaining errors are my own.

†Imperial College Business School, Tanaka Building, Exhibition Road, London SW7 2AZ; Email: d.wei12@imperial.ac.uk.

Abstract

We propose a model of macroprudential interventions in financial markets. We solve a heterogeneous-agent asset pricing model and show that trading on the difference in beliefs causes distorted and more volatile individual consumption and leisure as well as deviated output. Ex ante however, these effects are not internalized in individual consumption-leisure-portfolio decisions,while macroprudential policy helps to offset those distortions. Our results show that macroprudential policy reduces equity premium and volatility, and enhances the social welfare under every possible reasonable probability measure by the belief-neutral welfare criteria in Brunnermeier, Simsek, and Xiong (2014).

1 Introduction

As the ultimate asset holders, households process new information and decide on their trades using competing models of economic fundamentals through financial intermediaries, e.g. commercial banks, hedge funds, ETFs, etc. Some of those households do not always attempt to share risks with others; rather, in periods of large disagreement of views they take bets on the relative accuracy of their models' predictions. However, speculative trading is a negative sum game because they may cause financial imbalances (Scheinkman and Xiong (2003)). Macroprudential policy is such a tool to regulate speculation. But how does it impact the social welfare and economic growth? This is the main question we are trying to answer in this paper.

Before looking at the macroprudential policy, we shall understand why trading due to the difference in belief would cause negative externalities. This point can be illustrated by the following story from Brunnermeier, Simsek, and Xiong (2014)[1]. Consider a bet between Joe and Bob who argue over the filling material of a pillow. Joe thinks the filling is natural down with probability 0.9 while Bob thinks it is artificial down. They decide to bet over by cutting this pillow: if the filling is natural down, then Bob gives Joe \$100; if it is artificial down, Joe gives Bob \$100 ; the winner buys a new pillow with a cost of \$50[2]. Clearly, both Joe and Bob will choose to enter the bet as their expectation of profit is $0.9 \times (\$100 - \$50) + 0.1 \times (-\$100) = \$35$ after replacement cost for the destroyed pillow. This bet is desirable from either Bob or Joe's perspective, hence the betting is Pareto optimal. However, the bet results nothing rather than a wealth transfer and a destroyed pillow. Of course this bet may be taken just for entertainment purpose, but a more realistic trigger is that each bettor thinks he would win and the other side would lose. If this bet is triggered by difference in beliefs, obviously it is a negative sum game, no matter what is the probability measure of the central planner takes to evaluate the social welfare. The welfare loss, or the externality caused by the bet, is the destroyed pillow.

Speculation may be such a negative sum betting in financial markets which are often associated with high leverage. The recent financial crisis and a long history of financial booms and busts have

[1]The original story is from p. 193 in Kreps (2012).

[2]The replacement cost may be state dependent, e.g. \$50 for a natural down pillow and \$20 for a artificial down pillow.

highlighted the failure to appreciate how soaring leverage levels against the background of robust macroeconomic performance and low interest rates support a massive growth in balance sheets in the financial system (Reinhart and Rogoff (2009)). The damaging real effects associated with the recent financial crisis have generated a broad agreement among academics and policy makers that financial regulation needs to acquire a macroprudential dimension, which ultimately aims to lessen the potentially damaging negative externalities from financial markets to the real sector (Hanson, Kashyap, and Stein (2011), Borio and Shim (2007) and Farhi and Werning (2013)).

Macroprudential policy affects financial markets and aims to provide a built-in stabilizer to internalize the endogenous risks and externalities with concern the wider economy. By developing a dynamic asset pricing model in a production economy with endogenous output, we explore the risk premium channel of macroprudential policy, as the recent crisis highlights the significant spill over effect from financial markets to the real sector. More specifically, we assume that the expected technological growth is latent to households, who have to rely on public signals for its estimation. Two types of households have identical preference but different precision on their priors (different variance of their estimation). The more confident (lower variance of estimation) households speculate on their accuracy against the less confident households through financial markets, resulting in potentially distorted and more volatile individual consumption and leisure, which are bad for individual household and the social welfare. Further more, distorted portfolio and consumption choice may lead to deviated aggregate labor supply and aggregate output from the natural level. Blanchflower and Oswald (2013) document a similar pattern that the housing market, where ordinary households invest most of their wealth, can produce negative 'externalities' upon the labor market[3].

Central banks can influence the speculation by regulating the funding channel, i.e. financial intermediaries. Those intermediaries are facing various regulations or constraints, among which some common market based instruments are

$$\text{Leverage ratio} = \frac{\text{Totol exposure}}{\text{Equity}}, \tag{1}$$

[3] Blanchflower and Oswald (2013) show that rises in home-ownership lead to fewer new businesses.

$$\text{Haircut} = \frac{\text{Assets*Price - Debt}}{\text{Assets*Price}} = \frac{\text{Equity}}{\text{Assets*Price}}, \tag{2}$$

and

$$\text{Loan to Value (LTV)} = \frac{\text{Debt}}{\text{Assets*Price}}. \tag{3}$$

Even though commercial banks, hedge funds, shadow banks, mortgagors have different types of constraints, comparing the different yardsticks for leverage, we find $\frac{1}{\text{Leverage ratio}} = \text{Haircut} = 1\text{- LTV}$ and we observe a wide range of LTV ratios across the financial system from 99 percent for special purpose vehicles to 80 percent for mortgages (Schoenmaker and Wierts (2015)). No matter what the cross-sectional difference in ratios requirement, macroprudential policy tools share the same target to regulate speculation and stabilize the financial market. In this paper, we consider the LTV constraint as the representative constraint of various macroprudential policy tools.

The difference in priors is the only friction in our economy besides the policy and the only heterogeneity between two types of households. It is also the key component to generate the externalities upon speculation. More specifically, the less confident households would adopt more perceived shocks into their Bayesian updating, thus the difference in beliefs between two types of households is persistent and volatile over a long time. Two types of household speculate against each other, and their individual consumption and leisure become more volatile, and the aggregate output would deviate from the natural level[4]. Unlike 'pecuniary externalities' which rely on distorted relative price changes by introducing incompleteness to the Arrow-Debreu construct, the externality in our economy is triggered by trading due to the difference in beliefs among households.

To offset the belief heterogeneity friction, central banks directly set a LTV constraint to influence speculation. Furthermore, even in the case where two types of households initially hold an identical estimation of technological growth, the dynamic effect of macroprudential policy forces the households to take into account of the speculation profit limit when they optimize their consumption and portfolio. We try to explore the effects of macroprudential policy on asset prices, social welfare and economic growth. We compare the outcomes macroprudential policy with different tightness

[4]Analogous to New Keynesian literature, we can term the output level in the case where there is no difference in beliefs as the *natural output*, and the the deviation from *natural output* is the *output gap*.

by solving the model numerically to capture the inherent path dependent effect. The results are following.

First, we solve for a decentralized economy and show that the macroprudential policy causes the stock price to increase and the stock volatility to decrease. The intuition is simple. In presence of heterogeneity of priors and learning, the difference in belief fluctuates in a large extent in sufficient long term. In a complete market where there is no policy and both agents can freely trade with each other, the households are exposed to the sentiment risk (speculative risk) which increases the equity premium and volatility comparing to the case where there is no belief heterogeneity (similar to David (2008)). While by restricting on the trading position, macroprudential policy offsets with the speculation motive and reduces households' exposure to the sentiment risk; thus it reduces the equity premium, stock volatility and boosts up the asset prices in terms of price dividend ratio.

Second, the presence of macroprudential policy is superior to no policy for the social welfare under the belief-neutral welfare criteria by Brunnermeier, Simsek, and Xiong (2014), and tighter policy enhances the social welfare in every possible reasonable probability measure. It is tempting to evaluate macroprudential policy under the true probability measure, which however is often extremely hard to be known even for the central planner. Given heterogeneous priors, the probability measures of different types of households are different. A policy is likely to be good for the social welfare under the probability measure of one type of households but bad under another. Therefore instead of taking stand on which (whose) belief is correct, a belief-neutral welfare criterion asserts that an allocation is belief-neutral efficient (inefficient) if it is efficient (inefficient) under any every possible reasonable probability measure. Our result shows that the resource allocation under macroprudential policy which constrains speculation is belief neutral superior to the allocation without policy. In other words, macroprudential policy increases the social welfare in any reasonable probability measure. The reason is that difference in beliefs twists the individual consumption-leisure-portfolio choice and produces 'externalities', e.g. the output gap and greater consumption volatility, which are mitigated by macroprudential policy.

Third, we show that a balanced allocation of consumption goods is optimal for the aggregate labor supply and economy as a result of consumption-leisure optimal choice. The aggregate labor supply is a function of consumption sharing. Under identical individual preferences assumption, we find that the aggregate labor supply is at maximum when consumption allocation is well-balanced. Moreover, our result shows that with the presence of difference in beliefs the consumption allocation deviates from balance. Analogously to the New Keynesian literature, we can refer to the *output gap* as the difference between this deviated output and the output (*natural output*) in the economy where there is no belief heterogeneity friction. Given exogenous technological growth, the output growth deviates from the natural level with the presence of the difference in beliefs. In other words, the belief heterogeneity results in an output gap in our economy. Macroprudential policy, in contrast, affects the output gap in the following two ways. First, by restricting on potential extreme portfolio choice, the macroprudential policy overturns individual consumption and leisure choice, making the current consumption allocation close to balance. Second, by setting a portfolio constraint, macroprudential policy maintains the wealth distribution in the future between two types of households by preventing the potential huge wealth transfer due to speculation. It helps to achieve a balanced consumption allocations and thus stabilize the economic growth.

The rest of the paper is organized as follow: Section 2 reviews the literature, Section 3 lays out the baseline model, Section 4 characterizes the equilibrium, Section 5 presents the numerical example of the impact of macroprudential policy on asset prices, social welfare and aggregate economy. Various cases are discussed in Section 6 and we conclude in Section 7.

2 Related Literature

This paper is closest to the literature that studies risk sharing, speculation and savings decisions of investors in the presence of heterogeneous beliefs. Blume et al. (2014) study an exchange economy and show that, the welfare might be higher in an incomplete market than a complete market if measured under a set of objective beliefs instead of the investor's beliefs. Heyerdahl-Larsen and Walden (2014) study hedging and speculation in a static model and show that the competitive equilibrium is always inefficient under their own efficiency criterion. Buss et al. (2016) also study the AK production economy in discrete time and show similar results. Our paper contributes to this literature by endogenizing the aggregate labor supply and studying the real effects of the speculation due to belief heterogeneity and the macroprudential policy.

'Pecuniary externality' is one of the main existing theoretical justification for macroprudential policy. In the pecuniary externality literature, it simply rises due to the market incompleteness introduced to the Arrow-Debreu construct. The relative price change which is induced by the redistribution of asset holding given incomplete asset market affects the spanning properties of the limited existing set of assets. This pecuniary externality is not internalized by competitive agents, resulting an constrained inefficient equilibrium. The planner can improve the equilibrium outcome by intervening in the financial markets (see e.g. Stiglitz (1982), Geanakoplos and Polemarchakis (1985) Geanakoplos et al. (1990)).

Farhi and Werning (2013) consider the nominal rigidities friction and zero lower bound of monetary policy. Instead of pecuniary externalities, they emphasize aggregate demand externalities. Our paper considers the behavioural heterogeneity and stresses on suboptimal individual choice and aggregate supply externalities. Elenev, Landvoigt, and Van Nieuwerburgh (2016) consider the leverage constraints of financial intermediary as the main friction, and evaluate the asset prices and output with the implicit support from governments.

This paper is also related to the financial sector modeling of the macro economy. Earlier works before financial crisis emphasis 'financial accelerator' and the balance sheet channel of monetary policy. Those works include Bernanke and Gertler (1989), Bernanke and Gertler (1995), Gertler and Karadi (2011), Bemanke, Gertler, and Gilchrist (1996), Bernanke, Gertler, and Gilchrist (1999), and many others. More recent research allows a richer setting of financial sector. He and Krishnamurthy (2013) study the asset pricing effect when the marginal investor is a financial intermediary who is facing an equity capital constraint, which is similar to us. Brunnermeier and Sannikov (2014) study the persistent endogenous risk driven by asset liquidity and its non-linear implication effects and show that the economy is prone to instability and occasionally enters volatile crisis episodes. He and Krishnamurthy (2014) consider systemic risk arises when shocks lead to states where a disruption in financial intermediation adversely affects the economy and feeds back into further disrupting financial intermediation. Drechsler, Savov, and Schnabl (2014) consider a dynamic heterogeneous agents asset pricing model in which monetary policy affects the risk premium component of the cost of capital. The effect of policy shocks are exemplified via bank balance sheet effect.

On the methodological side, our paper draws from the heterogeneous agent asset pricing literature. We model an economy where two types of agents with different priors, which gives rise to a credit market as in Scheinkman and Xiong (2003), Dumas, Kurshev, and Uppal (2009) and Xiong and Yan (2009). David (2008) resolves the equity premium puzzle by exploring the speculation on preciser estimation. Our model is also related the literature on collateral or margin constraints and their effect on asset prices (see Gromb and Vayanos (2002), Brunnermeier and Pedersen (2009), Gorton and Ordoñez (2014), Moreira and Savov (2014)). Basak and Cuoco (1998) solve the asset pricing implication with limited stock market participation, and Gallmeyer and Hollifield (2008) also examine asset prices in heterogeneous-agent model with a short constraint.

3 Model

We model an finite-horizon continuous time production economy in which a competitive final goods producer aggregates a continuum of intermediate inputs

$$Y_t = \left(\int_0^1 Y_{j,t}^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \tag{4}$$

where the intermediate goods $Y_{j,t}$ output by a continuum of monopolistically competitive intermediate goods producers. Firm j employs $n_{j,t}$ units of labor and technology A_t to produce

$$Y_{j,t} = A_t n_{j,t}. \tag{5}$$

The technology level is exogenous, and its dynamics follow

$$\frac{dA_t}{A_t} = \mu_A dt + \sigma_A dZ_{A,t}. \tag{6}$$

where μ_A, σ_A are constant and $Z_{A,t}$ is a standard Brownian motion.

3.1 Learning

This production economy is populated by a continuum of measure one of households who don't observe the expected technological growth rate μ_A and therefore have to use public signals to estimate its true value. We assume that there are two types of households (two agents) whose Gaussian priors are different, i.e. for agent i, $\mu^i_{A,t} \sim Normal\left(\hat{\mu}^i_{A,t}, v^i_t\right)$. Their forecast $\hat{\mu}^i_{A,t} = E^i\left[\mu^i_A|\mathcal{F}^A_t\right]$ follows

$$d\hat{\mu}^i_{A,t} = \frac{\bar{v}_i}{\sigma_A} dZ^i_{A,t} \tag{7}$$

where

$$dZ^i_{A,t} = \frac{1}{\sigma_A}\left(\frac{dA_t}{A_t} - \hat{\mu}^i_{A,t} i dt\right) \tag{8}$$

is the shocks agent i perceives and $\bar{v}_{i,t}$ is the variance of the estimation :

$$\bar{v}^i_t = \frac{\bar{v}^i_0 \sigma^2_A}{\bar{v}^i_0 t + \sigma^2_A}. \tag{9}$$

Without lost of generality, we assume $v^1_0 \leq v^2_0$, therefore the agent 2 reacts more to perceived shocks, and the difference in beliefs is stochastic.

Though the agents' estimation of the expected technological growth is time varying, we can still define their subjective probability measure $\mathbb{P}^i$ against the objective probability measure $\mathbb{P}$ by the following exponential martingale

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = \xi_{t,i}, \tag{10}$$

where

$$\xi_{t,i} = \xi_{t,0} \exp\left[\int_0^t -\sigma^2_{\xi,i} ds + \int_0^t \sigma_{\xi,i} dZ_{A,t}\right] \tag{11}$$

$$\sigma_{\xi,i} = \frac{\hat{\mu}^i_{A,t} - \mu_{A,t}}{\sigma_A}. \tag{12}$$

We quantify the level of disagreement by the process ξ_t, which is defined as:

$$\xi_t = \frac{\xi_{t,2}}{\xi_{t,1}}, \tag{13}$$

and hence

$$\frac{d\xi_t}{\xi_t} = \mu_\xi dt + \sigma_\xi dZ_{A,t}, \tag{14}$$

where

$$\mu_\xi = -\sigma_{\xi,1}\left(\sigma_{\xi,2} - \sigma_{\xi,1}\right) \tag{15}$$

$$\sigma_\xi = \sigma_{\xi,2} - \sigma_{\xi,1}. \tag{16}$$

As the difference in beliefs is persistent and stochastic in this economy, the level of disagreement and its volatility are stochastic, constituting the source of the sentiment risk. This feature of the model allow us to meet the twin challenges: (1) heterogeneity in beliefs and persistent disagreement between investors and (ii) the financial risk resides internally in the financial system rather than externally in the production system.

3.2 Preferences and Optimization

Instantaneous utility of type i representative agent is given by the following function that depends on the consumption $C_{i,u}$ and leisure $L_{i,t}$:

$$U(C_{i,t}, L_{i,t}) \equiv \log C_{i,t} + \frac{L_{i,t}^{1-\eta}}{1-\eta} \tag{17}$$

where η catches the smoothing motive for leisure component and takes value of 1 in our baseline model to match the standard log preference in the real business cycle literature (see Ljungqvist and Sargent (2004)). The leisure is defined as the total hours minus the working hours: $L_{i,t} \equiv \bar{N}_i - N_{i,t}$, and the aggregate total hours is $\bar{N} = \bar{N}_1 + \bar{N}_2$, in which we assume $\bar{N}_i = \beta_i \bar{N}$ and $\beta_1 = \beta_2 = \frac{1}{2}$.

In this economy, there are no frictions other than beliefs heterogeneity and policy intervention, and we assume that two types of agents shall invest their financial wealth into two securities: a real bond and a claim to dividend flow (equity). Both agents optimize the consumption, leisure and portfolio to maximize their expected utility, which is given by

$$V_{i,t} = \max_{C_{i,u}, N_{i,u}, \theta_{i,u} \in \Theta} E_t^i \left[\int_t^\infty e^{-\rho(u-t)} U_i(C_{i,u}, L_{i,u}) du \right] \tag{18}$$

subject to a dynamic budge constraint

$$dG_{i,t} = r_t G_{i,t} dt + \theta_{i,t} \left(dJ_t - r_t J_t + D_t dt \right) + \frac{W_t}{P_t} \left(\bar{N}_i - L_{i,t} \right) dt - C_{i,t} dt, \tag{19}$$

and the dynamics of consumption share

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_1} dt + \sigma_{\nu_1} dZ^i_{A,t}, \tag{20}$$

where $\nu_{1,t} = \frac{C_{1,t}}{Y_t}$, ρ is the constant subjective discount rate, P_t is the price level, $D_t = Y_t - \frac{W_t}{P_t} N_t$ is the dividend and $N_t = N_{1,t} + N_{2,t}$ is the aggregate labor supply, J_t is the price of the dividend claim, and $G_{i,t}$ represents their total wealth includes the financial assets and the human capital. $\theta_{i,u}$ is the portfolio weight of stock, and Θ represents the set of admissible portfolio strategies.

We assume both types of agents are regulated by the LTV constraint. But for simplicity we assume the more confident agent is not subject to LTV constraint as she reacts less to the signal. This allows us to apply the standard martingale approach to characterize the optimal consumption and portfolio for general price systems as in Cox and Huang (1989). The SDF of the more confident agent, summarizing the stock and bond price system, is given by

$$\frac{d\Lambda_{1,t}}{\Lambda_{1,t}} = -r_t dt - \kappa_{1,t} dZ^1_{A,t}, \tag{21}$$

$$\Lambda_{1,0} = 1. \tag{22}$$

Her consumption portfolio problem is

$$\max_{C_{1,t}, L_{1,t}} E^1_0 \left[\int_0^T e^{-\rho t} \left(\log C_{1,t} + \frac{L^{1-\eta}_{1,t}}{1-\eta} \right) dt \right] \tag{23}$$

$$\text{subject to } E^1 \left[\int_0^T \Lambda_{1,t} \left(C_{1,t} - \frac{W_t}{P_t} N_{1,t} \right) dt \right] \le x_1 J_0. \tag{24}$$

The FOC gives the optimal consumption and leisure choice:

$$\hat{C}_{1,t} = \frac{e^{-\rho t}}{y_1 \Lambda_{1,t}}, \tag{25}$$

$$\hat{L}_{1,t} = \left(\frac{e^{-\rho t}}{y_1 \Lambda_{1,t} \frac{W_t}{P_t}} \right)^{\frac{1}{\eta}}, \tag{26}$$

where $\kappa_{1,t}$ is the instantaneous risk price and y_1 is the Lagrange multiplier from the static budget constraint and solves following equation

$$E_0^1\left[\int_0^T e^{-\frac{\rho}{\eta}t}\Lambda_{1,t}^{\frac{\eta-1}{\eta}}\left(\frac{W_t}{P_t}\right)^{\frac{\eta-1}{\eta}}dt\right]y_1^{-\frac{1}{\eta}}+\frac{1}{\rho}\left(1-e^{-\rho T}\right)y_1^{-1}=E_0^1\left[\int_0^T\Lambda_{1,t}\frac{W_t}{P_t}\bar{N}_1dt\right]+x_1J_0. \quad (27)$$

The less confident agent faces an incomplete financial market, as her portfolio construction is complicated by the LTV constraint. Similar to Cvitanic and Karatzas (1992) and Chabakauri (2015), we solve the constrained agent's optimization problem in a fictitious unconstrained economy, in which the the price of the instantaneous bond and equity follow

$$\frac{dB_t}{B_t}=\left(r_t+\delta\left(v_t^*\right)\right)dt \quad (28)$$

$$\frac{dJ_t}{J_t}=\left(\mu_{J,t}^2+v_t^*+\delta\left(v_t^*\right)\right)dt+\sigma_{J,t}dZ_{A,t}^2 \quad (29)$$

where $\delta(\cdot)$ is the *support function* for the set of admissible portfolio weights Θ, defined as

$$\delta(v)=\sup_{\theta\in\Theta}\left(-v\theta\right) \quad (30)$$

which is obtained from complementary slackness condition $v_t^*\theta_t^*+\delta(v^*)=0$ (see Cvitanic and Karatzas (1992) and Karatzas and Shreve (1998)). The *adjustment* v_t^* can be interpreted as the shadow cost of the constraint. Therefore, the SDF of the Type 2 agent in the fictitious economy is:

$$\frac{d\Lambda_{2,t}}{\Lambda_{2,t}}=-\left(r_t+\delta\left(v_t^*\right)\right)dt-\left(\kappa_{2,t}+\frac{v_t^*}{\sigma_{J,t}}\right)dZ_{A,t}^2, \quad (31)$$

$$\Lambda_{2,0}=1, \quad (32)$$

and the less confident agent's consumption portfolio problem is

$$\max_{C_{2,t},L2,t}E^2\left[\int_0^T e^{-\rho t}\left(\log C_{2,t}+\frac{L_{2,t}^{1-\eta}}{1-\eta}\right)dt\right] \quad (33)$$

subject to the static budget constraint

$$E^2\left[\int_0^T\Lambda_{2,t}\left(C_{2,t}-\frac{W_t}{P_t}N_{2,t}\right)dt\right]\leq x_2J_0. \quad (34)$$

The FOC gives the optimal consumption and leisure choice:

$$\hat{C}_{2,t} = \frac{e^{-\rho t}}{y_2 \Lambda_{2,t}}, \tag{35}$$

$$\hat{L}_{2,t} = \left(\frac{e^{-\rho t}}{y_2 \Lambda_{2,t} \frac{W_t}{P_t}} \right)^{1/\eta}, \tag{36}$$

where $\kappa_{2,t}$ is the instantaneous risk price and y_2 is the Lagrange multiplier from the static budget constraint and solves following equation

$$E_0^2 \left[\int_0^T e^{-\frac{\rho}{\eta} t} \Lambda_{2,t}^{\frac{\eta-1}{\eta}} \left(\frac{W_t}{P_t} \right)^{\frac{\eta-1}{\eta}} dt \right] y_2^{-\frac{1}{\eta}} + \frac{1}{\rho} \left(1 - e^{-\rho T}\right) y_2^{-1} = E_0^2 \left[\int_0^T \Lambda_{2,t} \frac{W_t}{P_t} \bar{N}_2 dt \right] + x_2 J_0. \tag{37}$$

4 The Equilibrium

To determine the equilibrium price system, it suffices to find the two SDF processes that clear the consumption market. Define

$$\lambda_t = \frac{y_1 \Lambda_{1,t}}{y_2 \Lambda_{2,t}} \tag{38}$$

where $\lambda_0 = \frac{y_1}{y_2}$. Its dynamics follow

$$\frac{d\lambda_t}{\lambda_t} = -\mu_\lambda dt - \sigma_\lambda dZ_{D,t}^1, \tag{39}$$

where

$$\mu_{\lambda,t} = -\kappa_{2,t} \frac{v_t^*}{\sigma_{J,t}} - \left(\frac{v_t^*}{\sigma_{J,t}} \right)^2 - \delta\left(v_t^*\right), \tag{40}$$

$$\sigma_{\lambda,t} = \kappa_{1,t} - \kappa_{2,t} - \frac{v_t^*}{\sigma_{J,t}} \tag{41}$$

λ_t is the only state variable in the economy and essentially equivalent to the consumption allocation as the individual SDF is just the marginal utility with respect to consumption. It summaries the impact of difference in beliefs and the policy. In absence of policy, the shadow price of constraint is 0, and the dynamics of λ_t depend only on the difference in beliefs $\kappa_{1,t} - \kappa_{2,t} = \frac{\hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2}{\sigma_A}$. While the policy affects the drift and the volatility of λ_t and thus impacts optimal choices, asset prices and social welfare. We characterize the equilibrium by following proposition.

Proposition 1 *Assume that an equilibrium exists. The SDFs in the fictitious for two types of households are*

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c\left(Y_t, \lambda_t\right)}{U_c\left(Y_0, \lambda_0\right)}, \qquad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \tag{42}$$

where $U_c\left(Y_t, \lambda_t\right)$ is the marginal utility of a social planner which is defined as

$$U\left(Y_t, \lambda_t\right) = \max_{C_{1,t}+C_{2,t} \leq A_t\left(\bar{N}-L_{1,t}-L_{2,t}\right)=Y_t} U(C_{1,t}, L_{1,t}) + \lambda_t U(C_{2,t}, L_{2,t}). \tag{43}$$

The stock price J_0 is computed from the more confident agent's probability measure

$$J_0 = \frac{1}{\epsilon} E_0^1 \left[\int_0^T e^{-\rho s} \frac{U_c\left(Y_s, \lambda_s\right)}{U_c\left(Y_0, \lambda_0\right)} Y_s ds \right]. \tag{44}$$

The equilibrium consumption allocations are

$$C_{1,t} = \frac{1}{\epsilon U_c\left(Y_t, \lambda_t\right)}, \qquad C_{2,t} = \frac{\lambda_t}{\epsilon U_c\left(Y_t, \lambda_t\right)}. \tag{45}$$

The stochastic weighting process λ_t has the dynamics given by (39) where λ_0 clears the markets at time $t=0$, namely

$$C_1\left(\lambda_0, A_0\right) + C_2\left(\lambda_0, A_0\right) = A_0\left(\bar{N} - L_1\left(\lambda_0, A_0\right) - L_2\left(\lambda_0, A_0\right)\right) \tag{46}$$

Conversely, if there exists processes $\Lambda_{1,t}$, $\Lambda_{2,t}$, λ_t and J_t satisfying equations (42) - (46), with $\sigma_Y > 0$, the associated optimal consumption portfolio policies clear all markets.

Proof: see Appendix B.1.

The equilibrium consists of two regions which depend on whether the difference in beliefs is large enough to trigger the LTV constraint. In the first region, the difference in beliefs is small enough so that agent 2 is only willing to hold a small portion of stock without any leverage. Here the LTV constraint does not bind. The dynamics of the weighting process depends on the difference in beliefs only. In the second region, the difference in beliefs is large enough so that agent 2 is eager to lever up to hold more stock. Here the LTV constraint binds; the dynamics of the weighting process depends on the tightness of the LTV constraint as well as the difference in beliefs. The next proposition reports the shadow price of the constraint in these two regions.

Proposition 2 *In the above equilibrium, for LTV constraint $\theta \leq \tilde{\theta}$ with $\tilde{\theta} > 1$, adjustment v_t^* satisfies Kuhn-Tucker optimality conditions*

$$v_t^* \left(\frac{\theta^*}{\tilde{\theta}} - 1 \right) = 0, \tag{47}$$

$$v_t^* \tilde{\theta} \leq 0, \tag{48}$$

$$\frac{\theta^*}{\tilde{\theta}} - 1 \leq 0 \tag{49}$$

and is given by the following expression

$$v_t^* = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \tilde{\theta}\sigma_{J,t}^2 - \kappa_{2,t}\sigma_{J,t} & \text{if } \theta^* = \tilde{\theta}, \end{cases} \tag{50}$$

and

$$\delta(v_t^*) = -\tilde{\theta} v_t^*. \tag{51}$$

Proposition 2 shows that the shadow price of the policy constraint is determined by the tightness of policy and the endogenous stock price volatility. For simplicity we further assume that the constraint is inverse of the volatility to disentangle the shadow price and stock price volatility, and the result is given by the following corollary.

Corollary 1 *If the LTV constraint takes the form of inverse of the volatility $\tilde{\theta} = \frac{\bar{\kappa}}{\sigma_{J,t}}$. The adjustment*

$$v_t^* = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \bar{\kappa}\sigma_{J,t} - \kappa_{2,t}\sigma_{J,t} & \text{if } \theta^* = \tilde{\theta}, \end{cases} \tag{52}$$

and its support function is

$$\delta(v_t^*) = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \kappa_{2,t}\bar{\kappa} - \bar{\kappa}^2 & \text{if } \theta^* = \tilde{\theta}. \end{cases} \tag{53}$$

Proof: see Appendix B.2.

The closed form solutions for the support function relies on the assumption in Corollary 1 , which we also keep in the following sections.

4.1 Asset Prices

In this part, we discuss the asset pricing implications of macroprudential policy. First we compare the stock price with and without the sentiment risk (speculative risk).

Proposition 3 *Suppose $\bar{J}_0$ is the representative agent's marginal valuation of the stock without different in beliefs, then*

$$J_0 < \bar{J}_0. \tag{54}$$

Proof: see Appendix B.3.

Proposition 3 shows that the stock price without the sentiment risk is higher. The intuition is that the speculation due to the difference in beliefs increases the variation of the aggregate SDF and thus induce the sentiment risk whose existence increases the equity premium and lowers the stock price.[5] The macroprudential policy offsets with the sentiment risk, and thus raises the stock price.

Proposition 4 *Suppose $\tilde{J}_0$ is the constrained agent's marginal valuation of the stock defined as*

$$\tilde{J}_0 = E_0^2 \left[\int_0^T \frac{\Lambda_{2,s}}{\Lambda_{2,0}} D_s ds \right]. \tag{55}$$

Then

$$\begin{cases} J_0 > \tilde{J}_0, & \textit{if the constrait binds with positive probability for some } t' \geq t, \\ J_0 = \tilde{J}_0, & \textit{else.} \end{cases} \tag{56}$$

Proof: see Appendix B.4.

If the constraint does not bind the less confident agent, her stock holding variates accordingly with the speculation motives or the difference in beliefs fluctuates. If there is positive probability that the LTV constraint will bind at some time in the future, the stock price is higher than the

[5] Like Cochrane (2016) points out, the existence of the sentiment risk induces the time varying risk bearing capacity and thus increases the variation of the aggregate SDF.

less confident agent's valuation for the stock. The difference between the stock prices and the less confident agent's valuation can be interpreted as the speculative premium from the less confident agent[6]. The speculative premium exists because the constrained agent expects to be able to sell the stock to the unconstrained agent at a better price in the future.

The proposition above characterizes the consumption allocations and the stock price in equilibrium, then the following proposition shows the risk prices and risk-free rate.

Proposition 5 *The market prices of risk for two agents are*

$$\kappa_{1,t} = \begin{cases} \frac{1}{\nu_{1,t}}\sigma_{Y,t} - \frac{\nu_{2,t}}{\nu_{1,t}}\bar{\kappa} & \textit{if the constraint binds,} \\ \sigma_{Y,t} + \nu_{2,t}\bar{\mu}, & \textit{otherwise,} \end{cases} \tag{57}$$

$$\kappa_{2,t} = \begin{cases} \frac{1}{\nu_{1,t}}\sigma_{Y,t} - \frac{\nu_{2,t}}{\nu_{1,t}}\bar{\kappa} - \bar{\mu}_t & \textit{if the constraint binds,} \\ \sigma_{Y,t} - \nu_{1,t}\bar{\mu}, & \textit{otherwise.} \end{cases} \tag{58}$$

The instantaneous risk-free rate is

$$r_t = \begin{cases} \rho + \mu^1_{Y,t} - \left(\nu_{1,t}\kappa^2_{1,t} + \nu_{2,t}\bar{\kappa}^2\right) - \nu_{1,t}\delta\left(v^*_t\right), & \textit{if the constraint binds,} \\ \rho + \boldsymbol{\mu}_{Y,t} - \sigma^2_{Y,t}, & \textit{otherwise,} \end{cases} \tag{59}$$

where $\boldsymbol{\mu}_{Y,t}$ is the aggregate belief defined by

$$\boldsymbol{\mu}_{Y,t} = \nu_{1,t}\mu^1_{Y,t} + \nu_{2,t}\mu^2_{Y,t}. \tag{60}$$

Proof: see Appendix B.5.

Proposition 5 shows that the risk prices and the risk free rate in the unconstrained region depend on the difference in beliefs and the consumption allocation $\nu_{i,t}$, consistent with Basak (2000) and Bhamra and Uppal (2014). When the constraint is binding, the subjective risk price of the unconstrained agent depends on her own consumption share and the tightness of policy. Conditional on the consumption allocation, the subjective risk price of the unconstrained agent is higher in the constrained region, because the policy forces the unconstrained agent to bear additional risk despite her belief. As the output is endogenous, the macroprudential policy impacts the aggregate output volatility and the prices of risk as a consequence of trading constraint. In the appendix, we show that

[6] Similar to Morris (1996),Scheinkman and Xiong (2003) and Gallmeyer and Hollifield (2008).

the output volatility is determined by the difference in beliefs as well as the consumption allocation. The portfolio of less confident agent has a highly volatile exposure to the fundamental risk, as she put more weight on the shocks when she updates her estimation. While if she is constrained by the LTV cap, her risk exposure is also capped, making consumption allocation and labor supply less volatile and achieving stability in economic growth.

The equilibrium risk-free rate in our economy is endogenous. The rate in the unconstrained region depends on the difference in beliefs as well as the consumption allocation, consistent with Basak (2000) in terms of expression. The risk-free rate is lower when the LTV constraint caps the leverage of the less confident but more optimistic agent. Unlike the short sale constraint case in Gallmeyer and Hollifield (2008) in which the risk-free rate depends solely on the beliefs of one agent, it still depends on the difference in beliefs besides the consumption allocation. The reason is that the pessimist is completely banned from the market if she is short sale constrained, however when facing a macroprudential constraint, she can still partially share the risk.

With the price of risk, we can obtain the equity premium by compute the equity volatility which is reported in the following proposition.

Proposition 6 *The volatility of the equity is given by*

$$\sigma_{J,t} = \sigma_Y + \frac{\lambda_t}{1+\lambda_t}\sigma_\lambda + \frac{E_t^1\left[\int_t^T e^{-\rho s}\mathcal{D}_t^1\lambda_s ds\right]}{E_t^1\left[\int_t^T e^{-\rho s}\left(1+\lambda_s\right)ds\right]} \tag{61}$$

where $\mathcal{D}_t\lambda_s$ is the Malliavin derivative for λ_s and defined in the proof of this proposition.

Proposition 6 shows how the stock price reacts to technological shocks through changes in λ_t under macroprudential policy. The first term of (61) measures the effect of a technological shock on the stock price through Y_t output. The second term shows the instantaneous impact of a technological shock on the SDF through λ_t, and illustrates the effect of macroprudential policy on the volatility. The third term measures a path dependent the long term effect of the technological shocks on the SDF in the future. The derivative $\mathcal{D}_t\lambda_s$ measures how perturbing $Z_{A,t}^1$ affects instantaneous change of λ_s for $s \geq t$. The second and third terms shows how difference in beliefs and macroprudential policy impacts the volatility. When beliefs are identical and the constraint does not bind, the λ_t is constant, the second and the third term vanish, and the stock price volatility reduces to the output volatility. In presence of modest heterogeneity in beliefs, the equity volatility depends on the difference in beliefs only. While the equity volatility is affected by macroprudential constraint if the difference in beliefs is large enough.

Given that the expectation structure in (61), the effect of macroprudential policy on the asset prices cannot be characterized in closed form. However intuitively, the macroprudential policy prevents the speculation and reduces the volatility of λ_t (and the volatility of consumption allocation and wealth distribution) and therefore the equity and output volatility, achieving a more resilient financial market and economy.

4.2 Welfare Analysis

We have so far analyzed the asset pricing implications of macroprudential policy. In this part we evaluate its impact on the social welfare under the belief neutral welfare criterion.

In our economy, both types of agents benefit from the consumption as well as the leisure time, which is affected by the macroprudential policy as a result of offsetting with the sentiment risk Whether the policy is beneficial to the social welfare becomes an important question. While it is attempting to evaluate the social welfare under the true probability measure, which however is unlikely to be known even for the social planner. Therefore we evaluate the policy impact under a belief neutral criterion.

Brunnermeier, Simsek, and Xiong (2014) propose a welfare criterion for economies in which agent have heterogeneously distorted beliefs. In fact, in our model, the conflicting beliefs of the more confident agent and the less confident agent induce a form of externality. First, the difference in priors (or simply confidence level in presence of learning) induces a bias in consumption leisure choice in our model, resulting a deviation in aggregate labor supply and total output. Second, consumption and leisure become more volatile due to heterogeneous risk exposure in their investment.

Hence we consider the following criterion to evaluate effect of macroprudential policy robustly, disregarding whose belief is correct.

Definition 1 *Consider two social allocations, x and y. If the expected social welfare of allocation x dominates that of allocation y for every reasonable probability measure $\mathbb{P}^h$, where $\mathbb{P}^h$ denotes the convex combination of the two agents' beliefs with weight $h > 0$ and $1 - h > 0$,*

$$W\left(E_0^h\left[V_1(x)\right], E_0^h\left[V_2(x)\right]\right) \geq W\left(E_0^h\left[V_1(y)\right], E_0^h\left[V_2(y)\right]\right) \tag{62}$$

with the inequality holding strictly for at least one reasonable measure, then allocation x is belief-neutral superior to allocation y.

As both types of agents are endowed with identical wealth, labor (time) and preference, we construct a simple social welfare function as a utilitarian welfare function, and it is as following:

$$W\left(E_t^h\left[\mathcal{V}_{1,t}\right], E_t^h\left[\mathcal{V}_{2,t}\right]\right) = E_t^h\left[\mathcal{V}_{1,t}\right] + E_t^h\left[\mathcal{V}_{2,t}\right], \tag{63}$$

where

$$\mathcal{V}_{i,t} = \int_t^T e^{-\rho(u-t)} U(C_{i,u}, L_{i,t}) du. \tag{64}$$

We compute the social welfare to show the macroprudential policy is beneficial, and the details are in the next section.

4.3 Macroprudential Policy and Aggregate Economy

To endogenize the economic growth, the next proposition characterizes the linkage between households consumption-leisure-portfolio choice to the aggregate labor supply.

Proposition 7 *In equilibrium, the aggregate labor supply is a function of consumption allocation $\nu_{i,t}$, and it satisfies the following polynomial equation*

$$\left(\nu_{1,t}^{\frac{1}{\eta}} + \nu_{2,t}^{\frac{1}{\eta}}\right) N_t^{\frac{1}{\eta}} + \left(\frac{W_t}{A_t P_t}\right)^{\frac{1}{\eta}} N_t - \left(\frac{W_t}{A_t P_t}\right)^{\frac{1}{\eta}} \bar{N} = 0. \tag{65}$$

Proof: see Appendix B.6

Proposition 7 shows that the aggregate labor supply depends on the consumption share, which is determined by the state variable λ_t. The solution of labor can be solved by the polynomial theory if $\eta \in \left\{1, 2, 3, 4, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$, or solved in closed-from by using the hypergeometric function if $1/\eta \in \mathbb{N}$. The equilibrium labor is concave and symmetric in consumption share when $0 < \eta < 1$, and it is at maximum when $\nu_{1,t} = 0.5$. Hence the equilibrium labor and hence the aggregate economic output are maximized if the consumption goods are equally allocated. We plot the closed form result in Figure 1. Individual agent's labor supply is bounded by the maximum working hours, and it is decreasing in her consumption share. This is intuitive as if she is poor and her consumption level is low, she needs to work harder to earn salary to cover the expense for consumption. While she is rich and her consumption level is high, she can mainly rely on her capital income and enjoy the leisure hours.

More importantly, this proposition reveals the risk premia channel via which macroprudential policy impacts the aggregate labor supply and the real economy. Macroprudential policy constrains the portfolio weight and the effective risk premium for the type 2 agent. As agents are forward looking, the restriction on speculation also corrects the initial consumption-leisure choice. In addition, the long run impact of macroprudential is summarized by the dynamics of λ_t, in which the volatility component and thus the evolution path are affected by the LTV constraint. As λ_t is a positive martingale and is expected to decrease in the probability measure of the more confident agent. By reducing its volatility component, macroprudential policy impacts expected long term trend of λ_t which determines the long run consumption allocation and thus affects the expected long term trend of aggregate labor and output.

5 Numerical Results

In this section we further analyze the effects of macroprudential policy by setting parameter values and solving for the resulting equilibrium. This requires solving optimization of the two types of agents simultaneously. Since these results rely on solving a path dependent problem and do not

Table 1: Choice of Parameter Values and Benchmark Values of the State Variables

Name	Symbol	Value
Volatility of technological development	σ_A	0.02
Time preference of both agents	ρ	0.02
Curvature Parameter	η	1
Agent 1's expected technological growth rate	$\hat{\mu}^1_{A,0}$	0.02
Variance of Agent 1's prior	$\bar{v}_{1,0}$	0.01
Variance of Agent 1's prior	$\bar{v}_{2,0}$	0.0001
Difference of Agent 2's belief and Agent 1's	$g_t = \hat{\mu}^1_{A,0} - \hat{\mu}^2_{A,0}$	0.01

permit closed form solutions, we apply numerical methods, specifically Monte Carlo simulation. Conditional on an initial guess for λ_0, we simulate the weighting process and output process to solve for the stock price through (44). We iterate to find a λ_0 satisfying the equilibrium conditions.

5.1 Baseline Model

We provide numerical examples of the effect of imposing macroprudential policy on asset prices and social welfare. In baseline model we set the curvature parameter for the leisure component to be 1 so that agents have logarithmic utility flow over both consumption and leisure. This preference is also consistent with the textbook setting in neo-classcial growth literature. The reason to do so is to simply focus on the difference in beliefs and the policy implication in discount rate channel by shutting down the impact on the dividend flow and aggregate output. We will also show the main result is robust to other parametrization. Table 1 displays benchmark parameters. We set volatility of the technological development to 2% to match the volatility of aggregate GDP growth. We can set the $g_t = \hat{\mu}^1_{A,0} - \hat{\mu}^2_{A,0}$ to be a slightly positive number, i.e. 0.01 to match the equity volatility and risk free rate. The disagreement process g_t is time varying due to learning under different confidence level, which both will converge to zero given sufficient long learning time.

Figure 2 shows that the macroprudential policy increases the asset prices and the social welfare in baseline parameterization. The upper two panels illustrate the impact over asset prices, specifically the PD ratio and the equity volatility. Think of the policy-free economy as the benchmark case in which the policy tightness parameter $\bar{\kappa}$ is infinity, the upper left panel shows that the tighter policy increases the asset prices in terms of PD ratio. The intuition behind the increase in asset prices is quite simple. In presence of difference in confidence level, the difference in beliefs and sentiment risk are persistent. Speculation on beliefs is also persistent over long time until both agents reach a consensus on their forecast. It's well known that the sentiment risk increases the risk premium (e.g. Dumas, Kurshev, and Uppal (2009), Buraschi and Jiltsov (2006)), as the trading triggered by the difference in beliefs increases the variation in aggregate SDF; while by overturning the portfolio, the macroprudential policy prevents large wealth gains and losses due to speculation and thus reduces large variation of the consumption allocation and the aggregate discount factor. In other words, macroprudential policy mitigates the sentiment risk and reduces the equity premium, and thus increases the asset prices. The right upper panel shows a very similar pattern to the equity volatility. With tighter macroprudential policy to offset against the speculation motive, the equity volatility decreases.

The bottom left panel shows how macroprudential policy impacts the social welfare by plotting the two extreme boundaries of social welfare at all possible reasonable probability measures against macroprudential policy with different tightness level. The blue line represents the social welfare evaluated under the agent 1's probability measure, while the red line is the social welfare evaluated under the agent 2's probabilty measure. As the social welfare under any possible reasonable probability measure should be a convex combination of probability measures of two agents, these two lines represent the two extreme values of the social welfare against different tightness of macroprudential policy. The figure shows that the boundaries increase in policy tightness. In the baseline model, a decrease in the consumption and leisure volatilities is the contributing factor to welfare improve, due to the aggregate labor supply is fixed given log preference over leisure, as shown in the right bottom panel in Figure (2). We shall show the result of other parameterizations in the following sections, in which case the the policy impacts the aggregate output.

5.2 Macroprudential Policy and Aggregate Output

In the baseline model we have shown the case $\eta = 1$ to focus on the discount rate channel. While $\eta \neq 1$, Proposition 7 shows the aggregate labor supply as the function of consumption allocation.

We set $\eta = \frac{1}{2}$ and Figure 1 plots the analytical solution where $\eta = \frac{1}{2}$. From this figure, we can see the aggregate labor supply is at maximum when the consumption allocations for two agents are equal .

In comparison to the previous case, the macroprudential policy has impact over the aggregate output when the aggregate labor supply is a function of consumption allocation. Given the identical individual preference setting, if there is no difference in beliefs through the path, the endowment (wealth) allocation and the consumption allocation should be equal, and the aggregate labor should be at maximum. The heterogeneity of beliefs, however, affects the consumption-leisure-portfolio choice over time and makes the consumption allocation deviated from equal and the aggregate labor supply and output away from the maximum. By twisting the portfolio choice of the less confident agent who has stronger reaction to the signal, the macroprudential policy overturns the consumption allocation and impacts the aggregate economy. The result is shown in the right bottom panel in Figure 3. As the policy becomes tighter, the aggregate labor supply and aggregate output increases then decreases. The intuition is following: moderate policy helps to reduce the distortion in portfolio selection thus the impact of the heterogeneity of beliefs. However, the over-tight policy over-correct the portfolio and may set too much restriction over the less confident agent who would have to reduce his consumption. Over tight policy creates an imbalanced consumption allocation and is bad for output maximization.

Figure 3 also shows the impact of macroprudential policy over the asset prices and the social welfare with this alternative parametrization. The tighter policy increases the PD ratio and the social welfare in any possible reasonable probability measure, reduces the equity volatility, same as the baseline case. Hence, the results show a trade off of policy conduction. Tighter policy enhances the social welfare, while does not always increase the aggregate output and thus the economic growth.

6 Discussion

We are now discuss how our conclusions are affected by changes in various model setting and parameter choices.

6.1 Speculation and Risk Sharing

With the identical preference setting all the trading between two types of households are due to heterogeneity in their prospects. The difference in beliefs indeed contributes a significant portion of daily trades, meanwhile the risk sharing should also be an important motive for trading. For this reason, we extend the baseline model by assuming two agents also differ in their risk aversion besides the priors (see Appendix C). The results are shown in Figure 4, and the main implications still hold: the macroprudential policy improves the social welfare unambiguously. The intuition behind this result is as follows. The macroprudential policy has only a small effect on the risk sharing in absence of the sentiment risk and only induces a small welfare loss. In contrast, in the economy with differences in beliefs, the macroprudential policy substantially prevents the speculation motives. So the welfare gains from consumption and leisure smoothing offsets the negative welfare loss from impairing the risk sharing. Even though with extremely tight policy the welfare gains do not offset the welfare loss, the main result does still hold. Similar intuition explains that the PD ratio still increases in the tightness of policy as in the left upper panel in Figure 4.

6.2 Preference Assumption, Policy and Output Prospects

The previous results have shown the impact of macroprudential policy on aggregate output depends on the agents' preference. Figure (1) shows the aggregate labor as the function of the consumption allocation between two types of households if we assume $0 < \eta < 1$. This assumption is reasonable in that η determines the desire for leisure smoothing, which is reasonably less important than consumption smoothing for households. While if $\eta > 1$, the aggregate labor and thus the aggregate output are concave functions of the consumption allocation, implying the income and wealth inequality stimulates aggregate labor supply and output growth. This implication is still controversial and beyond this research. But even though we allow $\eta > 1$, the consumption allocation distortion caused by the sentiment risk and speculation induces overheat in the economy, while macroprudential policy offsets with the sentiment risk and speculation motive, achieving balanced economic growth.

7 Conclusion

We propose a model of macroprudential interventions in financial markets. We solve a heterogeneous-agent asset pricing model and show that trading on the difference in beliefs causes distorted and more volatile individual consumption and leisure as well as deviated output. Ex ante however, these effects are not internalized in individual consumption-leisure-portfolio decisions. While macroprudential policy helps to offset those distortions. Our results show that macroprudential policy reduces equity premium and volatility, and enhances the social welfare under every possible reasonable probability measure.

References

Basak, S. 2000. A model of dynamic equilibrium asset pricing with heterogeneous beliefs and extraneous risk. *Journal of Economic Dynamics and Control* 24:63–95.

Basak, S., and D. Cuoco. 1998. An equilibrium model with restricted stock market participation. *Review of Financial Studies* 11:309–41.

Bemanke, B., M. Gertler, and S. Gilchrist. 1996. The financial accelerator and the flight to quality. *The Review of Economics and Statistics* 78:1–15.

Bernanke, B., and M. Gertler. 1989. Agency costs, net worth, and business fluctuations. *American Economic Review* 79:14–31.

Bernanke, B. S., M. Gertler, and S. Gilchrist. 1999. The financial accelerator in a quantitative business cycle framework. *Handbook of macroeconomics* 1:1341–93.

Bernanke, B. S., and M. L. Gertler. 1995. Inside the black box: The credit channel of monetary policy transmission. *Journal of Economic Perspectives* 9:27–48.

Bhamra, H. S., and R. Uppal. 2014. Asset prices with heterogeneity in preferences and beliefs. *Review of Financial Studies* 27:519–80.

Blanchflower, D. G., and A. J. Oswald. 2013. Does high home-ownership impair the labor market? Working Paper, National Bureau of Economic Research.

Blume, L. E., T. Cogley, D. A. Easley, T. J. Sargent, V. Tsyrennikov, et al. 2014. The case for incomplete markets .

Borio, C. E., and I. Shim. 2007. What can (macro-) prudential policy do to support monetary policy? .

Brunnermeier, M. K., and L. H. Pedersen. 2009. Market liquidity and funding liquidity. *Review of Financial studies* 22:2201–38.

Brunnermeier, M. K., and Y. Sannikov. 2014. A macroeconomic model with a financial sector. *The American Economic Review* 104:379–421.

Brunnermeier, M. K., A. Simsek, and W. Xiong. 2014. A welfare criterion for models with biased beliefs. *Quarterly Journal of Economics* .

Buraschi, A., and A. Jiltsov. 2006. Model uncertainty and option markets with heterogeneous beliefs. *The Journal of Finance* 61:2841–97.

Buss, A., B. Dumas, R. Uppal, and G. Vilkov. 2016. The intended and unintended consequences of financial-market regulations: A general-equilibrium analysis. *Journal of Monetary Economics* .

Chabakauri, G. 2015. Asset pricing with heterogeneous preferences, beliefs, and portfolio constraints. *Journal of Monetary Economics* 75:21–34.

Cochrane, J. H. 2016. Macro-finance .

Cox, J. C., and C.-f. Huang. 1989. Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of Economic Theory* 49:33–83.

Cvitanic, J., and I. Karatzas. 1992. Convex duality in constrained portfolio optimization. *Annals of Applied Probability* 2:767–818.

David, A. 2008. Heterogeneous beliefs, speculation, and the equity premium. *The Journal of Finance* 63:41–83.

Drechsler, I., A. Savov, and P. Schnabl. 2014. A model of monetary policy and risk premia .

Dumas, B., A. Kurshev, and R. Uppal. 2009. Equilibrium portfolio strategies in the presence of sentiment risk and excess volatility. *The Journal of Finance* 64:579–629.

Elenev, V., T. Landvoigt, and S. Van Nieuwerburgh. 2016. A macroeconomic model with financially constrained producers and intermediaries. *Available at SSRN* .

Farhi, E., and I. Werning. 2013. A theory of macroprudential policies in the presence of nominal rigidities. Working Paper, National Bureau of Economic Research.

Gallmeyer, M., and B. Hollifield. 2008. An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy. *Review of Finance* 12:323–64.

Geanakoplos, J., M. Magill, M. Quinzii, and J. Dreze. 1990. Generic inefficiency of stock market equilibrium when markets are incomplete. *Journal of Mathematical Economics* 19:113–51.

Geanakoplos, J., and H. M. Polemarchakis. 1985. Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete. Working Paper, Cowles Foundation for Research in Economics, Yale University.

Gertler, M., and P. Karadi. 2011. A model of unconventional monetary policy. *Journal of monetary Economics* 58:17–34.

Gorton, G., and G. Ordoñez. 2014. Collateral crises. *The American Economic Review* 104:343–78.

Gromb, D., and D. Vayanos. 2002. Equilibrium and welfare in markets with financially constrained arbitrageurs. *Journal of Financial Economics* 66:361–407.

Hanson, S. G., A. K. Kashyap, and J. C. Stein. 2011. A macroprudential approach to financial regulation. *The Journal of Economic Perspectives* 25:3–28.

He, Z., and A. Krishnamurthy. 2013. Intermediary asset pricing. *The American Economic Review* 103:732–70.

———. 2014. A macroeconomic framework for quantifying systemic risk .

Heyerdahl-Larsen, C., and J. Walden. 2014. Efficiency and distortions in a production economy with heterogeneous beliefs. *Working Paper* .

Karatzas, I., J. P. Lehoczky, and S. E. Shreve. 1990. Existence and uniqueness of multi-agent equilibrium in a stochastic, dynamic consumption/investment model. *Mathematics of Operations research* 15:80–128.

Karatzas, I., and S. Shreve. 1998. *Methods of mathematical finance*. New York: Springer-Verlag.

Kreps, D. M. 2012. *Microeconomic foundations i: choice and competitive markets*, vol. 1. Princeton University Press.

Ljungqvist, L., and T. J. Sargent. 2004. *Recursive macroeconomic theory*. The MIT Press.

Moreira, A., and A. Savov. 2014. The macroeconomics of shadow banking. Working Paper, National Bureau of Economic Research.

Morris, S. 1996. The logic of belief and belief change: A decision theoretic approach. *Journal of Economic Theory* 69:1–23.

Nualart, D. 2006. *The malliavin calculus and related topics*, vol. 1995. Springer.

Reinhart, C. M., and K. S. Rogoff. 2009. *This time is different: eight centuries of financial folly.* princeton university press.

Scheinkman, J. A., and W. Xiong. 2003. Overconfidence and speculative bubbles. *Journal of political Economy* 111:1183–220.

Schoenmaker, D., and P. Wierts. 2015. Regulating the financial cycle: An integrated approach with a leverage ratio. *Economics Letters* 136:70–2.

Stiglitz, J. E. 1982. The inefficiency of the stock market equilibrium. *Review of Economic Studies* 49:241–61.

Xiong, W., and H. Yan. 2009. Heterogeneous expectations and bond markets. *Review of Financial Studies* .

A Figures

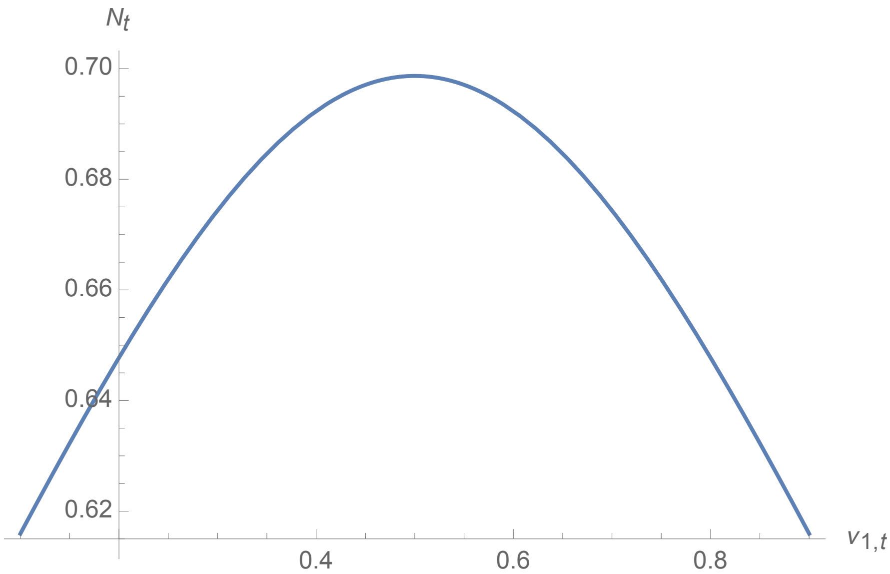

Figure 1: Labor in Equilibrium

This figure shows that the aggregate labor supply in equilibrium is symmetric in consumption share. We solve the aggregate labor supply in closed form and plot the solution where the risk aversion $\eta = \frac{1}{2}$.

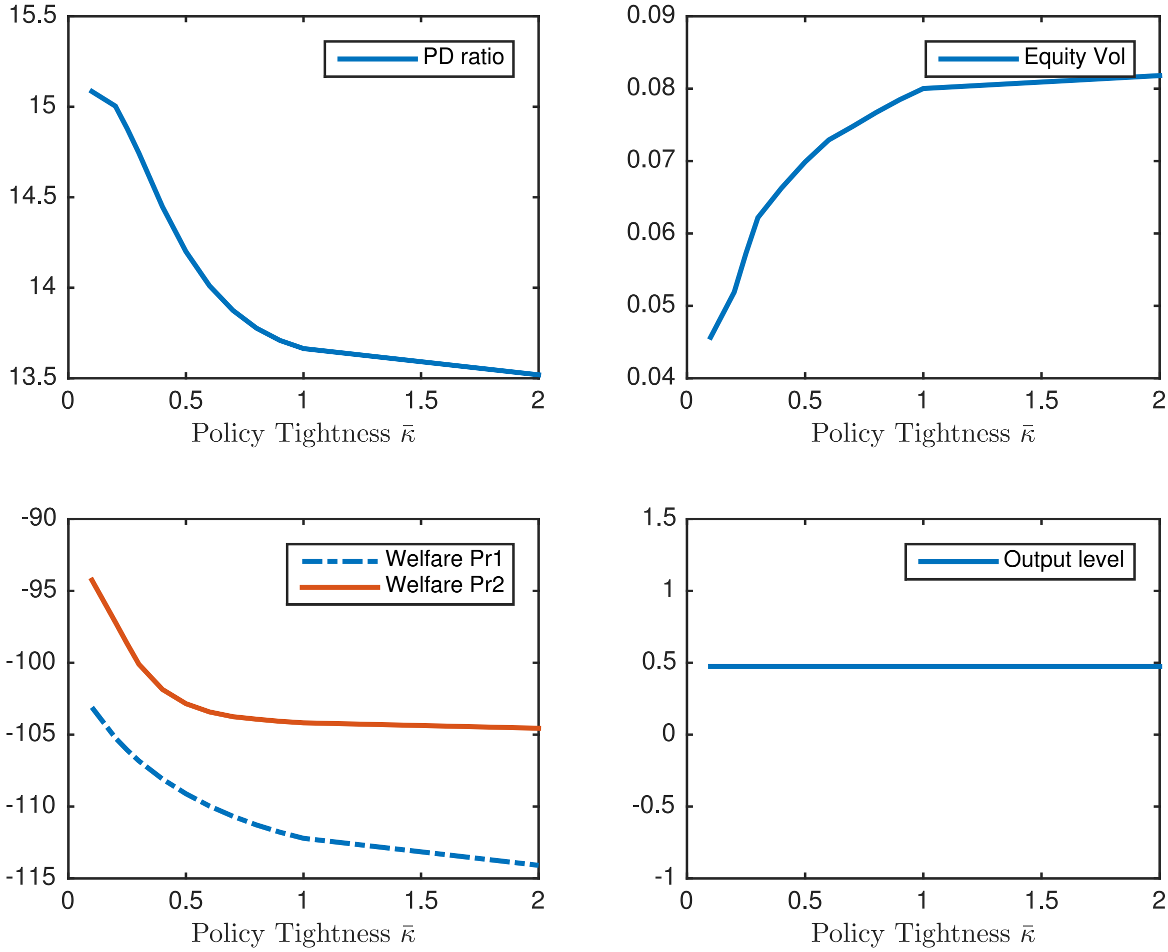

Figure 2: Baseline model

This figure plots the aggregate welfare, Price-dividend ratio and volatility of equity, and output level against different tightness of macroprudential policy ($\bar{\kappa}$ is smaller, the policy is tighter). In this baseline calibration, $\eta = 1$, so the individual utility depends on the logarithmic of consumption and leisure. We set $g_t = \hat{\mu}^1_{A,t} - \hat{\mu}^2_{A,t} = 0.01$. The figures show that tighter policy increases the asset prices and reduces the equity volatility. The left bottom panel plots the two boundaries of social welfare under any probability measures, showing that tighter policy increases the social welfare even without knowing the true probabilty measure. The output is flat in policy tightness because the aggregate labor is independent with the consumption allocation in the setting where the utility flow over leisure has identical function form (logarithmic) as consumption.

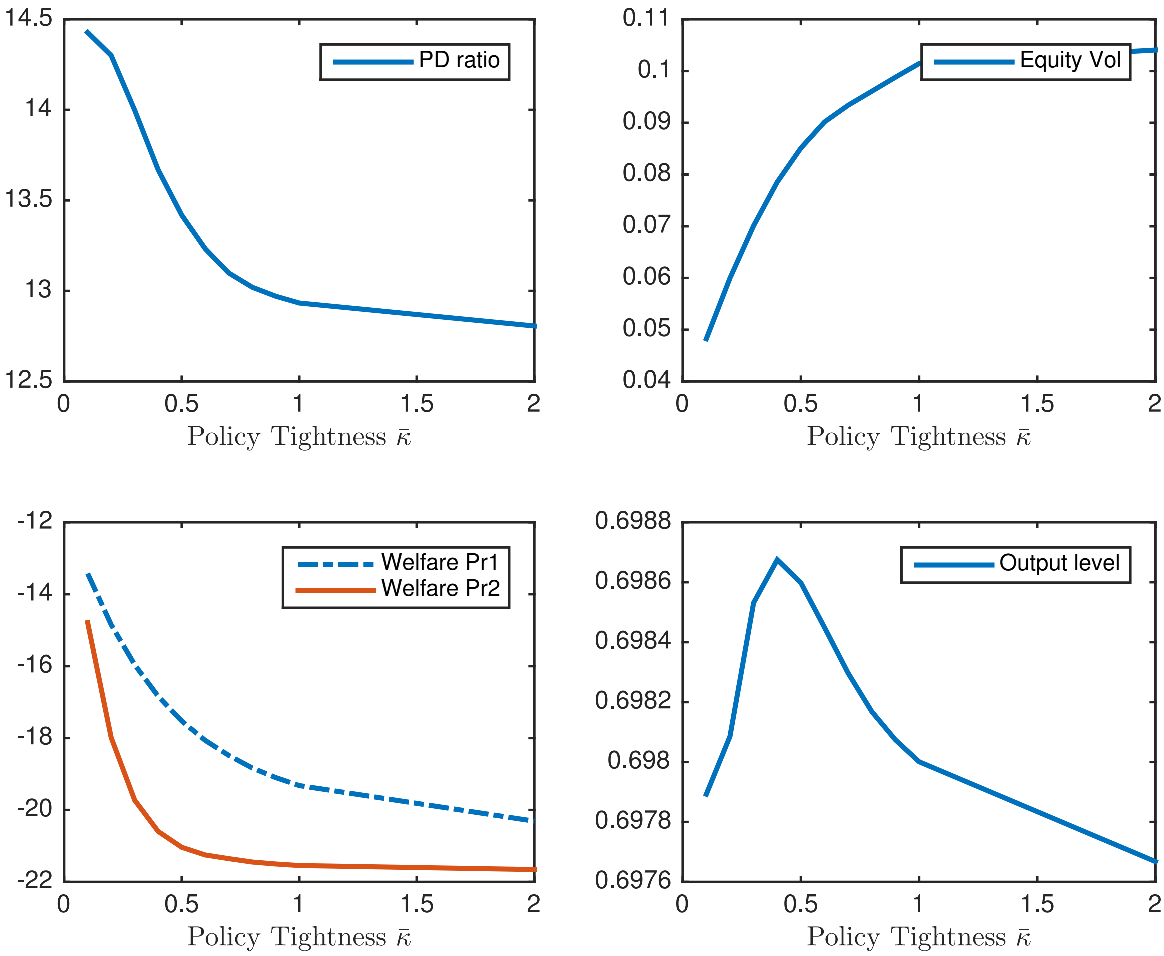

Figure 3: Baseline model

This figure plots the aggregate welfare, Price-dividend ratio and volatility of equity, and output level against different tightness of macroprudential policy ($\bar{\kappa}$ is smaller, the policy is tighter). We set $\eta = 0.5$ and $g_t = \hat{\mu}^1_{A,t} - \hat{\mu}^2_{A,t} = 0.01$. The figures show that tighter policy increases the asset prices and reduces the equity volatility. The left bottom panel plots the two boundaries of social welfare under any probability measures, showing that tighter policy increases the social welfare even without knowing the true probabilty measure. Unlike the baseline parameterization, the output increases then decreases in policy tightness.

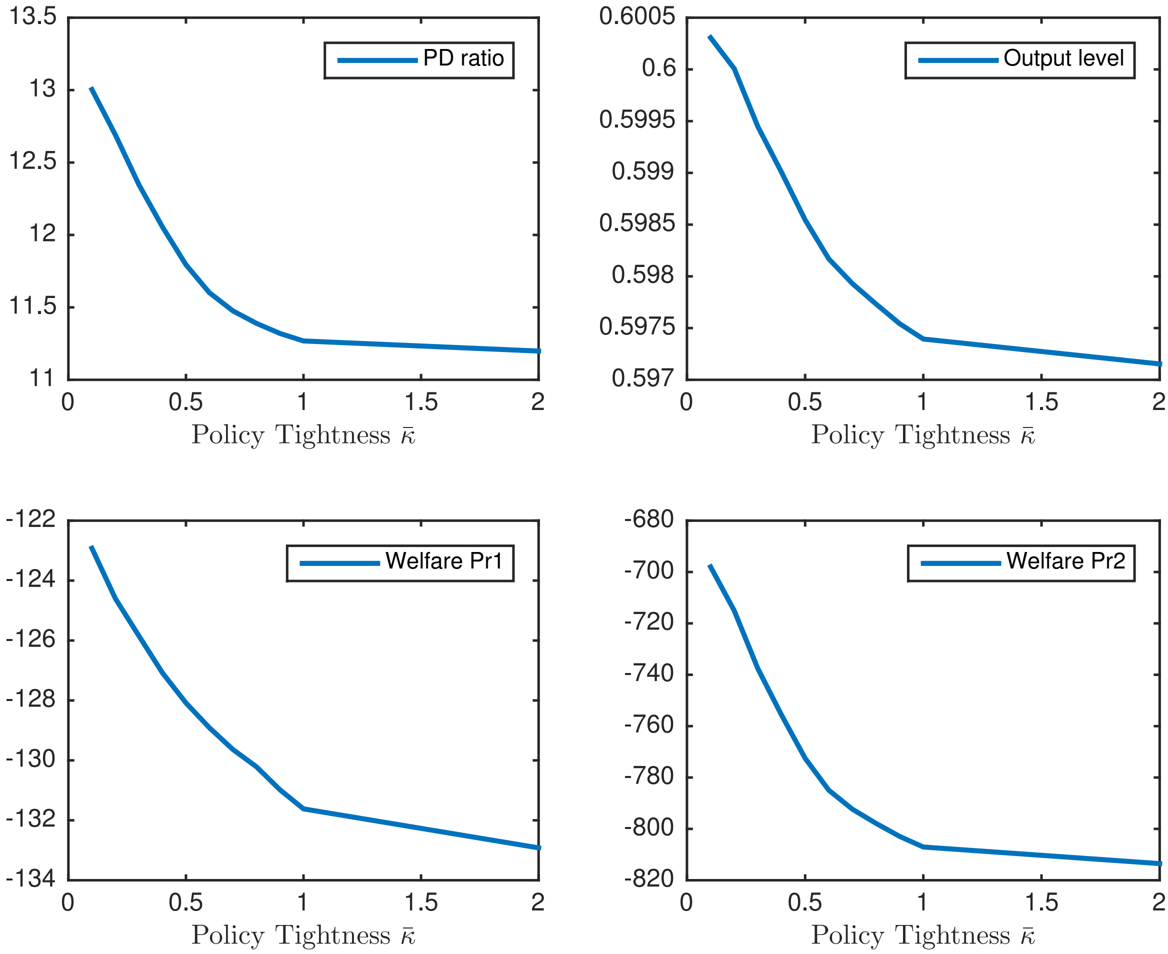

Figure 4: Extension: Different Risk Aversion

This figure plots the aggregate welfare, Price-dividend ratio and output level against different tightness of macroprudential policy ($\bar{\kappa}$ is smaller, the policy is tighter). Agents are different in their risk aversion. Agent 1 has a relative risk aversion $\gamma = 2$, while Agent 2 has log preference (relative risk aversion equals 1). We set $\eta = 1$ and $g_t = \hat{\mu}^1_{A,t} - \hat{\mu}^2_{A,t} = 0.01$. The figures show that tighter policy increases the asset prices and output level. The left bottom panel plots the social welfare under Agent 1's subjective probability, while the right bottom panel plots under Agent 2's subjective probability. These two plots showing that tighter policy increases the social welfare even without knowing the true probabilty measure.

B Proofs

B.1 Proof of Proposition 1

The proof follows Karatzas, Lehoczky, and Shreve (1990) with the appropriate modifications taken to accommodate for investors facing different state prices. From clearing in the consumption good market, state prices follow from the representative agent construction. And the equivalent SDF using the marginal utility function of the planner is:

$$U_c(Y_t, \Omega_t, \lambda_t) = \frac{\partial U_1}{\partial C_{1,t}} \frac{\partial C_{1,t}}{\partial Y_t} + \lambda_t \frac{\partial U_2}{\partial C_{2,t}} \frac{\partial C_{2,t}}{\partial Y_t} + \frac{\partial U_1}{\partial L_{1,t}} \frac{\partial L_{1,t}}{\partial Y_t} + \lambda_t \frac{\partial U_2}{\partial L_{2,t}} \frac{\partial L_{2,t}}{\partial Y_t} \tag{66}$$

$$= \frac{\partial U_1}{\partial C_{1,t}} + \frac{\partial U_1}{\partial L_{1,t}} \frac{\partial (L_{1,t} + L_{2,t})}{\partial Y_t} = \frac{\partial U_1}{\partial C_{1,t}} - \frac{1}{A_t} \frac{\partial U_1}{\partial L_{1,t}} = \frac{\partial U_1}{\partial C_{1,t}} - \frac{W_t}{A_t P_t} \frac{\partial U_1}{\partial C_{1,t}} \tag{67}$$

$$= \frac{1}{\epsilon} \frac{\partial U_1}{\partial C_{1,t}}. \tag{68}$$

Inserting the marginal utility yields the individual SDF. The Dividend flow in a monopolistic competition equilibrium is

$$D_t = Y_t - \frac{W_t}{P_t} N_t = Y_t \left(1 - \frac{W_t}{A_t P_t}\right) = \frac{1}{\epsilon} Y_t \tag{69}$$

Inserting into the fundamental pricing formula the stock price.

To show the converse, assume that there exists $\Lambda_{i,t}$, λ_t and J_t satisfying the pricing equation. Clearing in the consumption goods market follows from the solutions to the investors' consumption-portfolio problems. Clearing in bond market follows from the fact that the discount total wealth, $\Lambda_{2,t}(X_{1,t} + X_{2,t})$ where $X_{i,t}$ is the financial wealth of agent i, is a martingale since agents' optimal wealth process is a tradeable strategy, and they must agree on the prices of all tradeable strategies.

$$X_{1,t} + X_{2,t} = \frac{1}{\Lambda_{2,t}} E^1 \left[\int_t^T \Lambda_{2,s} \left(C_{1,s} - \frac{W_s}{P_s} N_{1,s} + C_{2,s} - \frac{W_s}{P_s} N_{2,s} \right) ds \right]. \tag{70}$$

By substituting clearing in the consumption goods market, clearing in the bond market results. Finally, clearing in the stock market follows by comparing the diffusion dynamics of $\Lambda_{2,t} J_t$ with $\Lambda_{2,t}(X_{1,t} + X_{2,t})$ using clearing in both consumption goods market and the bond market.

B.2 Proof of Corollary 1

The weight of the portfolio of the constrained agent with log preference in the fictitious economy is

$$\theta_t = \frac{1}{\sigma_{J,t}}\left(\kappa_{2,t} + \frac{v_t^*}{\sigma_{J,t}}\right) \leq \tilde{\theta} \tag{71}$$

$$v_t^* = \begin{cases} 0, & \text{if } \theta^* < \tilde{\theta} \\ \sigma_{J,t}\left(\bar{\kappa} - \kappa_{2,t}\right) & \text{if } \theta^* = \tilde{\theta} \end{cases} \tag{72}$$

and

$$\delta(v_t^*) = -\tilde{\theta} v_t^* = -\bar{\kappa}\left(\bar{\kappa} - \kappa_{2,t}\right). \tag{73}$$

B.3 Proof of Proposition 3

Without difference in beliefs, under the representative agent's marginal valuation for the stock given by

$$\frac{J_0}{D_0} = E_t^1\left[\int_0^T \frac{U_C\left(Y_s, \lambda_s\right)}{U_C\left(Y_0, \lambda_0\right)} \frac{D_s}{D_0} ds\right] = E_t^1\left[\int_0^T e^{-\rho s} \frac{1+\lambda_{1,s}}{1+\lambda_{1,0}} ds\right] \leq E_t^1\left[\int_0^T e^{-\rho s} ds\right] = \frac{\bar{J}_0}{D_0}. \tag{74}$$

The inequality is due to the fact that λ_t is a positive martingale and therefore a decreasing process for which the long run decrease rate is $\left(\hat{\mu}_{A,t}^1 - \hat{\mu}_{A,t}^2\right)^2$.

B.4 Proof of Proposition 4

Under type 2 agent's beliefs and her marginal valuation for the stock given by (55)

$$\frac{\tilde{J}_0}{D_0} = E_t^2\left[\int_0^T \frac{\Lambda_{2,s}}{\Lambda_{2,0}} \frac{D_s}{D_0} ds\right] = E_t^1\left[\int_0^T \frac{\xi_t \lambda_t^{-1}}{\xi_0 \lambda_0^{-1}} \frac{\Lambda_{1,s}}{\Lambda_{1,0}} \frac{D_s}{D_0} ds\right] \leq E_t^1\left[\int_0^T \frac{\Lambda_{1,s}}{\Lambda_{1,0}} \frac{D_s}{D_0} ds\right] = \frac{J_0}{D_0} \tag{75}$$

because that the stochastic process $\frac{\xi_t \lambda_t^{-1}}{\xi_0 \lambda_0^{-1}}$ is a positive martingale and that the covariance of $\frac{\xi_t \lambda_t^{-1}}{\xi_0 \lambda_0^{-1}}$ and $\frac{\Lambda_{1,s}}{\Lambda_{1,0}} \frac{D_s}{D_0}$ is negative when the LTV constraint binds.

B.5 Proof of Proposition 5

Applying Ito's lemma to each agent's first order conditions

$$e^{-\rho t}\frac{\partial U_i}{\partial C_{i,t}} = y_i \Lambda_{i,t} \tag{76}$$

and matching the diffusion term,

$$-\frac{\frac{\partial^2 U_1}{\partial C_{1,t}^2}}{\frac{\partial U_1}{\partial C_{1,t}}}\sigma_{C_1} = \sigma_{C_1} = \kappa_{1,t} \tag{77}$$

$$-\frac{\frac{\partial^2 U_2}{\partial C_{2,t}^2}}{\frac{\partial U_2}{\partial C_{2,t}}}\sigma_{C_2} = \sigma_{C_2} = \kappa_{2,t} + \frac{v_t^*}{\sigma_{J,t}} \tag{78}$$

where $C_{i,t}$ satisfies

$$dC_{i,t} = \mu_{C_i} dt + \sigma_{C_i} dZ_{A,t}^i. \tag{79}$$

Market clearing condition in consumption goods implies that

$$C_{1,t}\sigma_{C_1} + C_{2,t}\sigma_{C_2} = \sigma_{Y,t} Y_t, \tag{80}$$

which implies (57) and (58) in the unconstrained region. In the constrained region, LTV constraint, $\tilde{\theta} = \frac{\bar{\kappa}}{\sigma_{J,t}}$ allows

$$\sigma_{C_2} = \bar{\kappa}, \tag{81}$$

combine with the equation above, we obtain the constrained part of (57) and (58).

The risk free rate follows from applying Ito's lemma to the state-price density for each agent, changing to the true probability measure, matching the deterministic terms and rearranging. Risk premium can be calculated using the consumption share weighted risk prices times the stock volatility. And the $\mu_{Y,t}^1$ and $\mu_{Y,t}^2$ comes from applying Ito's lemma to $Y_t = A_t N_t$ with respect to agent 1 and agent 2's probability measure.

B.6 Dynamics of the Output

The output depends on the technological development as well as the aggregate labor supply. The individual optimization yields Leisure

$$C_{i,t} = \frac{e^{-\rho t}}{y^i \Lambda_{i,t}} \tag{82}$$

$$L_{i,t}^{-\eta} = \frac{e^{-\rho t}}{y_i \Lambda_{i,t} \frac{W_t}{P_t}} \tag{83}$$

Consumption goods market clear:

$$\bar{N} - \sum_{i=1,2} N_{i,t} = \left(\frac{W_t}{P_t}\right)^{-\frac{1}{\eta}} \sum_{i=1,2} C_{i,t}^{\frac{1}{\eta}} \tag{84}$$

$$\bar{N} - N_t = \left(\frac{W_t}{P_t}\right)^{-\frac{1}{\eta}} Y_t^{\frac{1}{\eta}} \sum_{i=1,2} \nu_{i,t}^{\frac{1}{\eta}} = \left(\frac{W_t}{A_t P_t}\right)^{-\frac{1}{\eta}} N_t^{\frac{1}{\eta}} \sum_{i=1,2} \nu_{i,t}^{\frac{1}{\eta}} \tag{85}$$

By Ito's lemma, the diffusion of Y_t

$$\sigma_Y = \sigma_A - \frac{\lambda_t}{N_t} \sigma_\lambda \frac{\partial N_t}{\partial \lambda_t} \tag{86}$$

As

$$\sigma_\lambda = \kappa_1 - \kappa_2 - \frac{v_t^*}{\sigma_{J,t}} = \begin{cases} \frac{1}{\nu_{1,t}} \sigma_{Y,t} - \left(1 + \frac{\nu_{2,t}}{\nu_{1,t}}\right) \bar{\kappa} & \text{if the constraint binds,} \\ \bar{\mu}_t, & \text{otherwise.} \end{cases} \tag{87}$$

hence

$$\sigma_Y = \begin{cases} \frac{\sigma_A + \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \left(1 + \frac{\nu_{2,t}}{\nu_{1,t}}\right) \bar{\kappa}}{1 + \frac{\lambda_t}{N_t} \frac{\partial N_t}{\partial \lambda_t} \frac{1}{\nu_{1,t}}}, & \text{if the constraint binds,} \\ \sigma_A - \frac{1}{N_t} \lambda_t \bar{\mu}_t \frac{\partial N_t}{\partial \lambda_t}, & \text{otherwise.} \end{cases} \tag{88}$$

Assume that an equilibrium exists. The SDFs in the fictitious for the two agents are

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c(Y_t, \lambda_t)}{U_c(Y_0, \lambda_0)}, \qquad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \tag{89}$$

where $U_c(Y_t, \lambda_t)$ is the marginal utility of a social planner which is defined as

$$U(Y_t, \lambda_t) = \max_{C_{1,t} + C_{2,t} \le A_t \left(\bar{N} - L_{1,t} - L_{2,t}\right) = Y_t} \log C_{1,t} + \frac{L_{1,t}^{1-\eta}}{1-\eta} + \lambda_t \left(\log C_{2,t} + \frac{L_{2,t}^{1-\eta}}{1-\eta}\right) \tag{90}$$

The stock price J_t is computed from the unconstrained agent's probability measure

$$J_t = E_t^1 \left[\int_t^T e^{-\rho s} \frac{U_c(Y_s, X_s, \lambda_s)}{U_c(Y_0, X_0, \lambda_0)} D_s ds \right]. \tag{91}$$

B.7 Proof of Proposition 6

We use the following lemma to prove Proposition Volatility.

Lemma 1 (Clark-Ocone Theorem - Nualart (2006)) *Let $F \in D^{1,2}$, where the space $D^{1,2}$ is the closure of the class of smooth random variables S with respect tot he norm*

$$||F||_{1,2} = \left[E\left[|F|^2\right] + E\left(||\mathcal{D}F||^2_{L^2(T)} \right) \right]^{1/2}. \tag{92}$$

Suppose that W_t is a multi-dimensional Brownian Motion. Then

$$F = E[F] + \int_0^T E\left(\mathcal{D}_s F | \mathcal{F}_s\right) dW_s. \tag{93}$$

Taking conditional expectations,

$$E[F|\mathcal{F}_t] = E[F] + \int_0^t E\left(\mathcal{D}_s F | \mathcal{F}_s\right) dW_s. \tag{94}$$

Proof of Lemma 1. See Nualart (2006).

Next we consider the equity pricing equation respect to probability measure of the Type 1 Agent:

$$J_t = \frac{1}{U_C(C_t, \lambda_t)} E^1 \left[\int_s^T e^{-\rho s} U_C(C_t, \lambda_t) \frac{Y_s}{\epsilon} ds | \mathcal{F}_t \right]. \tag{95}$$

Define the $\mathcal{L}^2\left(\mathcal{P}^1\right)$ martingale M_t

$$M_t = E_t^1 \left[\int_t^T e^{-\rho s} U_C(C_s, \lambda_s) Y_s ds \right]. \tag{96}$$

By the martingale representation theorem, there exist a process $\phi(t)$ such that $E_t^1 \left[\int_t^T |\phi(t)|^2 ds \right] < \infty$ and $M_t = M_0 + \int_0^T \phi(t) dZ^1_{A,t}$.

The value of the stock can be rewritten as

$$J_t = \frac{1}{\epsilon U_C\left(C_t, \lambda_t\right)}\left[M_t - \int_0^t e^{-\rho s} U_C\left(C_s, \lambda_s\right) Y_s ds\right] \tag{97}$$

Applying Ito's lemma to equation above and comparing the diffusion term earns the volatility

$$\sigma_{J,t} = -\frac{U_{CC}\left(C_t, \lambda_t\right)}{U_C\left(C_t, \lambda_t\right)} Y_t \sigma_Y + \frac{U_{C\lambda}\left(C_t, \lambda_t\right)}{U_C\left(C_t, \lambda_t\right)} \lambda_t \sigma_\lambda + \frac{1}{E_t^1\left[\int_t^T e^{-\rho s} U_C\left(C_s, \lambda_s\right) Y_s ds\right]} \phi(t) \tag{98}$$

$$= \sigma_Y + \frac{\lambda_t}{1+\lambda_t} \sigma_\lambda + \frac{1}{E_t^1\left[\int_t^T e^{-\rho s}\left(1+\lambda_s\right) ds\right]} \phi(t) \tag{99}$$

To complete the proof we need to use Malliavin derivatives to calculate the the response of M_t to fundamental shocks. By Lemma 1, the diffusion term of dynamics of M_t is

$$\phi(t) = E_t^1\left[\int_t^T e^{-\rho s} \mathcal{D}_t^1 \lambda_s ds\right] \tag{100}$$

where the Malliavin derivatives are

$$\frac{\mathcal{D}_t^1 \lambda_s}{\lambda_s} = -\sigma_\lambda(t) - \int_t^s \mathcal{D}_t^1 \sigma_\lambda(z)\left[\sigma_\lambda(z) dz + dZ_{A,z}^1\right] \tag{101}$$

$$\mathcal{D}_t^1 \sigma_\lambda(z) = \begin{cases} \mathcal{D}_t^1 \bar{\mu}(z) & \text{if } \sigma_\lambda = \bar{\mu}(z) \\ \frac{\partial \kappa_{1,z}}{\partial \lambda_z} \mathcal{D}_t^1 \lambda_z & \text{if } \sigma_\lambda = \kappa_{1,z} - \bar{\kappa} \end{cases} \tag{102}$$

$$\mathcal{D}_t^1 \bar{\mu}(z) = e^{-\int_t^z \frac{\bar{v}_u^2}{\sigma_A^2} du} \frac{1}{\sigma_A^2}\left(\bar{v}_t^1 - \bar{v}_t^2\right) \tag{103}$$

C Speculation and Risk Sharing

C.1 Heterogeneous Preferences

To rationalize the risk sharing motive, we assume two types of households differs in their preference besides the priors. Specifically, we assume the utility of the more confident agent follows Instantaneous utility of type i representative agent is given by the following function that depends on the consumption $C_{i,u}$ and leisure $L_{i,t}$:

$$U(C_{1,t}, L_{1,t}) \equiv \frac{C_{1,t}^{1-\gamma}}{1-\gamma} + \log L_{1,t}, \tag{104}$$

$$U(C_{2,t}, L_{2,t}) \equiv \log C_{2,t} + \log L_{2,t}. \tag{105}$$

Apart from the heterogeneity in risk aversion γ to induce risk sharing, We keeps all other settings identical to the baseline model. Both agents optimize the consumption, leisure and portfolio to maximize their expected utility, which is given by

$$V_{i,t} = \max_{C_{i,u}, N_{i,u}, \theta_{i,u} \in \Theta} E_t^i \left[\int_t^{\infty} e^{-\rho(u-t)} U_i(C_{i,u}, L_{i,u}) du \right] \tag{106}$$

subject to a dynamic budge constraint

$$dG_{i,t} = r_t G_{i,t} dt + \theta_{i,t} \left(dJ_t - r_t J_t + D_t dt \right) + \frac{W_t}{P_t} \left(\bar{N}_i - L_{i,t} \right) dt - C_{i,t} dt, \tag{107}$$

and the dynamics of consumption share

$$\frac{d\nu_{1,t}}{\nu_{1,t}} = \mu_{\nu_1} dt + \sigma_{\nu_1} dZ_t. \tag{108}$$

The FOC of agent 1's optimization problem gives the optimal consumption and leisure choice:

$$\hat{C}_{1,t} = \left(\frac{e^{-\rho t}}{y_1 \Lambda_{1,t}} \right)^{\frac{1}{\gamma}}, \qquad \hat{C}_{2,t} = \frac{e^{-\rho t}}{y_1 \Lambda_{2,t}} \tag{109}$$

$$\hat{L}_{1,t} = \frac{e^{-\rho t}}{y_1 \Lambda_{1,t} \frac{W_t}{P_t}}, \qquad \hat{L}_{2,t} = \frac{e^{-\rho t}}{y_2 \Lambda_{2,t} \frac{W_t}{P_t}} \tag{110}$$

y_i is the Lagrange multiplier from the static budget constraint and solves the following equations

$$E_0^1 \left[\int_0^T e^{-\frac{\rho}{\gamma} t} \Lambda_{1,t}^{1-\frac{1}{\gamma}} dt \right] y_1^{-\frac{1}{\gamma}} + \frac{1}{\rho} \left(1 - e^{-\rho T} \right) y_1^{-1} = E_0^1 \left[\int_0^T \Lambda_{1,t} \frac{W_t}{P_t} \bar{N}_1 dt \right] + x_1 J_0, \tag{111}$$

$$y_2 = \frac{\frac{2}{\rho} \left(1 - e^{-\rho T} \right)}{E_0^2 \left[\int_0^T \Lambda_{2,t} \frac{W_t}{P_t} \bar{N}_2 dt \right] + x_2 J_0}. \tag{112}$$

The labor market clearing condition leads to

$$A_t \left(\bar{N} - L_{1,t} - L_{2,t} \right) = C_{1,t} + C_{2,t} \tag{113}$$

$$A_t \left(\bar{N} - (1 + \lambda_t) L_{1,t} \right) = C_{1,t} + \lambda_t C_{1,t}^{\gamma} \tag{114}$$

$$\left(\bar{N} - (1 + \lambda_t) L_{1,t} \right) = \left(\frac{W_t}{A_t P_t} \right)^{\frac{1}{\gamma}} A_t^{\frac{1}{\gamma} - 1} L_{1,t}^{\frac{1}{\gamma}} + \lambda_t \frac{W_t}{A_t P_t} L_{1,t} \tag{115}$$

The following proposition characterizes the equilibrium in this new setting.

Corollary 2 *Assume that an equilibrium exists. The SDFs in the fictitious for two types of households are*

$$\Lambda_{1,t} = e^{-\rho t} \frac{U_c\left(Y_t, \lambda_t\right)}{U_c\left(Y_0, \lambda_0\right)}, \qquad \Lambda_{2,t} = \frac{\lambda_t}{\lambda_0} \Lambda_{1,t}, \tag{116}$$

where $U_c\left(Y_t, \lambda_t\right)$ is the marginal utility of a social planner which is defined as

$$U\left(Y_t, \lambda_t\right) = \max_{C_{1,t}+C_{2,t} \leq A_t\left(\bar{N}-L_{1,t}-L_{2,t}\right)=Y_t} U(C_{1,t}, L_{1,t}) + \lambda_t U(C_{2,t}, L_{2,t}). \tag{117}$$

The stock price J_0 is computed from the more confident agent's probability measure

$$J_0 = \frac{1}{\epsilon} E_0^1 \left[\int_0^T e^{-\rho s} \frac{U_c\left(Y_s, \lambda_s\right)}{U_c\left(Y_0, \lambda_0\right)} Y_s ds \right]. \tag{118}$$

The equilibrium consumption allocations are

$$C_{1,t} = \left(\frac{1}{\epsilon U_c\left(Y_t, \lambda_t\right)} \right)^{\frac{1}{\gamma}}, \qquad C_{2,t} = \frac{\lambda_t}{\epsilon U_c\left(Y_t, \lambda_t\right)}. \tag{119}$$

The stochastic weighting process λ_t has dynamics given by (39) where λ_0 clears the markets at time $t=0$, namely

$$C_1\left(\lambda_0, A_0\right) + C_2\left(\lambda_0, A_0\right) = A_0\left(\bar{N} - L_1\left(\lambda_0, A_0\right) - L_2\left(\lambda_0, A_0\right)\right) \tag{120}$$

Conversely, if there exists processes $\Lambda_{1,t}$, $\Lambda_{2,t}$, λ_t and J_t satisfying equations (116) - (120), with $\sigma_Y > 0$, the associated optimal consumption portfolio policies clear all markets.

Proof: Similar to B.1.

Corollary 3 *The market price of risk for two agents are*

$$\kappa_{1,t} = \begin{cases} \frac{\gamma}{\nu_{1,t}}\left(\sigma_{Y,t} - \nu_{2,t}\bar{\kappa}\right) & \textit{if the constraint binds,} \\ \boldsymbol{R}_t \sigma_{Y,t} + \nu_{2,t} \boldsymbol{R}_t \bar{\mu}, & \textit{otherwise,} \end{cases} \tag{121}$$

$$\kappa_{2,t} = \begin{cases} \frac{\gamma}{\nu_{1,t}}\left(\sigma_{Y,t} - \nu_{2,t}\bar{\kappa}\right) - \bar{\mu}_t & \textit{if the constraint binds,} \\ \boldsymbol{R}_t \sigma_{Y,t} - \frac{1}{\gamma}\nu_{1,t} \boldsymbol{R}_t \bar{\mu}, & \textit{otherwise,} \end{cases} \tag{122}$$

where $\boldsymbol{R}_t$ is the aggregate risk aversion defined by

$$\boldsymbol{R}_t = \left(\frac{\nu_{1,t}}{\gamma} + \frac{\nu_{2,t}}{1} \right)^{-1}. \tag{123}$$

By Ito's lemma, the diffusion of Y_t

$$\sigma_Y = \sigma_A - \frac{\lambda_t}{N_t}\sigma_\lambda \frac{\partial N_t}{\partial \lambda_t} + \frac{A_t}{N_t}\sigma_A \frac{\partial N_t}{\partial A_t} \tag{124}$$

As

$$\sigma_\lambda = \kappa_1 - \kappa_2 - \frac{v_t^*}{\sigma_{J,t}} = \begin{cases} \frac{\gamma}{\nu_{1,t}}\left(\sigma_{Y,t} - \nu_{2,t}\bar{\kappa}\right) - \bar{\kappa} & \text{if the constraint binds,} \\ \bar{\mu}_t, & \text{otherwise.} \end{cases} \tag{125}$$

hence

$$\sigma_Y = \begin{cases} \frac{\sigma_A + \frac{\lambda_t}{N_t}\frac{\partial N_t}{\partial \lambda_t}\left(1+\gamma\frac{\nu_{2,t}}{\nu_{1,t}}\right)\bar{\kappa} + \frac{A_t}{N_t}\sigma_A\frac{\partial N_t}{\partial A_t}}{1+\frac{\lambda_t}{N_t}\frac{\partial N_t}{\partial \lambda_t}\frac{\gamma}{\nu_{1,t}}}, & \text{if the constraint binds,} \\ \sigma_A - \frac{1}{N_t}\lambda_t\bar{\mu}_t\frac{\partial N_t}{\partial \lambda_t} + \frac{A_t}{N_t}\sigma_A\frac{\partial N_t}{\partial A_t}, & \text{otherwise.} \end{cases} \tag{126}$$