Pricing Foreign Equity Option with time-changed Lévy Process

Abstract. In this paper we propose a general foreign equity option pricing framework that unifies the vast foreign equity option pricing literature and incorporates the stochastic volatility into foreign equity option pricing. Under our framework, the time-changed Lévy processes are used to model the underlying assets price of foreign equity option and the closed form pricing formula is obtained through the use of characteristic function technology. Numerical tests indicate that stochastic volatility has a dramatic effect on the foreign equity option prices.
Keywords: Foreign equity option; Lévy process; Time-changed Lévy process; Fast Fourier transformation

1 Introduction

With the growth in globalization of investments and the continued liberalization of cross-border cash flows, the currency translated foreign equity options (cross-currency options) have gained wider popularity. Foreign equity options are contingent claims where the payoff is determined by an equity in one currency but the actual payoff is done in another currency. By a variety of combinations on linking foreign asset price and exchange rate, foreign equity options traded on international markets provide an efficient means of managing multidimensional risks.

Previous studies dealing with the currency translated foreign equity options usually model the dynamics of asset price and exchange rate with Brownian motions, see, for example, Wei (1992), Dravid, Richardson, and Sun (1993), Ho, Stapleton, and Subrahmanyam (1995), Reiner (1992), Toft and Reiner (1997), and Kwok and Wong (2000). Duan and Wei (1999) priced foreign currency and cross-currency options under GARCH model. However, despite the success of the Black-Scholes model based on Brownian motion and normal distribution, two empirical phenomenons can not be explained by Black-Scholes model: (1) the asymmetric leptokurtic features and (2) the volatility smile. Simultaneously, jumps are clearly identifiable from equity data, see, for example, Eraker (2004), Eraker, Johannes, and Poison (2003), and references therein. Many studies have been conducted to modify the Black-Scholes model, see, for example, Merton (1976), Heston (1993), Bakshi, Cao, and Chen (1997), Bates (2000), Duffie, Pan and Singleton (2000), Geman, Madan, and Yor (2001), Kou (2002), Carr and Wu (2004), Chen and Kulperger (2006), Lau and Siu (2008), and Xu et al. (2009, 2010). In the exchange rate modelling, Brownian motions are also contradicted with empirical phenomenon. Many studies indicate that jumps are important components of the exchange rate dynamics, see, for example, Xu, Wu and Li (2010), Xu et al. (2010), Jorion (1998), Johnson and Schneeweis (1994, 2002), Bates (1996a, 1996b), and Carr and Wu (2007).

Huang and Hung (2005) went beyond the traditional Black-Scholes framework and priced foreign equity options under Lévy processes. In Huang and Hung's paper, the exchange rate and foreign asset prices are modeled as multidimensional Lévy processes and the option value is calculated with the Fourier inverse transformation. The motivation of this study is that Huang and Hung (2005) assumed the volatility of underlying asset returns of foreign equity option constant. This assumption differs from many empirical study results that return volatilities vary stochastically over time. The purpose of this paper is to explore the use of time-changed Lévy processes as a way to capture this fact, and the closed form pricing formula of foreign equity option is obtained. Following Carr and Madan (1999), fast fourier transform of option prices is derived. The foreign equity option pricing model used by Huang and Hung (2005) is a special case of our pricing model when the stochastic clock on which the Lévy process is run becomes a calendar time. As in Carr and Wu (2004), we can regard the original clock as calendar time and the new random clock as business time. A more active business day implies a faster business clock, and randomness in business activity generates randomness in volatility.

This paper is organized as follows. In Section 2, we introduce the Lévy characteristics and types of Lévy processes. Section 3 presents the fundamental theorem simplifying the calculation of the characteristic function of the time-changed Lévy process. Section 4 shows foreign equity option pricing based on time-changed Lévy process. The concluding remarks are given in Section 5.

2 Lévy processes

Lévy processes constitute a wide class of stochastic processes whose sample paths can be continuous, mostly continuous with occasional discontinuities, and purely discontinuous. Generally, Lévy processes are a combination of a linear drift, a Brownian motion, and a jump process. The classic Black-Scholes (BS) model is characterized as the only continuous Lévy model. For a more complete presentation on the topic of Lévy processes see the books Cont and Tankov (2004).

2.1 Lévy characteristics

For the remainder of the paper, we fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a standard complete filtration $F = \{\mathscr{F}_t | t \geq 0\}$. The following definition formalizes the class of Lévy processes (see Cont and Tankov, 2004, P. 68).

Definition 1. [Lévy processes] *A right-continuous with left limits stochastic process $(X_t)_{t\geq 0}$ on $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that X_0=0 is called a Lévy process if it possesses the following properties:*

1. Independent increments: for every increasing sequence of times $t_0, \cdots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.

2. Stationary increments: the law of $X_{t+h} - X_t$ does not depend on t.

3. Stochastic continuity: $\forall t > 0$, $\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.

By the Lévy-Itô decomposition, any Lévy process X_t on $\mathbb{R}^d$ can be written as following representation form (see Cont and Tankov, 2004, Proposition 3.7):

$$X_t = \gamma t + B_t + \int_{|x|\geq 1, s\in[0,t]} x J_X(ds \times dx) + \lim_{\epsilon \downarrow 0} \int_{\epsilon \leq |x| < 1, s\in[0,t]} x\{J_X(ds \times dx) - \nu(dx)ds)\}, \quad (1)$$

where $\gamma \in \mathbb{R}^d$ is a constant vector, B_t is a d-dimensional Brownian motion with covariance matrix A, and J_X is a poisson random measure on $[0, \infty) \times \mathbb{R}^d$ with intensity $\nu(dx)dt$. In particular, Lévy measure ν is defined on $\mathbb{R}^d_0$ ($\mathbb{R}^d$ less zero) with

$$\int_{\mathbb{R}^d_0}(1 \wedge x^2)\nu(dx) < \infty,$$

and describes the arrival rates for jumps of every possible sizes for each component of X. The Lévy-Itô decomposition entails that every Lévy process is specified by the vector $\gamma \in \mathbb{R}^d$, the positive semi-definite matrix on $A \in \mathbb{R}^{d\times d}$, and the Lévy measure ν defined on $\mathbb{R}^d_0$. The triplet (γ, A, ν) is called characteristic triplet or Lévy triplet of the process X_t. By the Lévy-Khinchin representation theorem (see Cont and Tankov, 2004, Theorem 3.1), the characteristic function of X_t with characteristic triplet (γ, A, ν) has the form

$$\phi_{X_t}(z) = E[e^{iz\cdot X_t}] = e^{-t\psi_x(z)}, z \in \mathbb{R}^d, t \geq 0, \tag{2}$$

where the characteristic exponent $\psi_x(z)$ is given by

$$\psi_x(z) = -i\gamma \cdot z + \frac{1}{2}z \cdot Az + \int_{\mathbb{R}^d_0}(1 - e^{iz\cdot x} + iz \cdot x1_{|x|\leq 1})\nu(dx). \tag{3}$$

The characteristic function (2) is defined on the real space $z \in \mathbb{R}^d$. In many applications, it is convenient to extend the characteristic function parameter z to the complex space $z \in \mathbb{C}^d$, where the characteristic function is well defined. When characteristic function $\phi_{X_t}(z)$ is defined on the complex space, it is referred to as the generalized Fourier transform (see Titchmarsh, 1975)

2.2 Types of Lévy processes

Depending on differences in their jump component, Lévy processes used to model the financial asset price dynamics fall into two categories. The first category, called finite-activity models, are characterized by the feature:

$$\int_{\mathbb{R}_0}\nu(dx) = \lambda < \infty. \tag{4}$$

Intuitively speaking, a finite-activity process exhibits a finite number of jumps within any finite time interval. For such processes, the integral $\int_{\mathbb{R}_0}\nu(dx)$ defines the Poisson intensity λ. Obviously, one can choose any distribution function for the jump size and obtain the following Lévy measure:

$$\nu(dx) = \lambda F(dx). \tag{5}$$

The classical example of a finite-activity jump process is the compound Poisson jump-diffusion model of Merton (1976) (MJ). Conditional on one jump occurring, the MJ model assumes that the jump magnitude is normally distributed with mean μ_J and variance σ^2_J, and the Lévy measure is given by

$$\nu(dx) = \lambda\frac{1}{\sqrt{2\pi}\sigma_J}\exp(-\frac{(x-\mu_J)^2}{2\sigma^2_J})dx. \tag{6}$$

In another example, Kou (2002) assumes a double-exponential conditional distribution for the jump size, and the Lévy measure is given by

$$\nu(dx) = p\lambda\eta_1 e^{-\eta_1 x}1_{x>0} + (1-p)\lambda\eta_2 e^{-\eta_2|x|}1_{x<0}dx. \tag{7}$$

In the first category, called jump-diffusion models, the normal evolution of prices is given by a diffusion process, punctuated by jumps at random times.

By using the Lévy-Khintchine formula in (3), the characteristic exponent corresponding to these compound Poisson jump-diffusion model is given by

$$\psi_x(z) = -ibz + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}_0} (1 - e^{izx})\nu(dx). \tag{8}$$

where $b = \gamma - \int_{|x|\le 1} x\nu(dx)$.

The second category consists of models with infinite number of jumps within any finite time interval, which we will call infinite activity models. The integral of Lévy measure (4) is no longer finite. Examples in this class include the finite moment log-stable (FMLS) model of Carr and Wu (2003), the variance gamma (VG) model of Madan and Milne (1991) and Madan, Carr, and Chang (1998), the normal inverse Gaussian (NIG) model of Barndorff-Nielsen (1998), the CGMY model of Carr et al. (2002), and the generalized hyperbolic (GH) model of Eberlein, Keller, and Prause (1998). For more discussion of these models, see Cont and Tankov (2004). Table 1 lists the Lévy measures and characteristic exponents of the finite-activity and infinite-activity jump models. We also list the characteristic exponents of an arithmetic Brownian motion, which is the only purely continuous Lévy process. Note that the FMLS model can be regarded as a special case of α-stable model with $\alpha \in (1, 2]$, $\beta = -1$, and $x < 0$.

Table 1. Entries summarize the Levy measure and its corresponding characteristic exponent for finite-activity and infinite-activity jump models.

Models	Levy measures $\nu(dx)/dx$	Characteristic exponent $\psi_x(z)$
Pure continuous Levy process		
$\mu t + \sigma W_t$	—	$-i\mu z + \frac{1}{2}\sigma^2 z^2$
Finite-activity models		
Merton (1976)	$\frac{\lambda}{\sqrt{2\pi\sigma_J^2}} \exp\{-\frac{(x-\mu_J)^2}{2\sigma_J^2}\}$	$-ibz + \frac{\sigma^2 z^2}{2} - \lambda\{e^{i\mu_J z - \sigma_J^2 z^2/2} - 1\}$
Kou (2002)	$p\lambda\eta_1 e^{-\eta_1 x} 1_{x>0} + (1-p)\lambda\eta_2 e^{-\eta_2 \|x\|} 1_{x<0}$	$-ibz + \frac{\sigma^2 z^2}{2} - iz\lambda\{\frac{p}{\eta_1 - iz} - \frac{1-p}{\eta_2 + iz}\}$
Infinite-activity models		
VG	$\frac{1}{\kappa\|x\|} \exp\{\frac{\theta}{\sigma^2} - \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}\}$	$\frac{1}{\kappa} \ln(1 + \frac{z^2\sigma^2\kappa}{2} - i\theta\kappa z)$
NIG	$\exp(\beta x) \frac{\delta\alpha}{\pi\|x\|} K_1(\alpha\|x\|)$	$-\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iz)^2} - iz\mu)$
CGMY	$C\frac{\exp(-G\|x\|)}{\|x\|^{1+Y}}, x < 0; C\frac{\exp(-M\|x\|)}{\|x\|^{1+Y}}, x > 0$	$C\Gamma(-Y)[M^Y - (M - iz)^Y + G^Y - (G + iz)^Y]$
GH	$\frac{e^{\beta x}}{\|x\|}(\int_0^\infty \frac{\exp(-\sqrt{2y+\alpha^2}\|x\|)}{\pi^2 y(J_\lambda^2 \delta\sqrt{2y} + Y_\lambda^2 \delta\sqrt{2y})} dy$ $+ \lambda 1_{\lambda \ge 0} e^{-\alpha\|x\|})$	$-\ln[e^{i\mu z}(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iz)^2})^{\lambda/2} \frac{\kappa_\lambda \delta\sqrt{\lambda^2 - (\beta + iz)^2}}{\kappa_\lambda \delta\sqrt{\alpha^2 - \beta^2}}]$
α-stable model	$\frac{A}{x^{\alpha+1}} 1_{x>0} + \frac{B}{\|x\|^{\alpha+1}} 1_{x<0}$	$\exp\{-\sigma^\alpha \|z\|^\alpha (1 - i\beta \text{sgn}\, z \tan\frac{\pi\alpha}{2}) + i\mu z\}, \alpha \ne 1;$ $\exp\{-\sigma\|z\|(1 + i\beta\frac{2}{\pi}\text{sgn}\, z \log\|z\|) + i\mu z\}, \alpha = 1$

3 Time-changed Lévy processes

To capture the stochastic volatility from economic shocks, as in Carr and Wu (2004), we introduce a random time change to Lévy process. Let X_t denote a d-dimensional Lévy process and $t \to T_t (t \ge 0)$ be an increasing right-continuous process with left limits that satisfy the usual technical conditions, the time-changed Lévy process Y_t is defined by evaluating X at T_t, i.e.,

$$Y_t = X_{T_t}, t \ge 0. \tag{9}$$

Obviously, by specifying different Lévy characteristics for X_t and different random processes for T_t, we can generate a wide range of stochastic processes from this setup. Following Carr and Wu (2004), for simplicity we also characterize the random time change in terms of its local intensity $v(t)$,

$$T_t = \int_0^t v(s_-)ds. \tag{10}$$

Carr and Wu (2004) label $v(t)$ as the instantaneous (business) activity rate and regard T_t as business time at calendar time t. A more active business day, captured by a higher activity rate, generates higher volatility for asset returns. The randomness in business activity generates randomness in volatility.

If random time T_t is independent of X_t, the characteristic function of time-changed Lévy process $Y_t = X_{T_t}$ can be obtained directly by using Eq. (2),

$$\begin{aligned}\phi_{y_t}(z) = E[e^{iz.X_{T_t}}] &= E[E[e^{iz.X_u}]|T_t = u] \\ &= E[e^{-T_t\psi_x(z)}] \\ &= \mathscr{L}_{T_t}(\psi_x(z))\end{aligned} \tag{11}$$

Under independence, the characteristic function of Y_t is reduced to the Laplace transform of T_t evaluated at characteristic exponent of X_t. Hence, if the characteristic exponent of X_t and the Laplace transform of T_t are both available in closed form, the characteristic function of Y_t can be obtained in closed form. In principle, the characteristic exponent of X_t can be calculated by the Lévy-Khintchine theorem in (3). To obtain the Laplace transform of T_t in closed form, Carr and Wu (2004) show that one can adopt the vast literature in term structure modeling for the purpose of modeling the instantaneous activity rate $v(t)$ by regarding $\psi_x(z)v(t)$ as the instantaneous interest rate.

When the Lévy process and time-change are correlated, Carr and Wu (2004) propose a new measure transform method, named leverage-neutral measure, to generalize the reduction in (11) of the characteristic function to a bond pricing formula. This generalization is very important in option pricing model based on time-changed Lévy process, and allows us to easily capture the well-known leverage effect.

Theorem 1. [Carr and Wu, 2004, Theorem 1] *The problem of finding the generalized Fourier transform of the time-changed Lévy process $Y_t = X_{T_t}$ under measure $\mathbb{P}$ reduces to the problem of finding the Laplace transform of random time under the complex-valued measure $\mathbb{Q}(z)$, evaluated at the characteristic exponent $\psi_x(z)$ of X_t,*

$$\phi_{Y_t}(z) = E[e^{iz.Y_t}] = E^z[e^{-T_t\psi_x(z)}] = \mathscr{L}^z_{T_t}(\psi_x(z)), \tag{12}$$

where $E[\cdot]$ and $E^z[\cdot]$ denote expectations under measures $\mathbb{P}$ and $\mathbb{Q}(z)$, respectively. The new class of complex-valued measures $\mathbb{Q}(z)$ is absolutely continuous with respect to $\mathbb{P}$ and is defined by

$$\frac{d\mathbb{Q}(z)}{d\mathbb{P}}|_t = M_t(z), \tag{13}$$

with

$$M_t(z) = \exp(iz.Y_t + T_t\psi_x(z)), z \in \mathfrak{D} \in \mathbb{C}^d. \tag{14}$$

Theorem 1 generalizes the previous results on an independent time change to the case where the Lévy process and the time change can be correlated. When the leverage effect exists in the original economy, the expectation can be performed under this complex-valued measure proposed by Carr and Wu (2004) as if the economy were devoid of the leverage effect.

4 Foreign equity option pricing under time-changed Lévy processes

The underlying asset of foreign equity options are foreign equities, with the strike price being in either foreign or domestic currency, but with the payoff being transformed into domestic currency based on the exchange rate on expiration. The payoff of a foreign equity option stuck in a foreign currency is given by

$$\mathrm{FEO}_F = F_t(S_t - K_F)^+ \tag{15}$$

where F_t is the exchange rate at time t in domestic/foreign currency, S_t is the stock price in foreign currency on expiration, and K_F is the strike price in the foreign currency. In an alternative form of the foreign equity option, the strike price can also be expressed in domestic currency. This type of option is appropriate to an investor who wishes to make sure that the future payoff from the foreign market is meaningful when converted into his or her own currency. Such an option has the following payoff

$$\mathrm{FEO}_D = (F_t S_t - K_D)^+ \tag{16}$$

where K_D is the strike price in domestic currency. Foreign swap options differ from foreign equity options in providing investors with the right to exchange one foreign asset for another. That is, foreign swap options help investors protect themselves against not only exchange rate fluctuations but also further investment protection. The payoff of a foreign swap option is given by

$$\mathrm{Swap} = F_t(S_t^1 - S_t^2)^+. \tag{17}$$

To investigate the foreign equity option pricing, we consider a market model with one exchange rate and two foreign assets (F_t^1, S_t^2, S_t^3) given by

$$F_t^1 = F_0^1 e^{Y_t^1}, S_t^2 = S_0^2 e^{Y_t^2}, S_t^3 = S_0^3 e^{Y_t^3}, \tag{18}$$

where (F_0^1, S_0^2, S_0^3) denotes the price at time 0, and (Y_t^1, Y_t^2, Y_t^3) is a three dimensional time-changed Lévy process. We specify that under a risk neutral measure $\mathbb{Q}$, the logarithm of the exchange rate and two foreign assets follow a time-changed Lévy process,

$$\begin{pmatrix} Y_t^1 \\ Y_t^2 \\ Y_t^2 \end{pmatrix} = \begin{pmatrix} r_d - r_f \\ r_f \\ r_f \end{pmatrix} t + \begin{pmatrix} W^1_{T_t^1} \\ W^2_{T_t^2} \\ W^3_{T_t^3} \end{pmatrix} - \frac{1}{2} A \begin{pmatrix} T_t^1 \\ T_t^2 \\ T_t^3 \end{pmatrix} + (J_t - \bar{\mu} t) \tag{19}$$

where $(T_t^1, T_t^2, T_t^3)^\top$ is a 3-dimensional random time changes, and $(W^1_{T_t^1}, W^2_{T_t^2}, W^3_{T_t^3})^\top$ is a 3-dimensional Brownian motion. In order to obtain the explicit closed form solution for foreign equity option pricing, we assume that the Brownian, $W_t^i, i = 1, 2, 3$, is independent of each other and the covariance matrix A satisfy $(A)_{ii} = \sigma_i^2$, and $(A)_{ij} = 0, i \neq j$. The second term, $(r_d - r_f, r_f, r_f)^\top t$, is determined by no-arbitrage. The third term, $(W^1_{T_t^1}, W^2_{T_t^2}, W^3_{T_t^3})^\top - \frac{1}{2} A (T_t^1, T_t^2, T_t^3)^\top$, comes from the diffusion, with $\frac{1}{2} A (T_t^1, T_t^2, T_t^3)^\top$ as the concavity adjustment. The last term, $J_t - \bar{\mu} t$, represents the contribution from the jump component, with $\bar{\mu} t$ as the analogous concavity adjustment for J_t. Each jump component in J_t is assumed to be independent. Constant vector $\bar{\mu}$ is determined by the specification of the jump structure J_t. The compound Poisson jump process of Merton (1976) is used in this paper, which has been widely adopted by the finance literature.

Under this process, the last term on the right side of Eq. (19) becomes

$$J_t - \bar{\mu}t = \left(\begin{array}{c} \sum_{i=1}^{N_{1t}} \ln q_{1i} - \lambda_1 (e^{\mu_{1J} + \frac{1}{2}\sigma_{1J}^2} - 1)t \\ \sum_{i=1}^{N_{2t}} \ln q_{2i} - \lambda_2 (e^{\mu_{2J} + \frac{1}{2}\sigma_{2J}^2} - 1)t \\ \sum_{i=1}^{N_{3t}} \ln q_{3i} - \lambda_3 (e^{\mu_{3J} + \frac{1}{2}\sigma_{3J}^2} - 1)t \end{array} \right)$$

where $\{q_{\cdot i}\}$ is a sequence of independent identically distributed nonnegative conditional jump size random variables such that $\ln q_{\cdot}$ is normally distributed with mean $\mu_{\cdot J}$ and variance $\sigma_{\cdot J}^2$, and N_{it} are independent Poisson processes with constant intensity λ_i for $i = 1, 2, 3$. When the stochastic time changes T_t in Eq. (19) are a calendar time t, this Lévy process is just the one used by Huang and Hung (2005) in pricing foreign equity options. When the $\lambda = 0$ and $T_t = t$, the stochastic processes governing the asset dynamics become the Brownian motions which are used by Dravid, Richardson, and Sun (1993) and Kwok and Wong (2000) in pricing foreign equity options. In equation (19), the stochastic time changes can be applied to both the diffusion and jump martingale components. However, in order to show how to incorporate the stochastic volatility into the foreign equity option pricing, we just apply stochastic time changes to the diffusion component.

In order to have a tractable Laplace transform of the random time in this paper, we consider that the instantaneous activity rate follows the mean-reverting square-root process of Heston (1993). Under the risk-neutral measure $\mathbb{Q}$, the activity rate process, therefore satisfies the following stochastic differential equation

$$[v_1(t), v_2(t), v_3(t)]^\top = [\frac{\partial T_t^1}{\partial t}, \frac{\partial T_t^2}{\partial t}, \frac{\partial T_t^3}{\partial t}]^\top, \tag{20}$$

$$dv_i(t) = \kappa_i(\theta_i - v_i(t))dt + \sigma_{v_i}\sqrt{v_i(t)}dB_t^i, i = 1, 2, 3. \tag{21}$$

where $B_t^i, i = 1, 2, 3$, is a standard Brownian motion under $\mathbb{Q}$, which can be correlated with the standard Brownian W_t^i in the return process by $\rho_i dt = E^{\mathbb{Q}}[dW_t^i dB_t^i], i = 1, 2, 3$. κ, θ, and σ_v are respectively speed of adjustment, long-run mean, and volatility of volatility.

4.1 FEO_D options

To price FEO_D options driven by a time-changed Lévy process, we first derive the generalized Fourier transform of the asset return under the risk-neutral measure and then use the efficient fast Fourier transform (FFT) algorithm proposed by Carr and Madan (1999) to compute option prices.

4.1.1 Deriving the Fourier transform

Let $F_t^1 S_t^2$ be the underlying asset price of a FEO_D option at time t, and $F_0^1 S_0^2$ be the price at time 0. Then, we can specify the price process as an exponential affine function of the time-changed Lévy process Y_t,

$$F_t^1 S_t^2 = F_0^1 S_0^2 \exp(Y_t^1 + Y_t^2) \tag{22}$$

Let $s_t = \ln(F_t^1 S_t^2 / F_0^1 S_0^2)$ denote the log return of the asset. Then, by Theorem 1, the generalized Fourier transform of s_t under this specification is given by

$$\begin{aligned}
\phi_{s_t}(z) = E[e^{izs_t}] &= E[\exp\{izr_d t + iz(W^1_{T^1_t} + W^2_{T^2_t}) - iz\frac{1}{2}(T^1_t\sigma_1^2 + T^2_t\sigma_2^2) \\
&\quad + iz(\sum_{i=1}^{N_{1t}} \ln q_{1i} - \lambda_1(e^{\mu_{1J}+\frac{1}{2}\sigma_{1J}^2} - 1)t + \sum_{i=1}^{N_{2t}} \ln q_{2i} - \lambda_2(e^{\mu_{2J}+\frac{1}{2}\sigma_{2J}^2} - 1)t)\}] \\
&= e^{izr_d t} e^{-t(\psi_{1,J-\bar{\mu}} + \psi_{2,J-\bar{\mu}})} \mathscr{L}^z_{(T^1_t, T^2_t)}(\psi_{w^1+w^2}(z) + iz\frac{1}{2}(\sigma_1^2 + \sigma_2^2)), \qquad (23)
\end{aligned}$$

where

$$\begin{pmatrix} \psi_{1,J-\bar{\mu}} \\ \psi_{2,J-\bar{\mu}} \end{pmatrix} = \begin{pmatrix} \lambda_1[iz(e^{\mu_{1J}+\frac{1}{2}\sigma_{1J}^2} - 1) - (e^{iz\mu_{1J}-\frac{1}{2}z^2\sigma_{1J}^2} - 1)] \\ \lambda_2[iz(e^{\mu_{2J}+\frac{1}{2}\sigma_{2J}^2} - 1) - (e^{iz\mu_{2J}-\frac{1}{2}z^2\sigma_{2J}^2} - 1)] \end{pmatrix},$$

$\psi_{w_1+w_2}(z)$ is the characteristic exponent of $W_t^1 + W_t^2$, and $\mathscr{L}^z_{(T^1_t,T^2_t)}(\cdot)$ represents the Laplace transform of the stochastic time (T_t^1, T_t^2) under a new complex-valued measure $\mathbb{Q}(z)$. The measure $\mathbb{Q}(z)$ is absolutely continuous with respect to the risk-neutral measure $\mathbb{Q}$ and is defined by

$$\frac{d\mathbb{Q}(z)}{d\mathbb{Q}} = \exp\{iz(W^1_{T^1_t} + W^2_{T^2_t} - \frac{1}{2}(T^1_t\sigma_1^2 + T^1_t\sigma_2^2)) + T^1_t\psi_{w^1}(z) + T^2_t\psi_{w^2}(z) + \frac{1}{2}iz(T^1_t\sigma_1^2 + T^2_t\sigma_2^2)\}. \qquad (24)$$

Since the Laplace transform of the time change in Eq. (23) is defined under a new measure $\mathbb{Q}(z)$, we need to obtain the instantaneous activity rate processes under $\mathbb{Q}$. By the Girsanov's Theorem, under the measure $\mathbb{Q}(z)$, the diffusion part of $v_i(t)$ is unchanged, while the drift part of $v_i(t)$ is changed into

$$\mu^{\mathbb{Q}(z)}_{v_i}(t) = \kappa_i(\theta_i - v_i(t)) + iz\sigma_i\sigma_{v_i}\rho_i v_i(t), i = 1, 2, 3.$$

If, we assume that the three activity rates are independent of each other, then the Laplace transform in Eq. (23) becomes a product of two Laplace transform, one for the stochastic time T_t^1, and the other for the stochastic time T_t^2,

$$\mathscr{L}^z_{(T^1_t,T^2_t)}(\psi_{w^1+w^2}(z) + iz\frac{1}{2}(\sigma_1^2 + \sigma_2^2)) = \mathscr{L}^z_{T^1_t}(\psi_{w^1}(z) + iz\frac{1}{2}\sigma_1^2)\mathscr{L}^z_{T^2_t}(\psi_{w^2}(z) + iz\frac{1}{2}\sigma_2^2). \qquad (25)$$

By Proposition 1 of Carr and Wu (2004), the Laplace transform of the random time, $T_t^i = \int_0^t v_i(s-)ds$, i=1, 2, is an exponential-affine function of the Markov process $v_i(t)$:

$$\mathscr{L}^z_{T^i_t}(\lambda_i) = E^{\mathbb{Q}(z)}[e^{-\lambda_i T^i_t}] = \exp(-b_i(t)v_i(0) - c_i(t)), i = 1, 2, \qquad (26)$$

where

$$\lambda_1 = \psi_{w^1}(z) + iz\frac{1}{2}\sigma_1^2 = \frac{1}{2}(z^2\sigma_1^2 + iz\sigma_1^2), \qquad (27)$$

$$\lambda_2 = \psi_{w^2}(z) + iz\frac{1}{2}\sigma_2^2 = \frac{1}{2}(z^2\sigma_2^2 + iz\sigma_2^2), \qquad (28)$$

and $b_i(t)$ and $c_i(t)$ can be obtained by solving the following ordinary differential equations:

$$b_i'(t) = \lambda_i - (\kappa_i - iz\sigma_i\sigma_{v_i}\rho_i)b_i(t) - \frac{1}{2}\sigma_{v_i}^2 b_i^2(t), \qquad (29)$$

$$c_i'(t) = b_i(t)\kappa_i\theta_i, \qquad (30)$$

with the boundary conditions $b_i(0) = 0$ and $c_i(0) = 0$. For this one-factor case, the ordinary differential equations can be solved analytically,

$$b_i(t) = \frac{2\lambda_i(1 - e^{\eta_i t})}{2\eta_i - (\eta_i - \kappa_i^*)(1 - e^{-\eta_i t})}; \tag{31}$$

$$c_i(t) = \frac{\kappa_i\theta_i}{\sigma_{v_i}^2}[2\ln(1 - \frac{\eta_i - \kappa_i^*}{2\eta_i}(1 - e^{-\eta_i t})) + (\eta_i - \kappa_i^*)t], \tag{32}$$

with

$$\eta_i = \sqrt{(k_i^*)^2 + 2\sigma_{v_i}^2\lambda_i}, \ \kappa_i^* = \kappa_i - iz\rho_i\sigma_i\sigma_{v_i}. \tag{33}$$

4.1.2 Fast Fourier transform of option prices

It is convenient to represent the time-0 value of the FEO_D call option at maturity t as

$$C(k) = e^{-r_d t}E^{\mathbb{Q}}(e^{s_t} - e^k)1_{s_t \geq k}, \tag{34}$$

where $s_t = \ln F_t^1 S_t^2 / F_0^1 S_0^2$, $k = \ln K/F_0^1 S_0^2$, and $C(k) = C(K)/F_0^1 S_0^2$. Note that we drop the subscript D in the strike price K_D and maturity t as no confusion shall occur. In the following, we shall focus on computing the relative call price of $C(k)$, and the absolute call option price $C(K_D, t)$ can be easily obtained by multiplying it by the spot price $F_0^1 S_0^2$.

Let $z = z_r + iz_i, z_r, z_i \in \mathbb{R}$, the generalized Fourier transform of FEO_D option price $C(k)$ is

$$\begin{aligned}
\mathscr{G}(z) &= \int_{-\infty}^{\infty} e^{izk}C(k)dk = \int_{-\infty}^{\infty} e^{izk}E^{\mathbb{Q}}[e^{-r_d t}(e^{s_t} - e^k)1_{s_t \geq k}dk] \\
&= e^{-r_d t}E^{\mathbb{Q}}[\int_{-\infty}^{s} e^{izk}(e^{s_t} - e^k)dk] \\
&= e^{-r_d t}E^{\mathbb{Q}}[(\frac{e^{izk}e^{s_t}}{iz} - \frac{e^{(iz+1)k}}{iz+1})_{k=-\infty}^{k=s_t}]
\end{aligned} \tag{35}$$

For e^{izk} to be convergent at $k = -\infty$, the imaginary part of z needs $z_i < 0$. Under $z_i < 0$, the transform for the option price becomes

$$\mathscr{G}(z) = e^{-r_d t}E^{\mathbb{Q}}[\frac{e^{(1+iz)s_t}}{iz} - \frac{e^{(1+iz)s_t}}{iz+1}] = e^{-r_d t}\frac{\phi_{s_t(z-i)}}{iz(iz+1)}. \tag{36}$$

Given that $\mathscr{G}(z)$ is well defined, the option price is obtained via the inversion formula:

$$C(k) = \frac{1}{2\pi}\int_{iz_i-\infty}^{iz_i+\infty} e^{-izk}\mathscr{G}(z)dz = \frac{e^{z_i k}}{\pi}\int_0^{\infty} e^{-iz_r k}\mathscr{G}(z_r + iz_i)dz_r, \tag{37}$$

which can be approximated on a finite interval by

$$C(k) \approx \hat{C}(k) = \frac{e^{z_i k}}{\pi}\sum_{j=1}^{N} e^{-iz_r(j)k}\mathscr{G}(z_k(j) + iz_i)\Delta z_r \tag{38}$$

where $z_r(j)$ are the nodes of z_r and Δz_r is the spacing between nodes.

Following Carr and Madan (1999), we set $z_r(j) = \eta(j-1)$, $k_u = -b + \lambda(u-1), u = 1, \cdots, N$, and require $\eta\lambda = 2\pi/N$. Then we can write our call FEO_D option price as:

$$C(k_u) = \frac{e^{-z_i k_u}}{\pi}\sum_{j=1}^{N} e^{-i\lambda\eta(j-1)(u-1)}e^{ib\eta(j-1)}\mathscr{G}(\eta(j-1)+iz_i)\frac{\eta}{3}(3+(-1)^j-\delta_{j-1}), \ u = 1, \cdots, N, \tag{39}$$

where δ_j is the Kronecker delta function that is unity for $j = 0$ and zero otherwise. For the more discussions about computing the Fourier inversions and pricing option using fast Fourier transform, see Wu (2008).

4.2 Foreign swap options

The payoff of a foreign swap option depends on the value of three underlying assets, and the value is given by

$$\begin{aligned}\text{Swap} &= e^{-r_d t}E^{\mathbb{Q}}(F_t^1 S_t^2 - F_t^1 S_t^3)^+ \\ &= e^{-r_d t}E^{\mathbb{Q}}(F_0^1 S_0^2 e^{Y_t^1+Y_t^2} - F_0^1 S_0^3 e^{Y_t^1+Y_t^3})^+. \end{aligned} \tag{40}$$

If the underlying assets are modelled by the exponentials of Lévy processes, then the sum $F_t^1 S_t^2 - F_t^1 S_t^3$ will no longer be a Lévy process, and the Fourier method used for FEO_D option pricing can not be used directly. The Monte Carlo method can be used to price foreign swap options. Under the risk-neutral measure, the dynamics of the stock price under time-changed Lévy process are:

$$dS_t = (r - \bar{\mu})S_t dt + \sigma S_t dW_{T_t} + S_t(q-1)dN_t \tag{41}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma_v \sqrt{v_t}(\rho dB_t + \sqrt{1-\rho^2}dW_t). \tag{42}$$

Eq. (41) gives the dynamics of the stock price: $r - \bar{\mu}$ is the risk neutral drift with, $r = r_f$, for the foreign stock price, $r = r_d - r_f$, for the foreign exchange, $\bar{\mu} = \lambda(\exp(\mu_J + \frac{1}{2}\sigma_J^2) - 1)$, and $T_t = \int_0^t v(s)ds$ are the stochastic time changes. Eq. (42) gives the evolution of the activity rate process which follows the square-root process. W_t and B_t are two independent Brownian motion processes, and ρ represents the instantaneous correlation between the return process and volatility process. So far, various time-discretization and simulation schemes have been proposed to simulate the processes Eq. (41) and (42). For more discussions about using simulation method for option pricing under stochastic volatility and Lévy processes, see Broade and Kaya (2006) and Cont and Tankov (2004).

5 Numerical analysis

In this section, numerical analysis is performed to obtain an insight into the influences of volatility of volatility, the long-run mean, the mean and variance of jumps, and the arrival intensity of jumps from both the foreign equity and exchange rate on FEO_D option prices. The following values are set and will be used in the numerical analysis: $r_d = 0.03$, $r_f = 0.05$, $\kappa_1 = 1.5$, $\kappa_2 = 2$, $\theta_1 = 0.02$, $\theta_2 = 0.03$, $\sigma_{v_1} = 0.2$, $\sigma_{v_2} = 0.3$, $\sigma_1 = 1$, $\sigma_2 = 1$, $\mu_{1J} = 0.3$, $\mu_{2J} = 0.2$, $\sigma_{1J} = 0.2$, $\sigma_{2J} = 0.4$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\rho_1 = -0.5$, $\rho_2 = -0.5$, $v_1(0) = 0.001$, $v_2(0) = 0.003$, $F_0 = 1$, $S_0 = 2.2$, $K_D = 2$, $T = 0.5$. In applying fast fourier transform for pricing a European call foreign equity option, following Carr and Madan (1999) we set the $z_i = -1.25$, $N = 4096$, $c = 600$, $\eta = c/N$, $b = \pi/\eta$, and $\lambda = 2b/N$.

1) The impact of the long-run mean θ and volatility of volatility σ_v on the FEO_D prices

Figure 1 illustrates FEO_D option prices under different long-run mean θ and volatility of volatility σ_v. In our model, the variance from both the foreign equity and exchange rate drifts toward a long-run mean of θ, with mean-reversion speed determined by κ. We know that a higher variance $v(t)$ raises the prices of all options, just as it does in the Black-Scholes model. Hence, an increase in the average variance θ increases the prices of FEO_D. Figure 1 shows that the FEO_D option price is an increasing function of θ. The parameter σ_v controls the volatility of volatility. σ_v increases the kurtosis of spot returns and creates two fat tails in the distribution of spot returns. As Heston (1993) showed that this has effect of raising far-in-the-money and far-out-of-money option prices and lowering in-the-money and out-of-the-money option prices. Hence, Figure 1 displays that the FEO_D option price is a decreasing function of σ_v.

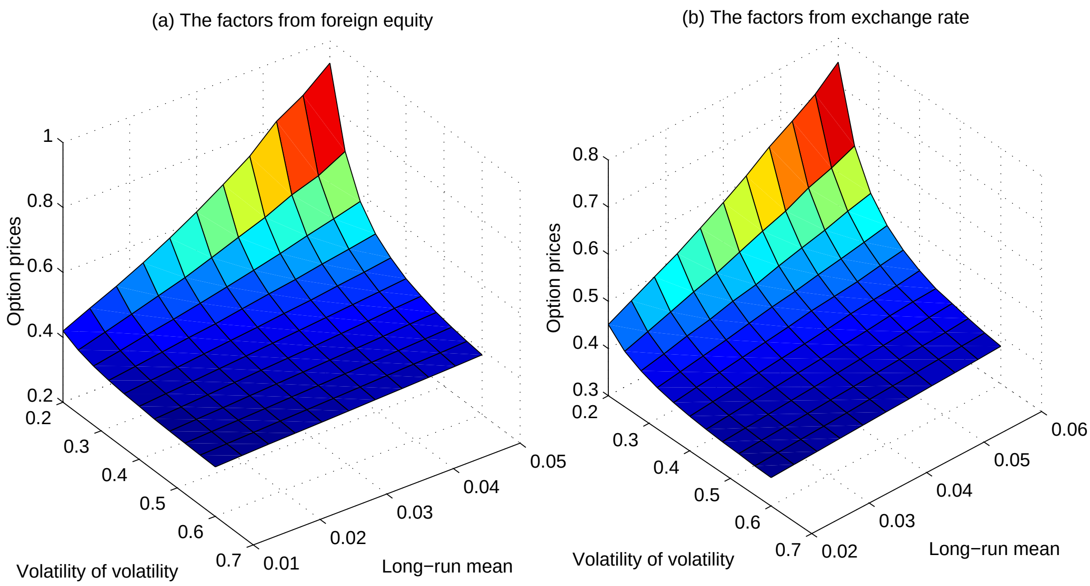

Figure 1 The impact of the long-run mean θ and the volatility of volatility σ_v on the $\mathrm{FEO_D}$ option price.

2) The impact of the mean jump size μ_J and the jump intensity λ on the FEO_D prices

Figure 2 displays FEO_D option prices under different mean jump size μ_J and jump intensity λ. FEO_D option price is an increasing function of λ and μ_J. The possibility of large upside benefits increases with large jump intensity and mean jump size, but the nonlinearity of payoffs of FEO_D options limits the down-side loss; consequently, FEO_D options are valuable under large λ and μ_J.

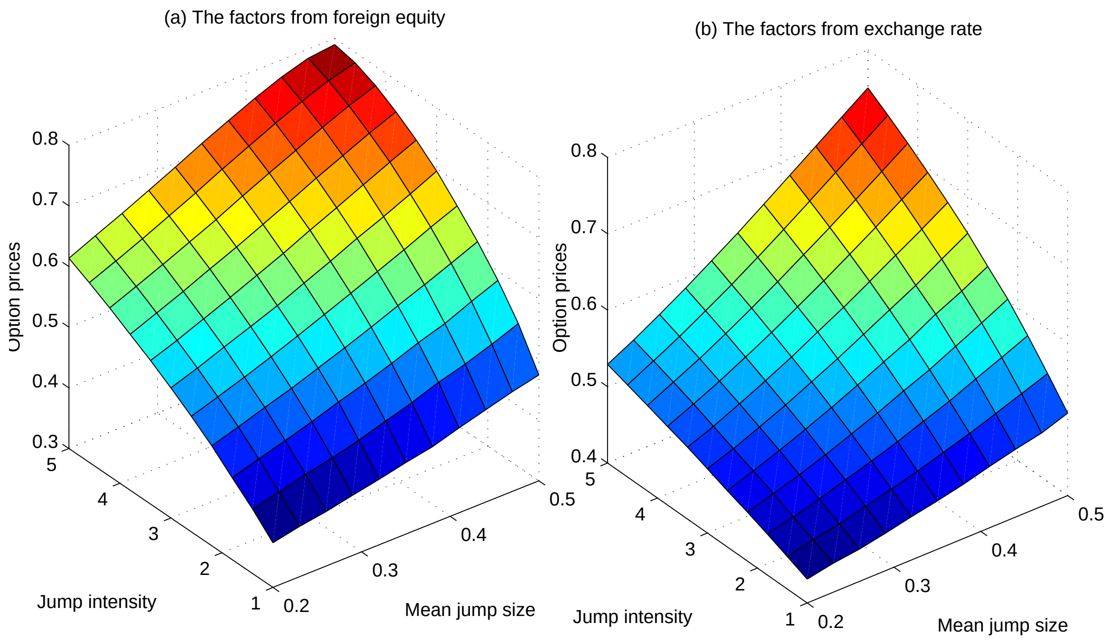

Figure 2 The impact of the mean jump size μ_J and the jump intensity λ on the $\mathrm{FEO_D}$ option price.

3)The impact of the jump size volatility v_J and the jump intensity λ on the $\mathrm{FEO_D}$ option prices

Figure 3 shows $\mathrm{FEO_D}$ option prices under different jump size volatility σ_J and jump intensity λ. Figure 3 displays that the $\mathrm{FEO_D}$ option price is an increasing function of σ_J and λ. The reasoning behind the phenomenon is the same as the 2). The effects of mean jump size, jump size volatility, and jump intensity on the $\mathrm{FEO_D}$ option prices obtained in this paper are the same as Huang and Hung (2005).

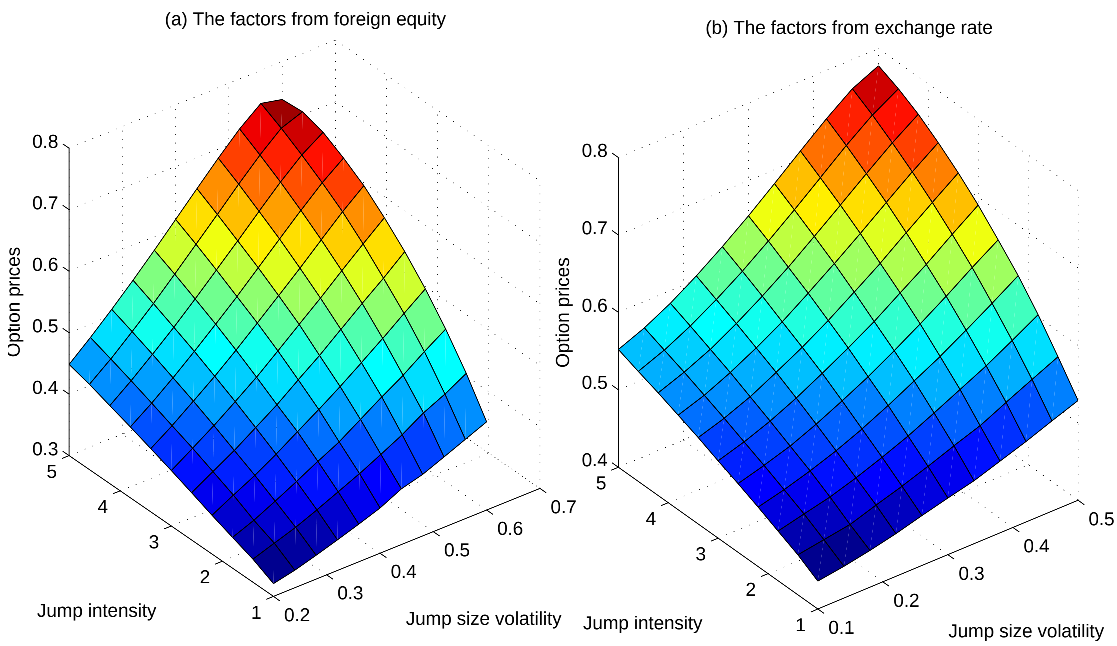

Figure 3 The impact of the variance of jumps v_J and the jump intensity λ on the $\mathrm{FEO_D}$ option price.

Figure 1-6 show the impact of various Lévy process parameters θ, σ_v, μ_J, σ_J, and λ on the $\mathrm{FEO_D}$ option prices. Our numerical results show that all the factors such as long-run mean, volatility of volatility, mean jump size, jump size volatility, and jump intensity from both the foreign equity and the exchange rate have significant impact on the $\mathrm{FEO_D}$ option prices. The numerical results show that our proposal to incorporate the stochastic volatility into foreign equity option pricing model is necessary and this can help us to model the option prices more precisely.

6 Conclusions

Foreign equity options are contingent claims where the payoff is determined by an equity in one currency but the actual payoff is done in another currency. Foreign equity option pricing requires us to consider both the foreign asset price process and the exchange rate process altogether. Huang and Hung (2005) priced foreign equity options using the Lévy processes. In Huang and Hung's paper, they considered jumps in the foreign asset prices and exchange rates and assumed the volatility as constant. In this paper, we propose a general foreign equity option pricing framework that unifies the vast foreign equity option pricing literature and captures the three key pieces of evidence on financial securities: (1) jumps, (2) stochastic volatility, and (3) the leverage effect. Under our framework, the exchange rate and foreign asset prices are governed by time-changed Lévy processes, and the closed form pricing formula for the foreign equity option is obtained through the use of characteristic function technology.

References

[1] Bakshi, G., Cao, C., & Chen, Z. (1997). Empirical performance of alternative option pricing models. *Journal of Finance*, 52, 2003-2049.

[2] Barndorff-Nielsen, O. E. (1998). Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2, 41-68.

[3] Bates, D. (1996a). Dollar jump fears 1984–1992: distributional abnormalities implicit in foreign currency futures options. *Journal of International Money and Finance*, 15, 65-93.

[4] Bates, D. (1996b). Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options. *Review of Financial Studies*, 9, 69-107.

[5] Bates, D. (2000). Post-'87 Crash fears in S&P 500 Futures option market. *Journal of Econometrices*, 94, 181-238.

[6] Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economics*, 81, 637-659.

[7] Broade, M., & Kaya, O. (2006). Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations Research*, 2, 217-231.

[8] Carr, P., & Madan, D. (1999). Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2, 61-73.

[9] Carr, P., Geman, H., Madan, D., & Yor, M. (2002). The fine structure of asset returns: An empirical investigation. *Journal of Business*, 75, 305-332.

[10] Carr, P., & Wu, L. R. (2003). Finite moment log stable process and option pricing. *Journal of Finance*, 58, 753-777.

[11] Carr, P., & Wu, L. R. (2004). Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 71, 113-141.

[12] Carr, P., & Wu, L. R. (2007). Stochastic skew in currency options. *Journal of Financial Economics*, 86, 213-247.

[13] Chen, Z., & Kulperger, R. (2006). Minimax pricing and Choquet pricing. *Insurance: Mathematics and Economics*, 38, 518-528.

[14] Cont, R., & Tankov, P. (2004). Financial Modelling with Jump Processes. Chapman & Hall/CRC Press.

[15] Dravid, A., Richardson, M., & Sun, T. (1993). Pricing foreign index contingent claims: An application to Nikkei index warrants. *Journal of Derivatives*, 1, 33-51.

[16] Duan, J., & Wei, J. Z. (1999). Pricing foreign-currency and cross-currency options under GARCH. *Journal of Derivatives*, 7, 51-63.

[17] Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and option pricing for affine jump-diffusions. *Econometrica*, 68, 1343-1376.

[18] Eberlein, E., Keller, U., & Prause, K. (1998). New insights into smile, mispricing, and value at risk: The hyperbolic model. *Journal of Business*, 71, 371-406.

[19] Eraker, B. (2004). Do stock prices and volatility jump? Reconciling evidence from spot and option prices. *Journal of Finance*, 59, 1367-1403.

[20] Eraker, B., Johannes, M., & Poison, N. (2003). The impact of jumps in equity index volatilityand returns. *Journal of Finance*, 58, 1269-1300.

[21] Geman, H., Madan, D. B., & Yor, M. (2001). Time changes for Lévy processes. *Mathematical Finance*, 11, 79-96.

[22] Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6, 327-344.

[23] Ho, T. S., Stapleton, R. C., & Subrahmanyam, M. G. (1995). Correlation risk, cross-market derivative products and portfolio performance. *Journal of European Financial Management*, 1, 105-124.

[24] Huang, S. C., & Hung, M. W. (2005). Pricing foreign equity options under lévy processes. *The Journal of Futures Markets*, 25, 917-944.

[25] Johnson, G., & Schneeweis, T. (1994). Jump-diffusion processes in the foreign exchange markets and the release of macroeconomic news. *Computational Economics*, 7, 309-329.

[26] Jorion, P. (1988). On jump processes in the foreign exchange and stock markets. *Review of Financial Studies*, 1, 427-445.

[27] Johnson, T. C. (2002). Volatility, momentum, and time-varying skewness in foreign exchange returns. *Journal of Business and Economic Statistics*, 20, 390-411.

[28] Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science*, 48, 1086-1101.

[29] Kwok, Y. K., & Wong, H. Y. (2000). Currency-translated foreign equity options with path dependent features and their multi-asset extensions. *International Journal of Theoretical and Applied Finance*, 3, 257-278.

[30] Lau, J. W., & Siu, T. K. (2008). On option pricing under a completely random measure via a generalized Esscher transform. *Insurance: Mathematics and Economics*, 43, 99-107.

[31] Madan, D., & Milne, F. (1991). Option pricing with VG martingale components. *Mathematical Finance*, 1, 39-56.

[32] Madan, D. B., Carr, P. P., & Chang, E. C. (1998). The variance gamma process and option pricing. *European Finance Review*, 2, 79-105.

[33] Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3, 125-144.

[34] Reiner, E. (1992). Quanto mechanics. *Risk*, 5, 59-63.

[35] Titchmarsh, E. C. (1975). Theory of Fourier Integrals. London: Oxford University Press.

[36] Toft, K. B., & Reiner, E. S. (1997). Currency-translated foreign equity options: The American case. *Advances in Futures and Options Research*, 9, 233-264.

[37] Wei, J. Z. (1992). Pricing forward contracts and options on foreign assets: Theories and empirical tests. PhD thesis, University of Toronto.

[38] Wu, L. R. (2008). Modeling financial security returns using Lévy process. In: Birge, J., Linetsky, V. (Eds.). Handbook in Operations Research and Management Science: Financial Engineering. Elsevier.

[39] Xu, W. D., Wu, C. F., Xu, W. J., & Li, H. Y. (2009). A jump-diffusion model for option pricing under uncertainty environments. *Insurance: Mathematics and Economics*, 44, 337-344.

[40] Xu, W. D., Wu, C. F., Xu, W. J., & Li, H. Y. (2010). Dynamic asset allocation with jump risk. *The Journal of Risk*, 12, 29-44.

[41] Xu, W. D., Wu, C. F., & Li, H. Y. (2010). Robust general equilibrium under stochastic volatility model. *Finance Research Letters*, doi: 10.1016/j.frl.2010.05.002.

[42] Xu, W. J., Xu, W. D., Li, H. Y., & Zhang, W. G. (2010). A study of Greek letters of currency option under uncertainty environments. *Mathematical and Computer Modelling*, 51, 670-681.