Mutually Exclusive Rival Options: Risk Evaluation

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Abstract

We evaluate the risk aspects of a complex portfolio of real options to switch or divest in a duopoly context. After summarizing the basic model, covering three sequences, four thresholds, and seven strategic and rival options, we look at four risk elements: delta, vega, rho, and epsilon, the conventional option Greeks. The value function of both the leader and follower is most sensitive to revenue variations (delta), which we view in terms of sensitivities (to incremental, and percentage changes) and partial derivatives. We ask what are the plausible and appropriate risk avoidance actions for each of these risk exposures. We are tentative about which risk measurements are most useful for risk managers of these complex real option portfolios, except that the risks of the present value of operations is unlikely to be the dominate real concern.

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1. Introduction

We investigate the plausible risk elements regarding the choice of divesting or switching to a lower operating cost technology in adjusting to a declining market in a game-theoretic world of two comparable rivals. Following the approach of Adkins et al. (2023) (MERO) there are three significant sequential changes over three regimes (stages) in this duopoly game[2]: (i) market share changes arising from the individual player actions, (ii) revenue changes due to a declining market size and a stochastically evolving price, and (iii) revenue changes arising from investing in an alternative technology having a more appropriate cost structure.[3] The divest and switch options are mutually-exclusive (joint). While the first-mover has a salvage value advantage, the second-mover has a temporary market share advantage after the leader downsizes. There are partly analytical solutions to the eleven equations in the model, including four action thresholds which form the boundaries for the five regimes. We evaluate the overall risk (change in the value function) for each firm, as inputs change, but the specific rival and standard option values have complex risk profiles.

Appropriate extensions to the MERO are to propose management actions (hedging, games) at each stage, with the current and prospective parameter values. After covering the appropriate areas of risk identification, measurement and evaluation, the plausible actions include dealing with external parties or internal rivals. External or exogenous such as the derivative markets, governments, regulators, and the legal system offer several fields for actions.[4] Actions involving internal (endogenous) actual or potential rivals include collusion, industry marketing programs, and pooled risk (both price and quantity) sharing. These actions are focused on the value functions (total value for the L or F), which consist of the value of operations, and the value of the respective portfolio of options. As a side issue, the value of each option is also considered separately. Note that each

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[2] The complete model derivation is shown in Appendix B. Appendix A shows the assumed base case parameter values and model solutions.

[3] Many other configurations of market shares, salvage values, and revenue and operating cost changes can be designed, some suitable for our specific model, others requiring model redesign, appropriate for future research.

[4] Borenstein, S., J. Bushnell and E. Mansura (2023), "The Economics of Electricity Reliability", *Journal of Economic Perspectives*, 37:4, page 196, price risk "can provide a stronger incentive for retailers to procure, or hedge, their energy in forward markets. Some retailers physically hedge this risk by vertically integrating between generation and retailing functions. Others, however, benefit from bankruptcy laws by offering a fixed retail price and not hedging."

risk (of a downside value loss) might also be considered an upside opportunity. Probably a single measurement like a "real value at risk" is not sufficient to view these risks/opportunities.

The switch option is treated like a call option and the divest option is treated like a put option. The consideration of such alternative options was first raised by Dias (2004) (who provided solutions using finite differences) and developed further by Décamps et al. (2006) for a monopoly market. Décamps et al. (2006) study irreversible investments in alternative projects and show that when firms hold the option to switch from a smaller scale to a larger scale project, a hysteresis region between the investment region can persist even if the uncertainty of the output price increases. Nishihara and Ohyama (2008) model R&D competition in alternative technologies. There are other applications of the theory of mutually exclusive options, such as Bakke et al. (2016), and of real competitive strategies, such as Comincioll et al. (2020), but apparently not joint competitive strategies. Adkins et al. (2022, 2023) extend the mutually exclusive option framework to a duopoly market, thus considering the effect of competition on thresholds and values.

Joaquin and Butler (2000) consider the first mover advantage of lower operating costs. Tsekrekos (2003) suggests both temporary and pre-emptive permanent market share advantages for the leader in a sequential investment pattern. Paxson and Pinto (2003) model a leader with an initial market share advantage, which then evolves as new customers arrive (birth) and existing customers depart (death)[5]. Paxson and Melmane (2009) provide a two-factor model where the leader starts with a larger market share, applied to show that (by foresight) Google was likely to be undervalued compared to Yahoo at the Google IPO. Bobtcheff and Mariotti (2013) consider a pre-emptive game of two innovative competitors, whose existence may be revealed only by first mover investment. Azevedo and Paxson (2014) review the literature on developing such real option games.

Perhaps advances in technology will inspire first movers to switch technology. But what if the first mover experiences a temporary loss in market share (or editors who resist articles like this being composed by Claude rather than by aging professors who are slow but never "hallucinate)?[6] Due

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[5] Appendix G reviews the innovations in these two articles regarding some analytical partial derivatives (delta, and alpha), and discussions of the respective leader/follower choices and actions.

[6] Korinek, A. (2023), "Generative AI for Economic Research: Use Cases and Implications for Economists", *Journal of Economic Literature*, 61: 1281-1317.

to both pollution concerns and competition from natural gas, some coal power plants are being shut down, possibly lacking cheaper emission control. Given the possible global warming, some U.S. states, such as Arizona and California, have experienced water shortages, and are considering alternative actions (limiting the use of water for agriculture, restricting water flows of the Colorado river in/out of Lake Powell or Lake Mead) which have different implications for the states competing in using that water.

There are at least four plausible views of these options: **I** sensitivities to percentage changes in the parameter values; **II** partial derivatives, analytical and numerical; **III** tables and figures of the option values across a range of parameter values; and **IV** comparison of set alternative values (an arbitrary +/- from the base parameter values). Which risk expression is most useful for the Chief Real Options Manager CROM ?

Our key contribution is the consideration of the overall risk exposure to a host of changing input parameter values, showing the composition of that risk (on the present value of operations, and on each separate option). Possibly unique are the mostly analytical partial derivatives (delta, vega, rho, epsilon) of the value functions, with illustrative numerical results.

The critical findings are (i) that delta is the most important risk exposure for this set of parameter values and for this particular model; (ii) switching, divestment, and rival options have different sensitivities to revenue, volatility, rate and yield changes, providing a rich field for decision analysis; and (iii) since the signs and dimensions of risk exposure for the values of the leader and the follower change over different regimes (revenue levels), risk evaluation and hedging are challenging activities, offering lots of possibilities for interesting future research.

The rest of the paper is organized as follows. Section 2 summarizes the divestment and the switching models for the joint formulation. Section 3 shows the numerical results, discusses some of the option characteristics, and provides a sensitivity analysis. Section 4 concludes the work and provides some suggestions for further research and applications.

2. Mutually Exclusive Option Duopoly Model

We consider a duopoly market with two active and ex-ante symmetric rationale firms (holding the same parameter values) operating with an incumbent high operating cost technology, referred to as policy X, producing the same product output in perpetuity with a market price $p(t)$ subject to uncertainty and facing a declining market volume $q(t)$. Each firm holds a perpetual option to abandon production and receive a salvage value while in the incumbent X stage. A first-mover divestment advantage exists such that first-mover receives the full salvage amount Z while the second-mover receives only the partial amount λZ where $0 \leq \lambda < 1$. Once the divestment option is exercised, the firm exits the market which is referred to as policy O. Alternatively, while operating at X, each firm holds a perpetual option to switch to a more appropriate lower operating cost technology referred to as policy Y, but incurs a positive irrecoverable investment cost denoted by K. Note that there is a salvage value when firms switch from policy X to policy Y or divest, with no divestment after the switch.

The two players in the duopoly game are designated the leader and the follower, referred to as L and F, respectively. This implies that the leader is always first to enact a policy change from X to either O or Y, and that the follower always enacts the identical policy change as the leader but subsequently. $D_{F|Y,X}$ denotes the market share of the follower given that the leader is pursuing policy Y and the follower policy X.

We assume the market price p follows a gBm process described by:

$$dp = \alpha p dt + \sigma p dW \tag{1}$$

where α is the constant instantaneous conditional expected price change per unit of time, σ is its constant instantaneous conditional standard deviation per unit of time, and dW is the increment of a standard Wiener process. For convergence purposes $\delta = r - \alpha > 0$, where r is the riskless interest rate and δ the convenience yield. The market volume flow q is described by:

$$dq = -\theta q dt \tag{2}$$

where $\theta > 0$ denotes a known constant market depletion rate. Using Ito's lemma, the firm value G satisfies the differential equation with $v = pq$, :

$$\frac{1}{2}\sigma^2 v^2 \frac{\partial^2 G(v)}{\partial v^2} + (r - \delta - \theta) v \frac{\partial G(v)}{\partial v} - rG(v) + D(v - f) = 0 \tag{3}$$

with the following solution:

$$G(v) = A_1 v^{\beta_1} + A_2 v^{\beta_2} + \frac{Dv}{\delta+\theta} - \frac{Df}{r} \tag{4}$$

where:

$$\beta_{1,2} = \left(\frac{1}{2} - \frac{r-\delta-\theta}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{r-\delta-\theta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{5}$$

$A_1 \geq 0$ and $A_2 \geq 0$ are two unknown variables to be determined from the context, (both are relevant for values of v between the divest and switching joint thresholds).

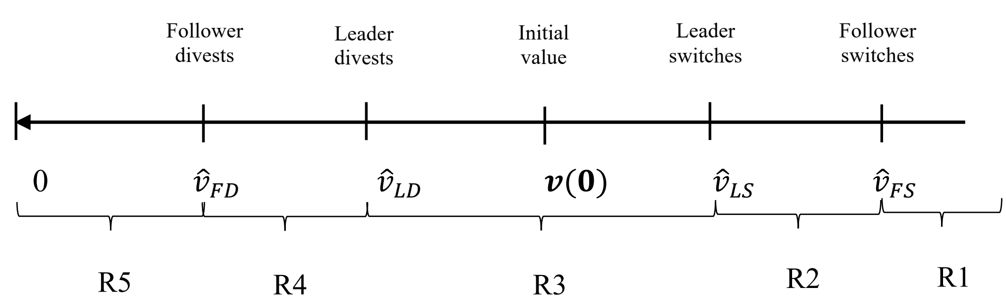
The nature of the duopoly game is that the leader always commits to a policy change ahead of the follower. Further, for the current context, the switch threshold is always greater than the divestment threshold, Décamps et al. (2006). We denote the switching thresholds for the leader and follower by $\hat{v}_{LS}$ and $\hat{v}_{FS}$, respectively, the divestment thresholds for the leader and follower by $\hat{v}_{LD}$ and $\hat{v}_{FD}$, respectively, so the threshold order, with the initial revenue value $v(0)$ within the leader's thresholds, is:

$$\hat{v}_{FD} < \hat{v}_{LD} < v(0) < \hat{v}_{LS} < \hat{v}_{FS} \tag{6}$$

The revenue regimes are illustrated in Figure 1.

Figure 1: Leader and Follower Thresholds for a Random Revenue (v)

This figure shows the revenue (v) regimes and the switch and divest thresholds, R1 is the region where both firms have switched to policy *Y*; R2 is the region where the leader has switched to policy *Y* and the follower operates with policy *X*; R3 is the region where both firms operate with policy *X*; R4 is the region where the leader has divested and the follower operates with policy X; and R5 is the region where both firms have divested.



The value function under the joint formulation for the leader is:

$$V_L(v) = \begin{cases} D_{L|Y,Y}\left(\frac{v}{\delta+\theta} - \frac{f_Y}{r}\right) & if\ v \geq \hat{v}_{FS}\ R1 \\ D_{L|Y,X}\left(\frac{v}{\delta+\theta} - \frac{f_Y}{r}\right) + A_{1LSS}v^{\beta_1} & if\ \hat{v}_{LS} \leq v < \hat{v}_{FS}\ R2 \\ D_{L|X,X}\left(\frac{v}{\delta+\theta} - \frac{f_X}{r}\right) + A_{1LS}v^{\beta_1} + A_{2LD}v^{\beta_2} & if\ \hat{v}_{LD} < v < \hat{v}_{LS}\ R3 \\ Z & if\ v \leq \hat{v}_{LD}\ R4 \end{cases} \tag{7}$$

The value function under the joint formulation for the follower is:

$$V_F(v) = \begin{cases} D_{F|Y,Y}\left(\frac{v}{\delta+\theta} - \frac{f_Y}{r}\right) & if\ v \geq \hat{v}_{FS}\ R1 \\ D_{F|Y,X}\left(\frac{v}{\delta+\theta} - \frac{f_X}{r}\right) + A_{1FS}v^{\beta_1} + A_{2FD}v^{\beta_2} & if\ \hat{v}_{LS} \leq v < \hat{v}_{FS}\ R2 \\ \quad D_{F|X,X}\left(\frac{v}{\delta+\theta} - \frac{f_X}{r}\right) + A_{1FS}v^{\beta_1} + A_{2FD}v^{\beta_2} & \\ \ +A_{1FSS}v^{\beta_1} + A_{2FDD}v^{\beta_2} & if\ \hat{v}_{LD} < v < \hat{v}_{LS}\ R3 \\ D_{F|O,X}\left(\frac{v}{\delta+\theta} - \frac{f_X}{r}\right) + A_{1FS}v^{\beta_1} + A_{2FD}v^{\beta_2} & if\ \hat{v}_{FD} \leq v < \hat{v}_{LD}\ R4 \\ \lambda Z & if\ v < \hat{v}_{FD}\ R5 \end{cases} \tag{8}$$

The boundary conditions in the thresholds (value matching and smooth pasting) along with value functions (7) and (8) create a set of equations from which the solutions to the unknown thresholds and coefficients are obtainable. There are four unknown thresholds signalling the leader's and follower's switching and divesting policies, $\hat{v}_{LS}$, $\hat{v}_{FS}$, $\hat{v}_{LD}$, and $\hat{v}_{FD}$, respectively, four unknown option coefficients associated with the leader's and follower's switching and divesting policies, A_{1LS}, A_{2LD}, A_{1FS}, and A_{2FD}, respectively, and three unknown rival option coefficients associated with the leader's value when the follower switches, A_{1LSS}, with the follower's value accruing when the leader switches, A_{1FSS}, and divests, A_{2FDD}.

The solutions for the follower's two thresholds $\hat{v}_{FS}$ and $\hat{v}_{FD}$ are:

$$\hat{v}_{FD}{}^{\beta_2}\left(\hat{v}_{FS}\frac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta}\frac{\beta_1-1}{\beta_1} - \frac{D_{F|Y,Y}f_Y - D_{F|Y,X}f_X}{r} - (K-\lambda Z)\right) - \hat{v}_{FS}{}^{\beta_2}\left(\lambda Z - \frac{D_{F|O,X}\hat{v}_{FD}}{\delta+\theta}\frac{\beta_1-1}{\beta_1} + \frac{D_{F|O,X}f_X}{r}\right) = 0 \tag{9}$$

$$\hat{v}_{FD}{}^{\beta_1}\left(\hat{v}_{FS}\frac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta}\frac{\beta_2-1}{\beta_2} - \frac{D_{F|Y,Y}f_Y - D_{F|Y,X}f_X}{r} - (K-\lambda Z)\right) - \hat{v}_{FS}{}^{\beta_1}\left(\lambda Z - \frac{D_{F|O,X}\hat{v}_{FD}}{\delta+\theta}\frac{\beta_2-1}{\beta_2} + \frac{D_{F|O,X}f_X}{r}\right) = 0 \tag{10}$$

The follower's switching and divestment option coefficients are, respectively:

$$A_{1FS} = \frac{1}{\beta_1 \Delta_F}\left(\hat{v}_{FS}\frac{D_{F|Y,Y} - D_{F|Y,X}}{\delta+\theta}\hat{v}_{FD}{}^{\beta_2} + \hat{v}_{FD}\frac{D_{F|O,X}}{\delta+\theta}\hat{v}_{FS}{}^{\beta_2}\right) \tag{11}$$

$$A_{2FD} = \frac{1}{\beta_2 \Delta_F}\left(-\hat{v}_{FS}\frac{D_{F|Y,Y} - D_{F|Y,X}}{\delta+\theta}\hat{v}_{FD}{}^{\beta_1} + \hat{v}_{FD}\frac{D_{F|O,X}}{\delta+\theta}\hat{v}_{FS}{}^{\beta_1}\right) \tag{12}$$

where $\Delta_F = \hat{v}_{FS}{}^{\beta_1}\hat{v}_{FD}{}^{\beta_2} - \hat{v}_{FS}{}^{\beta_2}\hat{v}_{FD}{}^{\beta_1}$.

The solutions for the leader's two thresholds $\hat{v}_{LS}$ and $\hat{v}_{LD}$ are:

$$\hat{v}_{LD}{}^{\beta_2}\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta}\frac{\beta_1 - 1}{\beta_1} - \frac{D_{L|Y,X}f_Y - D_{L|X,X}f_X}{r}\right) - (K - Z) - \left[\hat{v}_{LS}{}^{\beta_2}\left(Z - \frac{D_{L|X,X}\hat{v}_{LD}}{\delta+\theta}\frac{\beta_1 - 1}{\beta_1} + \frac{D_{L|X,X}f_X}{r}\right)\right] = 0 \tag{13}$$

$$\hat{v}_{LD}{}^{\beta_1}\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta}\frac{\beta_2 - 1}{\beta_2} - \frac{D_{L|Y,X}f_Y - D_{L|X,X}f_X}{r} + A_{1LS}\hat{v}_{LS}{}^{\beta_1}\frac{\beta_2 - \beta_1}{\beta_2} - (K - Z)\right) - \hat{v}_{LS}{}^{\beta_1}\left(Z - \frac{D_{L|X,X}\hat{v}_{LD}}{\delta+\theta}\frac{\beta_2 - 1}{\beta_2} + \frac{D_{L|X,X}f_X}{r}\right) = 0 \tag{14}$$

The leader's switching and divestment option coefficients are, respectively:

$$A_{1LS} = \frac{1}{\beta_1 \Delta_L}\left(\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} + \beta_1 A_{1LSS}\hat{v}_{LS}{}^{\beta_1}\right)\hat{v}_{LD}{}^{\beta_2} + \hat{v}_{LD}\frac{D_{L|X,X}}{\delta+\theta}\hat{v}_{LS}{}^{\beta_2}\right) \tag{15}$$

$$A_{2LD} = -\frac{1}{\beta_2 \Delta_L}\left(-\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} + \beta_1 A_{1LSS}\hat{v}_{LS}{}^{\beta_1}\right)\hat{v}_{LD}{}^{\beta_1} - \hat{v}_{LD}\frac{D_{L|X,X}}{\delta+\theta}\hat{v}_{LS}{}^{\beta_1}\right) \tag{16}$$

where $\Delta_L = \hat{v}_{LS}{}^{\beta_1}\hat{v}_{LD}{}^{\beta_2} - \hat{v}_{LS}{}^{\beta_2}\hat{v}_{LD}{}^{\beta_1}$.

The solutions for the three rival options are:

$$A_{1FSS} = \left(D_{F|Y,X} - D_{F|X,X}\right)\left(\frac{\hat{v}_{LS}}{\delta+\theta} - \frac{f_X}{r}\right)\frac{\hat{v}_{LD}{}^{\beta_2}}{\Delta_L} - \left(D_{F|O,X} - D_{F|X,X}\right)\left(\frac{\hat{v}_{LD}}{\delta+\theta} - \frac{f_X}{r}\right)\frac{\hat{v}_{LS}{}^{\beta_2}}{\Delta_L} \tag{17}$$

$$A_{2FDD} = -\left(D_{F|Y,X} - D_{F|X,X}\right)\left(\frac{\hat{v}_{LS}}{\delta+\theta} - \frac{f_X}{r}\right)\frac{\hat{v}_{LD}{}^{\beta_1}}{\Delta_L} + \left(D_{F|O,X} - D_{F|X,X}\right)\left(\frac{\hat{v}_{LD}}{\delta+\theta} - \frac{f_X}{r}\right)\frac{\hat{v}_{LS}{}^{\beta_1}}{\Delta_L} \tag{18}$$

$$A_{1LSS} = \left(\frac{\hat{v}_{FS}}{\delta+\theta} - \frac{f_Y}{r}\right)\left(D_{L|Y,Y} - D_{L|Y,X}\right)\hat{v}_{FS}{}^{-\beta_1} \tag{19}$$

2.1 Partial Derivatives

$$\frac{\partial V_L(v)}{\partial v}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial v}=D_{L|Y,Y}\,\dfrac{1}{\delta+\theta} \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial v}=D_{L|Y,X}\,\dfrac{1}{\delta+\theta}+\beta_1 A_{1LSS}\,v^{\beta_1-1} \text{ for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial v}=D_{L|X,X}\,\dfrac{1}{\delta+\theta}+\beta_1 A_{1LS}\,v^{\beta_1-1}+\beta_2 A_{2LD}\,v^{\beta_2-1} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial v}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases} \tag{20}$$

$$\frac{\partial V_L(v)}{\partial \sigma}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial \sigma}=0 \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial \sigma}=v^{\beta_1}\,\dfrac{\partial A_{1LSS}}{\partial \sigma}+A_{1LSS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial \sigma} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial \sigma}=v^{\beta_1}\,\dfrac{\partial A_{1LS}}{\partial \sigma}+A_{1LS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial \sigma}\\ \qquad +v^{\beta_2}\,\dfrac{\partial A_{2LD}}{\partial \sigma}+A_{2LD}\,v^{\beta_2}\,log(v)\dfrac{\partial \beta_2}{\partial \sigma} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial \sigma}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases} \tag{21}$$

$$\frac{\partial V_L(v)}{\partial r}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial r}=D_{L|Y,Y}\,\dfrac{f_Y}{r^2} \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial r}=D_{L|Y,X}\,\dfrac{f_Y}{r^2}+v^{\beta_1}\,\dfrac{\partial A_{1LSS}}{\partial r}+A_{1LSS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial r} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial r}=D_{L|X,X}\,\dfrac{f_Y}{r^2}+v^{\beta_1}\,\dfrac{\partial A_{1LS}}{\partial r}+A_{1LS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial r}\\ \qquad +v^{\beta_2}\,\dfrac{\partial A_{2LD}}{\partial r}+A_{2LD}\,v^{\beta_2}\,log(v)\dfrac{\partial \beta_2}{\partial r} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial r}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases} \tag{22}$$

$$\frac{\partial V_L(v)}{\partial \delta} = \begin{cases} \dfrac{\partial V_{L1}(v)}{\partial \delta} = -D_{L|Y,Y} \dfrac{v}{(\delta+\theta)^2} \quad \text{for } v \geq \hat{v}_{FS} \\ \dfrac{\partial V_{L2}(v)}{\partial \delta} = -D_{L|Y,X} \dfrac{v}{(\delta+\theta)^2} + v^{\beta_1} \dfrac{\partial A_{1LSS}}{\partial \delta} + A_{1LSS}\, v^{\beta_1} \log(v) \dfrac{\partial \beta_1}{\partial \delta} \quad \text{for } \hat{v}_{LS} \leq v < \hat{v}_{FS}, \\ \dfrac{\partial V_{L3}(v)}{\partial \delta} = -D_{L|X,X} \dfrac{v}{(\delta+\theta)^2} + v^{\beta_1} \dfrac{\partial A_{1LS}}{\partial \delta} + A_{1LS}\, v^{\beta_1} \log(v) \dfrac{\partial \beta_1}{\partial \delta} \\ \qquad + v^{\beta_2} \dfrac{\partial A_{2LD}}{\partial \delta} + A_{2LD}\, v^{\beta_2} \log(v) \dfrac{\partial \beta_2}{\partial \delta} \quad \text{for } \hat{v}_{LD} < v < \hat{v}_{LS}, \\ \dfrac{\partial V_{L4}(v)}{\partial \delta} = 0 \quad \text{for } v \leq \hat{v}_{LD}. \end{cases}$$

(23)[7]

In this study, **I** involves a sensitivity analysis of the sensitivity of all outputs to a 1 % in the parameter inputs. **II** analytical partial derivatives (for v), and a mix of partial derivatives for some of the other critical inputs (numerical where the thresholds are also affected by changes in input values) are shown in detail at interim points for each regime or stage. **III** shows the decomposition of the value function (VF) across a v range, crossing all of the relevant regimes. **IV** shows the function values for a somewhat arbitrary +/- increment around the illustrative base case. Plausibly, the measures of risk exposure are best served by one of these formats. We provide these four risk measurements for delta, vega, rho, and epsilon[8], first assessing which of these real option "Greeks" is likely to be critical over different regimes.

3. Numerical Evaluations

We use numerical evaluations using base case parameter values given in Table 1. The values of β_1 and β_2 for the base case are 2.2656 and -1.7656, respectively.

Table 1: Base Case Parameter Values

Definition	Notation	Value
Risk-free rate	r	0.08
Convenience yield	δ	0.03
Market depletion rate	θ	0.04

[7] Full partial derivatives for the follower, and numerical solutions at v=6, 7.5, 9.5 and 12 are in Appendix F.
[8] Eventually kappa and alpha, ("alpha" is an abbreviation of final market share, teliki agora, in Greek)

Market price volatility	σ	0.20
Follower's divestment proportion	λ	0.20
Unadjusted periodic operating cost for policy ***X***	f_X	10.0
Unadjusted periodic operating cost for policy ***Y***	f_Y	2.0
Leader's divestment value	Z	25.0
Switching investment cost to policy ***Y***	K	35.0
Leader's market share given both leader and follower pursue policy ***X***	$D_{L\|X,X}$	0.50
Leader's market share given both leader and follower pursue policy ***Y***	$D_{L\|Y,Y}$	0.50
Leader's market share given leader pursues policy ***Y*** and follower policy ***X***	$D_{L\|Y,X}$	0.425
Leader's market share given leader exits and follower pursues policy ***X***	$D_{L\|O,X}$	0.00

3.1 Numerical Results

With the base case values, we present the numerical solutions for the leader's and follower's various thresholds and coefficients in Table 2.

Table 2: Values for the Various Thresholds and Option Coefficients

	Leader		Follower	
	$\hat{v}_{LD}$	6.0924	$\hat{v}_{FD}$	5.7392
DIVEST	A_{2LD}	862.9820	A_{2FD}	1034.8147
			A_{2FDD}	-643.7031
	$\hat{v}_{LS}$	8.2585	$\hat{v}_{FS}$	12.2631
SWITCH	A_{1LS}	0.1412	A_{1FS}	0.0132
	A_{1LSS}	0.0385	A_{1FSS}	0.1252

Table 3 is a "big picture of the risk exposure" of the leader and follower to changes in critical inputs. The inputs are chosen according to the conventional Greeks for options (delta, vega, rho, and epsilon, Δ,υ,ρ,δ). From the complete sensitivities table in Appendix C and D across the most important two middle regimes, delta is typically the most important, followed by epsilon. While vega and rho are not so critical, those Greeks are of conventional interest concerning traded options.[9] Note that the partial derivatives of the VF with respect to changing v does not involve

[99] The conventional derivatives text Hull,J. (2022), *Options, Futures and Other Derivatives*, Pearson/Prentice Hall focuses on delta, theta (time sensitivity), gamma, vega and rho (in that order), but ignores kappa and epsilon.

any change in the thresholds or option coefficients. Note that the partial derivatives of the VF with respect to changing σ does not involve any change in the PV of operations, but in the thresholds and option coefficients. Note that the partial derivatives of the VF with respect to changing r involves changing the present value of operating costs, and also in the thresholds and option coefficients. Note that the partial derivatives of the VF with respect to changing δ involves changing the present value of revenue, and also in the thresholds and option coefficients.

The obvious observation from Table 3 is that the VF L is a smooth function across the discrete v, while the VF F jump around the leader switch threshold, changing from being more sensitive at v below the threshold, to less sensitive above, as the leader shifts from an initial market share of 50% to a reduced middle market share of 42.5% with lower operating costs. Apart from the jumps, the deltas (Δv) are positive and increasing over v for both L and F. When should the follower buy protective puts on v, or short v to lock in a price, as v increases?

There is a similar although opposite effect for epsilon (Δδ) across increasing v increments. Apart from the jumps, the epsilons are negative and decreasing over v for both L and F. When (and how) should the follower buy protective puts on δ, as v increases?

There is a similar effect for vega (Δσ) across increasing v increments. Apart from the jumps, the vegas are negative and decreasing over v for both L and F. When should the follower buy protective volatility puts, as v increases?

Although rho (Δr) is not critical, it is possibly easier to hedge through interest rate derivatives. However, while the VF L benefits from an increased interest rate as v increases, the follower benefits only above the L switch threshold. Should the follower protect its VF from interest rate increases below the L switch threshold, suddenly reversing the hedge position after the L switches?

Table 3

Sensitivities of the Value Functions to 1% Increase in Critical Inputs

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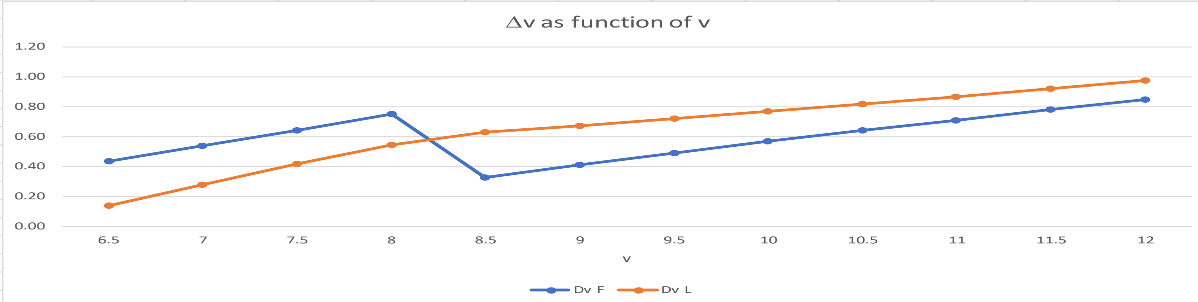
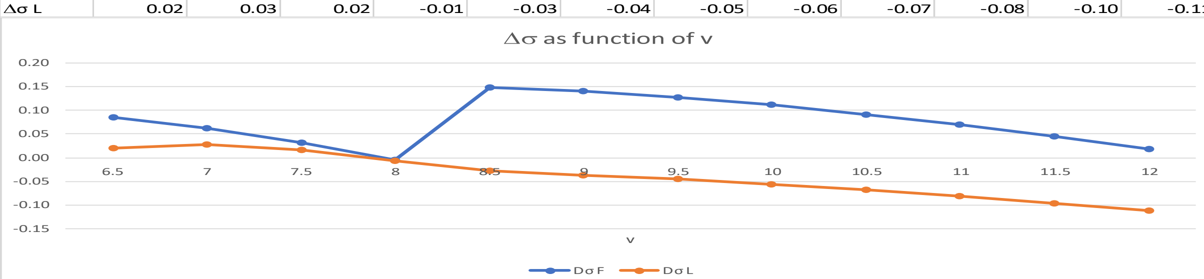
Theta is not relevant for the perpetual options in our model, but gamma is considered in the proofs that the differential equations are solved, Appendix E.

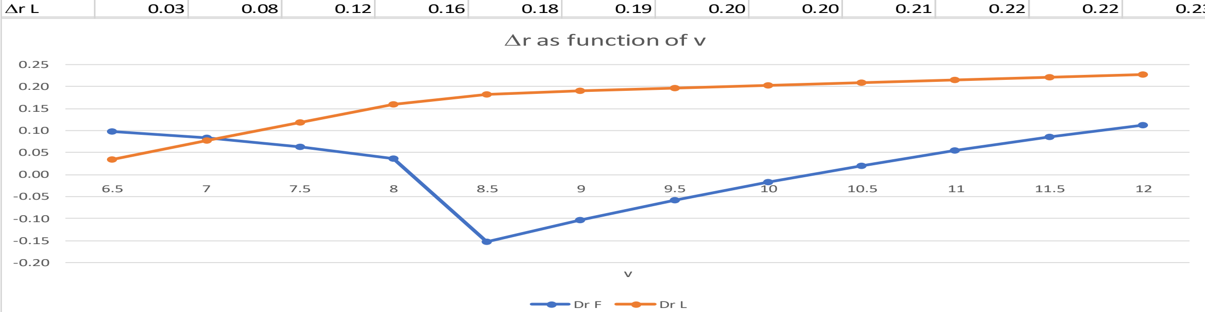
v	6.5	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5	12
Δv F	0.43	0.54	0.64	0.75	0.32	0.41	0.49	0.57	0.64	0.71	0.78	0.85
Δv L	0.14	0.28	0.42	0.55	0.63	0.67	0.72	0.77	0.82	0.87	0.92	0.97



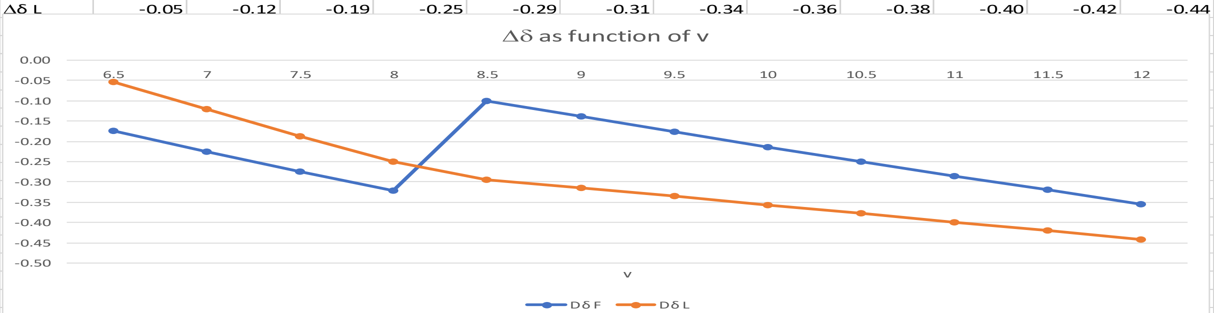
$\Delta\sigma$ F	0.09	0.06	0.03	0.00	0.15	0.14	0.13	0.11	0.09	0.07	0.04	0.02
$\Delta\sigma$ L	0.02	0.03	0.02	-0.01	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.10	-0.11



Δr F	0.10	0.08	0.06	0.04	-0.15	-0.10	-0.06	-0.02	0.02	0.05	0.09	0.11
Δr L	0.03	0.08	0.12	0.16	0.18	0.19	0.20	0.20	0.21	0.22	0.22	0.23



$\Delta\delta$ F	-0.17	-0.22	-0.27	-0.32	-0.10	-0.14	-0.18	-0.21	-0.25	-0.29	-0.32	-0.35
$\Delta\delta$ L	-0.05	-0.12	-0.19	-0.25	-0.29	-0.31	-0.34	-0.36	-0.38	-0.40	-0.42	-0.44



Now, why do these changes in the value functions occur? Are they due to the present value and/or the option portfolio value changes, and why are they different over the regimes?

3.2. DELTA

I Sensitivities to changes in v are extracted from the Appendix D1 & D2, Complete Sensitivities Tables for R3 & R2

Table 4

The Effect of 1% Revenue Increase on the Value Functions in a Duopoly

Absolute Δ R3 v=7.5					Absolute Δ R2 v=9.5	
	BASE CASE	v up 1%			BASE CASE	v up 1%
VF F	15.5103	0.6445		VF F	27.7577	0.4911
F 3 PV OPS	-8.9286	0.5357		F 2 PV OPS	6.1607	0.7804
F 3 S	1.2644	0.0288		F 2 S	2.1600	0.0492
F 3 D	29.5043	-0.5138		F 2 D	19.4370	-0.3385
F 3 SS	12.0232	0.2741				
F 3 DD	-18.3531	0.3196				
VF L	29.2381	0.4164		VF L	43.3704	0.7208
L 3 PV OPS	-8.9286	0.5357		L 2 PV OPS	47.0536	0.5768
L 3 S	13.5616	0.3092		L 2 SS	6.3168	0.1440
L 3 D	24.6051	-0.4285		L 2 -(K-Z)	-10	0

An increase from v=7.5 to 7.575 (1%) results in an absolute increase of the VF F of nearly .22 more than for the L, mostly due to the increase in the F's rival options, SS and DD, since the increase in the PV OPS of both is the same, with the same equal market share before either divests or switches. An increase from v=9.5 to 9.595 (1%) results in an increase of the VF L of over .22 more than for the F, in spite of the greater increase for the F's OPS (with then a 57.5% market share), due to the decrease in the F's divest option value. So, while both benefit from a v (mostly price) increase, the effect on rival and divest option values is quite different. This confirms the adage, even if you are ahead (leader) or behind (follower), watch the value of your rival and strategic options as v increases. The PV of operations is not everything.

II Partial derivatives Numerical Results

Table 5

Partial Derivatives at Base Values, v=7.5, 9.5

R3, v=7.5		R2, v=9.5	
δVL3/δv	δVF3/δv	δVL2/δv	δVF2/δv
5.4472	8.5316	7.75779	5.1171
δVL3/δσ	δVF3/δσ	δVL2/δσ	δVF2/δσ
8.2421	15.6286	-22.9924	63.3195
δVL3/δr	δVF3/δr	δVL2/δr	δVF2/δr
-476.12	-544.3535	247.8524	-797.2131
δVL3/δδ	δVF3/δδ	δVL2/δδ	δVF2/δδ
-630.9151	-921.0211	-1123.035	-596.631

The deltas are all positive, Δ F3>ΔL3, ΔF2<ΔL2, consistent with Table 4. What the absolute number should be used for is challenging, since it is obviously not for delta hedging at a particular level of v. Note that the deltas for each element of the value function are shown in Appendix F. In line with conventional option pricing theory, it could be argued that

$$\frac{\partial V_{L3}(v)}{\partial v} = D_{L|X,X}\frac{1}{\delta+\theta} + \beta_1 A_{1LS}\, v^{\beta_1 - 1} + \beta_2 A_{2LD}\, v^{\beta_2 - 1} = 7.1429+4.0966-5.7922=5.4472 \tag{24}$$

$$\beta_1 A_{1LS}\, v^{\beta_1 - 1} = 4.0966$$

$$\beta_2 A_{2LD}\, v^{\beta_2 - 1} = -5.7922$$

a short position 4.1/7.5=55% in v should be used to delta hedge the switch option, and a long position 5.8/7.5=77% should be used to delta hedge the divest option when v=7.5, but this point is not well presented in the literature.[10]

III Table 6 shows the composition of the VFs for L and F across a v range 5.5-12.5 by .5 increments, with a closer focus in Tables 7 & 8.

Table 6

Follower's Value Function as Function of v, Across Regimes															
v	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00	10.50	11.00	11.50	12.00	12.50
Regime	L 5	L 4	L 3	L 3	L 3	L 3	L 2	L 2	L 2	L 2	L 2	L 2	L 2	L 2	L 1
F Value SUM	5.0000	5.2278	7.8934	11.4606	15.5103	19.9728	23.2798	25.3484	27.7577	30.4481	33.3735	36.4974	39.7911	43.2311	46.7857
F Op PV		-39.2857	-16.0714	-12.5000	-8.9286	-5.3571	-2.0536	2.0536	6.1607	10.2679	14.3750	18.4821	22.5893	26.6964	76.7857
$A1FS\ v^{\beta 1}$		0.7626	0.9143	1.0814	1.2644	1.4634	1.6789	1.9110	2.1600	2.4262	2.7098	3.0110	3.3300	3.6671	
$A2FD\ v^{\beta 2}$		43.7508	37.9850	33.3263	29.5043	26.3269	23.6545	21.3839	19.4370	17.7541	16.2887	15.0043	13.8718	12.8676	
$A1FSS\ v^{\beta 1}$			8.6940	10.2834	12.0232	13.9162									
$A2FDD\ v^{\beta 2}$			-23.6284	-20.7305	-18.3531	-16.3765									
InvestCost	5.0000														-30.0000

Leader's Value Function as Function of v, Across Regimes															
v	5.50	6.00	6.50	7.00	7.50	8.00	8.50	9.00	9.50	10.00	10.50	11.00	11.50	12.00	12.50
Regime	L 5	L 4	L 3	L 3	L 3	L 3	L 2	L 2	L 2	L 2	L 2	L 2	L 2	L 2	L 1
L Value SUM	25.0000	25.0000	25.4125	26.8916	29.2381	32.2950	35.8919	39.6064	43.3704	47.1845	51.0495	54.9660	58.9347	62.9561	76.7857
L Op PV	0.0000	0.0000	-16.0714	-12.5000	-8.9286	-5.3571	40.9821	44.0179	47.0536	50.0893	53.1250	56.1607	59.1964	62.2321	76.7857
$A1LSS\ v^{\beta 1}$							4.9098	5.5886	6.3168	7.0952	7.9245	8.8053	9.7383	10.7240	
$A1LS\ v^{\beta 1}$			9.8064	11.5992	13.5616	15.6969									
$A2LD\ v^{\beta 2}$			31.6775	27.7924	24.6051	21.9552									
InvestCost	25.0000	25.0000					-10.0000	-10.0000	-10.0000	-10.0000	-10.0000	-10.0000	-10.0000	-10.0000	

[10] Appendix E shows these results in an Excel Spreadsheet. Also, the Δ & Γ for v=6, 7.5, 9.5 and 12.5, are a compliment for Table 6.

IV Table 7 compares the VF composition in R3, base v=7.5 down v=7, up v=8.

Table 7

	Values Over Three v (R3)			Change
v	7	7.5	8	1
Regime L3				
F Value SUM	11.4606	15.5103	19.9728	8.5122
F Op PV	-12.5000	-8.9286	-5.3571	7.1429
$A1FS\ v^{\beta 1}$	1.0814	1.2644	1.4634	0.3820
$A2FD\ v^{\beta 2}$	33.3263	29.5043	26.3269	-6.9995
$A1FSS\ v^{\beta 1}$	10.2834	12.0232	13.9162	3.6328
$A2FDD\ v^{\beta 2}$	-20.7305	-18.3531	-16.3765	4.3540
L Value SUM	26.8916	29.2381	32.2950	5.4033
L Op PV	-12.5000	-8.9286	-5.3571	7.1429
$A1LS\ v^{\beta 1}$	11.5992	13.5616	15.6969	4.0977
$A2LD\ v^{\beta 2}$	27.7924	24.6051	21.9552	-5.8372

In regime R3, L benefits less than F by a v increase, even though the effect on the PV OPS is the same, given equal market shares and operating costs, because F benefits from the increase in the value of rival options. L could demand 1 from the F, so the L net value change is 6.4, and the F reduced value change is then 7.5, for encouraging a price increase, a win-win compromise.

Table 8 compares the VF composition in R2, base v=9.5 down v=9, up v=10.

Table 8

	Values Over Three v (R2)			Change
v	9	9.5	10	1
Regime L2				
F Value SUM	25.3484	27.7577	30.4481	5.0997
F Op PV	2.0536	6.1607	10.2679	8.2143
$A1FS\ v^{\beta 1}$	1.9110	2.1600	2.4262	0.5152
$A2FD\ v^{\beta 2}$	21.3839	19.4370	17.7541	-3.6298
L Value SUM	39.6064	43.3704	47.1845	7.5781
L Op PV	44.0179	47.0536	50.0893	6.0714
$A1LSS\ v^{\beta 1}$	5.5886	6.3168	7.0952	1.5067
INVEST	-10.0000	-10.0000	-10.0000	0.0000

In regime R2, L benefits more than F by a v increase. The effect on the PV OPS is not the same, given the L market shares is less than for the F, but the F suffers from a decrease in the value of the divest option. L could offer the F 1 (or pay common marketing costs), so the L net value change is 6.57, the F net value is 6.10, in order to encourage a price increase, a win-win compromise.

How might these measures of risk exposure be used in practice? First of all, the "big picture" shows what is important, in this case v (or Δ). **I** Table 4 provides a convenient view of the absolute $ comparison of the F and L gains, and decomposition of those changes at v=7.5, 9.5. Both benefit (or lose in the case of a 1% in v). The change in the PV OPS is not the major focus when v=7.5, but is dominate at higher v. **II** the use and mis-use of analytical partial derivatives (for v) is a challenge for future research, but meanwhile the signs and comparative dimensions are consistent with Table 4 and 7 & 8. **III** Table 6 shows the VF across a range of v, and provides a convenient format for any particular extract, such as **IV** shows the function values for arbitrary +.5/-.5 increment around the illustrative base case. Also, **IV** can be used to view whether the downside loss is symmetric (opposite sign, the result is usually not exactly the same in size) with the upside gain. Which format is the most visually convenient is perhaps a matter of taste and presentation clarity, probably not expressible by one number such as Value at Risk VaR, or the numerous alternatives developed for traded options.

3.3 VEGA

I Sensitivities to a 1 % change in the base case volatility of 20% are shown in Table 9

Table 9

Absolute Δ R3 v=7.5					**Absolute Δ R2 v=9.5**	
	BASE CASE	σ up 1%			BASE CASE	σ up 1%
VF F	15.5103	0.0321		VF F	27.7577	0.1277
F 3 PV OPS	-8.9286	0.0000		F 2 PV OPS	6.1607	0.0000
F 3 S	1.2644	-0.0684		F 2 S	2.1600	-0.1237
F 3 D	29.5043	0.2129		F 2 D	19.4370	0.2514
F 3 SS	12.0232	-0.0635				
F 3 DD	-18.3531	-0.0489				
VF L	29.2381	0.0168		VF L	43.3704	-0.0457
L 3 PV OPS	-8.9286	0.0000		L 2 PV OPS	47.0536	0.0000
L 3 S	13.5616	-0.1580		L 2 SS	6.3168	-0.0457
L 3 D	24.6051	0.1747		L 2 -(K-Z)	-10.0000	0.0000

Increases in volatility reduce the divest thresholds for both the L and F, and reduce all of the divest option coefficients, but increase the switch thresholds and increase the switch option coefficients, except for the F switch option coefficient. Table 9 shows that the VF F increases by a very small absolute amount in R3. Although the divest coefficient decreases, the power β_2 increases, so the overall effect is the divest option value increases, offset by decreases in the other three option values. There is a similar pattern at the R2 stage (v=9.5), except the absolute changes are somewhat larger. Generally, although some of the separate option values are sensitive to changes in volatility, the portfolio of options for the F is not, at least at the R3 stage. The value functions for the L are not very sensitive to small changes in volatility.

II Partial derivatives "vega" are shown in Table 5. The signs are consistent with Table 9, all value function vegas are positive, except for :

$$\frac{\partial V_L(v)}{\partial \sigma} = \begin{cases} \frac{\partial V_{L2}(v)}{\partial \sigma} = v^{\beta_1} \frac{\partial A_{1LSS}}{\partial \sigma} + A_{1LSS} v^{\beta_1} log(v) \frac{\partial \beta_1}{\partial \sigma} \quad \text{for } \hat{v}_{LS} \le v < \hat{v}_{FS}, \end{cases} \tag{25}$$

$$\begin{aligned} \frac{\partial V_{L2}(v)}{\partial \sigma} &= .4763 v^{2.2656} + .0385 * -7.1127 v^{2.2656} LN(9.5) \quad \text{for } v = 9.5 \\ &= 78.1649 - 101.1513 = -22.9924 \end{aligned} \tag{26}$$

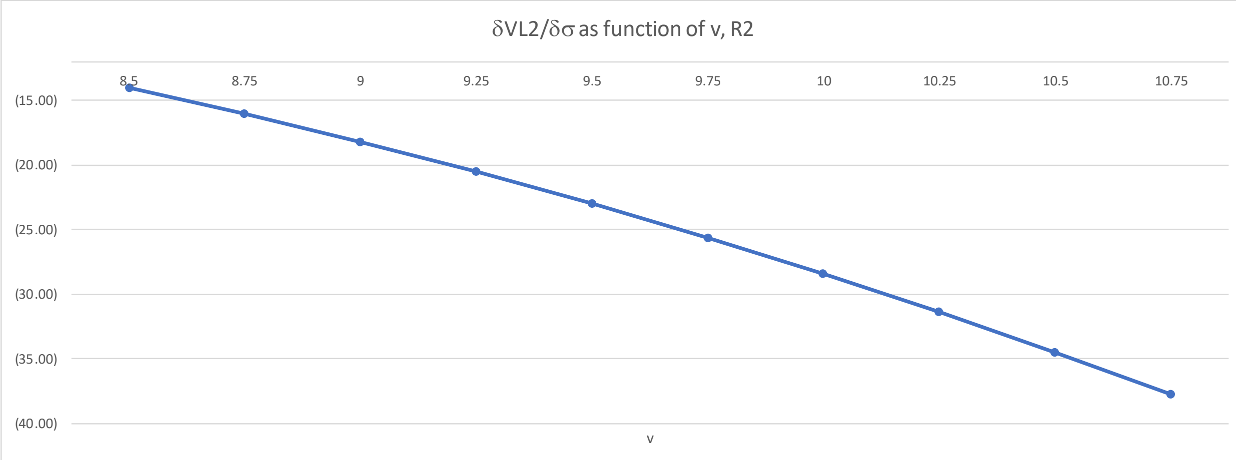
$\frac{\partial \beta_1}{\partial \sigma}$<0, so the second part of the partial derivative is negative.

III Table 10 shows the vegas across various ranges in R2

Suppose the actual market volatility is 20%, and v is between vLS and vFS, Regime 2.

Table 10

v R2	8.5	8.75	9	9.25	**9.5**	9.75	10	10.25	10.5	10.75
δVL2/δσ	(13.99)	(16.02)	(18.19)	(20.52)	(22.99)	(25.62)	(28.41)	(31.37)	(34.48)	(37.77)
(δA1LSS/δσ)v^β1	60.75	64.87	69.15	73.58	**78.16**	82.90	87.79	92.84	98.05	103.42
ROLSS*(δβ1/δσ)*(v^b1)*LOG(v)	(74.73)	(80.89)	(87.34)	(94.09)	(101.15)	(108.52)	(116.20)	(124.21)	(132.53)	(141.19)
ROLSS*(δβ1/δσ)	(0.27)	(0.27)	(0.27)	(0.27)	(0.27)	(0.27)	(0.27)	(0.27)	(0.27)	(0.27)
v^β1	127.54	136.20	145.18	154.47	164.10	174.04	184.32	194.92	205.86	217.13



The negative vega $\frac{\partial VF_2}{\partial \sigma}$ becomes more negative with v across this range, implying that although this rival option is not of large value, after the leader switches there is no advantage for further increases in volatility. The effect of the vega on the volatility risk exposure over the R2 range of v appears to be linear. The leader benefits if vFS falls due to a decline in volatility, but it is unlikely that the leader can do much alone to reduce volatility for the follower. If the follower is myopic and ignorant of real options, perhaps the leader can persuade the follower that lower risk is best all around. Naturally, such a leader would discourage publication and circulation of this article, indeed.

IV Now, we turn to the exposure of the leader and follower to changes in the "effective price volatility" from 15% to 25% as indicated in Table 11, where the last seven rows are the derived option coefficients.

Table 11 Derived Thresholds and Option Coefficients for σ=15%, 20%, 25%

Table 11

σ	**0.1500**	**0.2000**	**0.2500**
β_1	2.7228	2.2656	1.9757
β_2	(2.6117)	(1.7656)	(1.2957)
vFD	6.3599	5.7392	5.1441
vFS	9.9311	12.2631	16.5486
vLD	6.4101	6.0924	5.7394
vLS	7.6662	8.2585	9.0899
A1FS= SO F S	0.0138	0.0132	-0.0032
A2FD= SO F D	4641.9220	1034.8147	472.5265
A1LSS=RO L SS	0.0169	0.0385	0.0620
A1LS= SO L S	0.0752	0.1412	0.1865
A2LD=SO L D	3824.5225	862.9820	390.8268
A1FSS=RO F SS	0.0591	0.1252	0.2005
A2FDD=RO F DD	-3330.4886	-643.7031	-267.8460

In Regime 3, when v=7.5, Table 12 shows that the leader's switch option $SO\ L\_S$ decreases with an increase in p volatility -8.1665 for 15% to 25%, while the leader's divest option $SO\ L\_D$ increases 8.8935, for a net gain of 0.727. The F's divest option SO F_D increases lots with an increase in p volatility, more than offsetting the decrease in the other three options when volatility increases from 15% to 25%.

Table 12: R3, $v = 7.5$

R3, v=7.5				
Volatility	**15%**	**20%**	**25%**	Change
VF L	29.0510	29.2381	29.7780	0.7270
L 3 PV OPS	-8.9286	-8.9286	-8.9286	0.0000
L 3 SO L S	18.1557	13.5616	9.9892	-8.1665
L 3 SO L D	19.8239	24.6051	28.7174	8.8935
VF F	15.4749	15.5103	16.6836	1.2087
F 3 PV OPS	-8.9286	-8.9286	-8.9286	0.0000
F 3 SO F S	3.3416	1.2644	-0.1691	-3.5108
F 3 SO F D	24.0608	29.5043	34.7205	10.6598
F 3 RO F SS	14.2642	12.0232	10.7417	-3.5225
F 3 RO F DD	-17.2631	-18.3531	-19.6809	-2.4178

This is a **differential** result, that is both leader and follower benefit from a volatility increase, but in differential amounts. Perhaps the leader and follower could share (perhaps proportion to benefits) the expense of promoting more p volatility.

The consequences are reversed for the leader if $v = 9.5$, above the leader's switching threshold (below the follower's) for R2, as shown in Table 13.

Table 13: R2, $v = 9.5$

Regime 2, vLS<v<vFS	v=9.5			
Volatility	**15%**	**20%**	**25%**	Change
VF L	44.8213	43.3704	42.3497	-2.4716
L 2 PV OPS	47.0536	47.0536	47.0536	0.0000
L2 RO L SS	7.7677	6.3168	5.2961	-2.4716
L 2 K-Z	-10.0000	-10.0000	-10.0000	0.0000
VF F	25.4986	27.7577	31.4511	5.9525
F 2 PV OPS	6.1607	6.1607	6.1607	0.0000
F 2 SO F S	6.3605	2.1600	-0.2698	-6.6303
F 2 SO F D	12.9774	19.4370	25.5602	12.5828

This is a **contrast** result, since the leader would prefer less volatility (the ROLSS decreases with an increase in volatility), but the follower benefits from more volatility (the SOFD increases more than the SOFS decreases, for a net increase benefitting the VF F).

3.4 RHO

I Sensitivities to changes in r are extracted from the Appendix D1 & D2, Complete Sensitivities

Table 14

The Effect of 1% Rate Increase on the Value Functions in a Duopoly

Absolute Δ R3 v=7.5					**Absolute Δ R2 v=9.5**	
	BASE CASE	r up 1%			BASE CASE	r up 1%
VF F	15.5103	0.0631		VF F	27.7577	-0.0580
F 3 PV OPS	-8.9286	0.6188		F 2 PV OPS	6.1607	0.7116
F 3 S	1.2644	0.0041		F 2 S	2.1600	0.0006
F 3 D	29.5043	-0.9842		F 2 D	19.4370	-0.7702
F 3 SS	12.0232	-0.2201				
F 3 DD	-18.3531	0.6445				
VF L	29.2381	0.1190		VF L	43.3704	0.1962
L 3 PV OPS	-8.9286	0.6188		L 2 PV OPS	47.0536	0.1052
L 3 S	13.5616	0.1435		L 2 SS	6.3168	0.0910
L 3 D	24.6051	-0.6433		L 2 -(K-Z)	-10	0.0000

An increase from r=.08 to .0808 (1%) results in a small increase of the VF F when v=7.5, due to a decline in the PV of operating costs, balanced against a loss in the divest option value; when v=9.5, the PV of fx also increases, but this is not enough to offset the loss in the divest option value. An increase in r results in an increase of the VF L, due to a decrease in the operating cost, which naturally is less as f falls from 10 to 2 in R2. In this case, the PV of operations is important but not everything.

II Partial derivatives "rho" are shown in Table 5. The signs are consistent with Table 15 & 16, all value function are negative, except for :

$$\begin{cases} \dfrac{\partial V_{L2}(v)}{\partial r} = D_{L|Y,X}\dfrac{f_Y}{r^2} + v^{\beta_1}\dfrac{\partial A_{1LSS}}{\partial r} + A_{1LSS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial r} \quad \text{for } \hat{v}_{LS} \le v < \hat{v}_{FS}, \\ D_{L|Y,X}\dfrac{f_Y}{r^2} + v^{\beta_1}\dfrac{\partial A_{1LSS}}{\partial r} + A_{1LSS}\,v^{\beta_1}\,log(9.5)\dfrac{\partial \beta_1}{\partial r} = 247.8524 \end{cases} \tag{27}$$

$\dfrac{\partial \beta_1}{\partial \sigma}$<0, but the third part of the partial derivative, which is negative, is outweighed by the positive first and second parts.

III

Table 15

Rho as a function of v

	A	B	C	D	E	F	G	H	I	J	K
31	v	8.5	8.75	9	9.25	**9.5**	9.75	10	10.25	10.5	10.75
32	δVL2/δr	191.29	203.24	215.56	228.26	**241.33**	254.77	268.58	282.77	297.34	312.28
33	(δA1LSS/δr)v^β1	262.92	280.77	299.27	318.44	**338.27**	358.78	379.96	401.82	424.37	447.60
34	ROLSS*(δβ1/r)*(v^b1)*LOG(v)	(71.63)	(77.53)	(83.71)	(90.18)	**(96.95)**	(104.01)	(111.38)	(119.05)	(127.03)	(135.33)
35	ROLSS*(δβ1/δr)	(0.60)	(0.60)	(0.60)	(0.60)	(0.60)	(0.60)	(0.60)	(0.60)	(0.60)	(0.60)
36	v^β1	127.54	136.20	145.18	154.47	164.10	174.04	184.32	194.92	205.86	217.13

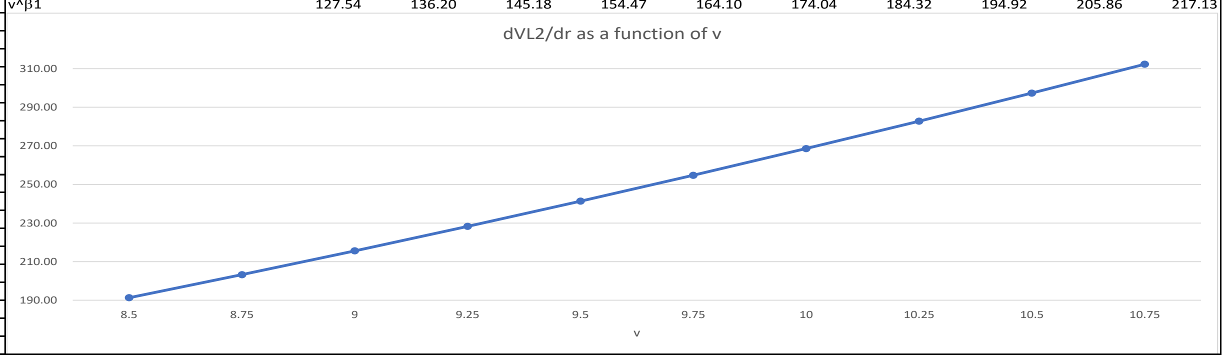


Table 16 shows that the effect of rho on the rate exposure over the R2 range appears to be linear.[11]

[11] This focus on the options only ignores the interest rate effect on the PV of operations.

IV

Table 16

REGIME 3	vLD<v<vLS	v=7.5		
RATE	**7%**	**8%**	**9%**	Change
VF L	27.7649	29.2381	30.6976	2.9327
L 3 PV OPS	-17.8571	-8.9286	-1.9841	15.8730
L 3 SO L S	11.3808	13.5616	15.0162	3.6354
L 3 SO L D	34.2412	24.6051	17.6655	-16.5757
VF F	14.4371	15.5103	16.1213	1.6842
F 3 PV OPS	-17.8571	-8.9286	-1.9841	15.8730
F 3 SO F S	1.0121	1.2644	1.1105	0.0984
F 3 SO F D	44.6767	29.5043	19.1466	-25.5302
F 3 RO F SS	14.6354	12.0232	9.2998	-5.3356
F 3 RO F DD	-28.0300	-18.3531	-11.4515	16.5785
vFD	6.0358	5.7392	5.4697	-0.5660
vFS	13.9822	12.2631	11.1691	-2.8131
vLD	6.3019	6.0924	5.9290	-0.3729
vLS	8.7285	8.2585	7.9009	-0.8276

In Regime 3, both the leader and follower benefit from an increase in interest rates, since the PV of operations at the high operating costs is negative, an increase in the rate decreases the negative PV. However, the leader benefits from a rate increase somewhat more than the follower (thus a differential example) because the net decline in the leader's two SO at that stage is somewhat less than the decline in the value of the follower's two SO and two RO at that stage.

Table 17

Regime 2, vLS<v<vFS		v=9.5		
Interest Rate	**7%**	**8%**	**9%**	Change
VF L	40.5421	43.3704	45.5410	4.9988
L 2 PV OPS	45.5357	47.0536	48.2341	2.6984
L2 RO L SS	5.0064	6.3168	7.3068	2.3004
L 2 K-Z	-10.0000	-10.0000	-10.0000	0.0000
VF F	29.5064	27.7577	27.5765	-1.9299
F 2 PV OPS	-4.1071	6.1607	14.1468	18.2540
F 2 SO F S	1.8004	2.1600	1.8336	0.0333
F 2 SO F D	31.8132	19.4370	11.5961	-20.2172

In Regime 2, the leader benefits but the follower does not benefit from an increase in interest rates (a contrast example). The PV of operations at the lower operating cost is positive, but an increase in the rate decreases the value of that operating cost. The vFS increases with increasing interest rates, so the L2 ROLSS increases. It is curious that the F SO FS first increases, then decreases, with increased interest rates.

3.5 EPSILON

I Table 18

Absolute Δ R3 v=7.5				Absolute Δ R2 v=9.5		
	BASE CASE	δ up 1%			BASE CASE	δ up 1%
VF F	15.5103	-0.2739		VF F	27.7577	-0.1767
F 3 PV OPS	-8.9286	-0.2286		F 2 PV OPS	6.1607	-0.3330
F 3 S	1.2644	-0.0388		F 2 S	2.1600	-0.0620
F 3 D	29.5043	0.2852		F 2 D	19.4370	0.2183
F 3 SS	12.0232	-0.1519				
F 3 DD	-18.3531	-0.1399				
VF L	29.2381	-0.1866		VF L	43.3704	-0.3355
L 3 PV OPS	-8.9286	-0.2286		L 2 PV OPS	47.0536	-0.2461
L 3 S	13.5616	-0.2325		L 2 SS	6.3168	-0.0894
L 3 D	24.6051	0.2745		L 2 -(K-Z)	-10.0000	0.0000

The decrease in the VF F in R3 is due to the decline in the PV of v, and the rival option values, not being offset by an increase in F3 D. Note the decline in the VF F compared to the VF L is greater when v is low (R3), reversed when v is high (R2).

II All of the epsilon partial derivatives are negative, consistent with Table 19, where all of the VF decline with an increase of δ.

III Table 19

v R2	8.5	8.75	9	9.25	**9.5**	9.75	10	10.25	10.5	10.75
δVL2/δδ	(247.79)	(260.34)	(273.07)	(285.98)	(299.06)	(312.29)	(325.68)	(339.21)	(352.88)	(366.68)
(δA1LSS/δδ)v^β1	(543.05)	(579.91)	(618.13)	(657.71)	**(698.68)**	(741.03)	(784.77)	(829.93)	(876.50)	(924.49)
ROLSS*(δβ1/δδ)*(v^b1)*LN(v)	295.26	319.57	345.06	371.74	399.62	428.74	459.09	490.72	523.62	557.82
ROLSS*(δβ1/δδ)	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08	1.08
v^β1	127.54	136.20	145.18	154.47	164.10	174.04	184.32	194.92	205.86	217.13

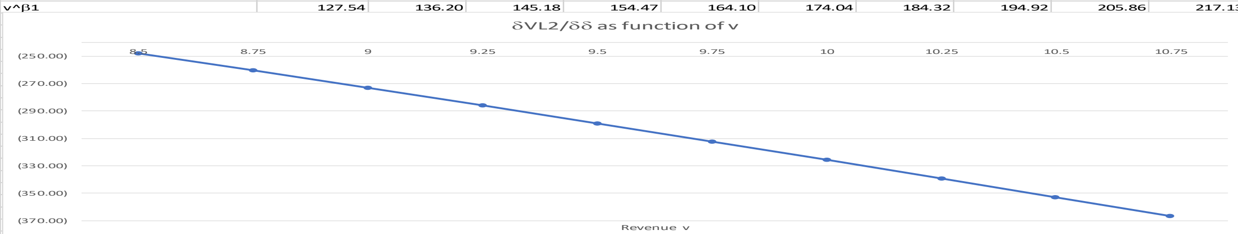


Table 20 shows that the effect of rho on the yield exposure of the option values over the R2 range appears to be linear.[12]

IV

Table 20

REGIME 3	vLD<v<vLS	v=7.5		
YIELD	2%	3%	4%	Change
VF L	39.0312	29.2381	25.4580	-13.5732
L 3 PV OPS	0.0000	-8.9286	-15.6250	-15.6250
L 3 SO L S	23.1488	13.5616	7.2729	-15.8759
L 3 SO L D	15.8824	24.6051	33.8101	17.9277
VF F	27.6937	15.5103	8.6799	-19.0138
F 3 PV OPS	0.0000	-8.9286	-15.6250	-15.6250
F 3 SO F S	3.1226	1.2644	0.3652	-2.7574
F 3 SO F D	20.1409	29.5043	38.8570	18.7162
F 3 RO F SS	17.7849	12.0232	7.6082	-10.1767
F 3 RO F DD	-13.3547	-18.3531	-22.5255	-9.1708
vFD	5.0852	5.7392	6.3098	1.2246
vFS	9.8830	12.2631	15.2435	5.3605
vLD	5.2116	6.0924	6.9777	1.7661
vLS	6.8968	8.2585	9.7166	2.8198

In Regime 3, both the leader and follower suffer from an increase in yields, since the PV of v is reduced with higher δ. However, the leader suffers from a yield increase somewhat less than the follower (thus a differential example) because the net decline in the leader's two SO at that stage is somewhat less than the net decline in the value of the follower's two SO and two RO.

Table 21

Regime 2, vLS<v<vFS	v=9.5			
YIELD	**2%**	**3%**	**4%**	Change
VF L	56.3490	43.3704	33.5415	-22.8075
L 2 PV OPS	56.6667	47.0536	39.8438	-16.8229
L2 RO L SS	9.6823	6.3168	3.6977	-5.9846
L 2 K-Z	-10.0000	-10.0000	-10.0000	0.0000
VF F	36.7299	27.7577	23.9386	-12.7913
F 2 PV OPS	19.1667	6.1607	-3.5938	-22.7604
F 2 SO F S	5.0101	2.1600	0.6692	-4.3409
F 2 SO F D	12.5532	19.4370	26.8631	14.3100

[12] This focus on the options only ignores the yield effect on the PV of operations.

As noted under I, in Regime 2, the leader suffers more than the follower from an increase in yield (a contrast example). The PV of revenue is positive, but an increase in the yield decreases the value of that revenue. The vFS increases with increasing yield, so the L2 ROLSS decreases. Exactly how either the L or F can alter the return shortfall, or convenience yields, is a challenge (perhaps by going short in a nearby futures contract and long in a dated contract, or vice versa, but rolling over such a calendar or temporal spread would involve transaction costs).

4. Conclusion

I Tables 4, 9, 14 and 18 provide a convenient view of the absolute comparison of the F and L gains/losses as v, σ, r, δ change. **IV** Tables 7-8, 12-13, 16-17 and 20-21 show the decomposition of those changes at v=7.5, 9.5, for arbitrary +/- increments around the illustrative base cases. The change in the PV OPS is not the major focus when v=7.5, but is often dominate at higher v. **II** the use and mis-use of analytical partial derivatives is a challenge for future research, but meanwhile the signs and comparative dimensions should be seen to be consistent with the numerical tables. **III** Table 6 shows the VF across a range of v, and provides a convenient format for any particular extract. Appendix E, Tables 10, 15 and 19 show the partial derivatives across a selection of v (focusing on R2).

These numerical results provide a rich format for suggesting, and evaluating, risk reduction and enhancement activities. Duque and Paxson (1993, 1994) studied using Greeks for finite traded options, using delta hedging and options with different moneyness and expirations, and interest rate derivatives, to reduce the equivalent value at risk in option portfolios. Typically adding appropriate opposite positions to hedge one type of risk, alters the other risk elements, so option hedging is complicated. Similar techniques are challenging regarding real option portfolios. Possibly long/short positions in v futures might reduce delta risk, if v prices are traded commodities. Similarly, long/short positions in options on v futures (or physical, through negotiated contracts) might be used to reduce volatility risk, and naturally in interest rate futures and options. Then in the real options context, arrangements with governments, rivals, and third parties provide a wide field for risk reduction or enhancement. Real option games can be viewed, and played, starting perhaps with the formats provided herein.

A key contribution of our paper is the consideration of the overall risk exposure to a host of changing input parameter values, showing the composition of that risk (on the present value of operations, and on each separate option). Possibly unique are the mostly analytical partial derivatives (delta, vega, rho, epsilon) of the value functions and each separate option, with illustrative numerical results.

The critical findings are (i) that delta is the most important risk exposure for this set of parameter values and for this particular model, but risks in packages of options sometimes reduce, sometimes compliment the present value risk (which appears to be the focus of lots of corporate hedging); (ii) switching, divestment, and rival options have different sensitivities to revenue, volatility, rate and yield changes; and (iii) since the signs and dimensions of risk exposure for the values of the leader and the follower change over different regimes (revenue levels), risk evaluation and hedging are challenging activities, offering lots of possibilities for interesting future research.

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Supplementary Appendix

Mutually Exclusive Rival Options: Risk Evaluation

Appendix A Joint Solution Formulae

	A	B	C	D
1			*JOINT Complete Solution*	*EQ*
2	*INPUT*	*Table A1*		MERO
3	*r*	*0.08*		*RISK*
4	*θ*	*0.04*		
5	*fX*	*10*		
6	*fY*	***2***		
7	*Z*	*25*		
8	*K*	***35***		
9	*σ*	***0.20***		
10	*λ*	***0.20***		
11	*δ*	*0.03*		
12	*D LXX*	*0.50*		
13	*D FXX*	*0.50*		
14	*D LOX*	*0.00*		
15	*D FOX*	*1.00*		
16	*D LYX*	*0.425*		
17	*D FYX*	*0.575*		
18	*D LYY*	*0.500*		
19	*D FYY*	*0.500*		
20	*OUTPUT*	*0.3533*	*B25-B23*	
21	β_1	*2.2656*	*0.5-(B3-B11-B4)/B9^2+SQRT((0.5-(B3-B11-B4)/B9^2)^2+2*B3/B9^2)*	
22	β_2	*(1.7656)*	*0.5-(B3-B11-B4)/B9^2-SQRT((0.5-(B3-B11-B4)/B9^2)^2+2*B3/B9^2)*	
23	*vFD*	*5.7392*	*yes*	
24	*vFS*	*12.2631*	*yes*	
25	*vLD*	*6.0924*		
26	*vLS*	*8.2585*		
27	*A1FS*	0.0132	(B24*(B19-B17)*(B23^B22)/(B11+B4)+B23*B15*(B24^B22)/(B11+B4))/(B21*B40)	11
28	*A2FD*	1034.8147	-(B24*(B19-B17)*(B23^B21)/(B11+B4)+B23*B15*(B24^B21)/(B11+B4))/(B22*B40)	12
29	*A1LSS*	0.0385	(B24/(B11+B4)-B6/B3)*(B18-B16)*(B24^(-B21))	19
30	*A1LS*	0.1412	((B26*(B16-B12)/(B11+B4)+B21*B29*(B26^B21))*(B25^B22)+B25*B12*(B26^B22)/(B11+B4))/(B21*B46)	15
31	*A2LD*	862.9820	(-((B26*(B16-B12)/(B11+B4)+B21*B29*(B26^B21))*(B25^B21))-B25*B12*(B26^B21)/(B11+B4))/(B22	16
32	*A1FSS*	0.1252	(B17-B13)*(B26/(B11+B4)-B5/B3)*(B25^B22)/B46-(B15-B13)*(B25/(B11+B4)-B5/B3)*(B26^B22)/B46	17
33	*A2FDD*	*-643.7031*	*-(B17-B13)*(B26/(B11+B4)-B5/B3)*(B25^B21)/B46+(B15-B13)*(B25/(B11+B4)-B5/B3)*(B26^B21)/B46*	18
34			*NUMERICAL SOLUTION*	
35	ΔF	*12.7542*	*(B24^B21)*(B23^B22)-(B24^B22)*(B23^B21)*	
36	ΔL	*3.4744*	*(B26^B21)*(B25^B22)-(B26^B22)*(B25^B21)*	
37	*9*	*0.0000*	*(B24*(B19-B17)*(B21-1)/(B21*(B11+B4))-(B19*B6-B17*B5)/B3-(B8-B10*B7))*(B23^B22)-(B10*B7-B15*B23*(B21-1)/(B21*(B11+B4))+*	9
38	*10*	*0.0000*	*(B24*(B19-B17)*(B22-1)/(B22*(B11+B4))-(B19*B6-B17*B5)/B3-(B8-B10*B7))*(B23^B21)-(B10*B7-B15*B23*(B22-1)/(B22*(B11+B4))+*	10
39	*13*	*0.0000*	*(B26*(B16-B12)*(B21-1)/(B21*(B11+B4))-(B16*B6-B12*B5)/B3-(B8-B7))*(B25^B22)-(B7-B25*B12*(B21-1)/(B21*(B11+B4)*	13
40	*14*	*0.0000*	*(B26*(B16-B12)*(B22-1)/(B22*(B11+B4))-(B16*B6-B12*B5)/B3+B29*(B26^B21)*(B22-B21)/B22-(B8-B7))*(B25^B21)-(B7-B12*B25*(B22-1)/(B22*(B11+*	14
41	Solver	0.00000	*Set SUM(B37:B40)=B41=0, Changing B23:B26*	

	A	B	C
1	**JOINT VF Formulae**		Table A2
2	INPUT		
3	r	0.08	
4	θ	0.04	
5	fX	10.00	
6	fY	2.00	
7	z	25.00	
8	k	35.00	
9	σ	0.20	
10	λ	0.20	
11	δ	0.03	
12	D LXX	0.50	
13	D FXX	0.50	
14	D LOX	0.00	
15	D FOX	1.00	
16	D LYX	0.425	
17	D FYX	0.575	
18	D LYY	0.50	
19	D FYY	0.50	
20	OUTPUT		
21	β_1	*2.2656*	*0.5-(B3-B11-B4)/B9^2+SQRT((0.5-(B3-B11-B4)/B9^2)^2+2*B3/B9^2)*
22	β_2	*(1.7656)*	*0.5-(B3-B11-B4)/B9^2-SQRT((0.5-(B3-B11-B4)/B9^2)^2+2*B3/B9^2)*
23	vFD	*5.7392*	
24	vFS	*12.2631*	
25	vLD	*6.0924*	
26	vLS	*8.2585*	
27	A1FS	0.0132	(B24*(B19-B17)*(B23^B22)/(B11+B4)+B23*B15*(B24^B22)/(B11+B4))/(B21*B34)
28	A2FD	1034.8147	-(B24*(B19-B17)*(B23^B21)/(B11+B4)+B23*B15*(B24^B21)/(B11+B4))/(B22*B34)
29	A1LSS	0.0385	(B24/(B11+B4)-B6/B3)*(B18-B16)*(B24^(-B21))
30	A1LS	0.1412	((B26*(B16-B12)/(B11+B4)+B21*B29*(B26^B21))*(B25^B22)+B25*B12*(B26^B22)/(B11+B4))/(B21*B35)
31	A2LD	862.9820	(-((B26*(B16-B12)/(B11+B4)+B21*B29*(B26^B21))*(B25^B21))-B25*B12*(B26^B21)/(B11+B4))/(B22
32	A1FSS	0.1252	(B17-B13)*(B26/(B11+B4)-B5/B3)*(B25^B22)/B35-(B15-B13)*(B25/(B11+B4)-B5/B3)*(B26^B22)/B35
33	A2FDD	-643.7031	-(B17-B13)*(B26/(B11+B4)-B5/B3)*(B25^B21)/B35+(B15-B13)*(B25/(B11+B4)-B5/B3)*(B26^B21)/B3
34	Delta_F	12.7542	(B24^B21)*(B23^B22)-(B24^B22)*(B23^B21)
35	Delta_L	3.4744	(B26^B21)*(B25^B22)-(B26^B22)*(B25^B21)
36	v	7.0000	
37	F Value	11.4606	IF(B36>=B24,B39,IF(AND(B36<B24,B36>=B26),B40,IF(AND(B36<B26,B36>=B25),B41,IF(AND(B36<B25,B36>=B23),B42,B43
38	L Value	26.8916	IF(B36>=B24,B44,IF(AND(B36<B24,B36>=B26),B45,IF(AND(B36<B26,B36>=B25),B46,B47)))
39	F 1 Row	7.5000	B19*(B36/(B4+B11)-B6/B3)-(B8-B10*B7)
40	F 2 Row	20.0327	B17*(B36/(B4+B11)-B5/B3)+B27*(B36^B21)+B28*(B36^B22)
41	F 3 Row	11.4606	B13*(B36/(B4+B11)-B5/B3)+B27*(B36^B21)+B28*(B36^B22)+B32*(B36^B21)+B33*(B36^B22)
42	F 4 Row	9.4077	B15*(B36/(B4+B11)-B5/B3)+B27*(B36^B21)+B28*(B36^B22)
43	F 5 Row	5.0000	B7*B10
44	L 1 Row	37.5000	B18*(B36/(B4+B11)-B6/B3)
45	L 2 Row	25.0375	B16*(B36/(B4+B11)-B6/B3)+B29*(B36^B21)-(B8-B7)
46	L 3 Row	26.8916	B12*(B36/(B4+B11)-B5/B3)+B30*(B36^B21)+B31*(B36^B22)
47	L 4 Row	25.0000	B7
48	F 1 Term1	37.5000	B19*(B36/(B4+B11)-B6/B3)
49	F 1 Term2	-30.0000	-(B8-B10*B7)
50	F 2 Term1	-14.3750	B17*(B36/(B4+B11)-B5/B3)
51	F 2 Term2 S	1.0814	B27*(B36^B21)
52	F 2 Term3 D	33.3263	B28*(B36^B22)
53	F 3 Term1	-12.5000	B13*(B36/(B4+B11)-B5/B3)
54	F 3 Term2 S	1.0814	B27*(B36^B21)
55	F 3 Term3 D	33.3263	B28*(B36^B22)
56	F 3 Term4 SS	10.2834	B32*(B36^B21)
57	F 3 Term5 DD	-20.7305	B33*(B36^B22)
58	F 4 Term1	-25.0000	B15*(B36/(B4+B11)-B5/B3)
59	F 4 Term2 S	1.0814	B27*(B36^B21)
60	F 4 Term3 D	33.3263	B28*(B36^B22)
61	F 5 Row	5.0000	B7*B10
62	L 1 Term1	37.5000	B18*(B36/(B4+B11)-B6/B3)
63	L 2 Term1	31.8750	B16*(B36/(B4+B11)-B6/B3)
64	L2 Term2 SS	3.1625	B29*(B36^B21)
65	L 2 Term 3	-10.0000	-(B8-B7)
66	L 3 Term1	-12.5000	B12*(B36/(B4+B11)-B5/B3)
67	L 3 Term2 S	11.5992	B30*(B36^B21)
68	L 3 Term3 D	27.7924	B31*(B36^B22)
69	L 4 Row	25.0000	B7

Appendix B Derivation of Joint Solution

The follower's value-matching relationships can be expressed as:

$$\begin{pmatrix} \hat{v}_{FS}^{\beta_1} & \hat{v}_{FS}^{\beta_2} \\ \hat{v}_{FD}^{\beta_1} & \hat{v}_{FD}^{\beta_2} \end{pmatrix}\begin{pmatrix} A_{1FS} \\ A_{2FD} \end{pmatrix} = \begin{pmatrix} \hat{v}_{FS}\dfrac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta} - \dfrac{D_{F|Y,Y}f_Y - D_{F|Y,X}f_X}{r} - \left(K-\lambda Z\right) \\ \lambda Z - \dfrac{D_{F|O,X}\hat{v}_{FD}}{\delta+\theta} + \dfrac{D_{F|O,X}f_X}{r} \end{pmatrix}, \tag{B1}$$

so, the solutions for the two option coefficients are given by:

$$\begin{pmatrix} A_{1FS} \\ A_{2FD} \end{pmatrix} = \begin{pmatrix} \hat{v}_{FS}^{\beta_1} & \hat{v}_{FS}^{\beta_2} \\ \hat{v}_{FD}^{\beta_1} & \hat{v}_{FD}^{\beta_2} \end{pmatrix}^{-1}\begin{pmatrix} \hat{v}_{FS}\dfrac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta} - \dfrac{D_{F|Y,Y}f_Y - D_{F|Y,X}f_X}{r} - \left(K-\lambda Z\right) \\ \lambda Z - \dfrac{D_{F|O,X}\hat{v}_{FD}}{\delta+\theta} + \dfrac{D_{F|O,X}f_X}{r} \end{pmatrix}, \tag{B2}$$

$$\begin{aligned} A_{1FS} &= \frac{\hat{v}_{FD}^{\beta_2}}{\Delta_F}\left(\hat{v}_{FS}\frac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta} - \frac{D_{F|Y,Y}f_Y - D_{F|Y,X}f_X}{r} - \left(K-\lambda Z\right)\right) \\ &\quad - \frac{\hat{v}_{FS}^{\beta_2}}{\Delta_F}\left(\lambda Z - \frac{D_{F|O,X}\hat{v}_{FD}}{\delta+\theta} + \frac{D_{F|O,X}f_X}{r}\right), \\ A_{2FD} &= -\frac{\hat{v}_{FD}^{\beta_1}}{\Delta_F}\left(\hat{v}_{FS}\frac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta} - \frac{D_{F|Y,Y}f_Y - D_{F|Y,X}f_X}{r} - \left(K-\lambda Z\right)\right) \\ &\quad + \frac{\hat{v}_{FS}^{\beta_1}}{\Delta_F}\left(\lambda Z - \frac{D_{F|O,X}\hat{v}_{FD}}{\delta+\theta} + \frac{D_{F|O,X}f_X}{r}\right), \end{aligned} \tag{B3}$$

where $\Delta_F = \hat{v}_{FS}^{\beta_1}\hat{v}_{FD}^{\beta_2} - \hat{v}_{FS}^{\beta_2}\hat{v}_{FD}^{\beta_1}$. (B4)

The associated smooth-pasting conditions are:

$$\begin{pmatrix} \beta_1\hat{v}_{FS}^{\beta_1-1} & \beta_2\hat{v}_{FS}^{\beta_2-1} \\ \beta_1\hat{v}_{FD}^{\beta_1-1} & \beta_2\hat{v}_{FD}^{\beta_2-1} \end{pmatrix}\begin{pmatrix} A_{1FS} \\ A_{2FD} \end{pmatrix} = \begin{pmatrix} \dfrac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta} \\ -\dfrac{D_{F|O,X}}{\delta+\theta} \end{pmatrix}, \tag{B5}$$

so, the solutions for the two option coefficients are given by:

$$\begin{pmatrix} A_{1FS} \\ A_{2FD} \end{pmatrix} = \begin{pmatrix} \beta_1\hat{v}_{FS}^{\beta_1} & \beta_2\hat{v}_{FS}^{\beta_2} \\ \beta_1\hat{v}_{FD}^{\beta_1} & \beta_2\hat{v}_{FD}^{\beta_2} \end{pmatrix}^{-1}\begin{pmatrix} \hat{v}_{FS}\dfrac{D_{F|Y,Y}-D_{F|Y,X}}{\delta+\theta} \\ -\hat{v}_{FD}\dfrac{D_{F|O,X}}{\delta+\theta} \end{pmatrix}, \tag{B6}$$

$$A_{1FS} = \frac{1}{\beta_1 \Delta_F}\left(\hat{v}_{FS} \frac{D_{F|Y,Y} - D_{F|Y,X}}{\delta+\theta} \hat{v}_{FD}^{\beta_2} + \hat{v}_{FD} \frac{D_{F|O,X}}{\delta+\theta} \hat{v}_{FS}^{\beta_2} \right),$$

$$A_{2FD} = \frac{1}{\beta_2 \Delta_F}\left(-\hat{v}_{FS} \frac{D_{F|Y,Y} - D_{F|Y,X}}{\delta+\theta} \hat{v}_{FD}^{\beta_1} + \hat{v}_{FD} \frac{D_{F|O,X}}{\delta+\theta} \hat{v}_{FS}^{\beta_1} \right). \tag{B7}$$

The two solutions for A_{1FS}, A_{2FD} yield the following two non-linear simultaneous equations for the unknown follower's thresholds $\hat{v}_{FS}, \hat{v}_{FD}$:

$$\hat{v}_{FD}^{\beta_2}\left(\hat{v}_{FS} \frac{D_{F|Y,Y} - D_{F|Y,X}}{\delta+\theta} \frac{\beta_1 - 1}{\beta_1} - \frac{D_{F|Y,Y} f_Y - D_{F|Y,X} f_X}{r} - (K - \lambda Z) \right)$$
$$= \hat{v}_{FS}^{\beta_2}\left(\lambda Z - \frac{D_{F|O,X} \hat{v}_{FD}}{\delta+\theta} \frac{\beta_1 - 1}{\beta_1} + \frac{D_{F|O,X} f_X}{r} \right), \tag{B8}$$

$$\hat{v}_{FD}^{\beta_1}\left(\hat{v}_{FS} \frac{D_{F|Y,Y} - D_{F|Y,X}}{\delta+\theta} \frac{\beta_2 - 1}{\beta_2} - \frac{D_{F|Y,Y} f_Y - D_{F|Y,X} f_X}{r} - (K - \lambda Z) \right)$$
$$= \hat{v}_{FS}^{\beta_1}\left(\lambda Z - \frac{D_{F|O,X} \hat{v}_{FD}}{\delta+\theta} \frac{\beta_2 - 1}{\beta_2} + \frac{D_{F|O,X} f_X}{r} \right). \tag{B9}$$

The leader's value-matching relationship can be expressed as:

$$\begin{pmatrix} \hat{v}_{LS}^{\beta_1} & \hat{v}_{LS}^{\beta_2} \\ \hat{v}_{LD}^{\beta_1} & \hat{v}_{LD}^{\beta_2} \end{pmatrix} \begin{pmatrix} A_{1LS} \\ A_{2LD} \end{pmatrix} = \begin{pmatrix} \hat{v}_{FS} \frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} - \frac{D_{L|Y,X} f_Y - D_{L|X,X} f_X}{r} + A_{1LSS} \hat{v}_{LS}^{\beta_1} - (K - Z) \\ Z - \frac{D_{L|X,X} \hat{v}_{LD}}{\delta+\theta} + \frac{D_{L|X,X} f_X}{r} \end{pmatrix},$$

(B10)

so, the solutions for the two option coefficients are given by:

$$\begin{pmatrix} A_{1LS} \\ A_{2LD} \end{pmatrix} = \begin{pmatrix} \hat{v}_{LS}^{\beta_1} & \hat{v}_{LS}^{\beta_2} \\ \hat{v}_{LD}^{\beta_1} & \hat{v}_{LD}^{\beta_2} \end{pmatrix}^{-1} \begin{pmatrix} \hat{v}_{LS} \frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} - \frac{D_{L|Y,X} f_Y - D_{L|X,X} f_X}{r} + A_{1LSS} \hat{v}_{LS}^{\beta_1} - (K - Z) \\ Z - \frac{D_{L|X,X} \hat{v}_{LD}}{\delta+\theta} + \frac{D_{L|X,X} f_X}{r} \end{pmatrix},$$

(B11)

$$A_{1LS} = \frac{\hat{v}_{LD}^{\beta_2}}{\Delta_L}\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} - \frac{D_{L|Y,X}f_Y - D_{L|X,X}f_X}{r} + A_{1LSS}\,\hat{v}_{LS}^{\beta_1} - (K-Z)\right)$$
$$-\frac{\hat{v}_{LS}^{\beta_2}}{\Delta_L}\left(Z - \frac{D_{L|X,X}\hat{v}_{LD}}{\delta+\theta} + \frac{D_{L|X,X}f_X}{r}\right),$$
$$A_{2LD} = -\frac{\hat{v}_{LD}^{\beta_1}}{\Delta_L}\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} - \frac{D_{L|Y,X}f_Y - D_{L|X,X}f_X}{r} + A_{1LSS}\,\hat{v}_{LS}^{\beta_1} - (K-Z)\right)$$
$$+\frac{\hat{v}_{LS}^{\beta_1}}{\Delta_L}\left(Z - \frac{D_{L|X,X}\hat{v}_{LD}}{\delta+\theta} + \frac{D_{L|X,X}f_X}{r}\right), \tag{B12}$$

where $\Delta_L = \hat{v}_{LS}^{\beta_1}\hat{v}_{LD}^{\beta_2} - \hat{v}_{LS}^{\beta_2}\hat{v}_{LD}^{\beta_1}$. (B13)

The associated smooth-pasting conditions are:

$$\begin{pmatrix} \beta_1\hat{v}_{LS}^{\beta_1-1} & \beta_2\hat{v}_{LS}^{\beta_2-1} \\ \beta_1\hat{v}_{LD}^{\beta_1-1} & \beta_2\hat{v}_{LD}^{\beta_2-1} \end{pmatrix}\begin{pmatrix} A_{1LS} \\ A_{2LD} \end{pmatrix} = \begin{pmatrix} \dfrac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} + \beta_1 A_{1LSS}\,\hat{v}_{LS}^{\beta_1-1} \\ -\dfrac{D_{L|X,X}}{\delta+\theta} \end{pmatrix}, \tag{B14}$$

so, the solutions for the two option coefficients are given by:

$$\begin{pmatrix} A_{1LS} \\ A_{2LD} \end{pmatrix} = \begin{pmatrix} \beta_1\hat{v}_{LS}^{\beta_1} & \beta_2\hat{v}_{LS}^{\beta_2} \\ \beta_1\hat{v}_{LD}^{\beta_1} & \beta_2\hat{v}_{LD}^{\beta_2} \end{pmatrix}^{-1}\begin{pmatrix} \hat{v}_{LS}\dfrac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} + \beta_1 A_{1LSS}\,\hat{v}_{LS}^{\beta_1} \\ -\hat{v}_{LD}\dfrac{D_{L|X,X}}{\delta+\theta} \end{pmatrix}, \tag{B15}$$

$$A_{1LS} = \frac{1}{\beta_1\Delta_L}\left(\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} + \beta_1 A_{1LSS}\,\hat{v}_{LS}^{\beta_1}\right)\hat{v}_{LD}^{\beta_2} + \hat{v}_{LD}\frac{D_{L|X,X}}{\delta+\theta}\hat{v}_{LS}^{\beta_2}\right),$$
$$A_{2LD} = \frac{1}{\beta_2\Delta_L}\left(-\left(\hat{v}_{LS}\frac{D_{L|Y,X} - D_{L|X,X}}{\delta+\theta} + \beta_1 A_{1LSS}\,\hat{v}_{LS}^{\beta_1}\right)\hat{v}_{LD}^{\beta_1} - \hat{v}_{LD}\frac{D_{L|X,X}}{\delta+\theta}\hat{v}_{LS}^{\beta_1}\right). \tag{B16}$$

The two solutions for A_{1LS}, A_{2LD} yield the following two non-linear simultaneous equations for the unknown follower's thresholds $\hat{v}_{LS}, \hat{v}_{LD}$:

$$
\begin{aligned}
&\hat{v}_{LD}^{\beta_2}\left(\hat{v}_{LS}\frac{D_{L|Y,X}-D_{L|X,X}}{\delta+\theta}\frac{\beta_1-1}{\beta_1}-\frac{D_{L|Y,X}f_Y-D_{L|X,X}f_X}{r}-\left(K-Z\right)\right)\\
&\qquad=\hat{v}_{LS}^{\beta_2}\left(Z-\frac{D_{L|X,X}\hat{v}_{LD}}{\delta+\theta}\frac{\beta_1-1}{\beta_1}+\frac{D_{L|X,X}f_X}{r}\right),\\
&\hat{v}_{LD}^{\beta_1}\left(\hat{v}_{LS}\frac{D_{L|Y,X}-D_{L|X,X}}{\delta+\theta}\frac{\beta_2-1}{\beta_2}-\frac{D_{L|Y,X}f_Y-D_{L|X,X}f_X}{r}+A_{1LSS}\,\hat{v}_{LS}^{\beta_1}\frac{\beta_2-\beta_1}{\beta_2}-\left(K-Z\right)\right)\\
&\qquad=\hat{v}_{LS}^{\beta_1}\left(Z-\frac{D_{L|X,X}\hat{v}_{LD}}{\delta+\theta}\frac{\beta_2-1}{\beta_2}+\frac{D_{L|X,X}f_X}{r}\right).
\end{aligned}
\tag{B17}
$$

The coefficients A_{1FSS}, A_{2FDD} can be expressed as:

$$
\begin{pmatrix}\hat{v}_{LS}^{\beta_1} & \hat{v}_{LS}^{\beta_2}\\ \hat{v}_{LD}^{\beta_1} & \hat{v}_{LD}^{\beta_2}\end{pmatrix}\begin{pmatrix}A_{1FSS}\\ A_{2FDD}\end{pmatrix}=\begin{pmatrix}\dfrac{D_{F|Y,X}-D_{F|X,X}}{\delta+\theta}\hat{v}_{LS}-\dfrac{D_{F|Y,X}-D_{F|X,X}}{r}f_X\\ \dfrac{D_{F|O,X}-D_{F|X,X}}{\delta+\theta}\hat{v}_{LD}-\dfrac{D_{F|O,X}-D_{F|X,X}}{r}f_X\end{pmatrix}. \tag{B18}
$$

Then:

$$
\begin{pmatrix}A_{1FSS}\\ A_{2FDD}\end{pmatrix}=\begin{pmatrix}\hat{v}_{LS}^{\beta_1} & \hat{v}_{LS}^{\beta_2}\\ \hat{v}_{LD}^{\beta_1} & \hat{v}_{LD}^{\beta_2}\end{pmatrix}^{-1}\begin{pmatrix}\left(D_{F|Y,X}-D_{F|X,X}\right)\left(\dfrac{\hat{v}_{LS}}{\delta+\theta}-\dfrac{f_X}{r}\right)\\ \left(D_{F|O,X}-D_{F|X,X}\right)\left(\dfrac{\hat{v}_{LD}}{\delta+\theta}-\dfrac{f_X}{r}\right)\end{pmatrix}, \tag{B19}
$$

$$
\begin{aligned}
A_{1FSS}&=\left(D_{F|Y,X}-D_{F|X,X}\right)\left(\frac{\hat{v}_{LS}}{\delta+\theta}-\frac{f_X}{r}\right)\frac{\hat{v}_{LD}^{\beta_2}}{\Delta_L}-\left(D_{F|O,X}-D_{F|X,X}\right)\left(\frac{\hat{v}_{LD}}{\delta+\theta}-\frac{f_X}{r}\right)\frac{\hat{v}_{LS}^{\beta_2}}{\Delta_L},\\
A_{2FDD}&=-\left(D_{F|Y,X}-D_{F|X,X}\right)\left(\frac{\hat{v}_{LS}}{\delta+\theta}-\frac{f_X}{r}\right)\frac{\hat{v}_{LD}^{\beta_1}}{\Delta_L}+\left(D_{F|O,X}-D_{F|X,X}\right)\left(\frac{\hat{v}_{LD}}{\delta+\theta}-\frac{f_X}{r}\right)\frac{\hat{v}_{LS}^{\beta_1}}{\Delta_L}.
\end{aligned}
\tag{B20}
$$

Appendix C Sensitivity of a 1% Increase in the Base Inputs on the VFs Across v Range

	R3	R3	R3	R3	R2	R2	R2	R2	R2	R2	R2	R2				
v	6.5	7	7.5	8	8.5	9	9.5	10	10.5	11	11.5	12	MEAN	STDEV	MAX	MIN
VF F	7.89	11.46	15.51	19.97	23.28	25.35	27.76	30.45	33.37	36.50	39.79	43.23	26.21	11.15	43.23	7.89
Δr F	0.10	0.08	0.06	0.04	-0.15	-0.10	-0.06	-0.02	0.02	0.05	0.09	0.11	**0.02**	**0.08**	**0.11**	**-0.15**
θ	-0.23	-0.30	-0.36	-0.43	-0.13	-0.18	-0.23	-0.28	-0.33	-0.38	-0.43	-0.47	-0.31	0.11	-0.13	-0.47
fX	-0.08	0.00	0.09	0.18	0.05	0.02	0.00	-0.02	-0.02	-0.02	-0.02	-0.01	0.01	0.07	0.18	-0.08
fY	-0.08	-0.12	-0.16	-0.20	-0.05	-0.05	-0.06	-0.07	-0.08	-0.09	-0.11	-0.12	-0.10	0.04	-0.05	-0.20
Z	0.04	0.06	0.07	0.09	0.04	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.05	0.02	0.09	0.04
K	-0.22	-0.35	-0.47	-0.60	-0.13	-0.15	-0.18	-0.20	-0.23	-0.26	-0.29	-0.33	-0.28	0.14	-0.13	-0.60
Δσ F	0.09	0.06	0.03	0.00	0.15	0.14	0.13	0.11	0.09	0.07	0.04	0.02	**0.08**	**0.05**	**0.15**	**0.00**
λ	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.05	0.05	0.04	0.00	0.05	0.04
Δδ F	-0.17	-0.22	-0.27	-0.32	-0.10	-0.14	-0.18	-0.21	-0.25	-0.29	-0.32	-0.35	**-0.24**	**0.08**	**-0.10**	**-0.35**
D L/XX	0.00	0.02	0.03	0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.03	0.00
D L/YX	0.10	0.16	0.23	0.30	0.09	0.08	0.06	0.05	0.04	0.03	0.02	0.01	0.10	0.09	0.30	0.01
Δ L/YY F	0.01	-0.02	-0.05	-0.08	-0.27	-0.32	-0.38	-0.44	-0.50	-0.57	-0.63	-0.71	-0.33	0.25	0.01	-0.71
Δv F	0.43	0.54	0.64	0.75	0.32	0.41	0.49	0.57	0.64	0.71	0.78	0.85	**0.59**	**0.16**	**0.85**	**0.32**
VF L	25.41	26.89	29.24	32.29	35.89	39.61	43.37	47.18	51.05	54.97	58.93	62.96	42.32	12.82	62.96	25.41
Δr L	0.03	0.08	0.12	0.16	0.18	0.19	0.20	0.20	0.21	0.22	0.22	0.23	**0.17**	**0.06**	**0.23**	**0.03**
θ	-0.07	-0.16	-0.25	-0.33	-0.39	-0.42	-0.45	-0.47	-0.50	-0.53	-0.56	-0.59	-0.39	0.16	-0.07	-0.59
fX	-0.01	-0.02	-0.01	0.01	0.03	0.03	0.04	0.04	0.04	0.05	0.05	0.06	0.03	0.03	0.06	-0.02
fY	-0.03	-0.06	-0.09	-0.12	-0.14	-0.14	-0.15	-0.15	-0.16	-0.16	-0.17	-0.17	-0.13	0.05	-0.03	-0.17
Z	0.24	0.24	0.24	0.25	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.25	0.01	0.26	0.24
K	-0.07	-0.17	-0.26	-0.35	-0.42	-0.42	-0.43	-0.44	-0.46	-0.47	-0.48	-0.49	-0.37	0.14	-0.07	-0.49
Δσ L	0.02	0.03	0.02	-0.01	-0.03	-0.04	-0.05	-0.06	-0.07	-0.08	-0.10	-0.11	**-0.04**	**0.05**	**0.03**	**-0.11**
λ	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.00	0.01	0.00
Δδ L	-0.05	-0.12	-0.19	-0.25	-0.29	-0.31	-0.34	-0.36	-0.38	-0.40	-0.42	-0.44	**-0.30**	**0.12**	**-0.05**	**-0.44**
D L/XX	-0.01	-0.01	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.01
D L/YX	0.02	0.05	0.08	0.11	0.12	0.11	0.09	0.08	0.06	0.03	0.01	-0.02	0.06	0.04	0.12	-0.02
Δ L/YY L	0.04	0.10	0.15	0.20	0.25	0.28	0.32	0.36	0.40	0.44	0.49	0.54	0.30	0.16	0.54	0.04
Δv L	0.14	0.28	0.42	0.55	0.63	0.67	0.72	0.77	0.82	0.87	0.92	0.97	**0.65**	**0.26**	**0.97**	**0.14**

Note that while the VF of both the F and L are sensitive to changes in the quantity decline rate θ, K and L/YY, and the VF L is sensitive to changes in the salvage value Z, these factors are left for further research.

Appendix D1 Complete Sensitivities of the Value Functions at v=7.5

7.5	Sensitivities	r	θ	fX	fY	Z	K	σ	λ	δ	D L/XX	D L/YX	D L/YY	v
R3, v=7.5	Panel A													7.575
	vFD	-0.39%	0.43%	0.82%	0.04%	0.03%	0.11%	-0.43%	0.03%	0.32%	0.00%	-0.07%	0.24%	
	vLD	-0.24%	0.58%	0.14%	0.20%	0.07%	0.59%	-0.22%	-0.01%	0.44%	0.06%	-0.17%	-0.33%	
	A2FD	**2.18%**	-0.48%	**2.50%**	0.05%	0.09%	0.14%	**-4.02%**	0.09%	-0.36%	0.00%	-0.08%	0.30%	
	A2LD	**2.94%**	-0.28%	1.19%	0.27%	0.49%	0.82%	**-4.03%**	-0.01%	-0.21%	0.69%	-0.24%	-0.46%	
	A2FDD	**-1.99%**	**0.75%**	**-3.71%**	0.27%	-0.24%	**0.88%**	**4.46%**	0.00%	0.56%	-1.08%	-0.51%	-0.03%	
	Panel B													
	vFS	-0.87%	0.86%	-0.51%	0.43%	-0.09%	**1.23%**	**1.00%**	-0.09%	0.64%	0.00%	0.30%	**1.41%**	
	vLS	-0.39%	0.68%	-0.30%	0.34%	-0.12%	**1.09%**	0.34%	-0.01%	0.51%	-0.07%	-0.43%	-0.28%	
	A1FS	**2.88%**	**-6.23%**	**10.64%**	**-3.39%**	**1.01%**	**-9.39%**	**-2.69%**	**1.01%**	**-4.70%**	0.00%	**5.82%**	**-20.28%**	
	A1LS	**3.63%**	**-4.47%**	**2.40%**	**-1.13%**	0.91%	**-3.40%**	**1.68%**	0.04%	**-3.37%**	-0.64%	**1.02%**	**1.94%**	
	A1FSS	0.67%	**-3.89%**	**4.33%**	**-1.47%**	0.65%	**-4.59%**	**2.34%**	0.02%	**-2.93%**	**1.13%**	**2.29%**	**1.03%**	
	Panel C													
VF F	15.51	0.41%	**-2.35%**	0.57%	-1.00%	0.48%	**-3.02%**	0.21%	0.27%	**-1.77%**	0.17%	**1.48%**	-0.33%	**4.16%**
F 3 Term1	-8.93	**6.93%**	**-3.41%**	**-7.00%**	0.00%	0.00%	0.00%	0.00%	0.00%	**-2.56%**	**1.00%**	0.00%	0.00%	**6.00%**
F 3 Term2 S	1.26	0.32%	**-4.07%**	**10.64%**	**-3.39%**	1.01%	**-9.39%**	**-5.41%**	1.01%	**-3.07%**	0.00%	**5.82%**	**-20.28%**	**2.28%**
F 3 Term3 D	29.50	**-3.34%**	**1.29%**	**2.50%**	0.05%	0.09%	0.14%	0.72%	0.09%	0.97%	0.00%	-0.08%	0.30%	**-1.74%**
F 3 Term4 SS	12.02	**-1.83%**	**-1.68%**	**4.33%**	**-1.47%**	0.65%	**-4.59%**	-0.53%	0.02%	-1.26%	1.13%	**2.29%**	1.03%	**2.28%**
F 3 Term5 DD	-18.35	3.51%	-1.02%	**-3.71%**	0.27%	-0.24%	0.88%	-0.27%	0.00%	-0.76%	**-1.08%**	-0.51%	-0.03%	**1.74%**
VF L	29.24	0.41%	-0.85%	-0.02%	-0.30%	0.83%	-0.89%	0.06%	0.01%	-0.64%	-0.02%	0.27%	0.51%	**1.42%**
L 3 Term1	-8.93	**6.93%**	**-3.41%**	**-7.00%**	0.00%	0.00%	0.00%	0.00%	0.00%	**-2.56%**	**-1.00%**	0.00%	0.00%	**6.00%**
L 3 Term2 S	13.56	**1.06%**	**-2.28%**	**2.40%**	-1.13%	0.91%	**-3.40%**	-1.16%	0.04%	**-1.71%**	-0.64%	1.02%	**1.94%**	**2.28%**
L 3 Term3 D	24.61	**-2.61%**	**1.49%**	1.19%	0.27%	0.49%	0.82%	0.71%	-0.01%	1.12%	0.69%	-0.24%	-0.46%	**-1.74%**
	Panel D	Absolute Change												
VF F	15.51	0.06	-0.36	0.09	-0.16	0.07	-0.47	0.03	0.04	-0.27	0.03	0.23	-0.05	0.64
F 3 Term1	-8.93	**0.62**	-0.30	-0.63	0.00	0.00	0.00	0.00	0.00	**-0.23**	0.09	0.00	0.00	**0.54**
F 3 Term2 S	1.26	0.00	-0.05	0.13	-0.04	0.01	-0.12	-0.07	0.01	-0.04	0.00	0.07	-0.26	0.03
F 3 Term3 D	29.50	**-0.98**	0.38	0.74	0.01	0.03	0.04	**0.21**	0.03	**0.29**	0.00	-0.02	0.09	**-0.51**
F 3 Term4 SS	12.02	**-0.22**	-0.20	0.52	-0.18	0.08	-0.55	-0.06	0.00	-0.15	0.14	0.27	0.12	**0.27**
F 3 Term5 DD	-18.35	**0.64**	-0.19	-0.68	0.05	-0.04	0.16	-0.05	0.00	-0.14	-0.20	-0.09	-0.01	**0.32**
VF L	29.24	0.12	-0.25	-0.01	-0.09	0.24	-0.26	0.02	0.00	-0.19	-0.01	0.08	0.15	0.42
L 3 Term1	-8.93	**0.62**	-0.30	-0.63	0.00	0.00	0.00	0.00	0.00	**-0.23**	-0.09	0.00	0.00	0.54
L 3 Term2 S	13.56	0.14	-0.31	0.33	-0.15	0.12	-0.46	-0.16	0.01	**-0.23**	-0.09	0.14	0.26	**0.31**
L 3 Term3 D	24.61	**-0.64**	0.37	0.29	0.07	0.12	0.20	**0.17**	0.00	**0.27**	0.17	-0.06	-0.11	**-0.43**

Appendix D2 Complete Sensitivities of the Value Functions at v=9.5

9.5	Sensitivities	r	θ	fX	fY	Z	K	σ	λ	δ	D L/XX	D L/YX	D L/YY	v
R2, v=9.5	Panel A													**8.585**
	vFD	-0.39%	0.43%	0.82%	0.04%	0.03%	0.11%	-0.43%	0.03%	0.32%	0.00%	-0.07%	0.24%	
	A2FD	**2.18%**	-0.48%	**2.50%**	0.05%	0.09%	0.14%	**-4.02%**	0.09%	-0.36%	0.00%	-0.08%	0.30%	
	Panel B													
	vFS	-0.87%	0.86%	-0.51%	0.43%	-0.09%	**1.23%**	**1.00%**	-0.09%	0.64%	0.00%	0.30%	**1.41%**	
	A1FS	**2.88%**	**-6.23%**	**10.64%**	**-3.39%**	**1.01%**	**-9.39%**	**-2.69%**	**1.01%**	**-4.70%**	0.00%	**5.82%**	**-20.28%**	
	A1LSS	**4.33%**	**-4.34%**	0.56%	-0.64%	0.10%	**-1.33%**	**2.48%**	0.10%	**-3.27%**	0.00%	**-5.98%**	**5.03%**	
	Panel C	Percentage Change												
VF F	23.28	-0.66%	-0.57%	0.22%	-0.19%	0.17%	-0.54%	0.64%	0.17%	-0.43%	0.00%	0.40%	-1.16%	1.39%
F 2 Term1	-2.05	**-34.65%**	**19.32%**	**35.00%**	0.00%	0.00%	0.00%	0.00%	0.00%	**14.51%**	0.00%	-0.74%	0.00%	-34.00%
F 2 Term2 S	1.68	0.17%	**-3.94%**	**10.64%**	**-3.39%**	1.01%	**-9.39%**	**-5.58%**	1.01%	**-2.96%**	0.00%	**5.82%**	**-20.28%**	**2.28%**
F 2 Term3 D	23.65	**-3.67%**	**1.40%**	**2.50%**	0.05%	0.09%	0.14%	1.02%	0.09%	1.05%	0.00%	-0.08%	0.30%	-1.74%
VF L	35.89	0.51%	-1.09%	0.08%	-0.38%	0.71%	-1.16%	-0.08%	0.01%	-0.82%	0.00%	0.32%	0.69%	1.75%
L 2 Term1	40.98	0.26%	-0.72%	0.00%	-0.26%	0.00%	0.00%	0.00%	0.00%	-0.54%	0.00%	1.00%	0.00%	1.26%
L2 Term2 SS	4.91	**1.58%**	**-2.01%**	0.56%	-0.64%	0.10%	-1.33%	-0.57%	0.10%	-1.51%	0.00%	**-5.98%**	**5.03%**	**2.28%**
L 2 Term 3	-10.00	0.00%	0.00%	0.00%	0.00%	**2.50%**	**-3.50%**	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
	Panel D	Absolute Change												
VF F	23.28	-0.15	-0.13	0.05	-0.05	0.04	-0.13	0.15	0.04	-0.10	0.00	0.09	-0.27	0.32
F 2 Term1	-2.05	**0.71**	-0.40	-0.72	0.00	0.00	0.00	0.00	0.00	**-0.30**	0.00	0.02	0.00	**0.70**
F 2 Term2 S	1.68	0.00	-0.07	0.18	-0.06	0.02	-0.16	-0.09	0.02	-0.05	0.00	0.10	-0.34	0.04
F 2 Term3 D	23.65	**-0.87**	0.33	0.59	0.01	0.02	0.03	**0.24**	0.02	**0.25**	0.00	-0.02	0.07	**-0.41**
VF L	35.89	0.18	-0.39	0.03	-0.14	0.26	-0.42	-0.03	0.01	-0.29	0.00	0.12	0.25	0.63
L 2 Term1	40.98	0.11	-0.29	0.00	-0.11	0.00	0.00	0.00	0.00	**-0.22**	0.00	0.41	0.00	**0.52**
L2 Term2 SS	4.91	0.08	-0.10	0.03	-0.03	0.01	-0.07	-0.03	0.01	-0.07	0.00	-0.29	0.25	0.11
L 2 Term 3	-10.00	0.00	0.00	0.00	0.00	0.25	-0.35	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Appendix E Δ & Γ of the Leader and Follower Value Functions

	A	B	C	D
36	v	7.5		
37	**Leader Divest VF v=7.5**	29.2381		B12*(B36/(B4+B11)-B5/B3)+B30*(B36^B21)+B31*(B36^B22)
38	ODE L3	0.0000		0.5*(B9^2)*(B36^2)*B40+(B3-B4-B11)*B36*B39-B3*B37+B12*(B36-B5)
39	G'(v)	**5.4472**		B12*(1/(B4+B11))+B30*B21*(B36^(B21-1))+B31*B22*(B36^(B22-1))
40	G''(v)	2.8271		B30*B21*(B21-1)*(B36^(B21-2))+B31*B22*(B22-1)*(B36^(B22-2))
41	G(vLD)	25.0000		B12*(B25/(B4+B11)-B5/B3)+B30*(B25^B21)+B31*(B25^B22)
42	**Z**	25.0000		
43	SP	0.0000		B12*(1/(B4+B11))+B30*B21*(B25^(B21-1))+B31*B22*(B25^(B22-1))
44	**Follower Divest VF v=6**	5.2278		B15*(B82/(B4+B11)-B5/B3)+B27*(B82^B21)+B28*(B82^B22)
45	ODE L4	0.0000		0.5*(B9^2)*(B82^2)*B47+(B3-B4-B11)*B82*B46-B3*B44+(B82-B5)
46	G'(v)	**1.6995**		B15*(1/(B4+B11))+B27*B21*(B82^(B21-1))+B28*B22*(B82^(B22-1))
47	G''(v)	5.9948		B27*B21*(B21-1)*(B82^(B21-2))+B28*B22*(B22-1)*(B82^(B22-2))
48	**G(vFD)**	5.0000		B15*(B23/(B4+B11)-B5/B3)+B27*(B23^B21)+B28*(B23^B22)
49	**λZ**	5.0000		B10*B7
50	SP	0.0000		B15*(1/(B4+B11))+B27*B21*(B23^(B21-1))+B28*B22*(B23^(B22-1))
51	**Leader Switch VF v=9.5**	53.3704		B16*(B83/(B4+B11)-B6/B3)+B29*(B83^B21)
52	ODE L2	0.0000		0.5*(B9^2)*(B83^2)*B54+(B3-B4-B11)*B83*B53-B3*B51+B16*(B83-B6)
53	G'(v)	**7.5779**		B16*(1/(B4+B11))+B29*B21*(B83^(B21-1))
54	G''(v)	0.2007		B29*B21*(B21-1)*(B83^(B21-2))
55	G(vLS)	34.1152		B16*(B26/(B4+B11)-B6/B3)+B29*(B26^B21)-(B8-B7)
56	V*	34.1152		B12*(B26/(B4+B11)-B5/B3)+B30*(B26^B21)+B31*(B26^B22)
57	SP	0.0000		B16*(1/(B4+B11))+B29*(B21)*(B26^(B21-1))-(B12*(1/(B4+B11))+B30*B21*(B26^(B21-1))+B31*B22*(B26^(B22-1)))
58	**Follower Before L S/D**	15.5103		B13*(B36/(B4+B11)-B5/B3)+B27*(B36^B21)+B28*(B36^B22)+B32*(B36^B21)+B33*(B36^B22)
59	ODE v=7.5 L3	0.0000		0.5*(B9^2)*(B36^2)*B61+(B3-B4-B11)*B36*B60-B3*B58+B13*(B36-B5)
60	G'(v)	**8.5316**		B13*(1/(B4+B11))+B27*B21*(B36^(B21-1))+B28*B22*(B36^(B22-1))+B32*B21*(B36^(B21-1))+B33*B22*(B36^(B22-1))
61	G''(v)	1.6453		B27*B21*(B21-1)*(B36^(B21-2))+B28*B22*(B22-1)*(B36^(B22-2))+B32*B21*(B21-1)*(B36^(B21-2))+B33*B22*(B22-1)*(
62	G(vLS)	22.4248		B17*(B26/(B4+B11)-B5/B3)+B27*(B26^B21)+B28*(B26^B22)
63	V*	22.4248		B13*(B26/(B4+B11)-B5/B3)+B27*(B26^B21)+B28*(B26^B22)+B32*(B26^B21)+B33*(B26^B22)
64	SP1	-6.3414		(B17-B13)/(B4+B11)-B32*B21*(B26^(B21-1))-B33*B22*(B26^(B22-1))
65	GF(vLD)	5.4100		B15*(B25/(B4+B11)-B5/B3)+B27*(B25^B21)+B28*(B25^B22)
66	V*	5.4100		B13*(B25/(B4+B11)-B5/B3)+B27*(B25^B21)+B28*(B25^B22)+B32*(B25^B21)+B33*(B25^B22)
67	SP2	-3.3258		(B15-B13)/(B4+B11)-B32*B21*(B25^(B21-1))-B33*B22*(B25^(B22-1))
68	**Follower After L Switch**	27.7577		B17*(B83/(B4+B11)-B5/B3)+B27*(B83^B21)+B28*(B83^B22)
69	ODE L2 v=9.5	0.0000		0.5*(B9^2)*(B83^2)*B71+(B3-B4-B11)*B83*B70-B3*B68+B17*(B83-B5)
70	G'(v)	**5.1171**		B17*(1/(B4+B11))+B27*B21*(B83^(B21-1))+B28*B22*(B83^(B22-1))
71	G''(v)	1.1202		B27*B21*(B21-1)*(B83^(B21-2))+B28*B22*(B22-1)*(B83^(B22-2))
72	G(vFS)	45.0932		B19*(B24/(B4+B11)-B6/B3)-(B8-B10*B7)
73	V*	45.0932		B17*(B24/(B4+B11)-B5/B3)+B27*(B24^B21)+B28*(B24^B22)
74	SP	0.0000		B17*(1/(B4+B11))+B27*B21*(B24^(B21-1))+B28*B22*(B24^(B22-1))-(B19*(1/(B4+B11)))
75	**Leader After F Switch**	76.7857		B18*(B84/(B4+B11)-B6/B3)
76	ODE L1 v=12.5	0.0000		0.5*(B9^2)*(B84^2)*B78+(B3-B4-B11)*B84*B77-B3*B75+B18*(B84-B6)
77	G'(v)	**7.1429**		B18*(1/(B4+B11)
78	G''(v)	0.0000		0.0000
79	G(vFS)	75.0932		B18*(B24/(B4+B11)-B6/B3)
80	V*	75.0932		B16*(B24/(B4+B11)-B6/B3)+B29*(B24^B21)
81	SP	1.0096		(B16-B18)*(1/(B4+B11))+B29*B21*(B24^(B21-1))
82	v	6.0000		
83	v	9.5000		
84	v	12.5000		
85	**Follower After Switch L1**	**7.1429**		B19*(1/(B4+B11))

Appendix F Analytical and Numerical Vegas, v=7.5, 9.5

Differentiate the leader's value function with respect to revenue v yields:

$$\frac{\partial V_L(v)}{\partial v}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial v}=D_{L|Y,Y}\dfrac{1}{\delta+\theta} \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial v}=D_{L|Y,X}\dfrac{1}{\delta+\theta}+\beta_1 A_{1LSS}\, v^{\beta_1-1} \text{ for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial v}=D_{L|X,X}\dfrac{1}{\delta+\theta}+\beta_1 A_{1LS}\, v^{\beta_1-1}+\beta_2 A_{2LD}\, v^{\beta_2-1} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial v}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases} \tag{F1}$$

Differentiate the leader's value function with respect to volatility σ yields:

$$\frac{\partial V_L(v)}{\partial \sigma}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial \sigma}=0 \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial \sigma}=v^{\beta_1}\dfrac{\partial A_{1LSS}}{\partial \sigma}+A_{1LSS}\, v^{\beta_1}\, log(v)\dfrac{\partial \beta_1}{\partial \sigma} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial \sigma}=v^{\beta_1}\dfrac{\partial A_{1LS}}{\partial \sigma}+A_{1LS}\, v^{\beta_1}\, log(v)\dfrac{\partial \beta_1}{\partial \sigma}\\ \qquad +v^{\beta_2}\dfrac{\partial A_{2LD}}{\partial \sigma}+A_{2LD}\, v^{\beta_2}\, log(v)\dfrac{\partial \beta_2}{\partial \sigma} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial \sigma}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases} \tag{F2}$$

Differentiation of the leader's value function with respect to the interest rate r yields:

$$\frac{\partial V_L(v)}{\partial r}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial r}=D_{L|Y,Y}\dfrac{f_Y}{r^2} \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial r}=D_{L|Y,X}\dfrac{f_Y}{r^2}+v^{\beta_1}\dfrac{\partial A_{1LSS}}{\partial r}+A_{1LSS}\, v^{\beta_1}\, log(v)\dfrac{\partial \beta_1}{\partial r} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial r}=D_{L|X,X}\dfrac{f_Y}{r^2}+v^{\beta_1}\dfrac{\partial A_{1LS}}{\partial r}+A_{1LS}\, v^{\beta_1}\, log(v)\dfrac{\partial \beta_1}{\partial r}\\ \qquad +v^{\beta_2}\dfrac{\partial A_{2LD}}{\partial r}+A_{2LD}\, v^{\beta_2}\, log(v)\dfrac{\partial \beta_2}{\partial r} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial r}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases} \tag{F3}$$

Differentiate the leader's value function with respect to the yield δ yields:

$$\frac{\partial V_L(v)}{\partial \delta}=\begin{cases}\dfrac{\partial V_{L1}(v)}{\partial \delta}=-D_{L|Y,Y}\,\dfrac{v}{(\delta+\theta)^2} \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{L2}(v)}{\partial \delta}=-D_{L|Y,X}\,\dfrac{v}{(\delta+\theta)^2}+v^{\beta_1}\,\dfrac{\partial A_{1LSS}}{\partial \delta}+A_{1LSS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial \delta} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{L3}(v)}{\partial \delta}=-D_{L|X,X}\,\dfrac{v}{(\delta+\theta)^2}+v^{\beta_1}\,\dfrac{\partial A_{1LS}}{\partial \delta}+A_{1LS}\,v^{\beta_1}\,log(v)\dfrac{\partial \beta_1}{\partial \delta}\\ \qquad +v^{\beta_2}\,\dfrac{\partial A_{2LD}}{\partial \delta}+A_{2LD}\,v^{\beta_2}\,log(v)\dfrac{\partial \beta_2}{\partial \delta} \quad \text{for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{L4}(v)}{\partial \delta}=0 \quad \text{for } v\leq \hat{v}_{LD}.\end{cases}$$

(F4)

Differentiate the follower's value function with respect to v yields:

$$\frac{\partial V_F(v)}{\partial v}=\begin{cases}\dfrac{\partial V_{F1}(v)}{\partial v}=D_{F|Y,Y}\,\dfrac{1}{\delta+\theta} \quad \text{for } v\geq \hat{v}_{FS}\\ \dfrac{\partial V_{F2}(v)}{\partial v}=D_{F|Y,X}\,\dfrac{1}{\delta+\theta}+\beta_1 A_{1FS}\,v^{\beta_1-1}+\beta_2 A_{2FD}\,v^{\beta_2-1} \text{ for } \hat{v}_{LS}\leq v<\hat{v}_{FS},\\ \dfrac{\partial V_{F3}(v)}{\partial v}=D_{F|X,X}\,\dfrac{1}{\delta+\theta}+\beta_1 A_{1FS}\,v^{\beta_1-1}+\beta_2 A_{2FD}\,v^{\beta_2-1}+\beta_1 A_{1FSS}\,v^{\beta_1-1}+\beta_2 A_{2FDD}\,v^{\beta_2-1} \text{ for } \hat{v}_{LD}<v<\hat{v}_{LS},\\ \dfrac{\partial V_{F4}(v)}{\partial v}=D_{F|O,X}\,\dfrac{1}{\delta+\theta}+\beta_1 A_{1FS}\,v^{\beta_1-1}+\beta_2 A_{2FD}\,v^{\beta_2-1} \text{ for } \hat{v}_{FD}\leq v<\hat{v}_{LD}\\ \dfrac{\partial V_{F5}(v)}{\partial v}=0 \quad \text{for } v\leq \hat{v}_{FD}.\end{cases}$$

(F5)

Differentiation of the follower's value function with respective to volatility yields:

$$\frac{\partial V_F(v)}{\partial \sigma}=\begin{cases}\frac{\partial V_{F1}(v)}{\partial \sigma}=0 \quad \text{for } v \geq \hat{v}_{FS} \\ \frac{\partial V_{F2}(v)}{\partial \sigma}=\frac{\partial A_{1FS}}{\partial \sigma}v^{\beta_1}+\frac{\partial A_{2FD}}{\partial \sigma}v^{\beta_2} \\ \quad +A_{1FS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial \sigma}+A_{2FD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial \sigma} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS}, \\ \frac{\partial V_{F3}(v)}{\partial \sigma}=\frac{\partial A_{1FS}}{\partial \sigma}v^{\beta_1}+\frac{\partial A_{2FD}}{\partial \sigma}v^{\beta_2}+\frac{\partial A_{1FSS}}{\partial \sigma}v^{\beta_1}+\frac{\partial A_{2FDD}}{\partial \sigma}v^{\beta_2} \\ \quad +A_{1FS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial \sigma}+A_{1FSS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial \sigma} \\ \quad +A_{2FD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial \sigma}+A_{2FDD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial \sigma} \quad \text{for } \hat{v}_{LD}\leq v<\hat{v}_{LS}, \\ \frac{\partial V_{F4}(v)}{\partial \sigma}=\frac{\partial A_{1FS}}{\partial \sigma}v^{\beta_1}+\frac{\partial A_{2FD}}{\partial \sigma}v^{\beta_2} \\ \quad +A_{1FS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial \sigma}+A_{2FD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial \sigma} \quad \text{for } \hat{v}_{FD}\leq v<\hat{v}_{LD}, \\ \frac{\partial V_{F5}(v)}{\partial \sigma}=0 \quad \text{for } v<\hat{v}_{FD}. \end{cases} \tag{F6}$$

Differentiation of the follower's value function with respective to interest rate changes yields:

$$\frac{\partial V_F(v)}{\partial r}=\begin{cases}\frac{\partial V_{F1}(v)}{\partial r}=D_{F|Y,Y}\frac{f_Y}{(r)^2} \quad \text{for } v \geq \hat{v}_{FS} \\ \frac{\partial V_{F2}(v)}{\partial r}=D_{F|Y,X}\frac{f_X}{(r)^2}+\frac{\partial A_{1FS}}{\partial r}v^{\beta_1}+\frac{\partial A_{2FD}}{\partial r}v^{\beta_2} \\ \quad +A_{1FS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial r}+A_{2FD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial r} \quad \text{for } \hat{v}_{LS}\leq v<\hat{v}_{FS}, \\ \frac{\partial V_{F3}(v)}{\partial r}=D_{F|X,X}\frac{f_X}{(r)^2}+\frac{\partial A_{1FS}}{\partial r}v^{\beta_1}+\frac{\partial A_{2FD}}{\partial r}v^{\beta_2}+\frac{\partial A_{1FSS}}{\partial r}v^{\beta_1}+\frac{\partial A_{2FDD}}{\partial r}v^{\beta_2} \\ \quad +A_{1FS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial r}+A_{1FSS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial r} \\ \quad +A_{2FD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial r}+A_{2FDD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial r} \quad \text{for } \hat{v}_{LD}\leq v<\hat{v}_{LS}, \\ \frac{\partial V_{F4}(v)}{\partial r}=D_{F|O,X}\frac{f_X}{(r)^2}+\frac{\partial A_{1FS}}{\partial r}v^{\beta_1}+\frac{\partial A_{2FD}}{\partial r}v^{\beta_2} \\ \quad +A_{1FS}\,v^{\beta_1}\,log(v)\frac{\partial \beta_1}{\partial r}+A_{2FD}\,v^{\beta_2}\,log(v)\frac{\partial \beta_2}{\partial r} \quad \text{for } \hat{v}_{FD}\leq v<\hat{v}_{LD}, \\ \frac{\partial V_{F5}(v)}{\partial r}=0 \quad \text{for } v<\hat{v}_{FD}. \end{cases} \tag{F7}$$

Differentiation of the follower's value function with respective to conyield changes yields:

$$\frac{\partial V_F(v)}{\partial \delta} = \begin{cases} \frac{\partial V_{F1}(v)}{\partial \delta} = -D_{F|Y,Y} \frac{v}{(\delta+\theta)^2} \quad \text{for } v \geq \hat{v}_{FS} \\ \frac{\partial V_{F2}(v)}{\partial \delta} = -D_{F|Y,X} \frac{v}{(\delta+\theta)^2} + \frac{\partial A_{1FS}}{\partial \delta} v^{\beta_1} + \frac{\partial A_{2FD}}{\partial \delta} v^{\beta_2} \\ \quad + A_{1FS} v^{\beta_1} log(v) \frac{\partial \beta_1}{\partial \delta} + A_{2FD} v^{\beta_2} log(v) \frac{\partial \beta_2}{\partial \delta} \quad \text{for } \hat{v}_{LS} \leq v < \hat{v}_{FS}, \\ \frac{\partial V_{F3}(v)}{\partial \delta} = -D_{F|X,X} \frac{v}{(\delta+\theta)^2} + \frac{\partial A_{1FS}}{\partial \delta} v^{\beta_1} + \frac{\partial A_{2FD}}{\partial \delta} v^{\beta_2} + \frac{\partial A_{1FSS}}{\partial \delta} v^{\beta_1} + \frac{\partial A_{2FDD}}{\partial \delta} v^{\beta_2} \\ \quad + A_{1FS} v^{\beta_1} log(v) \frac{\partial \beta_1}{\partial r} + A_{1FSS} v^{\beta_1} log(v) \frac{\partial \beta_1}{\partial r} \\ \quad + A_{2FD} v^{\beta_2} log(v) \frac{\partial \beta_2}{\partial r} + A_{2FDD} v^{\beta_2} log(v) \frac{\partial \beta_2}{\partial r} \quad \text{for } \hat{v}_{LD} \leq v < \hat{v}_{LS}, \\ \frac{\partial V_{F4}(v)}{\partial \delta} = -D_{F|O,X} \frac{v}{(\delta+\theta)^2} + \frac{\partial A_{1FS}}{\partial \delta} v^{\beta_1} + \frac{\partial A_{2FD}}{\partial \delta} v^{\beta_2} \\ \quad + A_{1FS} v^{\beta_1} log(v) \frac{\partial \beta_1}{\partial \delta} + A_{2FD} v^{\beta_2} log(v) \frac{\partial \beta_2}{\partial \delta} \quad \text{for } \hat{v}_{FD} \leq v < \hat{v}_{LD}, \\ \frac{\partial V_{F5}(v)}{\partial \delta} = 0 \quad \text{for } v < \hat{v}_{FD}. \end{cases} \tag{F8}$$

DELTA

The derivatives with respect to v=9.5 or 7.5 for the value function for the leader are:

$$\begin{aligned} \frac{\partial V_{L2}(v)}{\partial v} &= D_{L|Y,X} \frac{1}{\delta+\theta} + \beta_1 A_{1LSS} v^{\beta_1 - 1} = 6.0714 + 1.5064 = 7.5779, \\ \frac{\partial V_{L3}(v)}{\partial v} &= D_{L|X,X} \frac{1}{\delta+\theta} + \beta_1 A_{1LS} v^{\beta_1 - 1} + \beta_2 A_{2LD} v^{\beta_2 - 1} = 7.1429+4.0966-5.7922=5.4472 \end{aligned} \tag{F9}$$

The derivatives with respect to v=9.5 or 7.5 for the value function for the follower are:

$$\frac{\partial V_{F2}(v)}{\partial v} = D_{F|Y,X}\frac{1}{\delta+\theta} + \beta_1 A_{1FS}\, v^{\beta_1-1} + \beta_2 A_{2FD}\, v^{\beta_2-1} = 8.2143 + .5151 - 3.6123 = 5.1171 \quad v = 9.5$$

$$\frac{\partial V_{F3}(v)}{\partial v} = D_{F|X,X}\frac{1}{\delta+\theta} + \beta_1 A_{1FS}\, v^{\beta_1-1} + \beta_2 A_{2FD}\, v^{\beta_2-1} + \beta_1 A_{1FSS}\, v^{\beta_1-1} + \beta_2 A_{2FDD}\, v^{\beta_2-1} =$$

$$7.1429 + .3819 - 6.9456 + 3.6319 + 4.3206 = 8.5316 \quad v = 7.5$$

(F10)

VEGA

The derivatives with respect to σ =20% for the value function for the leader are:

$$\frac{\partial V_{L2}(v)}{\partial \sigma} = .4763\, v^{2.2656} + .0385 * -7.1127 v^{2.2656} LN(v) = -22.9924 \quad \text{for } v = 9.5$$

$$\frac{\partial V_{L3}(v)}{\partial \sigma} = 1.1971\, v^{2.2656} - 17989\, v^{-1.7656}$$

$$+ .1412 * -7.1127\, v^{2.2656} LN(v) + 10453\, v^{-1.7656} LN(v) = 8.2421 \quad \text{for } v = 7.5 \qquad \text{(F11)}$$

The derivatives with respect to σ =20% for the value function for the follower are:

$$\frac{\partial V_{F2}(v)}{\partial \sigma} = -.1702\, v^{2.2656} - 21519\, v^{-1.7656}$$

$$+ .0132(-7.1127) v^{2.2656} LN(v) + 1034(12.1227) v^{-1.7656} LN(v) = 63.3195 \quad \text{for } v = 9.5$$

$$\frac{\partial V_{F3}(v)}{\partial \sigma} = -.1702 v^{2.2656} - .0936\, v^{2.2656} LN(v) + 21519.10\, v^{-1.7656} + 12534.39\, v^{-1.7656} LN(v) \qquad \text{(F12)}$$

$$+1.4598 v^{2.2656} - .8903\, v^{2.2656} LN(v) + 14856.16\, v^{-1.7656} - 7796.98\, v^{-1.7656} LN(v) = 15.6286 \quad \text{for } v = 7.5$$

RHO

The derivatives with respect to r=8% for the value function for the leader are:

$$\frac{\partial V_{L2}(v)}{\partial r} = 132.8125 + 2.0614 v^{2.2656} + .0385 * -15.6974 v^{2.2656}\, LN(v) = 247.8524 \quad \text{for v=9.5,}$$

$$\frac{\partial V_{L3}(v)}{\partial r} = 156.2500 + 6.3643 v^{2.2656} - 2.2163\, v^{2.2656} LN(v) \qquad \text{(F13)}$$

$$+ 31070.6485 v^{-1.7656} - 29602.5303\, v^{-1.7656} LN(v) = -476.1200 \quad \textit{for} \quad v = 7.5$$

The derivatives with respect to r=8% for the value function for the follower are:

$$\frac{\partial V_{F2}(v)}{\partial r} = 179.6875 + .4872v^{2.2656} + 27711.8476v^{-1.7656}$$
$$-.2066v^{2.2656}LN(v) - 35496.8416v^{-1.7656}LN(v) = -797.2131 \quad \text{for } v = 9.5$$
$$\frac{\partial V_{F3}(v)}{\partial r} = 156.2500 + .4872v^{2.2656} - .2066v^{2.2656}LN(v) + 27711.8476v^{-1.7656} - 35496.8416v^{-1.7656}LN(v)$$
$$+1.0976v^{2.2656} - 1.9649v^{2.2656}LN(v) - 15850.7179v^{-1.7656} + 22080.6935v^{-1.7656}LN(v) = -544.3535 \quad \text{for } v = 7.5$$

(F14)

EPSILON

The derivatives with respect to δ=3% for the value function for the leader are:

$$\frac{\partial V_{L2}(v)}{\partial \delta} = -823.9796 - 4.2578v^{2.2656} + 1.0817v^{2.2656}\,LN(v) = -1{,}123.035 \quad \text{for v=9.5},$$
$$\frac{\partial V_{L3}(v)}{\partial \delta} = -765.3061 - 16.1154v^{2.2656} + 3.9676v^{2.2656}LN(v) \qquad \text{(F15)}$$
$$-6006.1197v^{-1.7656} + 18898.5553v^{-1.7656}LN(v) = -630.9151 \quad \textit{for} \quad v = 7.5$$

The derivatives with respect to δ=3% for the value function for the follower are:

$$\frac{\partial V_{F2}(v)}{\partial \delta} = -1114.7959 - 345.4775v^{2.2656} - 231.2765v^{-1.7656}$$
$$+.3699v^{2.2656}LN(v) + 22661.5433v^{-1.7656}LN(v) = -596.631 \quad \text{for } v = 9.5$$
$$\frac{\partial V_{F3}(v)}{\partial \delta} = -765.3061 - 2.1053v^{2.2656} + .3699v^{2.2656}LN(v) - 12313.0468v^{-1.7656} + 22661.5433v^{-1.7656}LN(v)$$
$$-12.3804v^{2.2656} + 3.5175v^{2.2656}LN(v) + 12007.2107v^{-1.7656} - 14006.5386v^{-1.7656}LN(v) = -921.0211 \quad \text{for } v = 7.5$$

(F16)

	A	B	C	D
79	**Partial**	Analytical	LN(9.5)	2.2513
80	δA1LS/δσ	1.1971	δβ1/δσ	-7.1127
81	δA1LD/δσ	-17989.0420	δβ2/δσ	12.1127
82	δA1FSS/δσ	1.4598		
83	δA2FDD/δσ	14856.1553		
84	δA1FS/δσ	-0.1702		
85	δA2FD/δσ	-21519.1000		
86	δA1LSS/δσ	0.4763		
87	δVL2/δσ	-22.9914	B86*(B36^B22)+B30*D80*(B36^B22)*LN(B36)	
88	**Numerical Derivative**			
89	δA1LS/δσ	1.1971	(C30-B30)/(C9-B9)	
90	δA1LD/δσ	-17989.0350	(C31-B31)/(C9-B9)	
91	δA1FSS/δσ	1.4598	(C32-B32)/(C9-B9)	
92	δA2FDD/δσ	14856.1497	(C33-B33)/(C9-B9)	
93	δA1FS/δσ	-0.1702	(C27-B27)/(C9-B9)	
94	δA2FD/δσ	-21519.1406	(C28-B28)/(C9-B9)	
95	δA1LSS/δσ	0.4763	(C29-B29)/(C9-B9)	
96	**Num - Partial**			
97	δA1LS/δσ	0.0000	B88-B79	
98	δA1LD/δσ	0.0070	B89-B82	
99	δA1FSS/δσ	0.0000	B90-B81	
100	δA2FDD/δσ	-0.0056	B91-B82	
101	δA1FS/δσ	0.0000	B92-B83	
102	δA2FD/δσ	-0.0406	B93-B84	
103	δA1LSS/δσ	0.0000	B94-B85	
104	**R2**			
105	δVL2/δσ	-22.9914	B86*(B36^B22)+B30*D80*(B36^B22)*LN(B36)	
106		78.16	B86*(B36^B22)	
107		(101.15)	B30*D80*(B36^B22)*LN(B36)	
108		(22.9914)	SUM(B106:B107)	
109	δVF2/δσ	63.3195	B84*(B36^B22)+B85*(B36^B23)+B28*E22*(B36^B22)*LN(B36)+B29*E23*(B36^B23)*LN(B36)	
110		-27.9290	B84*(B36^B22)	
111		-404.1942	B85*(B36^B23)	
112		-34.5880	B28*E22*(B36^B22)*LN(B36)	
113		530.0306	B29*E23*(B36^B23)*LN(B36)	
114		63.3195	SUM(B108:B111)	
115	**R3**			
116	δVL3/δσ	8.2421	B80*(B70^B22)+B31*D80*(B70^B22)*LN(B70)+B81*(B70^B23)+B32*D81*(B70^B23)*LN(B70)	
117	δVL3/δσ	8.2421	SUM(B117:B120)	
118		114.9880	B80*(B70^B22)	
119		(194.3569)	B31*D80*(B70^B22)*LN(B70)	
120		(512.8980)	B81*(B70^B23)	
121		600.5090	B32*D81*(B70^B23)*LN(B70)	
122	δVF3/δσ	15.6286	B84*(B70^B22)+B28*D80*(B70^B22)*LN(B70)+B85*(B70^B23)+B29*D81*(B70^B23)*LN(B70)+B82*(B70^B22)+B33*D80*(B70^B22)*LN(B70)+B83*(B70^B23)+B34*D81*(B70^B23)*LN(B70)	
123		-16.3481	B84*(B70^B22)	
124		-18.1200	B28*D80*(B70^B22)*LN(B70)	
125		-613.5459	B85*(B70^B23)	
126		720.0794	B29*D81*(B70^B23)*LN(B70)	
127		140.2215	B82*(B70^B22)	
128		-172.3093	B33*D80*(B70^B22)*LN(B70)	
129		423.5741	B83*(B70^B23)	
130		-447.9230	B34*D81*(B70^B23)*LN(B70)	
131		15.6286	SUM(B123:B130)	
132	**R4**	6.0000	v	
133	δVF3/δσ	20.1417	B84*(B132^B22)+B28*D80*(B132^B22)*LN(B132)+B85*(B132^B23)+B29*D81*(B132^B23)*LN(B132)	
134		-9.8608	B84*(B132^B22)	
135	-	-9.7192	B28*D80*(B132^B22)*LN(B132)	
136		-909.8042	B85*(B132^B23)	
137	-	949.5259	B29*D81*(B132^B23)*LN(B132)	
138		20.1417	SUM(B134:B137)	

Appendix G Literature Review of Some Competitive Real Option Portfolio Partial Derivatives

Paxson & Pinto (2003) assume that market share reflects new customers entering (birth) at a rate λ and old customers departing (death) at a rate υ, so the population size is asymmetrically distributed at the rate $\rho=\lambda/\upsilon$. The market yields a net revenue flow x with a constant drift and volatility σ, but is adjusted by a multiplier **a***ρ, where **a** is the leader's initial market share (IMS). The value functions for the L and F are determined from value matching and smooth pasting conditions. The option to invest in such a market project is sensitive to changes in σ, shown in Figure 12.1 (positive vega), and to changes in **a** and ρ in Figures 12.2/12.3. The value function of the L is somewhat more complicated. In the next to last section, 12.3 these authors provide the analytical partial derivatives of the VF to **a** (labelled MS Δ) , ρ (labelled Ratio Δ), σ (labelled vega) and x (labelled Δ) for before and after x reaches the respective threshold for the L and F, along with Excel diagrams showing MS Δ across a range of x revenues, Ratio Δ across a range of x, and thresholds across a range of σ.

$$\frac{\partial V_L(x)}{\partial a} = \begin{cases} \dfrac{\beta_1 K \rho[(\beta_1 - 1)\delta + \rho(-1 + (\beta_1(1 - ar + au))]}{\delta(\beta_1 - 1)(ar - 1)^2} (\dfrac{x}{x_F})^{\beta_1} & \text{for } x < \hat{x}_F, \\ \dfrac{x\rho}{\delta} \quad \text{for } \hat{x}_F < x \end{cases}$$

$$\frac{\partial V_F(x)}{\partial x} = \begin{cases} \dfrac{1}{\delta} - \dfrac{\beta_1 (\dfrac{x}{x_F})^{\beta_1 - 1}}{\delta} + \dfrac{(a\beta_1\rho(\dfrac{x}{x_F})^{\beta_1 - 1}}{\gamma} & \text{for } x < \hat{x}_F, \\ \dfrac{a}{\delta}\rho \quad \text{for } \hat{x}_F < x \end{cases}$$

The discussion notes the opposite sensitivities of the VFs to changes in **a**, so "pre-emption is obvious and seems to justify what is described in the literature as the fear of pre-emption." Also, the relative VF confirm "the adage, if you're ahead, watch the competition."

Tsekrekos (2003) shows the sensitivity of the L and F value functions to market share, assumed to be constant after the F enters (exercises an investment opportunity). The market yields a net revenue flow x with a constant drift and volatility s. The leader receives **a** x, where **a** is the leader's initial market share, while the follower receives (1-**a**)x. The value functions for the L and F are determined from value matching and smooth pasting conditions. There are analytical solutions for the partial derivative of the value functions to **a**, along with diagrams of the market share derivatives across a revenue range, before and after the F invests.

$$\frac{\partial V_L(x)}{\partial a} = \begin{cases} -\frac{\beta_1 x}{\delta}\left(\frac{(1-a)(\beta_1-1)x}{\beta_1 K\delta}\right)^{\beta_1-1} & \text{for } x < \hat{x}_F, \\ -\frac{x}{\delta} \quad \text{for } \hat{x}_F < x \end{cases}$$

The author provides a page discussion regarding **a**: "an increase in a has an opposing effect on the value for the L and F…a higher **a** value increases the market share the L retains after the F enters, but also augments the period of time that the L earns monopolistic rents, by delaying the optimal F entry."

There are also analytical solutions for the partial derivative of the value functions to x, termed delta, along with diagrams of the delta derivatives across a range of market shares a.

$$\frac{\partial V_L(x)}{\partial x} = \begin{cases} \frac{1}{\delta} - \frac{\beta_1^2 K}{(\beta_1-1)x}\left(\frac{(1-a)(\beta_1-1)x}{\beta_1 K\delta}\right)^{\beta_1} & \text{for } x < \hat{x}_F, \\ \frac{a}{\delta} \quad \text{for } \hat{x}_F < x \end{cases}$$

Following the conventional L-F pattern, the L delta is first a positive function of increasing x, until before reaching the F threshold, it is negative, and then positive (and constant) after the threshold. "Intuitively, the larger **a**, the more sensitive the VF L is to increasing x. The author does not conclude that the L might encourage increasing x up to an optimum level x*, and for a while leave encouraging further x increases to the F, who benefits more until the F threshold from x increases.