Optimal Strategy for a Fund Manager with Option Compensation

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Abstract

I consider the problem of portfolio optimization for a manager whose compensation is given by the sum of a constant and a variable term. The constant term is a fixed percentage of the managed funds that is payed to the manager independently of his performance. The variable term is a premium that is proportional to the profit earned by the manager over a benchmark at a certain evaluation date. I find the optimal strategy and the optimal wealth in the Black-Scholes setting when the benchmark is a linear combination of the risky asset and the money market account. I also provide an approximated formula for the optimal strategy, based on a univariate Fourier inversion, that can be applied to more general dynamics.

keywords: Investment Analysis; Portfolio Management; Optimal Control; Fourier Transform

1 Introduction

I treat the problem of finding the optimal strategy for a fund manager subject to a performance based compensation. I consider a compensation contract given by the sum of a constant and a variable term. The constant term is a fixed amount that is paid to the manager independently of his performance. The variable term is equal to a call option where the underlying is the managed fund and the strike is the value of the benchmark at maturity. This is

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a stylized version of the usual compensation of fund managers who receive a fixed amount, usually based on a percentage of the asset under management and a variable prize, based on the over-performance with respect to a benchmark. For example, in the hedge fund industry, it has been a common practice, especially before the most recent financial crisis, to set up a constant of 2% of the asset under management and a 20% of the over-performance. I assume a Black-Scholes setting, with a risky asset following a Geometric Brownian Motion and a money market account paying a constant interest rate. The benchmark to which the compensation of the manager is linked is a linear combination of the risky asset and the money market account. The manager, endowed with a Constant Relative Risk Aversion (CRRA) utility function, selects the trading strategy that optimizes the expected utility of the final compensation, subject to a budget constraint.

This problem is an extension of the classical Merton model [12] that is interesting from several point of views. It is an interesting optimization problem as it presents some non trivial issues, like an option with a random strike price, written on a managed portfolio, and a non-concave objective function. It is also very relevant in practice because it addresses the question of how compensation affects the behavior of the managers. My work is based on the path-breaking paper by Carpenter [7] who showed that such problems can be solved by combining the martingale approach by Cox and Huang [8] and the concavification argument proposed by Aumann and Perles in [1]. Carpenter in [7] illustrated the application of her method to the cases of a benchmark that is either a constant or the minimum variance portfolio. My contribution is the computation in closed form of the optimal strategy when the benchmark is any combination of the two assets in the Black Scholes model. Moreover I provide an approximated formula for the optimal policy that is based on the inversion of a one-dimensional Fourier Transform. Such a formula has the advantage that can be applied also for more general dynamics. It is only required the knowledge of the characteristic function of the joint process of the state variables.

The approach proposed in [7] has been repeatedly used in the literature of managerial compensations. Cuoco and Kaniel in [9] compute the equilibrium in an economy populated by managers and investors. They solve a problem with the same objective function, but they assume that the process for the underlying is not given, but is endogenous to the model. Basak, Pavlova and Shapiro in [2] consider the incentives induced by an increasing and convex relationship of fund flows to relative performance. They consider the same market model, with a risky asset following a Geometric Brownian Motion, but with a different objective function. In fact they modeled the payoff function of a mutual fund manager, whose final wealth depends on the convex relation between performance and money inflow. Nicolosi, Herzel and Angelini in [13] extend the result in [2] to models with mean reversion, either in the market price of risk, or in the volatility of the asset prices. Herzel and Nicolosi, in [11], show how the optimal strategy of an investor, who may choose to allocate between a risk-free asset, a risky asset and a managed fund, is affected by different kinds of incentives for the manager. Basak, Shapiro, and Teplá in [3] consider a problem where a CRRA manager optimizes a linear payoff with the additional constraint of not underperforming the benchmark with a given level of probability. Barucci and Marazzina, in [4] investigate the case of a manager who is remunerated trough a High Water Mark incentive scheme, while in [5] study the problem of non-convex remuneration under a regime switching framework.

The rest of the paper is organized as follows: Section 2 presents the optimization problem and the dynamics of the model. Section 3 provides analytical formulas for the optimal strategy in the one dimensional Black-Scholes setting. Section 4 implements the optimal policy under different choices of the parameters and studies the impact of the incentives on it. Section 5 provides in closed form an approximation for the optimal policy. A comparison between the optimal strategy and the approximated one is also presented. Section 6 concludes. The Appendix contains technical details that may be skipped at a first reading.

2 Model setting

Let us define a complete probability space $(\Omega, \mathcal{F}, P)$, with filtration $\{\mathcal{F}_t\}$ $t \geq 0$ generated by a standard Brownian motion Z. The market is composed by a money market account B, providing constant interest rate r, and a risky asset S, traded continuously in time. The risk asset's price process follows

$$dS_t = S_t \left(\mu dt + \sigma dZ_t\right)$$

where μ and σ are constants.

A fund manager dynamically allocates the fund's wealth, initially valued at W_0, through a self-financing strategy. The value of the portfolio W follows

$$\frac{dW_t}{W_t} = (r + \theta_t(\mu - r))dt + \theta_t \sigma dZ_t \tag{1}$$

where θ_t is the fraction of the portfolio invested in the risky asset at time t.

The manager is compensated depending on his performance with respect to a benchmark Y over an investment period T according to the payoff function:

$$\Pi(W_T, Y_T) = \alpha(W_T - Y_T)^+ + K \tag{2}$$

where $K > 0$ represents a fixed compensation, and $\alpha > 0$ is the percentage of positive profit that the manager receives.

The benchmark Y is a constant portfolio consisting of a non-negative fraction β invested in the asset S and $1 - \beta$ invested in the risk-free money market. The dynamics of the benchmark is

$$\frac{dY_t}{Y_t} = (r + \beta(\mu - r))dt + \beta \sigma dZ_t.$$

The manager, endowed with a constant relative risk aversion utility function

$$u(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \tag{3}$$

solves the problem

$$\max_{\theta_T} E[u(\alpha(W_T - Y_T)^+ + K)],$$

subject to the dynamic budget constraint (1).

Since we are considering a dynamically complete and arbitrage free market model we can follow the martingale approach proposed by Cox and Huang in [8] and solve the static problem

$$\begin{aligned} &\max_{W_T} E[u(\alpha(W_T - Y_T)^+ + K)], \\ &E\left[\frac{\xi_T}{\xi_0} W_T\right] = W_0 \end{aligned} \tag{4}$$

where ξ is the state price density whose dynamics is

$$\begin{aligned} \frac{d\xi_t}{\xi_t} &= -rdt - \frac{\mu - r}{\sigma} dZ_t \\ \xi_0 &= 1 \end{aligned} \tag{5}$$

3 The optimal strategy

The optimal wealth at time T can be found by following [7] and it is:

$$W_T^* = \left\{ \left[I\left(\frac{\lambda^* \xi_T}{\alpha}\right) - K \right] \frac{1}{\alpha} + Y_T \right\} \mathbf{I}_{\left[I\left(\frac{\lambda^*\xi_T}{\alpha}\right)-K\right]\frac{1}{\alpha}+Y_T>\hat{W}(Y_T)} \tag{6}$$

where $I(x) = (u')^{-1}(x)$ is the inverse function of the marginal utility, $\lambda^* > 0$ is the Lagrange multiplier ensuing that the budget constraint (4) is satisfied, $\mathbf{I}_{\mathcal{A}}$ is the indicator function over the support $\mathcal{A}$, and $\hat{W}(y) > y > 0$ solves:

$$u\left(\alpha\left(\hat{W}(y) - y\right) + K\right) = u(K) + \alpha u'\left(\alpha\left(\hat{W}(y) - y\right) + K\right)\hat{W}(y). \tag{7}$$

The following proposition provides the optimal wealth and the optimal strategy at any time $t \le T$, when $\beta < \frac{\mu - r}{\sigma^2}$.

Proposition 3.1 *For any $\beta < \frac{\mu-r}{\sigma^2}$, the optimal wealth W_t^* and the optimal strategy θ_t^* are*

$$W^*(t, \xi_t; \hat{\xi}) = C_1 \xi_t^{-\frac{1}{\gamma}} \mathcal{N}(d_1) + C_2 \mathcal{N}(d_2) + C_3 \xi_t^{-\frac{\beta\sigma^2}{\mu - r}} \mathcal{N}(d_3) \tag{8}$$

and

$$\begin{aligned}
\theta^*(t, \xi_t; \hat{\xi}) &= \theta^M + \frac{\mu - r}{\sigma^2} \frac{1}{W_t^*} \left(-\frac{1}{\gamma} C_2 \mathcal{N}(d_2) + \left(\frac{\beta\sigma^2}{\mu - r} - \frac{1}{\gamma} \right) C_3 \xi_t^{-\frac{\beta\sigma^2}{\mu-r}} \mathcal{N}(d_3) \right. \\
&\quad \left. + \frac{\sigma C_1 \xi_t^{-\frac{1}{\gamma}} e^{-\frac{1}{2}d_1^2}}{(\mu - r)\sqrt{2\pi(T-t)}} + \frac{\sigma C_2 e^{-\frac{1}{2}d_2^2}}{(\mu - r)\sqrt{2\pi(T-t)}} + \frac{\sigma C_3 \xi_t^{-\frac{\beta\sigma^2}{\mu-r}} e^{-\frac{1}{2}d_3^2}}{(\mu - r)\sqrt{2\pi(T-t)}} \right)
\end{aligned} \tag{9}$$

where

$$\theta^M = \frac{\mu - r}{\gamma\sigma^2} \tag{10}$$

is the Merton strategy of the problem without incentives and

$$
\begin{aligned}
C_1 &= \frac{1}{\alpha}\left(\frac{\lambda^*}{\alpha}\right)^{-\frac{1}{\gamma}} e^{\left(\frac{1}{\gamma}-1\right)\left(r+\frac{1}{2\gamma}\left(\frac{\mu-r}{\sigma}\right)^2\right)(T-t)} \\
C_2 &= -\frac{K}{\alpha}e^{-r(T-t)} \\
C_3 &= Y_0 A(T) e^{\left(\frac{\beta\sigma^2}{\mu-r}-1\right)\left(r+\frac{1}{2}(\mu-r)\beta\right)(T-t)} \\
d_1 &= \frac{\ln\left(\frac{\hat{\xi}}{\xi_t}\right)+\left(r-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\left(1-\frac{2}{\gamma}\right)\right)(T-t)}{\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}} \\
d_2 &= \frac{\ln\left(\frac{\hat{\xi}}{\xi_t}\right)+\left(r-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)(T-t)}{\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}} \\
d_3 &= \frac{\ln\left(\frac{\hat{\xi}}{\xi_t}\right)+\left(r-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\left(1-\frac{2\beta\sigma^2}{\mu-r}\right)\right)(T-t)}{\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}}
\end{aligned}
$$

with

$$A(t) = e^{\left(r+\frac{1}{2}\beta(\mu-r)-\frac{1}{2}\beta^2\sigma^2-\frac{r\beta\sigma^2}{\mu-r}\right)t} \tag{11}$$

and $\hat{\xi}$ solves the equation

$$\hat{\xi} = f(\hat{\xi}) \tag{12}$$

where

$$f(\xi) = \frac{\alpha}{\lambda^*}u'\left(\alpha\left(\hat{W}(Y_0\xi^{-\frac{\beta\sigma^2}{\mu-r}}A(T)) - Y_0\xi^{-\frac{\beta\sigma^2}{\mu-r}}A(T)\right)+K\right).$$

Proof. The support of the indicator function in (6) can be written as

$$\xi_T < h(Y_T), \tag{13}$$

where

$$h(Y_T) = \frac{\alpha}{\lambda^*}u'\left(\alpha\left(\hat{W}(Y_T) - Y_T\right)+K\right). \tag{14}$$

Now I can express the benchmark in terms of the state price density. Indeed, for any t:

$$Y_t = Y_0 A(t)\xi_t^{-\frac{\beta\sigma^2}{\mu-r}}, \tag{15}$$

where $A(t)$ is defined in (11). In terms of ξ_T, the inequality (13) may be written as $\xi_T < \hat{\xi}$ where $\hat{\xi}$ satisfies equation (12). The existence and uniqueness of $\hat{\xi}$ is ensured by assuming that $\beta < \frac{\mu-r}{\sigma^2}$ (see the Appendix A for details). Hence the optimal wealth (6) at time T reads:

$$W_T^* = \left\{ \left[I\left(\frac{\lambda^* \xi_T}{\alpha}\right) - K \right] \frac{1}{\alpha} + Y_0 A(T) \xi_T^{-\frac{\beta\sigma^2}{\mu-r}} \right\} \mathbf{I}_{\xi_T < \hat{\xi}}. \tag{16}$$

The optimal wealth at time $t < T$ is then found using the fact that in a complete and free of arbitrage market, any tradable asset multiplied by the state price density is a martingale. Hence:

$$W_t^* = \frac{1}{\xi_t} E_t[\xi_T W_T^*], \tag{17}$$

where $E_t[\cdot]$ is the expectation conditioned at information at time t. Such an expectation is computed by using (16) for the optimal final wealth, together with the explicit expression of the state price density, coming from solving equation (5)

$$\xi_T = \xi_t e^{\left(-r-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2\right)(T-t)-\frac{\mu-r}{\sigma} Z_{T-t}},$$

and by using that

$$\begin{aligned} E_t\left[\xi_T^a \mathbf{I}_{\xi_T<\hat{\xi}}\right] &= \xi_t^a e^{-a\left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(1-a)\right)(T-t)} \\ &\times \mathcal{N}\left(\frac{\ln\left(\frac{\hat{\xi}}{\xi_t}\right) + \left(r+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(1-2a)\right)(T-t)}{\left(\frac{\mu-r}{\sigma}\right)\sqrt{T-t}}\right), \end{aligned}$$

for any real number a.

As for the optimal strategy θ_t^*, it is found by equating the diffusive coefficient in (1) with the diffusive coefficient obtained by applying Ito's lemma to $W^*(t, \xi_t; \hat{\xi})$, that is

$$\theta^* = -\frac{\mu-r}{\sigma^2} \frac{\xi_t}{W_t^*} \frac{dW^*}{d\xi}.$$

The expression (9) is recovered by computing the first derivative of (8) with respect to state variable ξ and rearranging the terms. □

When $\beta > \frac{\mu-r}{\sigma^2}$, the solution to (12) may not be unique. I performed many experiments and I found either one or two solutions. I have never

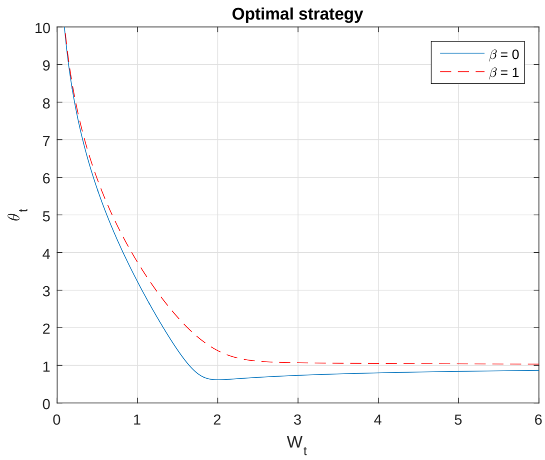

Figure 1: Optimal strategy as a function of the optimal wealth at time $t = 1$ year. The continuous line represents the case when the benchmark is the risk-free asset, while the dashed line shows the results when the benchmark is the risky asset. The contract parameters are $K = 3\%$, $\alpha = 15\%$. The time to maturity is $T - t = 1$ year. The initial value of the wealth and of the benchmark is $W_0 = Y_0 = 1$. The risk aversion parameter is $\gamma = 2$. The parameters defining the dynamics are: $\mu = 0.08$, $r = 0$, and $\sigma = 0.2$.

found more than two solutions. In the case of two solutions $\hat{\xi}_1 < \hat{\xi}_2$, the support of the indicator function in (6) is equivalent to $\mathbf{I}_{\xi_T < \hat{\xi}_2} - \mathbf{I}_{\xi_T < \hat{\xi}_1}$ and hence the optimal wealth and optimal strategy read respectively

$$\begin{aligned} W^*(t, \xi_t; \hat{\xi}_1, \hat{\xi}_2) &= W^*(t, \xi_t; \hat{\xi}_2) - W^*(t, \xi_t; \hat{\xi}_1) \\ \theta^*(t, \xi_t; \hat{\xi}_1, \hat{\xi}_2) &= \frac{W^*(t, \xi_t; \hat{\xi}_2)}{W^*(t, \xi_t; \hat{\xi}_1, \hat{\xi}_2)} \theta^*(t, \xi_t; \hat{\xi}_2) - \frac{W^*(t, \xi_t; \hat{\xi}_1)}{W^*(t, \xi_t; \hat{\xi}_1, \hat{\xi}_2)} \theta^*(t, \xi_t; \hat{\xi}_1) \end{aligned}$$

where $W^*(t, \xi_t; \hat{\xi}_i)$ and $\theta^*(t, \xi_t; \hat{\xi}_i)$ for $i = 1, 2$ are given respectively in Equation (8) and (9) when $\hat{\xi}$ is equal to $\hat{\xi}_i$.

4 Analysis of the optimal strategy

In this section I will compute the optimal strategy for different sets of the contract parameters.

Figure 1 shows the optimal strategy as a function of the optimal wealth at time $t = 1$ year. The contract and the dynamics are set as in [7] for comparability reason. The contract parameters are $K = 3\%$, $\alpha = 15\%$. The time to maturity is $T - t = 1$ year. The initial value of the wealth and of the benchmark is $W_0 = Y_0 = 1$. The risk aversion parameter is $\gamma = 2$. The dynamics parameters are: $\mu = 0.08$, $r = 0$, and $\sigma = 0.2$. The continuous line represents the case when the benchmark is the risk-free asset, that is the same case studied in [7]; the dashed line shows the results when the benchmark is the risky asset. In both cases the optimal strategy converges to the Merton level (10), for high values of the wealth. The Merton level is the constant proportion of the risky asset held in the portfolio by a manager who is paid a linear share of the profit. When the wealth is far above the benchmark level, that is when it is very likely that the option defining the variable premium of the contract finishes in the money, the manager's payoff is effectively linear in the profit and hence the standard Merton level is recovered. On the other hand, for low values of the portfolio value, it is optimal for the manager to deviate from the Merton level to boost the portfolio volatility in order to increase the probability to beat the benchmark and finish in the money. When the benchmark is the risk-free asset, there are some states where the optimal exposure is below the Merton level. This behavior, that has been explained in [7] with a reduction of the wealth volatility to compensate the leverage effect of the option component, is instead not observed when the benchmark is the risky asset. The reason is that in this case the leverage effect of the option component is weaker than in the other case as the volatility of the benchmark may reduce the volatility of the option.

Figure 2 shows the influence of α, the percentage of positive profit $(W_t - Y_t)^+$ received by the manager. The two panels report the optimal strategy as a function of the profit at time $t = 1$ year respectively for the case when the benchmark is the risky asset (left panel) and the case when it is the risk-free asset (right panel). The results are relative to $\alpha = 5\%$ (dashed line), $\alpha = 15\%$ (continuous line), and $\alpha = 30\%$ (dotted line). The other parameters are set as in Figure 1. For a higher number α of options held by the manager, it is optimal for the manager to reduce his exposure. This behavior, that at a first sight may be counterintuitive, has already been observed in [7] when the benchmark is the risk free asset. Here we observe the same effect when the benchmark is the risky asset. The reason of such a behavior is that the manager tries to maximize the utility of his personal portfolio (2) which consists of α shares of an option on the fund's value and cash (the constant

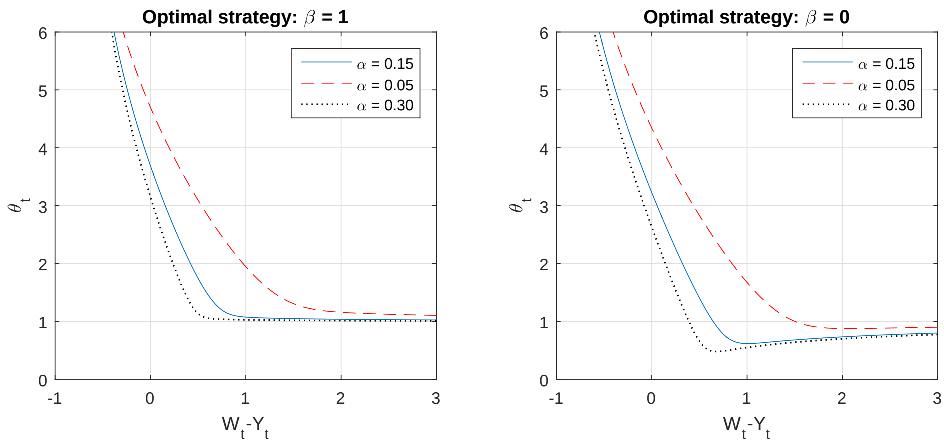

Figure 2: Optimal strategy as a function of the difference between the optimal wealth and the benchmark at time $t = 1$ year. The figure shows the results for different levels of the percentage defining the variable prize of the contract, $\alpha = 5\%$ (dashed line), $\alpha = 15\%$ (continuous line), and $\alpha = 30\%$ (dotted line), when the benchmark is either the risky asset, in the left panel, or the risk-free asset, in the right panel. The fixed premium of the contract is $K = 3\%$. The other parameters are set as in Figure 1, that is: $T - t = 1$ year, $W_0 = Y_0 = 1$, $\gamma = 2$, $\mu = 0.08$, $r = 0$ and $\sigma = 0.2$.

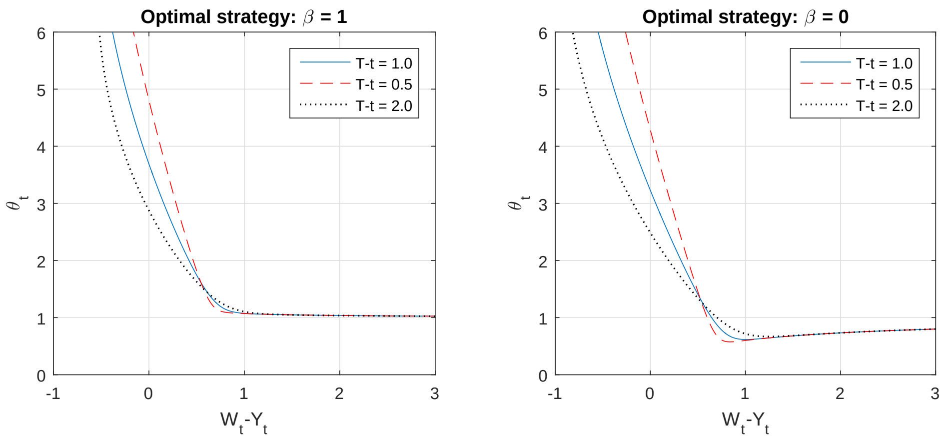

Figure 3: Optimal strategy as a function of the difference between the optimal wealth and the benchmark at time $t = 1$ year. The figure shows the results for different levels of the time to maturity, $T - t = 0.5$ years (dashed line), $T - t = 1$ year (continuous line), and $T - t = 2$ years (dotted line), when the benchmark is either the risky asset, in the left panel, or the risk-free asset, in the right panel. The other parameters are set as is Figure 1, that is: $\alpha = 15\%$, $K = 3\%$, $W_0 = Y_0 = 1$, $\gamma = 2$, $\mu = 0.08$, $r = 0$ and $\sigma = 0.2$.

part). A manager with a CRRA utility seeks to keep the volatility of his portfolio constant to the Merton level. Such a volatility by Ito's lemma is proportional to α and to the volatility of the assets under management $\theta_t \sigma$. Hence an increasing of the percentage α of the option causes a decreasing of the funds value volatility.

Figure 3 shows the influence of the time to maturity $T - t$. The two panels report the optimal strategy as a function of the profit $W_t - Y_t$ at time $t = 1$ year for different levels of the time to maturity, $T - t = 0.5$ years (dashed line), $T - t = 1$ year (continuous line), and $T - t = 2$ years (dotted line), when the benchmark is either the risky asset (left panel), or the risk-free asset (right panel). The other parameters are set as in Figure 1. In both the panels we observe that as the evaluation date is approaching, if the portfolio value is around the benchmark value, it is optimal for the manager to increase his exposure in the attempt to increase the likelihood of finishing in the money and get the variable part of compensation.

5 An approximated formula

In this section I derive approximated formulas for the optimal strategy and the optimal wealth. I show that in the setting described in Section 3 such a suboptimal solution provides a good approximation of the optimal one. The advantage of using this approach is that it can be implemented for more general dynamics, for which the characteristic function is known, to the case of N risky assets, and it is not restricted to the case $\beta < \frac{\mu - r}{\sigma^2}$.

Equation (13) defines the region where the manager's option ends in the money. To compute the optimal strategy W_t^* we have to integrate over such a region as in (17). The idea is to approximate function (14), delimitating the integration region, with a power-law function and then to use the Fourier Transform to express the indicator function as an integral representation. Then the optimal wealth at any time t is recovered by a univariate Fourier inversion. Similar techniques have already been used to price spread option. See for instance [6] or [10].

As an example of the approximation, Figure 4 left panel, shows the region where the option is in the money for the $\beta = 1$ case provided in Figure 1 and the corresponding approximated region. In this case the parameters of the power-law function aY_T^b approximating (14) are $a = 1.200$ and $b = -0.4795$ with a R^2 of the fit that is $R^2 = 0.9415$. The reason why the approximation should work is that aY_T^b approaches $h(Y_T)$ from above for some values of Y_T and from below for some other values of Y_T. Hence there could be a compensation of the error. Moreover, the computation of the optimal wealth and optimal strategy involves the computation of an expected value. Hence the behavior of the distribution's tails may mitigate the magnitude of the error.

Proposition 5.1 *Let us consider the following approximated integration region*

$$\tilde{\mathcal{A}} = \{(\xi_T, Y_T) \in R_+^2 : \xi_T < aY_T^b\}, \tag{18}$$

for some real numbers a and b. Moreover, let us define the characteristic function of the joint process $(\ln \xi_t, \ln Y_t)$

$$H_t(z_1, z_2) = E_t[e^{z_1 \ln \xi_T} e^{z_2 \ln Y_T}]. \tag{19}$$

Then the suboptimal wealth

$$\tilde{W}_t = \frac{1}{2\pi\xi_t} \sum_{j=1}^{3} A_j \int_{-\infty}^{+\infty} \hat{\varphi}(iu - R + c_j) H_t(-iu + R, b(iu - R + c_j) + d_j) du \tag{20}$$

where

$$\hat{\varphi}(z) = \frac{a^z}{z},$$

$$A_1 = \frac{1}{\alpha}\left(\frac{\lambda^*}{\alpha}\right)^{-1/\gamma}, \qquad A_2 = -\frac{K}{\alpha}, \qquad A_3 = 1,$$

$$c_1 = 1 - 1/\gamma, \qquad c_2 = c_3 = 1, \qquad d_1 = d_2 = 0, \qquad d_3 = 1,$$

$$R < 1 - 1/\gamma,$$

provides an approximation of the optimal wealth (8).

Proof. From Equations (6) and (13), and by using (17), we have:

$$W_t^* = \frac{1}{\xi_t}\sum_{j=1}^{3} A_j E_t[\xi_T^{c_j} Y_T^{d_j} \mathbf{I}_{\xi_T < h(Y_T)}]. \tag{21}$$

An approximation of (21) is obtained by considering the approximated integration region (18), that is

$$\tilde{W}_t = \frac{1}{\xi_t}\sum_{j=1}^{3} A_j E_t[\xi_T^{c_j} Y_T^{d_j} \mathbf{I}_{\xi_T < aY_T^b}].$$

Then I apply an inverse Fourier Transform to rewrite $\tilde{W}_t$ as

$$\tilde{W}_t = \frac{1}{2\pi\xi_t}\sum_{j=1}^{3} A_j E_t\left[\int_{-\infty}^{+\infty} \hat{\varphi}(iu - R + c_j) Y_T^{b(iu-R+c_j)+d_j} \xi_T^{-iu+R} du\right].$$

Equation (20) is then obtained by using Fubini's theorem and exchanging the order of integration. Condition $R < 1 - 1/\gamma$ is an integrability condition. □

Equation (20), provides an approximated formula for the optimal wealth. It can be implemented very efficiently by means of the Fast Fourier Transform (see the Appendix B for details). Let us highlight here that Equation (20) may be applied for any dynamics whose characteristic function is known analytically. Moreover, even in the case with N risky assets, (20) is based on a univariate Fourier Transform inversion. Also, such a formula holds for any choice of the parameters.

In the one-dimensional Black-Scholes setting, the characteristic function (19) reads

$$H_t(z_1, z_2) = \xi_t^{z_1} Y_t^{z_2} e^{C(z_1,z_2)(T-t)} \tag{22}$$

where

$$\begin{aligned} C(z_1, z_2) &= -rz_1 + \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 z_1(z_1 - 1) \\ &+ (r + \beta(\mu - r))z_2 + \frac{1}{2}\beta^2\sigma^2 z_2(z_2 - 1) - \beta(\mu - r)z_1 z_2. \end{aligned}$$

The corresponding strategy is then achieved by equating the diffusive coefficient in (1) with the diffusive coefficient obtained by applying Ito's lemma to $\tilde{W}(t, \xi_t, Y_t)$ [1]

$$\tilde{\theta}_t = -\frac{\mu - r}{\sigma^2}\frac{\xi_t}{\tilde{W}_t}\frac{\partial \tilde{W}}{\partial \xi} + \beta \frac{Y_t}{\tilde{W}_t}\frac{\partial \tilde{W}}{\partial Y}$$

and then by taking the derivatives under the integral sign in (20):

$$\begin{aligned} \frac{\partial \tilde{W}}{\partial \xi} &= \frac{1}{2\pi\xi_t^2}\sum_{j=1}^{3} A_j \int_{-\infty}^{+\infty} \hat{\varphi}(iu - R + c_j)(-iu + R - 1) \\ &\quad H_t(-iu + R, b(iu - R + c_j) + d_j)du \\ \frac{\partial \tilde{W}}{\partial Y} &= \frac{1}{2\pi\xi_t Y_t}\sum_{j=1}^{3} A_j \int_{-\infty}^{+\infty} \hat{\varphi}(iu - R + c_j)(b(iu - R + c_j) + d_j) \\ &\quad H_t(-iu + R, b(iu - R + c_j) + d_j)du. \end{aligned}$$

As an example, Figure 4, right panel, reports the optimal strategy as a function of the optimal wealth (continuous line) and the approximated strategy as a function of the approximated wealth (dotted line) at time $t = 1$ year when the benchmark is the risky asset and the other parameters are set

[1] Actually, in the one dimensional Black-Scholes setting, Equation (15) relates Y to ξ. Using such a relation $\tilde{\theta}_t = -\frac{\mu-r}{\sigma^2}\frac{\xi_t}{\tilde{W}_t}\frac{d\tilde{W}}{d\xi}$. For the N dimensional case, the information given by the state price density is no more equivalent to that of the benchmark and hence both the state variables are needed to compute the policy. For more general dynamics, we have to consider the partial derivatives of all the state variables defining the information at time t.

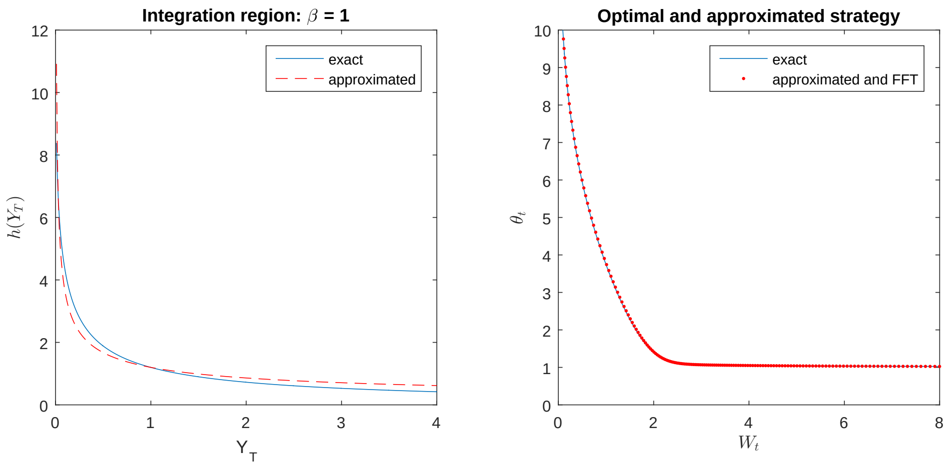

Figure 4: The left panel represents the region where the manager's option is in the money (the portion of the plane under the continuous line), and its approximation (the portion of the plane under the dashed line). The right panel shows the optimal strategy as a function of the optimal wealth (continuous line) and the approximated strategy as a function of the approximated wealth (dotted line) at time $t = 1$ year. The benchmark is the risky asset. The other parameters are set as in Figure 1.

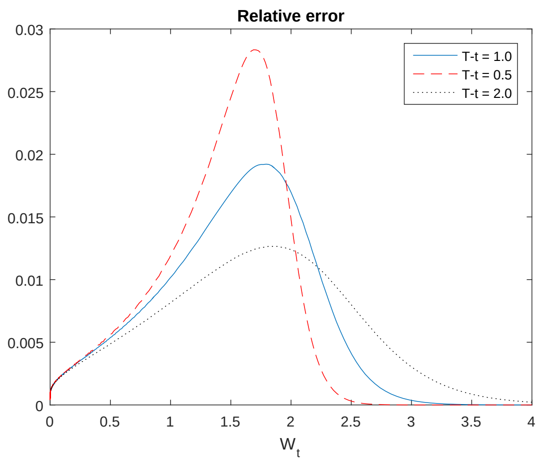

Figure 5: The figure shows the relative error $\tilde{\theta}_t/\theta_t^* - 1$ as a function of the optimal wealth when the time to maturity is $T-t = 1$ year (continuous line), $T-t = 0.5$ years (dashed line), and $T-t = 2$ years (dotted line) respectively. The benchmark is the risky asset. The other parameters are set as in Figure 1.

as in Figure 1. The approximated strategy $\tilde{\theta}_t$ is almost indistinguishable from the optimal strategy θ_t^* as it is also shown in Figure 5, where the relative error $\tilde{\theta}_t/\theta_t^* - 1$ is provided (continuous line). Figure 5 shows also the relative error when the time to maturity is $T-t = 0.5$ years (dashed line), and $T-t = 2$ years (dotted line) respectively. The time to maturity is the variable that has the strongest influence on the error. As it is shown in Figure 5, the error increases as the time to maturity decreases. The reason is that the approximation of the region where the option ends in the money is done at the evaluation date T. The more the evaluation date is distant in the future, the less the detail of the boundary of such a region influences the optimal solution at time t.

6 Conclusion

I computed the optimal strategy of a portfolio manager with power utility whose compensation is given by a fixed percentage of the managed funds and a variable part that depends on his performance. The variable term is proportional to an option on the portfolio value where the strike is the value of the benchmark at the evaluation date. This work is based on the results in [7] where the author showed how to solve the problem and presented the optimal policy in the Black-Scholes setting when the benchmark is either a constant or the minimum variance portfolio. I extended such results by providing in closed form the optimal strategy when the benchmark is any combination of the two assets.

I presented some case studies that illustrate the impact of the convex incentives on the optimal policy for different benchmark combinations. First of all, when the option is out of the money, the manager may increase a lot his risk exposure. And such an effect is stronger as the evaluation date is approaching. Second, a higher share of options induces the manager to take less risk. Third, when the benchmark is the risk-free asset, the manager can take a lower risk than the risk he would take without the incentive.

I proposed an approximated formula to compute the optimal policy, that is based on a power-law approximation of the boundary of the region where the option is in the money. The computation is then obtained by performing a one dimensional Fourier inversion. The approximated policy that I found matches very well the exact optimal policy in the setting presented in the paper. Furthermore, it has the advantage that it can be applied in a straight way also to cases with a more general model setting. The only requirement is the knowledge of the characteristic function of the joint process of the state variables. At the present stage of the work, I do not know how the approximation performs for different models. I leave this issue for further investigations.

A Existence and uniqueness of $\hat{\xi}$

We show here that for any $\beta < \frac{\mu - r}{\sigma^2}$ there exists a unique $\hat{\xi}$ satisfying Equation (12). This is obtained by rewriting Equation (13) in terms of $y = Y_T$, that is $y > f(y)$, and by solving

$$\hat{y} = f(\hat{y}) \tag{23}$$

where we defined

$$\begin{aligned} f(y) &= c\,(u'\,(\alpha g(y)+K))^{-a} \\ a &= \frac{\beta\sigma^2}{\mu - r} \\ c &= Y_0 A(T)\left(\frac{\alpha}{\lambda^*}\right)^{-a} \\ g(y) &= \hat{W}(y) - y \end{aligned}$$

and where we used relation (15) and definition (11).

As shown in [7], $g(y)$ is a positive function of y. Moreover $g(y)$ is an increasing function of y as it results by the sign of its derivative computed by expressing $\hat{W}(y)$ in terms of $g(y)$ in (7) and then by taking the derivative of (7) with respect to y:

$$g'(y) = \frac{-u'(\alpha g(y)+K)}{\alpha u''(\alpha g(y)+K)(g(y)+y)}. \tag{24}$$

In what follows we need to know the asymptotic behavior of $g(y)$. From (7) it is possible to understand that $lim_{y\to\infty} g(y) = \infty$. Then, using (3), we have that for large values of y and for $\gamma > 1$, after canceling the terms going to zero in the limit, Equation (7) gives

$$\alpha^{1-\gamma} g(y)^{-\gamma} y \sim \frac{K^{1-\gamma}}{\gamma - 1},$$

from which we see that $g(y) \sim y^{1/\gamma}$. On the other hand, when $\gamma < 1$, and for large values of y, Equation (7) reads

$$(\alpha g(y))^{1-\gamma}\left(\frac{\gamma}{1-\gamma} - \frac{y}{g(y)}\right) \sim \frac{K^{1-\gamma}}{1-\gamma}$$

which implies that $g(y) \sim y$.

Now we show that f is a strictly increasing function of y. By taking the derivative of f with respect to y, and using the definition of g' in (24), we obtain

$$f'(y) = \frac{a f(y)}{g(y)+y},$$

whose sign is positive as $f(y) > 0$. As concerning the concavity of f, using Equation (24), we have:

$$f''(y) = \frac{af(y)}{(g(y)+y)^2}(a - 1 - g'(y)).$$

Function f is concave when $g'(y) > a-1$. A sufficient condition ensuring the concavity of f for any value y is $a < 1$ as $g'(y) > 0$ for any y. Hence, for $a < 1$, that is for $\beta < \frac{\mu - r}{\sigma^2}$, f is a positive, increasing and concave function. We also show that for $a < 1$, f is asymptotically flat, that is $\lim_{y\to\infty} f'(y) \to 0$. This is ready understood by using that for large values of y: when $\gamma > 1$, $g(y) \sim y^{1/\gamma}$, therefore, $f'(y) \sim y^{a-1}$; for $\gamma < 1$, $g(y) \sim y$ hence $g(y) \sim y^{a\gamma - 1}$. Then a solution to (23) exists, and this is unique.

B The Fast Fourier Transform

The approximated wealth and strategy are defined through integrals of the form:

$$I(x) = \frac{e^{xR}}{2\pi}\int_{-\infty}^{\infty} \chi(u+iR)e^{-iux}du = \frac{e^{xR}}{\pi}\int_{0}^{\infty} \chi(u+iR)e^{-iux}du$$

where $x = \ln \xi_t$. This is because the conditional characteristic function (19) is written as in (22). The last equality in the above expression comes from the fact that the integral has to be real and from the symmetry properties of the real and imaginary part of χ.

We may use the Simpson's rule with N nodes $u_j = \eta(j-1)$ for $j = 1,\ldots,N$, and truncate the integral at $N\eta$, to approximate the integral as a sum:

$$I(x) \approx \frac{\eta e^{xR}}{\pi}\sum_{j=1}^{N} \chi(u_j + iR)e^{-iu_j x}w_j$$

where $w_j = \frac{1}{3}(3 + (-1)^j - \delta_{j-1})$ are the Simpson weights and the Kronecker delta δ_j is different from zero only for $j = 0$.

The FFT representation of the integral is obtained by a discretization of x as follows: $x_k = -b + v_k$ where $v_k = \lambda(k-1)$, for $k = 1,\ldots,N$, $b = \frac{N\lambda}{2}$, λ

is the size spacing, and $\eta\lambda = \frac{2\pi}{N}$

$$
\begin{aligned}
I(x_k) &\approx \frac{\eta e^{x_k R}}{\pi} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(k-1)} e^{ibu_j} \chi(u_j + iR) w_j \\
&= \frac{\eta e^{x_k R}}{\pi} FFT\left(e^{ibu_j} \chi(u_j + iR) w_j\right).
\end{aligned}
$$

In the cases presented in Section 5, the parameters used to implement the FFT algorithm are $N = 2^{12}$, $\eta = 0.125$, $R = -1/\gamma$.

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