Manipulation-Proof Performance Measure and the Cost of Tail Risk

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Abstract

This paper builds on the seminal Goetzmann, Ingersoll, Spiegel and Welch research on Manipulation-Proof Performance Measures (MPPM), with a different purpose. Manipulation of usual performance measures generally goes through taking risk which is not reflected in the second moment measure of return distribution, variance or volatility. This is particularly relevant for the hedge fund industry, which often aims at capturing risk premiums of the non-ordinary sort. It is also more and more relevant for all asset managers submitted to peer comparison.

The MPPM corrects for the impact of tail risk –negative skewness and kurtosis- taken by a fund manager (not necessarily with the explicit aim of manipulating the performance measures). In our paper, we try to quantify, using a Cornish Fisher technology allowing us to control for tail risk, the impact of such risk on the MPPM.

In that framework, we find that the MPPM effectively imposes a penalty on tail risk. This penalty increases nearly linearly with return kurtosis and return negative skewness. The size of the penalty is rather benign when return volatility is low or the risk parameter is low. It increases substantially for high volatilities and/or high risk parameters.

JEL classification: C02, G11, G12, G21

Key Words: Asset Allocation, Fund performance, Risk, Tail Risk, Cornish Fisher, Skewness, Kurtosis

1 – Introduction

It is now well known that fund managers are able to manipulate usual performance measures which take into account the first and second moments of the return distribution they achieve. An example of measure which is easily manipulated is the Sharpe Ratio. One way (among others) to manipulate is to sell out-of-the money puts on risky assets. This enhances average returns through the collection of premiums without substantially increasing the variance of returns.

However, such investments affect the higher order moments of return distributions, such as skewness and kurtosis. Apart from its manipulation-proof character, the measure proposed by Goetzmann & alii (2007) is able to capture the whole profile of return distributions, and hence the impact of skewness and kurtosis.

In this paper, we combine the manipulation-proof performance measure and the Cornish-Fisher technology properly implemented (Maillard, 2012) to assess the impact of tail risk in terms of a penalty on performance. Using the Cornish-Fisher transformation allows to explore a much wider field of skewness and kurtosis than other transformations such as Gramm-Charlier (Maillard, 2014).

2 – Manipulation-Proof Performance Measure (MPPM)

In their article, Goetzmann & alii (2007) show that the usual measures of mutual or fund performance – among them the Sharpe ratio, Jensen’s alpha, Treynor ratio…- may be manipulated by fund managers. The point had been previously noted by other authors, among them Leland (1999) and Lhabitant (2000) in the case of hedge funds.

Goetzmann & alii’s paper describes three general strategies for manipulating a performance measure:

- manipulation of the underlying distribution
- dynamic manipulation of measures assuming time stationnarity
- dynamic manipulation inducing estimation error.

As a way to counter all type of manipulation, they propose a Manipulation-Proof Performance measure (MPPM) which writes[2]:

$$\hat{\Theta} = \frac{1}{(1-\gamma)\Delta t} \ln\left(\frac{1}{T} \sum_{t=1}^{T} \left[\frac{1+r_t}{1+r_{ft}} \right]^{1-\gamma} \right)$$

T is the number of observations, γ is a parameter related to risk aversion, r_{ft} the risk-free rate for period t (assuming such thing still exists…), and Δt the length of the period (in years) on which the return is recorded. r_t is the return of the fund during period t. Implicitly, the risk-free rate acts as a benchmark against which the performance is measured. The ratio in the formula is one plus a geometric excess return x_t. The exponentiation by 1- γ of the relative performance is there to take risk into account.

As the authors state, the MPPM is very close to an expected utility, of the power or CRRA (Constant Relative Risk Aversion) form (with RRA equal to γ), of an end-of-period wealth, which an investor could like to optimize. By taking the logarithm and dividing by the length of the period and one minus the risk aversion parameter, they ensure that the measure is equivalent to an equivalent-certain (continuous) rate of return.

The MPPM is very close to the Morningstar Risk-Adjusted Return (MRAR) that this firm uses to compare the performance of various funds (the return computed by Morningstar is in traditional and not continuous form).

Goetzmann & alii suggest a risk aversion parameter of 2 our 3 in their examples, Morningstar selects a parameter of 3. Those values sit at the low end of what is generally considered relevant for a relative risk aversion (2-10, or more).

[2] We stick to the authors' notations, except that we substitute γ for ρ. γ is the usual symbol for denoting a relative risk aversion (RRA), and the parameter in the MPPM measure may be assimilated to a RRA (in the Goetzmann & alii's paper, γ is used for another purpose).

3 – MPMM and the cost of risk

In this section, we rewrite the formula using the geometric excess return.

$$\hat{\Theta} = \frac{1}{(1-\gamma)\Delta t}\ln\left(\frac{1}{T}\sum_{t=1}^{T}[1+x_t]^{1-\gamma}\right)$$

We will represent the (excess) return as:

$$\ln(1+x_t) = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\varepsilon_t{}'$$
$$E(\varepsilon_t) = 0 \quad E(\varepsilon_t^2) = 1$$

ε_t is i.i.d. but not necessarily Gaussian. The risk taken by a fund manager resides both in the level of volatility σ and in the higher moments[3].

If no risk is taken ($\sigma = 0$), one will obtain:

$$\hat{\Theta} = \frac{1}{(1-\gamma)\Delta t}\ln\left(\frac{1}{T}\sum_{t=1}^{T}\left[e^{\mu\Delta t}\right]^{1-\gamma}\right) = \frac{1}{(1-\gamma)\Delta t}\ln\left[e^{\mu\Delta t(1-\gamma)}\right] = \mu$$

If risk is taken, the MPPM writes:

$$\hat{\Theta} = \frac{1}{(1-\gamma)\Delta t}\ln\left(\frac{1}{T}\sum_{t=1}^{T}\left[e^{\left(\mu-\frac{\sigma^2}{2}\right)\Delta t+\sigma\sqrt{\Delta t}\varepsilon}\right]^{1-\gamma}\right) = \frac{1}{(1-\gamma)\Delta t}\ln\left(e^{\mu(1-\gamma)\Delta t}e^{-\frac{\sigma^2}{2}(1-\gamma)\Delta t}\,\frac{1}{T}\sum_{t=1}^{T}\left[e^{\sigma\sqrt{\Delta t}\varepsilon}\right]^{1-\gamma}\right)$$
$$\hat{\Theta} = \frac{\mu(1-\gamma)\Delta t}{(1-\gamma)\Delta t} + \frac{1}{(1-\gamma)\Delta t}\ln\left(e^{-\frac{\sigma^2}{2}(1-\gamma)\Delta t}\,\frac{1}{T}\sum_{t=1}^{T}\left[e^{\sigma\sqrt{\Delta t}\varepsilon}\right]^{1-\gamma}\right) = \mu - CR$$
$$where \quad CR = \frac{1}{(1-\gamma)\Delta t}\ln\left(e^{-\frac{\sigma^2}{2}(1-\gamma)\Delta t}\,\frac{1}{T}\sum_{t=1}^{T}\left[e^{\sigma\sqrt{\Delta t}\varepsilon}\right]^{1-\gamma}\right)$$

[3] Using μ-$\sigma^2/2$ for the drift term ensures in the Gaussian case that the expected return for the period is equal to μ. This is not necessarily true if the random component of the return is not Gaussian.

The performance measure will be the difference between the expected excess return of the strategy and the cost of risk, or penalty, imposed by the measure.

If risk is taken, of the Gaussian sort, it is easy to verify that, asymptotically (see Appendix 1), one obtains (a well-known result exactly true with normal log-returns and CRRA utility function):

$$CR = \gamma \frac{\sigma^2}{2}$$

The "cost of risk" taken by the fund manager is the RRA parameter times half the variance of return. Our objective in the next sections is to study how the MPPM, and the cost of risk, is influenced by higher order moments in the return distribution.

$$\hat{\Theta} = \hat{\Theta}(\mu, \sigma, \hat{S}, \hat{K}, ...)$$

where $\hat{S}$ is the skewness of the (log) return distribution and $\hat{K}$ its kurtosis (or rather kurtosis in excess of 3, corresponding to the kurtosis of a Gaussian distribution) [4,5].

Noting that CR does not depend on average return μ, we may write

$$CR = -\hat{\Theta}(0, \sigma, \hat{S}, \hat{K}, ...)$$

Finally, we will look at how kurtosis and skewness affect the cost of risk as an add-on (a geometric add-on) on the cost of risk in the Gaussian case.

$$CR = \gamma \frac{\sigma^2}{2} \left[1 + \Phi_{\gamma\sigma}(\hat{K})\right]\left[1 + \Psi_{\gamma\sigma\hat{K}}(\hat{S})\right]$$

[4] We use $\hat{S}$ and $\hat{K}$ to represent actual skewness and kurtosis to distinguish them from S and K, which are the notations currently used to represent the skewness parameter and the kurtosis parameter in the Cornish Fisher formula (see Maillard, 2012).

[5] Note that the values of skewness and kurtosis considered concern log-returns, as is common practice. Exponentiation to obtain common returns (which corresponds to the compounding of interest rates) modifies skewness and kurtosis: kurtosis increases and skewness increases algebraically).

4 – Methodology

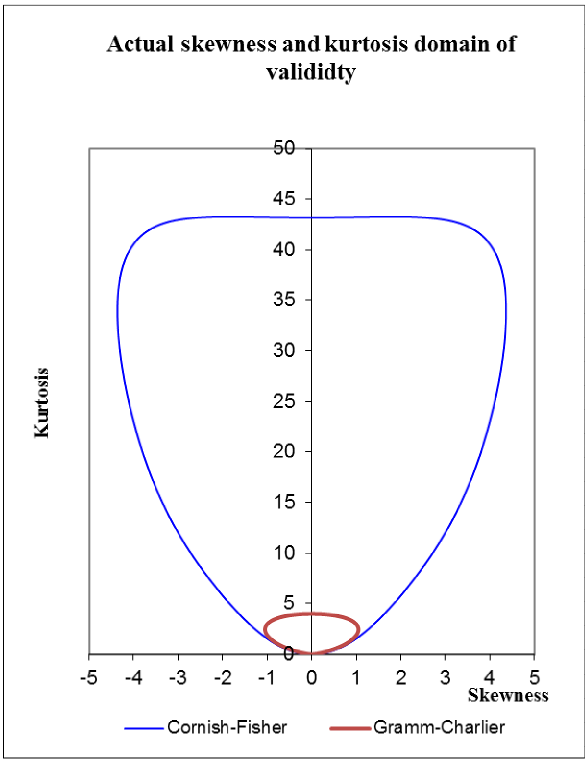
The Cornish-Fisher expansion, if properly used (Maillard, 2012), allows the generation of distributions with the desired volatility, skewness and kurtosis. It relies on the polynomial transformation of a normal standard distribution z into a distribution Z:

$$Z = z + (z^2 - 1)\frac{S}{6} + (z^3 - 3z)\frac{K}{24} - (2z^3 - 5z)\frac{S^2}{36}$$

S and K are parameters which determine skewness and kurtosis, but except for very low values do not coincide with skewness and kurtosis. The parameters will be computed to achieve the desired skewness $\hat{S}$ and kurtosis $\hat{K}$.

The Cornish-Fisher expansion has the advantage of displaying a broad domain of validitity (the transformation should lead to an always positive probability distribution, or increasing quantiles), much broader than other transformations such as Gramm-Charlier. It includes the values commonly encountered for skewness and kurtosis of portfolios' returns, as seen in Chart 1).

Chart 1



It has been used recently in various fields, such as option princing (Aboura & Maillard, 2016), Value-at-Risk computations (Fabozzi & al., 2012).

The actual value of the moments of distribution Z are given in Appendix. As *Z* is non standard (zero mean but variance slightly different from one), we will use the transformation leading to *Z'*:

$$Z' = \frac{z + (z^2 - 1)\frac{S}{6} + (z^3 - 3z)\frac{K}{24} - (2z^3 - 5z)\frac{S^2}{36}}{\sqrt{1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2}}$$

The computations of MPPM and cost of risk will be made asymptotically on distributions of 50,000 returns. As only one series of returns is used, we may rely on the 50,000 quantiles rather than on Monte-Carlo draws.

The evaluation of MPPM and the cost of risk will be made under the assumption of zero average performance. In order not to multiply the cases, we will also assume that Δt = 1, adjusting the time dimension of the certainty equivalent return. The impact of this time periodicity factor will be captured through the expression of volatility.

For instance, if the periodicity of return measures is monthly, which is common for hedge funds, a volatility input of 6% will correspond to an annualized volatility of 20.8%, which is standard for a diversified equity portfolio; a volatility input of 12% will correspond to an annualized volatility of 41.5%, which is usual for derivatives and hedge funds.

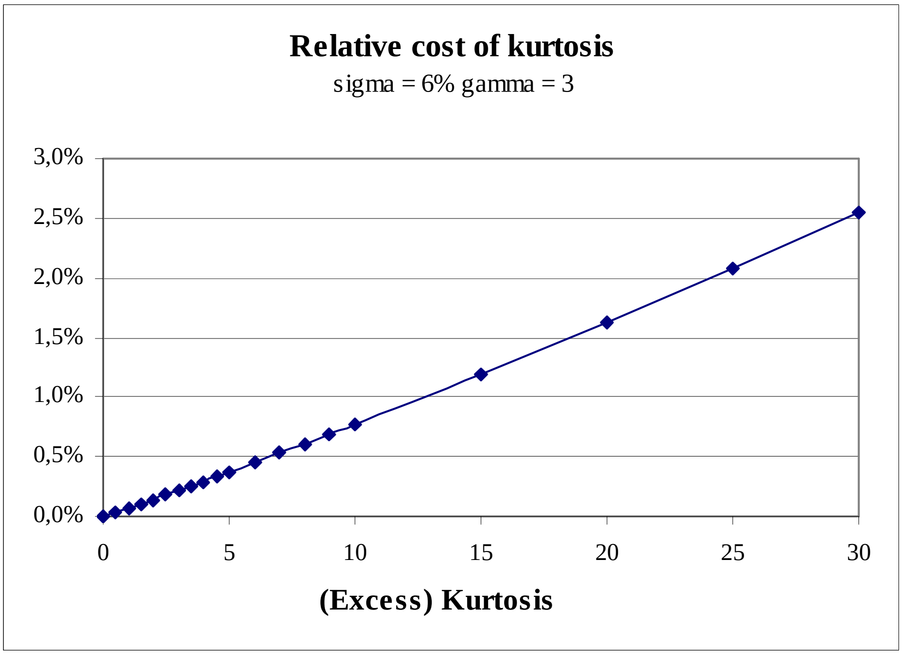
5 – The cost of kurtosis

In a Cornish Fisher framework, it is necessary to have positive excess kurtosis in order to have skewness. That induces us to start with an assessment of the impact of kurtosis on the cost of risk. In this section, we assume the absence of skewness.

Our base case will be defined by a RRA parameter equal to 3, as suggested by Goetzmann & al. and practised by Morningstar, and a period volatility equal to 6 %.

What we compute numerically thereafter is $\Phi_{3,6\%}(\hat{K})$.

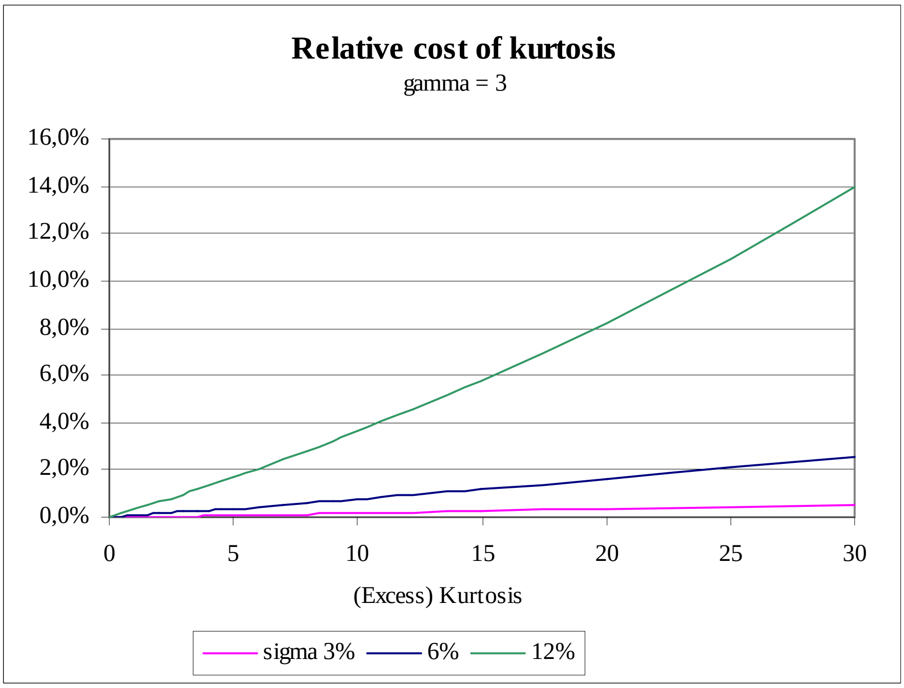
Chart 2



Kurtosis indeed has a cost, nearly proportional. In the base case, this cost is low: less than 0.1% per unity of kurtosis.

Allowing return volatility to vary leads to the following findings.

Chart 3

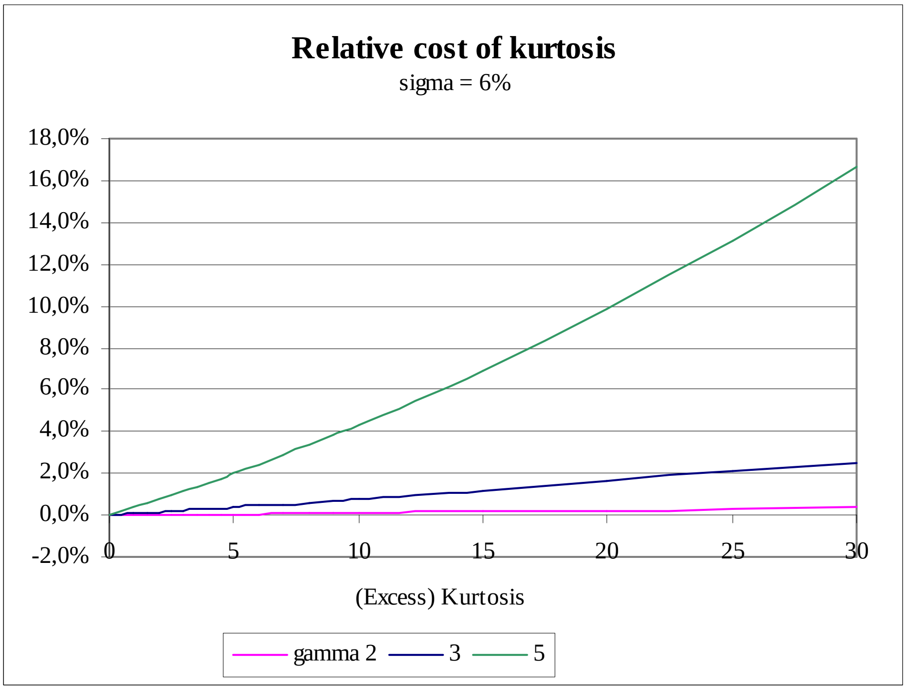


Quasi-linearity of the dependency of relative cost on kurtosis value is preserved, even at high volatilities (monthly 12% is more than 80% annualized).

When volatility doubles, the relative cost of kurtosis does more than quadruple. The sensitivity to volatility is thus huge.

Less intriguingly, relative cost of kurtosis is hugely sensitive to the relative risk aversion parameter.

Chart 4



To illustrate further the sensitivity of the relative cost of kurtosis to the risk aversion parameter and volatility, we plot the dependency for a given excess kurtosis of 8, 4 and 15.

Chart 5

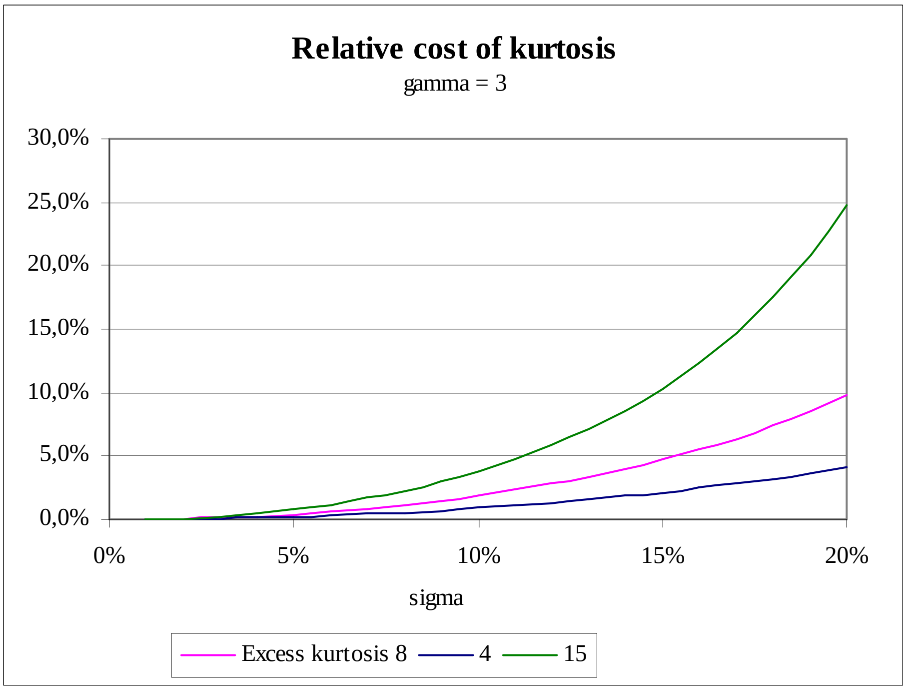
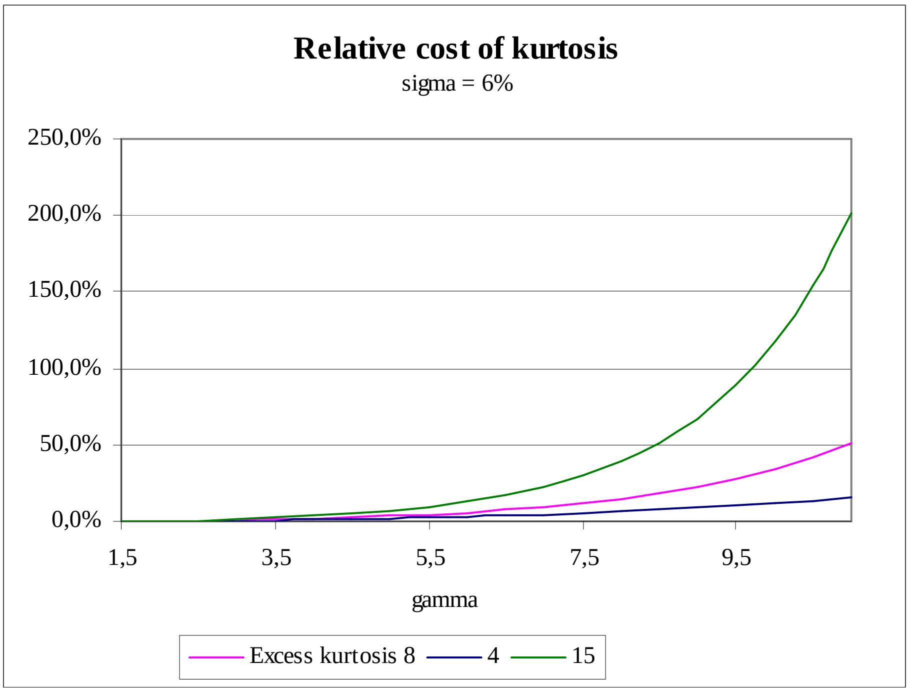


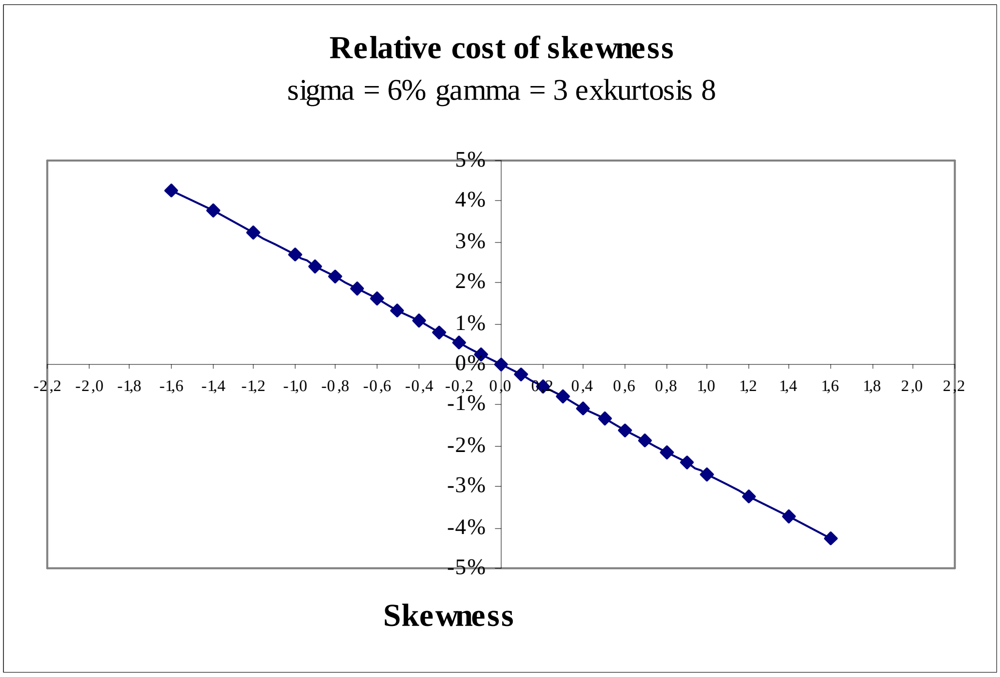
Chart 6



6 – The cost of skewness

Our base case will still be defined by a RRA parameter equal to 3 and a period volatility equal to 6 %. We add a third base parameter, choosing an excess kurtosis equal to 8.

Chart 7

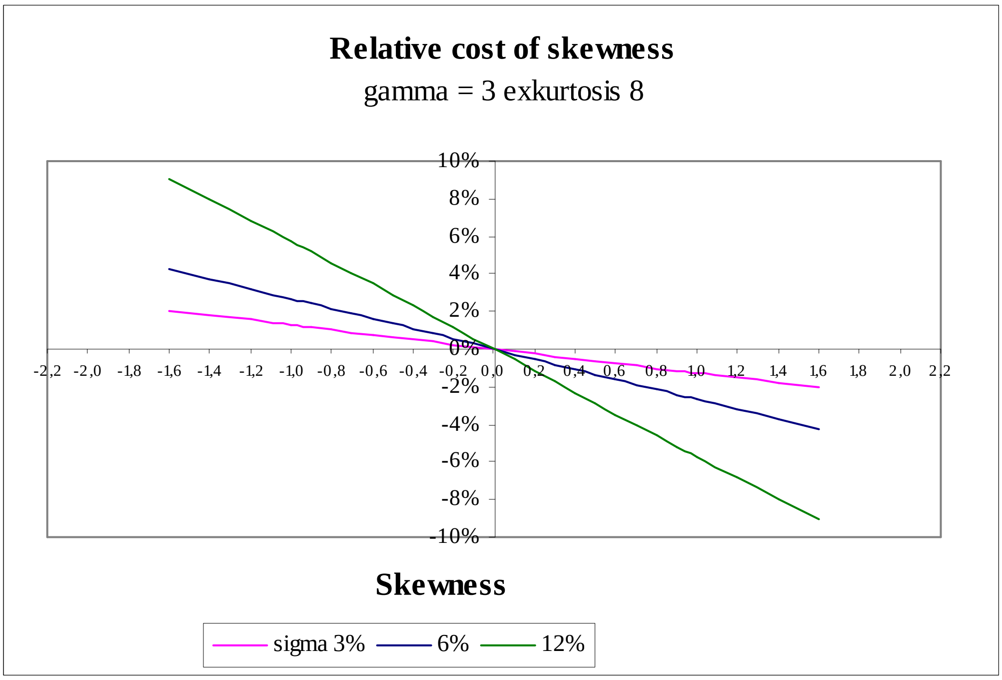


Negative skewness does indeed have a negative impact on the MPPM, and has a cost. The dependency is nearly perfectly linear. In the base case, one unit of negative kurtosis inflicts a penalty of 3% on the cost of risk.

Conversely, positive skewness is good for the performance measure, and decreases the cost of risk.

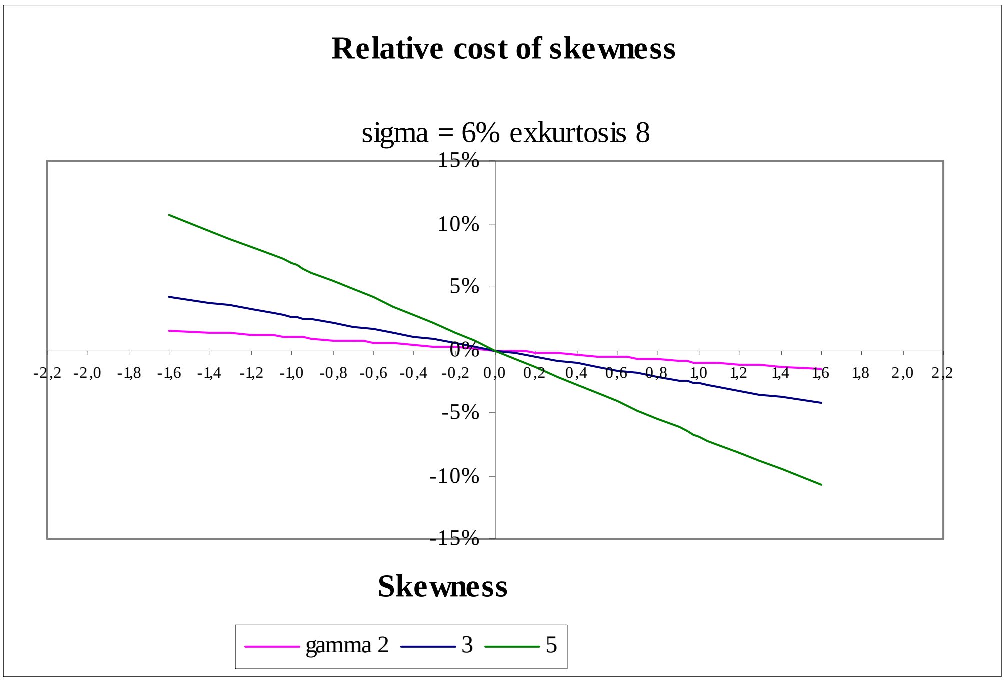
As for kurtosis, the impact of skewness on the cost of risk increases with the level of volatility, this time more or less linearly.

Chart 8



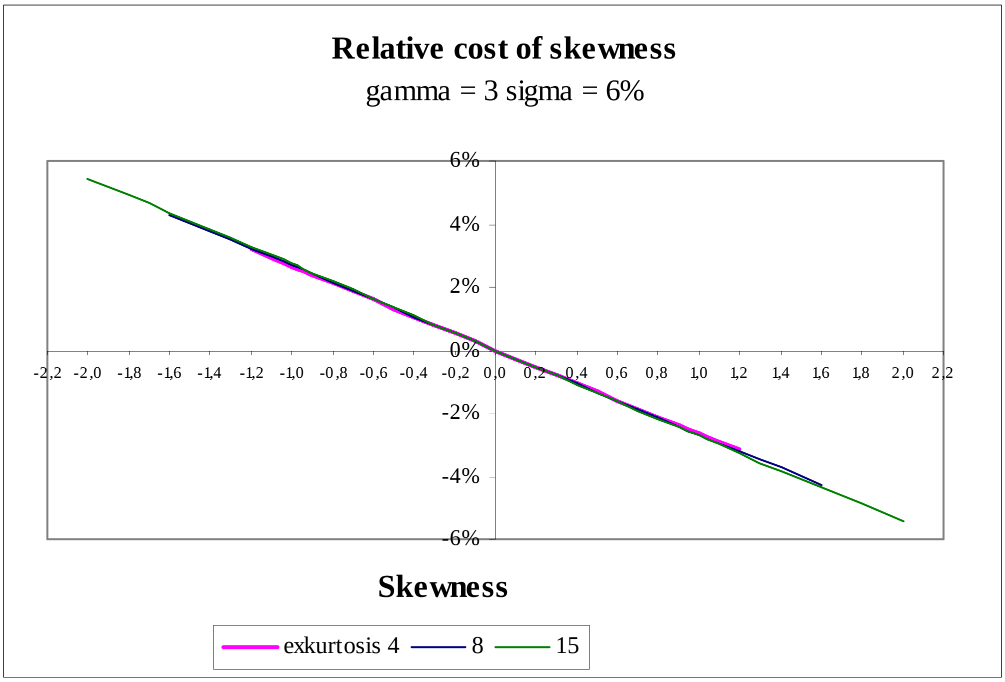
The relative cost of negative skewness also increases with the risk aversion parameter, more than proportionately.

Chart 9



Finally, the relative cost of negative skewness does not seem to depend significantly on the level of kurtosis, as illustrated below.

Chart 10



7 – Conclusions

Using a Cornish-Fisher framework to allow for controlled skewness and kurtosis, we find that the MPPM effectively does impose a penalty on tail risk It increases nearly linearly with return kurtosis and return negative skewness. The size of the penalty is rather benign when returns volatility is low and the risk parameter is low. It increases substantially for high volatilities and/or high risk parameters.

Those results hold for the Cornish Fisher framework as a way to capture skewness and kurtosis. It would be interesting to assess whether it is resilient to other distributions, which questions the potential impact of higher than fourth order moments. However, Cornish Fisher allows to explore a field of skewness and kurtosis which is much wider than other transformations such as Gramm-Charlier, and more in line with the skewnesses and kurtosis displayed in practice by financial assets returns.

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Appendix 1

Assuming that the empirical mean coincides asymptotically with the expected value,

$$CR = \frac{1}{(1-\gamma)\Delta t}\ln\left(e^{-\frac{\sigma^2}{2}(1-\gamma)\Delta t}\frac{1}{T}\sum_{t=1}^{T}\left[e^{\sigma\sqrt{\Delta t}\varepsilon}\right]^{1-\gamma}\right) = \frac{1}{(1-\gamma)\Delta t}\ln\left(e^{-\frac{\sigma^2}{2}(1-\gamma)\Delta t}E\left(\left[e^{\sigma\sqrt{\Delta t}\varepsilon}\right]^{1-\gamma}\right)\right)$$

$$CR = \frac{1}{(1-\gamma)\Delta t}\left\{\ln\left(e^{-\frac{\sigma^2}{2}(1-\gamma)\Delta t}\right) + \ln\left(E\left(e^{(1-\gamma)\sigma\sqrt{\Delta t}\varepsilon}\right)\right)\right\} = \frac{1}{(1-\gamma)\Delta t}\left\{-\frac{\sigma^2}{2}(1-\gamma)\Delta t + \ln\left(E\left(e^{(1-\gamma)\sigma\sqrt{\Delta t}\varepsilon}\right)\right)\right\}$$

For a normal standard random value ε,

$$E\left(e^{(1-\gamma)\sigma\sqrt{\Delta t}\varepsilon}\right) = e^{(1-\gamma)^2\sigma^2\Delta t/2}$$

$$CR = \frac{1}{(1-\gamma)\Delta t}\left\{-\frac{\sigma^2}{2}(1-\gamma)\Delta t + (1-\gamma)^2\sigma^2\Delta t/2\right\} = \frac{\sigma^2(1-\gamma)\Delta t}{2(1-\gamma)\Delta t}\{-1+(1-\gamma)\} = -\gamma\frac{\sigma^2}{2}$$

Appendix 2

The moments of the Cornish-Fisher distribution are computed in Maillard (2012).

The results are as follows.

$$M1 = 0$$

$$M2 = 1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2$$

$$M3 = S - \frac{76}{216}S^3 + \frac{85}{1296}S^5 + \frac{1}{4}KS - \frac{13}{144}KS^3 + \frac{1}{32}K^2 S$$

$$\begin{aligned} M4 = {} & 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8 \\ & - \frac{7}{12}KS^2 + \frac{113}{452}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2 \end{aligned}$$

$$\hat{S} = \frac{M3}{M2^{1.5}} = \frac{S - \frac{76}{216}S^3 + \frac{85}{1296}S^5 + \frac{1}{4}KS - \frac{13}{144}KS^3 + \frac{1}{32}K^2 S}{\left(1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2\right)^{1.5}}$$

$$\hat{K} = \frac{M4}{M2^2} - 3 = \frac{\left[\begin{aligned} & 3 + K + \frac{7}{16}K^2 + \frac{3}{32}K^3 + \frac{31}{3072}K^4 - \frac{7}{216}S^4 - \frac{25}{486}S^6 + \frac{21665}{559872}S^8 - \frac{7}{12}KS^2 \\ & + \frac{113}{452}KS^4 - \frac{5155}{46656}KS^6 - \frac{7}{24}K^2S^2 + \frac{2455}{20736}K^2S^4 - \frac{65}{1152}K^3S^2 \end{aligned}\right]}{\left(1 + \frac{1}{96}K^2 + \frac{25}{1296}S^4 - \frac{1}{36}KS^2\right)^2} - 3$$