A General Pricing Framework for No-Negative-Equity Guarantees with Equity-release Products: A Theoretical and Empirical Study

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ABSTRACT

We investigate stochastic house price returns, interest rates and mortality rates in the pricing of no-negative-equity guarantees (NNEGs) with the aim of identifying the risks involved in equity-release products, placing particular focus on the jump effect on house price returns. We propose an ARMA-GARCH jump model based on UK house returns data with significant jump persistence, autocorrelation and volatility clustering. Interest rate and mortality rate dynamics are respectively assumed to follow the CIR model (Cox et al. 1985) and the CBD model (Cairns et al. 2006) with the risk-neutral valuation framework for NNEG pricing being derived using the conditional Esscher transform technique (Bühlmann et al. 1996). Our numerical analyses reveal that the jump effect on house price returns, interest-rate risk and mortality-rate risk can affect the costs of NNEGs, with the impact being as significant as that for interest-rate risk. We also identify the model risk in the pricing of NNEGs by comparing various house price return models, including the Black-Scholes, Merton Jump, ARMA-GARCH, ARMA-EGARCH, Double Exponential Jump Diffusion and ARMA-GARCH Jump models.

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Introduction

The continuing global increase in life expectancy demands urgent consideration of the ways in which the retirement incomes of the elderly can be increased in order to ensure the maintenance of an acceptable standard of living. Although pension systems have long been the primary financial resource for elderly people, aging populations and increases in longevity on a global scale have put pension and annuity providers in untenable positions, such that the response by many providers has been unavoidable reductions in pension benefits (Antolin, 2007). About 75 per cent of the increasingly elderly populations around the world are now considered to have inadequate income upon their retirement; thus, governments are faced with the growing challenge of financing such aging populations. Clearly, therefore, development within the private markets of innovative financial products capable of increasing retirement income would be of significant benefit.

Many elderly people are considered to be “cash poor and equity rich” (McCarthy, et al. 2002; Rowlingson, 2006; Shan, 2011). In the UK, for example, the aggregate non-mortgaged equity owned by people over the age of 65 years was found to be £1,100 billion, whilst in the US, the median value of mortgage-free homes in the early part of the new century was found to be US$127,959, with more than 12.5 million elderly people having absolutely no mortgage debt (American Housing Survey, 2005). Home equity therefore offers a potential alternative financial resource capable of meeting current shortfalls in retirement income; and indeed, equity-release products are designed exactly for this purpose, with homeowners receiving a lump sum and/or annuity in exchange for the transfer of some, or all, of the value of their house to a financial institution upon their death. The loan value is ultimately determined by the age of the borrower, the interest rate and the value of the property. Such equity-release products are available in several developed countries, including the US, the UK, France, Australia, Canada and Japan, with the major advantage for homeowners being that they can receive cash without having to leave the property. Due to the trend of population aging, a number of studies have estimated the potential demand for equity-release products. For example, across Europe as a whole, the report by Towers Watson (2014) estimate that there is potential for over €20bn to be released from equity release products each year and over €20bn 10 years[4].

Equity-release products are widely offered by financial institutions, such as banks or insurance companies, but of course, there are risks involved for such institutions providing these products. The most obvious of these risks is the negative equity that such institutions may have to assume if the proceeds from the sale of the house prove to be less than the loan value paid out. Equity-release mortgages differ from traditional mortgages, since the loans and accrued interest are required to be repaid when the borrower dies or leaves the house. Given that the main risk factors involved in such products are the mortality of the homeowners, the interest rate and the underlying value of the property, the management of these risks has become a crucial element for equity-release product providers in the continuing development of this market.

The use of insurance or the writing of no-negative-equity guarantees (NNEGs) are the main methods used to deal with the associated risks in equity-release products. The Home Equity Conversion Mortgage (HECM) program in the US[5] is a typical example of the use of insurance, whereby a borrower pays an up-front fee of 2 per cent of the initial property value as the insurance fee, which then effectively protects

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[4] This estimate projected in 2030 is based on the following conservative assumptions: the elderly population in Europe is 124 million; the overall home ownership in the population is 71%; average house price is €210,000; a 22% loan-to value and annual sales of 1/2%; no inflation.

[5] The department of Housing and Urban Development (HUD) first introduced the HECM program in 1989.

the lenders against any losses arising if the loan balance exceeds the property value at the time of settlement. NNEGs are common practice within the UK, with such products protecting borrowers by capping the redemption amount of the mortgage at the lesser amount of the face value of the loan or the sale proceeds of the property; thus, NNEGs can be viewed as a European put option on the mortgaged property. Since the effective valuation of NNEGs has clearly become extremely important in developing an understanding of equity-release products, the primary aim of this study is to examine the risk factors involved in the pricing of NNEGs, taking into consideration house pricing, interest rates and mortality rates.

In the continuing development of the pricing of equity-release products, the primary concern, thus far, has been shown to be house price risk (Kau et al. 1995), with the assumption in a number of the prior studies being that house prices are driven by a Geometric Brownian Motion (GBM) for reverse mortgages, which thereby facilitates the application of the Black and Scholes (1973) option pricing formula to NNEG pricing.[6] Mortgage pricing models using the Black-Scholes approach have been introduced in several studies based upon the assumption that the house price process follows a standard stochastic process.[7] Further assuming that the house price index follows a GBM for derivative contracts based on the credit loss of mortgage portfolios, Duarte and McManus (2011) found that loss-based indices provided a better means of hedging credit risk in mortgage portfolios than indices based on house prices.

There are numerous similar examples to be found in the real estate literature; however, in many of the empirical investigations, two important properties have been found to be associated with house price return dynamics. Firstly, the log-return of

6 See, for example, Szymanoski (1994) and Wang et al. (2007).
7 Examples include Ambrose and Buttimer (2000), Bardhan et al. (2006) and Liao et al. (2008).

house prices is found to be autocorrelated, and secondly, the volatility of the log-return of house prices is found to be time-varying or volatility clustering. Li et al. (2010) and Chen et al. (2010) therefore turned to the use of 'Autoregressive moving average - generalized autoregressive conditional heteroskedasticity' (ARMA-GARCH) models as their approach to capturing house price dynamics in the UK equity-release market and the US HECM program. However, there must also be consideration of the fact that house price return dynamics have been subject to abnormal shocks over recent years, the most obvious example of which is the 2008 subprime mortgage crisis.

The UK house price quarterly returns from 1952 to 2012, based upon the fourth quarter of each year, are illustrated in Figure 1, with these details being obtained from the Nationwide House Price Index (HPI). As we can see, housing prices are found to have changed more than 5 per cent in a given quarter no less than 24 times over the sample period, thereby revealing significant jump risk when the quarterly housing price is found to have changed by more than three standard deviations. The most significant downward jump occurred in 2008, following the outbreak of the subprime mortgage crisis. Given that the effects of such a downward jump are both systematic and non-diversifiable, this can lead to enormous problems within the general real estate market; thus, the jump effects in house prices have attracted considerable attention and related investigations over recent years.

<Figure 1 is inserted about here>

Both Kau and Keenan (1996) and Chen et al. (2010) used the jump diffusion process to describe the changes in house prices, with the latter study demonstrating that abnormal shocks have significant impacts on mortgage insurance premiums. Chang et al. (2011) further extended the double exponential jump-diffusion model of Kou (2002) to consider the asymmetric jump risk in the pricing of mortgage insurance. On other hand, Eraker (2004), Duan et al. (2006, 2007), Maheu and McMcurdy (2004) and Daal et al. (2007) find that accommodating for jumps effect in the log return and volatility considerably improves the model's fit for the return data of equity markets.

Nevertheless, despite the jump risk having been taken into consideration in the modeling of house price dynamics in numerous prior studies, it appears that each of these studies has failed to consider the important properties of volatility persistence and autocorrelation in the log returns and allows time-variation in jump component of the log returns and volatility.[8] We therefore aim to fill the gap within the extant literature by taking these factors into consideration. In specific terms, we study the jump dynamics in house price returns based upon an ARMA-GARCH specification which allows for both constant and dynamic jumps. Following Chan and Maheu (2002) and Maheu and McMcurdy (2004)[9], we assume the distribution of jumps is to be Poisson with a time-varying conditional intensity parameter. In the empirical study, similar to the approach in Li et al. (2010), we focus on the UK equity-release market, using the Nationwide HPI to carry out our empirical analysis for the

8 Examples include Kau and Keenan (1996), Chen et al. (2010) and Chang et al. (2011).

9 Chan and Maheu (2002), Eraker (2004), Maheu and McMcurdy (2004), Duan et al. (2006, 2007) and Daal et al. (2007) all consider the GARCH jump model for dealing with equity returns and find that accommodating for jumps effect in the log return and volatility considerably improves the model's fit for the return data of equity markets. Among them, Eraker (2004) and Duan et al. (2006, 2007), Chan and Maheu (2002), Maheu and McMcurdy (2004) and Daal et al. (2007) consider a dynamic jump setting. Duan et al. (2006) extended theory developed by Nelson (1990) and Duan (1997) by considering limiting models for the GARCH-jump process. In additional, Duan et al. (2007) provide empirical test of GARCH-jump model to price options, using data on S&P 500 index and the set of European options written on S&P 500 index. Further, Daal et al. (2007) proposed asymmetric GARCH-jump models that synthesize autoregressive jump intensities and volatility feedback in the jump component to fit for the dynamics of the equity returns in the US and emerging Asian stock markets. However, different to these literatures, we deal with house price return dynamics instead of equity returns. Thus, we further consider the ARMA-GARCH jump framework.

selection of the jump dynamic specifications based upon actual house price returns data. In order to facilitate our investigation of the jump effects in house price return modeling, we carry out a comparison between the fitting accuracy of the proposed ARMA-GARCH jump model and various other jump diffusion models, such as the Merton (1976) and Kou (2002) models, as well as the models proposed within the prior literature relating to NNEG pricing, such as the GBM, ARMA-GARCH and ARMA-EGARCH models. Our empirical analyses, based upon three different National HPI data periods, reveal that the ARMA-GARCH jump model with dynamic jump specifications provides the best fit, according to both log-likelihood and Akaike information criteria (AIC). The ARMA-GARCH dynamic jump model shows significant persistence in the conditional jump, which indicates that when designing equity-release products, we cannot ignore the jump risk associated with house price returns.

Interest-rate risk is another important risk factor in the analysis of NNEGs, since interest rates are a fundamental economic variable within any economy, and cannot be treated as constant, particularly when relating to economic policies with long horizons; thus, the incorporation of the feature of stochastic interest rates in the valuation of contingent claims has been proposed in numerous studies within the extant financial literature.[10] It was also pointed out by Ho et al. (1997) that interest rate risk has become an increasingly important factor as a result of the term structure of interest rates affecting the value of options with long-term maturity. Kijima and Wong (2007) further considered the pricing of equity-indexed annuities with stochastic interest rates, noting their substantial effects on the valuing of insurance policies with long horizons. Since an NNEG is similar to writing a long-duration

[10] Examples include Merton (1973), Rabinovitch (1989), Turnbull and Milne (1991) and Amin and Jarrow (1992).

European put option on the mortgaged property, the stochastic interest rate in the NNEG pricing framework cannot be ignored. We therefore employ the well-known CIR term structure model (Cox et al. 1985) to capture the interest rate dynamics in the pricing of NNEGs.

The mortality risk factor has recently been considered in NNEG pricing as a result of the global decline in human mortality in the twentieth century. Given the uncertainty over improvement trends in long-term mortality, longevity risk has become a serious threat to lenders, since it increases the payout period and the risks involved in issuing equity-release products. In order to reflect this longevity risk, we also consider a stochastic mortality model, employing the well-known CBD model (Carins et al. 2006) for the valuation of NNEGs.

We provide a general valuation model in this study which allows for three stochastic components in the pricing of NNEGs, contributing to the extant literature on equity-release products in the following four significant ways. Firstly, our general valuation framework considers not only house price return dynamics, but also interest rate and mortality rate dynamics; we are therefore able to undertake separate analyses of the impacts of these three risk factors on NNEG costs. Secondly, we derive our risk-neutral valuation framework with house price return dynamics based upon an ARMA-GARCH jump process using the conditional Esscher transform technique (Bühlmann et al., 1996). Thirdly, our study addresses the model risk in NNEG pricing by comparing the costs based on various house price return models. Finally, our numerical findings reveal that ignoring the jump, interest rate and longevity risks will ultimately result in the underpricing of NNEGs.

The remainder of this paper is organized as follows. We construct an ARMA-GARCH jump model in Section 2 and then carry out empirical analyses to investigate the jump effect in house price returns. This is followed in Section 3 by the derivation of a risk-neutral valuation framework for NNEG pricing under ARMA-GARCH jump models. A numerical investigation of the effects of jump, interest rate and longevity risks is subsequently carried out on NNEG costs in Section 4. Finally, the conclusions drawn from this study are presented in Section 5.

Analysis of House Price Returns with Jumps

The ARMA(s,m)-GARCH(p,q) Jump Model

Analysis of the properties of volatility clustering and autocorrelation effects using house price return dynamics has already been undertaken by Chen, H. et al. (2010) and Li et al. (2010). We also consider the jump effect with house price return dynamics based upon an empirical investigation (see Figure 1); our analysis involves the construction of a house price return model capable of capturing the properties of volatility clustering and both jump and autocorrelation effects under Maheu and McCurdy (2004) framework.

We begin by investigating the house price returns data based upon time-series analysis, and then go on to develop the ARMA-GARCH jump model. Let $\left(\Omega;\Phi;P;\left(\Phi_t\right)_{t=0}^{T}\right)$ be a complete probability space, where P is the data-generating probability measure, with specifications for the conditional mean and conditional variance. Let H_t denote the UK house price index and Y_t represent the house price return at time t. Y_t is defined as $\ln(\frac{H_t}{H_{t-1}})$ and the proposed ARMA-GARCH jump model governing the return process is then expressed as:

$$Y_t = \ln(\frac{H_t}{H_{t-1}}) = \mu_t + \varepsilon_t, \tag{1}$$

The mean return follows an autoregressive moving average (ARMA) process as

$$u_t = c + \sum_{i=1}^{s} \vartheta_i Y_{t-i} + \sum_{j=1}^{m} \zeta_j \varepsilon_{t-j}, \quad (2)$$

where s is the order of the autocorrelation terms; m is the order of the moving average terms; ϑ_i is the i^{th}-order autocorrelation coefficient; ζ_j is the j^{th}-order moving average coefficient; ε_t is the total returns innovation observable at time t which is

$$\varepsilon_t = \varepsilon_{1,t} + \varepsilon_{2,t} \quad (3)$$

Extending from Maheu and McCurdy (2004)[11], we set two stochastic innovations in which the first component $(\varepsilon_{1,t})$ captures smoothly evolving changes in the conditional variance of returns and the second component $(\varepsilon_{2,t})$ causes infrequent large moves in returns and are denoted as jumps. $\varepsilon_{1,t}$ is set as a mean-zero innovation $\left(E\left[\varepsilon_{1,t} \mid \Phi_{t-1}\right] = 0\right)$ with a normal stochastic forcing process as

$$\varepsilon_{1,t} = \sqrt{h_t} z_t, \quad z_t \sim NID(0,1), \quad (4)$$

And h_t denote the conditional variance of the innovations, given an information set of Φ_{t-1},

$$h_t = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{p} \beta_j h_{t-j}, \quad (5)$$

where p is the order of the GARCH terms; q is the order of the ARCH term; α_i is the i^{th}-order ARCH coefficient; and β_j is the j^{th}-order GARCH coefficient. $\varepsilon_{1,t}$ is contemporaneously independent of $\varepsilon_{2,t}$. $\varepsilon_{2,t}$ is a jump innovation that is also conditionally mean zero $\left(E\left[\varepsilon_{2,t} \mid \Phi_{t-1}\right] = 0\right)$ and we describe $\varepsilon_{2,t}$ in next subsection.

[11] Maheu and McCurdy (2004) consider the jump setting under a constant conditional mean of GARCH model. We deal with a jump ARMA-GARCH model and the likelihood function for parameter estimation is reconstructed.

The Setting of Jump Dynamics

To capture the jump risk, the second component of innovation is employed to reflect the large change in price and modeled as

$$\varepsilon_{2,t} = \sum_{k=1}^{Nt} V_{t,k} - \phi\lambda_t \quad V_{t,k} \sim NID\left(\phi, \theta^2\right) \qquad \text{for } k = 1, 2, \cdots \tag{6}$$

where $V_{t,k}$ denotes the jump size for the k^{th} jump with the jump size following the normal distribution with parameters, (ϕ, θ^2) and N_t is the jump frequency from time $t-1$ to t, distributed as a Poisson process with a time-varying conditional intensity parameter (λ_t); that is:

$$P(N_t = j \mid \Phi_{t-1}) = \frac{\exp(-\lambda_t)\lambda_t^j}{j!},\ j = 0,1,2....,\tag{7}$$

where the parameter λ_t represents the mean and variance for the Poisson random variable, also referred to as the conditional jump intensity.

To facilitate our investigation of the jump effect on house price returns, we extend the work of Chan and Maheu (2002), Maheu and McCurdy (2004) and Daal et al. (2007) to specify λ_t as an ARMA form, which is

$$\lambda_t = \lambda_0 + \rho\lambda_{t-1} + \varsigma\psi_{t-1},\tag{8}$$

where ρ measures jumps persistence. Since the ς variable measures the sensitivity of the jump frequency (λ_t) to past shocks (ψ_{t-1}), with ψ_{t-1} representing the unpredictable component affecting our inference on the conditional mean of the counting process, N_{t-1}, then this suggests corresponding changes. We also investigate the constant jump effect, which represents a special case of Equation (8) with the restriction of constant jump intensity ($\lambda_t = \lambda_0$); this is imposed by setting $\rho = 0$ and $\varsigma = 0$.

The conditional jump intensity in this model is time-varying, with an unconditional value under certain circumstances. In order to derive the unconditional value of λ_t, we must first recognize that ψ_t is a martingale difference sequence with respect to Φ_{t-1}, because:

$$E\left[\psi_t \mid \Phi_{t-1}\right] = E\left[E\left[N_t \mid \Phi_t\right] \mid \Phi_{t-1}\right] - \lambda_t = \lambda_t - \lambda_t = 0, \qquad (9)$$

Thus, $E[\psi_t] = 0$ and $Cov(\psi_t, \psi_{t-i}) = 0$, $i > 0$.

Another way of interpreting this result is to note that, by definition, ψ_t is nothing more than the rational forecasting error associated with updating the information set; that is, $\psi_t = E[N_t \mid \Phi_t] - E[N_t \mid \Phi_{t-1}]$. There are several important features in the conditional intensity model as noted by Maheu and McCurdy (2004). First, if the conditional jump intensity is stationary, $\left(|\rho| < 1\right)$, then the unconditional jump intensity is equal to

$$E\left[\lambda_t\right] = \frac{\lambda_0}{1-\rho} \qquad (10)$$

Second, to forecast λ_{t+i}, the multi-period forecasts of the expected number of future jumps are

$$E\left[\lambda_{t+i} \mid \Phi_{t-1}\right] = \begin{cases} \lambda_t & i = 0 \\ \lambda_0\left(1+\rho+\cdots+\rho^{i-1}\right)+\rho^i\lambda_t & i \geq 1 \end{cases} \qquad (11)$$

Thus, the conditional jump intensity can be re-expressed as

$$\lambda_t = \lambda_0 + \left(\rho - \varsigma\right)\lambda_{t-1} + \varsigma E\left[N_{t-1} \mid \Phi_{t-1}\right] \qquad (12)$$

Because the jump intensity residual is defined as

$$\psi_{t-1} = E\left[N_{t-1} \mid \Phi_{t-1}\right] - \lambda_{t-1} = \sum_{j=0}^{\infty} jP\left(N_{t-1} = j \mid \Phi_{t-1}\right) - \lambda_{t-1} \qquad (13)$$

where $E\left[N_{t-1} \mid \Phi_{t-1}\right]$ is our ex post assessment of the expected number of jumps that occurred form $t-2$ to $t-1$, and $P\left(N_{t-1} = j \mid \Phi_{t-1}\right)$ is called the filter and is the ex post inference on N_{t-1} give time $t-1$ information.

Note that, a sufficient condition to ensure $\lambda_t \geq 0$, for all $t > 1$, is $\lambda_0 > 0, \rho \geq \varsigma$, and $\varsigma > 0$. In addition, to forecast the conditional jump intensity, the startup value of λ_0 and ψ_1 must be set. We follow Maheu and McCurdy (2004) to set λ_0 as the unconditional value shown in equation (10), and $\psi_1 = 0$. More details regarding the ARMA jump intensity can be referred to Maheu and McCurdy (2004).

Parameter Estimation

The parameters of the ARMA-GARCH jump model can be estimated using the maximum likelihood estimation (MLE) method. The construction of the likelihood function is described as follows. Let $F_n(\Theta)$ denote the log-likelihood function and Θ is the parameter set governing the ARMA-GARCH jump model, which implies $\Theta = (C, \vartheta_1, \vartheta_2, \vartheta_3, \zeta_1, \zeta_2, w, \alpha, \beta, \lambda_0, \rho, \varsigma, \phi, \theta)$ We aim to find the optimal parameters (Θ^*) to maximize the log-likelihood function. The log-likelihood function can be expressed as

$$F_n(\Theta) := \sum_{t=1}^{N} \log f(Y_t \mid \Phi_{t-1}, \Theta) \tag{14}$$

The conditional on j jumps occurring the conditional density of returns is Gaussian,

$$f(Y_t \mid N_t = j, \Phi_{t-1}, \Theta) = \frac{1}{\sqrt{2\pi(h_t + j\theta^2)}} \times \exp\left(-\frac{(Y_t - u_t + \phi\lambda_t - j\phi)^2}{2(h_t + j\theta^2)} \right). \tag{15}$$

In Equation (14), the conditional density of return at time t ($f(Y_t \mid \Phi_{t-1}, \Theta)$) for calculating log-likelihood function can be obtained by integrating out the number of jumps as

$$f\left(Y_t \mid \Phi_{t-1},\Theta\right)=\sum_{j=0}^{\infty} f\left(Y_t \mid N_t = j,\Phi_{t-1},\Theta\right)P\left(N_t = j \mid \Phi_{t-1},\Theta\right)$$
$$=\sum_{j=0}^{\infty}\frac{1}{\sqrt{2\pi(h_t + j\theta^2)}}\times\exp\left(-\frac{\left(Y_t - u_t + \phi\lambda_t - j\phi\right)^2}{2(h_t + j\theta^2)}\right).\frac{\exp(-\lambda_t)\lambda_t^j}{j!} \quad (16)$$

where the conditional density of N_t ($P\left(N_t = j \mid \Phi_{t-1},\Theta\right)$) is shown in Equation (7). Since we assume the time-varying conditional intensity parameter (λ_t) follow an ARMA form as shown in Equation (8), we need to work out the past shock(ψ_{t-1}) that affects the inference on the conditional mean of the counting process first. ψ_{t-1} is defined as

$$\psi_{t-1} = E\left[N_{t-1} \mid \Phi_{t-1},\Theta\right]-\lambda_{t-1}$$
$$=\sum_{j=0}^{\infty} jP\left(N_{t-1} = j \mid \Phi_{t-1},\Theta\right)-\lambda_{t-1} \quad (17)$$

where $E\left[N_{t-1} \mid \Phi_{t-1},\Theta\right]$ is given by Equation (13). This expression could be estimated if $P\left(N_{t-1} = j \mid \Phi_{t-1},\Theta\right)$ are known. Following Maheu and McCurdy (2004), the ex post probability of the occurrence of j jumps at time t-1 can be inferred using Bayes' formula as follows.

$$E\left[N_{t-1} \mid \Phi_{t-1},\Theta\right]=\sum_{j=0}^{\infty} jP\left(N_{t-1} = j \mid \Phi_{t-1},\Theta\right)$$
$$=\sum_{j=0}^{\infty} j\,\frac{f(Y_{t-1} \mid N_{t-1} = j,\Phi_{t-2},\Theta)P(N_{t-1} = j \mid \Phi_{t-2},\Theta)}{f(Y_{t-1} \mid \Phi_{t-2},\Theta)} \quad (18)$$
$$=\frac{\sum_{j=1}^{\infty}\frac{\exp(-\lambda_{t-1})\lambda_{t-1}^j}{j!}\frac{1}{\sqrt{2\pi(h_{t-1} + j\theta^2)}}\times\exp\left(-\frac{\left(Y_{t-1} - u_{t-1} + \phi\lambda_{t-1} - j\phi\right)^2}{2(h_{t-1} + j\theta^2)}\right)}{f(Y_{t-1} \mid \Phi_{t-2},\Theta)}$$

The details of Bayes' inference on calculating $E\left[N_{t-1} \mid \Phi_{t-1},\Theta\right]$ is presented in Maheu and McCurdy (2004) Thus, by iterating on (8), (16) and (18), we can construct the log-likelihood function and obtain the maximum likelihood estimators. In addition, Equations (16), (17) and (18) involves an infinite summation depending on the jumps.[12] We find that truncation of the infinite sum in the likelihood at 10 captures all the tail probabilities and gleans sufficient precision in the estimation procedure.

Empirical Analysis of Model Fit

We examine the performance of the ARMA-GARCH jump model using time-series data from the Nationwide HPI, placing particular focus on an investigation into whether the conditional jump intensity is time-varying or constant. Our quarterly data period runs from the fourth quarter of 1952 to the fourth quarter of 2012, thereby providing a total of 241 quarterly observations. As a check for robustness, we also examine the results for different data periods (from the fourth quarters of 1962 to 2012 and from the fourth quarters of 1972 to 2012).

The summary statistics on the levels and squares of the log-return series are reported in Table 1, from which there is clear evidence of time dependence using the modified Ljung-Box (LB) statistics (West and Cho, 1995). These statistics, which are reported for autocorrelations of up to 29 lags, are found to be robust to heteroskedasticity. The modified LB statistics show strong serial correlation in both the levels and the squares of the return series, a result which is consistent with those reported by Li et al. (2010), where the serial correlations in the Nationwide HPI returns were found to be significant.

<Table 1 is inserted about here>

We investigate the jump dynamics for both dynamic and constant jump models, using ARMA(3,2)-GARCH(1,1) models, with the parameters of these two ARMA-GARCH jump models being estimated by maximizing the conditional log-likelihood

[12] Equation (16), (17) and (18) involve an infinite sum over the possible number of jumps, N_t. In practice, for our model estimated we found that the conditional Poisson distribution had zero probability in the tail for values of $N_t \geq 10$ and the likelihood and the parameter estimates converge.

functions. The selection of the ARMA(3,2)-GARCH(1,1) models in the present study is based upon the Box-Jenkins approach.[13] Details on the evaluation of our ARMA(3,2)-GARCH(1,1) jump models and the relevant parameter estimates are presented in Table 2.

<Table 2 is inserted about here>

We evaluate the performance of the jump dynamics using log-likelihood, Akaike information criteria (AIC) and Bayesian information criteria (BIC).[14] The log-likelihood and AIC results indicate that the ARMA-GARCH dynamic jump model provides a better fit, with the persistence parameter (ρ) in this model being found to be 0.6380, with statistical significance. This finding suggests that a high probability of many (few) jumps will also tend to be followed by a similarly high probability of many (few) jumps.

Nevertheless, when taking into account the number of parameters involved in the evaluation of the various models, the BIC results suggest that the constant jump model has slight superiority over the dynamic jump model. Recall that ψ_t is the measurable shock constructed by econometricians using the ex post filter; thus, in a correctly-specified model, ψ_t should not display any systematic behavior.

In order to facilitate a thorough investigation in the present study of the importance of the jump effect in the modeling of house price returns, the existing models proposed in Chen, H. et al. (2010) and Li et al. (2010) – which include the GBM, ARMA-GARCH and ARMA-EGARCH models – are also fitted to exactly the same series of Nationwide HPI returns. We further compare the performance of the ARMA-GARCH jump model with other jump diffusion models, such as the

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13 Although not reported here, the parameter estimates of the models are available upon request.

14 AIC = −2/obs. ln(likelihood) + 2/ obs. × (No. of parameters) (Akaike, 1973); BIC = −2/ obs. ln(likelihood) + ([No. of parameters] × ln[obs.]) / obs.; obs. is the sample size.

Merton (1976) and Kou (2002) models, both of which allow for jump effects, but do not consider the effects of autocorrelation and volatility persistence.

The fitting results are presented for each of the different models in Table 3.[15] Our empirical results indicate the superiority of the ARMA-GARCH jump model over the existing house price return models, with the ARMA-GARCH dynamic jump model demonstrating further improvements on each of the other models based upon the log-likelihood and AIC values.

Although the jump effect is taken into consideration in the jump diffusion models, such as those proposed by Merton (1976) and Kou (2002), the performance of their models is nevertheless found to be inferior to that of the time-series models within which the effects of autocorrelation and volatility clustering are also taken into consideration; it therefore seems clear that a house price return model capable of simultaneously taking into consideration all three properties would represent an important contribution to this particular field of research.

<Table 3 is inserted about here>

As a check for the robustness of our results, we also investigate the model fit by considering different periods of the Nationwide HPI data. The results for the fourth quarters from 1962 to 2012 and quarters from 1972 to 2012 are shown in Table 4, For both sub-periods, the ARMA-GARCH dynamic jump model is still found to outperform each of the other models.

<Tables 4 are inserted about here>

The results reported in Tables 3 to 4 confirm that the addition of jump dynamics improves the specification of the conditional distribution, as compared with the GBM, Merton jump, double exponential jump diffusion, ARMA-GARCH, ARMA-

[15] The stochastic processes of these models are available upon request.

EGARCH and ARMA-GARCH constant jump models. In addition, the persistence parameter (ρ) governing the jump dynamic is statistically significant. It clearly indicates that jump risk in housing returns is significant and critical for pricing of NNEGs.[16]

Valuation of NNEGs under House Price Return, Interest Rate and Mortality Dynamics

The Payoff of NNEGs

Given that, for some considerable time, it has been accepted market practice within the UK for all equity-release products to include the provision of a no-negative-equity guarantee (NNEG), the effective valuation of NNEGs has clearly become an extremely important issue. NNEGs protect borrowers by capping the redemption amount of the mortgage at the lesser of the face amount of the loan or the sale proceeds of the property; thus, the provision of an NNEG is similar in effect to the writing of a European put option on the mortgaged property.

Let us consider, as an example, a 'roll-up' mortgage.[17] Let K_t denote the outstanding balance of the loan and H_t represent the value of the mortgaged property. The amount repayable (outstanding balance) at time T is the sum of the principal, K, plus the interest accrued at a roll-up rate[18], v_t; that is,

$$K_T = Ke^{\sum_{t=0}^{t=T-1} v_t}, \tag{19}$$

[16] The persistence parameter (ρ) governing the jump model is estimated to be around 0.6380, with statistical significance. We didn't report the entire parameter estimates here but they are available upon request.

[17] The most common types of payment options for equity-release products are lump sum (roll-up), terms, lines of credit, modified terms (combining lines of credit and term payments), tenure and modified tenure (combining lines of credit and tenure). Given that the roll-up mortgage has become the most popular payment option, our ongoing analysis focuses on this type of mortgage.

[18] The roll-up rate can be either fixed or floating. In addition, the initial principal is normally determined according to the value of the housing value.

At the time that the loan becomes repayable, time T, if $H_t < K_t$, then the borrower pays H_t, and if $H_t > K_t$,then the borrower pays K_t. Once the loan is repaid, the provider receives an amount, K_t, plus the NNEG payoff, which is:

$$-Max[K_t - H_t, 0], \tag{20}$$

or exactly the payoff of a short position on a European put option with strike price K_t written on an underlying mortgaged property, H_t. Nevertheless, the valuation of an NNEG is more complex than the valuation of a European equity put option, essentially because the house price returns are highly autocorrelated, with significant heteroskedasticity and jump effects. Neither the Black-Scholes nor the Merton jump option pricing formulae are appropriate for the valuation of NNEGs since the former assumes that the returns of the underlying asset follow a GBM, whilst the latter assumes that they follow a mixed-jump process. Thus, following the validation of the jump risk for house price returns, we go on to construct a valuation framework for NNEGs based upon the specifications of an ARMA-GARCH jump model and take the interest rate and mortality risk into account simultaneously.

The NNEG Valuation Framework

Let V(0, s) denote the no-arbitrage value of the NNEG which is due at time s. The NNEG becomes due when the borrower dies. Thus, for a person aged x at inception, the expected cost of the NNEG, denoted as $V_{NNEG}(0,x)$, can be expressed as a series of European put options with different maturity dates. Under a discrete time steps, the fair value of the expected cost of a NNEG is calculated as

$$V_{NNEG}(0,x) = \sum_{t=0}^{\omega-x-1} {}_s p^Q(0,x) q_s^Q(0,x) V(0,s), \tag{21}$$

where ω is the maximal age of the borrower; ${}_s p^Q(0,x)$ is the projected probability that a borrower aged x at inception will survive to age $x+s$ and $q_s^Q(0,x)$ is the mortality that a borrower aged x at inception will die during the future time interval s to s+1 under the risk adjusted probability measure Q, or referring to as the risk-neutral measure.

The no-arbitrage value of V(0, s) is calculated by discounting the payoff at time s under a risk-neutral measure Q, which is expressed as:

$$V(0,s) = E^Q\left[\exp\left(-\int_0^s r_t\ dt\right) Max\left[K_s - H_s, 0\right]\right]. \tag{22}$$

where r_t is the risk free interest rate at time t.

To deal with the no-arbitrage value of $V(0,s)$, we need to obtain the risk-neutral process of the underlying housing price return under the ARMA-GARCH jump model. We use the conditional Esscher transform technique to derive the corresponding risk-neutral pricing. In addition, we cannot ignore the effect of interest rate and mortality risk in pricing NNEG because of the long-horizon feature of NNEGs; thus, we extend the existing literature on NNEG pricing to incorporate a stochastic interest rate and mortality rate assumption by employing the CIR interest rate model (Cox et al., 1985) and CBD mortality model (Cairns et al., 2006) . The corresponding process to obtain the risk-neural valuation will be given in next Section

Risk-neutral Valuation

House price return dynamic: an ARMA-GARCH jump model

To price a NNEG, we derive the corresponding risk-neutral return dynamic under the proposed ARMA-GARCH jump model by employing the conditional Esscher transform technique (Bühlmann et al., 1996). This technique has been widely used in the pricing of financial and insurance securities in an incomplete market since its introduction in 1932.[19] Siu et al. (2004) use the conditional Esscher transform for pricing derivatives when the underlying asset returns were found to follow GARCH processes. Recently, such technique has been extended to deal with pricing reverse mortgage products (Li et al., 2010; Chen et al. , 2010; Yang, 2011; Lee et al., 2012). To introduce the conditional Esscher transform technique, we define a sequence $\left\{\Lambda_t \mid t = j\Delta t, j = 0,1,\cdots, T/\Delta t\right\}$ be a Φ_t - adapted stochastic process:

$$\Lambda_T = \prod_{t=\Delta t}^{T} \frac{\exp\left(aY_t\right)}{E\left[\exp\left(aY_t\right) \mid \Phi_{t-\Delta t}\right]} \tag{23}$$

where Y_t represents the house price return dynamic. The ARMA-GARCH jump model for capturing house price return under the real world measure can be referred to Equations (1)-(5). Bühlmann et al. (1996) has proved that $E\left(\Lambda_T\right) = 1$ and $E\left(\Lambda_T \mid \Phi_t\right) = \Lambda_t$. Equivalently, $\left\{\Lambda_t\right\}$ is a martingale under P. We define a new martingale measure Q by

$$\frac{dQ}{dP} \mid \Phi_t = \Lambda_T \tag{24}$$

Then, under a risk neutral measure, Q, the housing price return dynamic then becomes

$$Y_t = \ln\left(\frac{H_t}{H_{t-\Delta t}}\right) = r_{t-\Delta t}\Delta t - \frac{1}{2}h_t^* + \varepsilon_t^Q, \tag{25}$$

with $h_t^* = h_t + \left(\phi^2 + \theta^2\right)\lambda_t$ and $\varepsilon_t^Q = \varepsilon_t - a_t h_t^*$. ε_t^Q follows a normal distribution with mean 0 and variance h_t^* under measure Q. In other words, the house price return dynamic under measure Q is similar to the form under measure P, albeit with shifted

[19] See, for example, Gerber and Shiu (1994), Bühlmann et al. (1996), Siu et al. (2004), Li et al. (2010) and Chen et al. (2010).

parameters, that is. $Y_t \mid \Phi_{t-1} \sim N\left(r_{t-\Delta t}\Delta t - \frac{1}{2}h_t^*, h_t^*\right)$. See Appendix for the derivation of the risk-neutral ARMA-GARCH jump model in Equation (25).

Interest rate Dynamic: CIR model

To model interest rate risk, we employ the well-known CIR interest rate model (Cox et al. 1985) which results in the introduction of a 'square-root' term in the diffusion coefficient of the interest rate dynamics proposed by Vasicek (1977). The CIR model has been a benchmark in modeling interest rates for many years, essentially because of its analytical tractability, as well as the fact that, contrary to the Vasicek (1977) model, the interest rate is always positive. Under the CIR model, we assume that the time-t short rate, r_t, for a $\left(\Omega; \Phi; P; (\Phi_t)_{t=0}^T\right)$ is a complete probability space, governed by the following equation:

$$dr_t = \alpha_r(\mu_r - r_t)dt + \sqrt{r_t}\sigma_r dW_{r,t} \tag{26}$$

where $\{W_{r,t}, t \geq 0\}$ is a standard Brownian Motion with parameters $\theta_r \equiv (\alpha_r, \mu_r, \sigma_r)$. The drift function $\alpha_r(\mu_r - r_t)$ is linear with mean reversion property; that is, the interest rate, r_t, moves in the direction of its mean, μ_r, at speed α_r. The diffusion function, $r_t \sigma_r^2$, is proportional to the interest rate, r_t, which ensures that the process remains within a positive domain. Furthermore, if α_r, μ_r and σ_r are all positive, and if $2\alpha_r\mu_r \geq \sigma_r^2$ holds, then we can also assume that r_t will remain positive.

Under the risk-neutral probability measure, Q, the short rate defined in Equation (26) ensures that the discounted zero coupon bond price follows a martingale; that is:

$$P_{0,T} = E^Q\left[\frac{P_{T,T}}{B_T}\right] = E^Q\left[\exp\left(-\int_0^T r_s ds\right)\right] \tag{27}$$

where B_t refers to the money market account at time t which satisfies:

$$B_t = \exp\left(\int_0^t r_s ds\right). \tag{28}$$

According to the CIR model, a standard Brownian motion under the risk-neutral probability measure Q ($W_{r,t}^Q$) can be specified as:

$$dW_{r,t}^Q = dW_{r,t} + \frac{\vartheta_r \sqrt{r_t}}{\sigma_r} dt, \tag{29}$$

where ϑ_r is the risk premium parameter. Consequently, the short rate at time t becomes:

$$r_{t+\Delta t} - r_t = \left(\alpha_r^Q - \mu_r^Q r_t\right)\Delta t + \sqrt{r_t}\sigma_r \Delta W_{r,t}^Q, \tag{30}$$

where $\alpha_r^Q = \alpha_r \mu_r$ and $\mu_r^Q = \alpha_r + \vartheta_r$. We consider the correlation between the house price return and the short rate. Specifically, the correlation coefficient between $\frac{\varepsilon_t^Q}{\sqrt{h_t^*}}$ and $\frac{W_{r,t}^Q}{\sqrt{t}}$ is equal to $\rho_{Y,r}$. To incorporating the correlation for simulating the future dynamics of house price return and short rate, we use the Cholesky Decomposition method [20]. In addition, we apply the MLE technique to the actual British zero-coupon bond data to calibrate the CIR model. The data covers the same period as the housing price data from Q4 1952 to Q4 2012 based on DataStream database.

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[20] We let a random vector $x = \begin{bmatrix} \frac{\varepsilon_t}{\sqrt{h_t}} & \frac{W_{r,t}}{\sqrt{t}} \end{bmatrix}$ and $\Lambda_{\rho_{Y,r}}$ denote the correlation matrix as $\Lambda_{\rho_{Y,r}} = \begin{bmatrix} 1 & \rho_{r,Y} \\ \rho_{Y,r} & 1 \end{bmatrix}$. We can decompose $\Lambda_{\rho_{Y,r}}$ as LL^T, where L is the lower triangular matrix with real and positive diagonal entries as $L = \begin{bmatrix} 1 & 0 \\ \rho_{Y,r} & \sqrt{1-\rho_{Y,r}^2} \end{bmatrix}$. Finally, we can generate the new correlation random vector variable Z using the approach as $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = L * x' = \begin{bmatrix} \frac{\varepsilon_t}{\sqrt{h_t}} \\ \rho_{Y,r}\frac{\varepsilon_t}{\sqrt{h_t}} + \sqrt{1-\rho_{Y,r}^2}\frac{W_{r,t}}{\sqrt{t}} \end{bmatrix}$. The procedure is known as Cholesky Decomposition.

We make use of the zero-coupon bond with different time to maturities of 3-month, 6-month, 1-year, 5-year and 10-year to calibrate the model. The procedure of calibrating the CIR model follows Rémillard (2013).[21]

Due to we price the NNEG on a discrete time setup as shown in Equation (21), we extend the CIR model on a discrete basis. Following Lee et al.(2012), we assume that the short rate between t and $t+\Delta t$ is fixed at r_t, but still vary from one band to the next.

Mortality Dynamic: CBD model

To modeling mortality dynamics, as opposed to using the static mortality rate, we consider the longevity risk in NNEG pricing and employ the CBD model (Cairns et al, 2006) to project future mortality rates. The CBD model is attractive because it uses only a few parameters to obtain a good fit for the mortality probabilities of the elders; thus, this model has been widely adopted as a means of dealing with longevity risk for the elders (Wang et al., 2010, Yang, 2011). Since the reverse mortgage products are issued for the elders, we also adopt the CBD model. Under the CBD model, the mortality rate for a person aged x dying before x+1 valued in year t , denoted as q(t, x), is projected by:

$$\text{logit } q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \overline{x}), \tag{31}$$

where the parameter $\kappa_t^{(1)}$ represents the marginal effect of time on mortality rates; parameter $\kappa_t^{(2)}$ refers to the old age effect on mortality rates; and $\overline{x}$ is the mean age.[22] With the estimated values of ($\kappa_t^{(1)}, \kappa_t^{(2)}$), we can forecast the future mortality

[21] Their code in Matlab function "ESTCIR" is available on the Rémillard's website. http://www.brunoremillard.com/services.html

[22] We use the UK mortality data from 1950-2006 according to the human morality database (HMD) and

rates. In this study, we adopt Cairns et al. (2006)'s approach to estimate the parameters by using the least square method to fit the actual mortality curve and then project the ($\kappa_t^{(1)}, \kappa_t^{(2)}$) based upon a two-dimensional random walk with drift:

$$\kappa_{t+1} = \kappa_t + \mu + CZ_{t+1} \tag{32}$$

where $\kappa_t = \left[\kappa_t^{(1)}, \kappa_t^{(2)}\right]'$ and u is a constant 2×1 vector; C is a constant 2×2 upper triangular matrix; and Z_t is a two-dimensional standard Gaussian process.

Equation (32) describes the dynamics of the random walk process κ_t under the real world probability measure, P, for projecting the mortality rate shown in Equation (31). Let p(t, x) denote the projected one-year survival rate in year t based upon the CBD model, the projected probability in year t that a borrower aged x will survive to age $x + s$ is calculated by

$${}_s p(t, x) = p(t, x) p(t+1, x+1) \cdots p(t+s, x+s-1)\ . \tag{33}$$

To project the mortality rate under a risk-neutral probability measure Q, following Cairns et al. (2006), the dynamics become

$$\begin{aligned} \kappa_{t+1} &= \kappa_t + \mu + C\left(\tilde{Z}_{t+1} - \lambda_m\right) \\ &= \kappa_t + \tilde{\mu} + C\left(\tilde{Z}_{t+1}\right) \end{aligned} \tag{34}$$

where $\tilde{\mu} = \mu - C\lambda_m$.

$\tilde{Z}_{t+1}$ in Equation (34) is a standard two-dimensional normal random variable under Q. The vector $\lambda = (\lambda_{m1}, \lambda_{m2})$ represents the market price of the longevity risk associated with the respective processes of $Z_{1,t}$ and $Z_{2,t}$. where λ_{m1} is associated

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the data ages cover from age 60 to 100. Therefore, the mean age is 80 in our model calibration.

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with level shift in mortality and λ_{m2} is associated with a tilt in morality. As in Cairns et al. (2006), we assume that the market price of risk λ_m is not updated over time; however, since there is no liquid market for systematic longevity risk, it is difficult to calibrate the risk-neutral survival probabilities using empirical data. Therefore, we follow the approach of Cairns et al. (2006) to carry out the calibrations as the parameter value of $\lambda_m = [0.175,\ 0.175]'$ for the pricing of NNEGs in the present study.

Costs of No-Negative-Equity Guarantees

Example Setting and Assumptions

In this section, we study the impacts of different risk factors on NNEG costs. The no-arbitrage value of NNEGs depends upon the dynamics of house price returns and the interest and mortality rates. Taking these risk dynamics into account increases the degree of difficulty of pricing the NNEG in Equation (21) analytically; instead, we use Monte Carlo simulations. Thus, we first generate 100,000 sample paths of the risk-neutral house price returns, interest rates and mortality rates according to Equations (25), (30) and (34) separately and then calculate the value of NNEG ($V_{NNEG}(0,x)$)based on Equations (21) and (22). In addition, to implementing simulations, we assume that all deaths occur at midyear, and that δ is the average delay in the actual sale of the property in calculating the NNEG.

Regarding the mortgage product, we consider a floating roll-up mortgage which is the most popular equity-release product in the UK and the floating interest rate (v_t) is set as being equal to the risk-free interest rate(r_t) plus a constant spread (v_r), that is, v_t

$= r_t + v_r$.[23] For a comparison purpose, we follow Li et al. (2010) to set up the relevant assumptions for the NNEG and list the information in Table 5. In addition, the parameter estimates for the housing price return for the ARMA-GARCH jump model can refer to Table 2 (in Section 2), and for the interest rate and mortality rate models are in Section 3 as shown in Table 6 and Figure 2[24].

<Table 5 is inserted about here>

<Table 6 is inserted about here>

<Figure 2 is inserted about here>

House Price Risk Effects

Both the model risk and jump risk in house price dynamics are examined in this study by comparing the NNEG costs (reported in Table 7). Starting with our analysis of model risk, we calculate the NNEG costs under various house price return models, including the Black-Scholes, Merton jump diffusion, double exponential jump diffusion, ARMA-GARCH and ARMA-EGARCH models. The resultant NNEG values are expressed as a percentage of the total amount of cash advanced.

<Table 7 is inserted about here>

The results reveal that the GBM assumption yields the lowest value for NNEG costs; thus, these costs would tend to be significantly underpriced if we were to ignore the important properties of autocorrelation, volatility clustering and jump effects in house price dynamics. Our empirical analysis in Section 2 has already demonstrated that jump risk cannot be ignored when modeling house price dynamics, and indeed,

[23] A fixed roll-up mortgage was considered in Lee et al. (2012).

[24] To be consistent, we employ the three-month T-bill interest rates from Q4 1952 to Q4 2012 to estimate the parameters in CIR model.

we find that taking the jump effect into account increases the overall costs of NNEGs, with the ARMA-GARCH dynamic jump model exhibiting the most significant effect amongst all of the house price return models examined in this study.

Mortality Risk Effects

The expected NNEG costs based upon the proposed ARMA-GARCH jump model for different gender and age groups are reported in Table 8; the results facilitate our investigation of the mortality risk effects. An increase in the age of borrowers leads to a reduction in NNEG costs, although the costs vary for borrowers of different gender; we surmise that the NNEG costs will be greater for female borrowers than male borrowers, essentially because females have a longer life expectancy.

<Table 8 is inserted about here>

We also investigate the impact of longevity risk on NNEG pricing, and find that when such risk is taken into account, there is a slight increase in NNEG costs. For male borrowers at age 60, the cost as a proportion of the cash advanced increases from 5.12 per cent to 5.29 per cent under the ARMA-GARCH constant jump model, and from 5.19 per cent to 5.37 per cent under the ARMA-GARCH dynamic jump model.

Interest Rate Risk Effects

The results on the impact of interest rate risk are reported in Table 9, which shows that NNEG costs are increased by the stochastic interest rate. With all other parameters fixed, the NNEG rises from 5.80 per cent to 6.62 per cent for a female borrower, and from 4.50 per cent to 5.29 per cent for a male borrower, both at age 60. For older borrowers, interest rate risk has a less significant effect on NNEGs, since the implication is that contracts for older borrowers have shorter horizons. Clearly, lenders should be aware of this relationship when pricing NNEG contracts.

<Table 9 is inserted about here>

In summary, consideration of interest rates as a stochastic factor is important when assessing NNEG products, particularly floating roll-up mortgages; this finding is consistent with the prior literature on the pricing of derivatives and insurance contracts (Ho et al., 1997; Kijima and Wong, 2007).

Comparison of Different Risk Factor Effects on NNEG Costs

Finally, we compare the impacts on NNEG costs arising from the different jump risk factors involved in house price dynamics, interest rate risk and mortality risk. Separate analyses of the corresponding effects under ARMA-GARCH constant jump and ARMA-GARCH dynamic jump models are reported in Table 10, from which several interest findings arise.

Firstly, the impacts of the different risk factors on NNEG costs are consistent under both ARMA-GARCH constant jump and dynamic jump models, albeit in slightly differing degrees; for example, for a female borrower aged 60 under the ARMA-GARCH constant jump model, the impact on NNEG costs is found to be 0.82 per cent for interest rate risk, 0.58 per cent for jump risk and 0.22 per cent for mortality risk. Based upon the same borrower under the ARMA-GARCH dynamic jump model, the impacts on the NNEG costs become 0.8 per cent for interest rate risk, 0.67 per cent for jump risk and 0.24 per cent for mortality risk. Secondly with an increase in the age of borrowers, there is a corresponding reduction in the impact of these three risk factors on NNEG costs;

<Table 10 is inserted about here>

These findings clearly indicate that the greatest impact of NNEG costs stems from interest rate risk, as compared to jump risk.[25] However, our earlier analysis in

[25] Our analysis is based upon the jump model parameters. According to the empirical data, the jump

sub-section 4.1 indicated that the house price return model was the most significant factor determining NNEG costs.

Conclusions

In conjunction with the rapid growth in the equity-release market, there is growing demand for the development of effective risk management tools for these products. In the UK, equity-release products are commonly sold with no-negative-equity guarantee protection which caps the redemption amount at the lesser of the face amount of the loan or the sale proceeds. It therefore seems crucial for providers to have a firm understanding of the pricing of NNEGs, and the risk factors involved, particularly as house price, interest rate and mortality risks can affect NNEG pricing in differing degrees. We extend the current literature by considering these three risk factors in the pricing of NNEGs and by analyzing the corresponding effects.

Historical house price returns within the UK real estate market have experienced significant abnormal shocks, such as the subprime mortgage crisis in 2008, and since the providers of equity-release products assume substantial financial burdens when issuing NNEGs, it is extremely important for such providers to take into account the jump effects in house price returns when pricing these products. Despite this obvious requirement, this issue has not yet been dealt with in the prior literature; thus it is examined in the present study using an ARMA-GARCH jump model.

Furthermore, both interest rate and longevity risks can increase the probability of the home sale proceeds being less than the loan value paid out; hence, we also consider the CIR interest rate model and CBD mortality model to capture the respective interest rate and longevity risks in NNEG pricing, and then go on to develop a risk-neutral framework for NNEG pricing under these three risk factors. Based upon our numerical

parameter frequency is low which suggests that the jump risk is less significant for NNEGs than the interest rate risk.

analyses, we find that these three factors can affect the value of NNEGs, with the interest rate having the greatest impact, as compared to jump and mortality risks. Furthermore, the house price return models are found to be the most significant factor determining NNEG costs; thus, when issuing NNEGs, it is crucial for issuers to be able to identify the house price, interest rate and mortality risks.

We contribute to the extant literature on NNEG pricing in several ways. Firstly, having identified the jump risk as an intrinsic element of house price returns within the UK mortgage market, we go on to propose the use of an ARMA-GARCH jump model. Secondly, our estimation of this model reveals that it offers a better fit than the various other house price return models proposed within the prior literature. Thirdly, we derive a risk-neutral framework for NNEG pricing which allows for the analysis of jump, interest rate and longevity risks. Since equity-release products are becoming increasingly important in globally aging societies, financial institutions issuing such products need to understand the impact of the different risk factors on NNEG costs. We argue that the findings of our study can help such providers to manage the inherent risks.

In the light of our analysis, we consider the jump effect that allows for time-variation in the jump intensity, but it could be extended to different jump settings such as the stochastic volatility in jump innovation introduced in Daal et al. (2007). The estimation of jump process in different settings and the corresponding risk-neutral valuation differ greatly, so it would be worthwhile to investigate different jump process in the future.

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Table 1 Summary statistics, Q4 1952-2012

Variables	Mean	S.D.	Skewness[a,c]	Excess Kurtosis[a,c]	LB Q(29) Stats
Y_t	0.0185	0.0252	0.5108***	2.0034***	338.1603***
Y_t^2	0.0009	0.0018	4.0366***	21.3203***	233.8684***

Notes:

[a] The skewness and excess kurtosis statistics include a test of the null hypotheses that each is zero (the population values if the series is i.i.d. Normal.).

[b] The LB Q (29) statistics refer to the null hypothesis of no serial correlation with 29 lags.

[c] *** indicates significance at the 1% level.

Table 2 Parameter Estimates and model fit of constant and dynamic jump models, Q4 1952-2012

Parameters	ARMA(3,2)-GARCH(1,1) Models* Constant Jump		Dynamic Jump	
	Coeff.	S.E.	Coeff.	S.E.
Constant	0.0037***	0.0015	0.0045***	0.0010
ϑ_1	0.9050***	0.0279	0.9676***	0.0914
ϑ_2	−0.7489***	0.0279	−0.7628***	0.0510
ϑ_3	0.8748***	0.0286	0.8194***	0.0802
ζ_1	−0.1355***	0.0637	−0.4716***	0.1222
ζ_2	0.7799***	0.0612	0.7947***	0.0775
w	1.51e-06	8.49e-07	1.50e-05*	7.00e-06
α	0.1366***	0.0271	0.1163**	0.0453
β	0.6711***	0.0430	0.7500***	0.0723
λ_0	0.5361***	0.1030	0.0474*	0.0112
ρ	–	–	0.6380***	0.1999
ς	–	–	0.2763	0.2118
ϕ	0.0033**	0.0016	0.0356***	0.0060
θ	0.0142***	0.0017	0.0117***	0.0088
AIC	−5.7504		−5.7660	
BIC	−5.4666		−5.4618	
Log-likelihood	661.2973		677.5093	

Note: * indicates significance at the 10% level; ** indicates significance at the 5% level; and *** indicates significance at the 1% level.

Table 3 Model selection, Q4 1952-2012

Model	Log-Likelihood	AIC	BIC
Geometric Brownian Motion	543.0255	−4.5085	−4.4795
ARMA-GARCH	644.5349	−5.4854	−5.2763
ARMA-EGARCH	642.2203	−5.4657	−5.2333
Merton jump	558.0304	−4.6086	−4.5361
Double exponential jump diffusion	570.0125	−4.6381	−4.5555
ARMA-GARCH Constant jump	661.2973	−5.7504	−5.4666
ARMA-GARCH Dynamic jump	677.5093	−5.7660	−5.4618

Table 4 Robustness check of model selection

Model	Log-Likelihood	AIC	BIC
Panel A : Q4 1962-2012			
Geometric Brownian Motion	440.3251	−4.3833	−4.3503
ARMA-GARCH	516.5382	−5.3527	−5.1073
ARMA-EGARCH	482.4481	−4.9995	−4.7268
Merton jump	444.7692	−4.3977	−4.3552
Double exponential jump diffusion	462.3181	−4.4407	−4.5961
Constant jump ARMA-GARCH	525.2690	−5.4715	−5.1429
Dynamic jump ARMA-GARCH	532.0141	−5.4998	−5.1410
Panel B : Q4 1972-2012			
Geometric Brownian Motion	350.5096	−4.3564	−4.3179
ARMA-GARCH	403.0606	−5.2007	−4.9079
ARMA-EGARCH	394.7050	−5.0929	−4.7676
Merton jump	350.6187	−4.3602	−4.3200
Double exponential jump diffusion	365.4001	−4.4525	−4.3272
Constant jump ARMA-GARCH	425.0702	−5.5204	−5.1279
Dynamic jump ARMA-GARCH	431.0165	−5.6279	−5.1180

Table 5 Base assumption of parameter values for the pricing of NNEGs

Parameters	Notation	Value
Initial risk-free interest rate (%)	r_0	1.878
Interest rate spread (%)	v_r	2.000
Average delay in time (year)	δ	0.500

Market price of mortality risk	λ_m	0.175
Market price of interest rate	ϑ_r	−0.0712
Correlation between housing return and spot interest rate	$\rho_{Y,r}$	0.1595
Amount of loan advanced at inception	K	30,000
Initial property value for different ages, x , of borrowers (H_0)		
x = 60 Years		176,500
x = 70 Years		111,000
x = 80Years		81,000
x = 90 Years		60,000

Table 6 CIR model estimation reults,1952-2012

α_r	μ_r	σ_r	Log-likelihood
0.0864	0.0606	0.0740	3.3363

Table 7 NNEG costs under various house price return models

Unit: %

Model	Gender of Borrowers	Age of Borrowers (x): years			
		x = 60	x = 70	x = 80	x = 90
Geometric Brownian Motion	Male	3.05 (0.019)	2.28 (0.012)	1.17 (0.007)	0.69 (0.003)
	Female	4.69 (0.018)	3.76 (0.011)	1.89 (0.008)	1.41 (0.002)
ARMA-GARCH	Male	4.75 (0.023)	3.85 (0.018)	2.79 (0.011)	1.74 (0.007)
	Female	6.04 (0.022)	5.43 (0.019)	3.35 (0.011)	2.79 (0.007)
ARMA-EGARCH	Male	4.63 (0.025)	3.69 (0.020)	2.61 (0.012)	1.59 (0.008)
	Female	5.94 (0.026)	5.31 (0.019)	3.24 (0.011)	2.68 (0.008)
Merton jump model	Male	3.86 (0.019)	2.99 (0.013)	1.97 (0.007)	1.07 (0.003)
	Female	5.73 (0.018)	4.81 (0.011)	2.53 (0.007)	2.09 (0.004)
Double exponential jump diffusion	Male	3.29 (0.019)	2.41 (0.013)	1.34 (0.006)	0.85 (0.004)
	Female	5.18 (0.017)	4.35 (0.012)	2.11 (0.006)	1.79 (0.003)
ARMA-GARCH Constant jump	Male	5.29	4.20	2.97	1.87

		(0.027)	(0.023)	(0.017)	(0.010)
	Female	6.62	5.82	3.64	2.95
		(0.026)	(0.024)	(0.018)	(0.011)
ARMA-GARCH Dynamic jump	Male	5.37	4.32	3.09	1.89
		(0.027)	(0.024)	(0.017)	(0.011)
	Female	6.71	5.97	3.70	2.98
		(0.028)	(0.024)	(0.019)	(0.012)

Note: The standard error of the simulation is shown in the parentheses.

Table 8 Effects of mortality risk on NNEG costs

Unit: %

Model	Gender of Borrowers	Age of Borrowers (x years)			
		x = 60	x = 70	x = 80	x = 90
Panel A: Deterministic Mortality Assumption					
ARMA-GARCH Constant jump model	Male	5.12	4.07	2.85	1.76
		(0.026)	(0.023)	(0.017)	(0.010)
	Female	6.40	5.65	3.48	2.81
		(0.026)	(0.023)	(0.017)	(0.010)
ARMA-GARCH Dynamic jump model	Male	5.19	4.15	2.95	1.76
		(0.027)	(0.023)	(0.016)	(0.010)
	Female	6.47	5.77	3.53	2.83
		(0.027)	(0.024)	(0.018)	(0.011)
Panel B: CBD Model					
ARMA-GARCH Constant jump model	Male	5.29	4.20	2.97	1.87
		(0.027)	(0.023)	(0.017)	(0.010)
	Female	6.62	5.82	3.64	2.95
		(0.026)	(0.024)	(0.018)	(0.011)
ARMA-GARCH Dynamic jump model	Male	5.37	4.32	3.09	1.89
		(0.027)	(0.024)	(0.017)	(0.011)
	Female	6.71	5.97	3.70	2.98
		(0.028)	(0.024)	(0.019)	(0.012)

Note: The standard error of the simulation is shown in the parentheses.

Table 9 Impact of jumps on NNEG costs with stochastic and constant interest rates

Unit: %

Model	Gender of Borrowers	Age of Borrowers (x years)			
		x = 60	x = 70	x = 80	x = 90

Panel A: Stochastic Interest Rate					
ARMA-GARCH Constant jump model	Male	5.29	4.20	2.97	1.87
		(0.027)	(0.023)	(0.017)	(0.010)
	Female	6.62	5.82	3.64	2.95
		(0.026)	(0.024)	(0.018)	(0.011)
ARMA-GARCH Dynamic jump model	Male	5.37	4.32	3.09	1.89
		(0.027)	(0.024)	(0.017)	(0.011)
	Female	6.71	5.97	3.70	2.98
		(0.028)	(0.024)	(0.019)	(0.012)
Panel B: Constant Interest Rate					
ARMA-GARCH Constant jump model	Male	4.50	3.60	2.63	1.71
		(0.025)	(0.019)	(0.014)	(0.008)
	Female	5.80	5.18	3.24	2.75
		(0.024)	(0.020)	(0.016)	(0.009)
ARMA-GARCH Dynamic jump model	Male	4.60	3.73	2.70	1.72
		(0.026)	(0.019)	(0.015)	(0.009)
	Female	5.91	5.35	3.29	2.74
		(0.025)	(0.021)	(0.015)	(0.010)

Note: The standard error of the simulation is shown in the parentheses.

Table 10 Impact of risk effects on NNEG costs

Unit: %

Risk Factors	Gender of Borrowers	Age of Borrowers (x years)			
		x = 60	x = 70	x = 80	x = 90
Panel A: ARMA-GARCH Constant Jump Model					
Jump Risk	Male	0.54	0.35	0.18	0.13
	Female	0.58	0.39	0.29	0.16
Interest Rate Risk	Male	0.79	0.60	0.34	0.16
	Female	0.82	0.64	0.40	0.20
Mortality Risk	Male	0.17	0.13	0.12	0.11
	Female	0.22	0.17	0.16	0.14
Panel B: ARMA-GARCH Dynamic Jump Model					
Jump Risk	Male	0.62	0.47	0.30	0.15
	Female	0.67	0.54	0.35	0.19
Interest Rate Risk	Male	0.77	0.59	0.39	0.17
	Female	0.80	0.62	0.41	0.24
Mortality Risk	Male	0.18	0.17	0.14	0.13
	Female	0.24	0.20	0.17	0.15

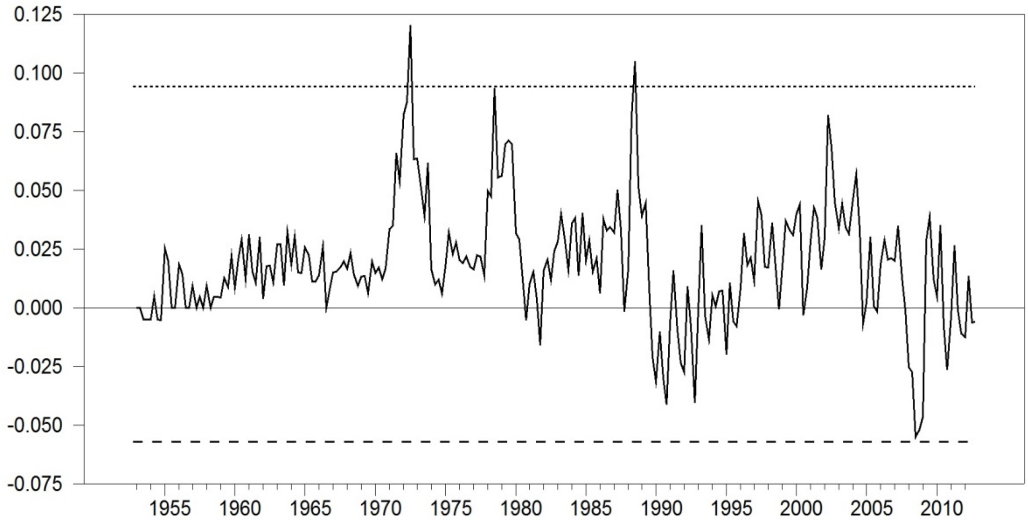

Figure 1 Historical quarterly returns of UK nationwide house price index

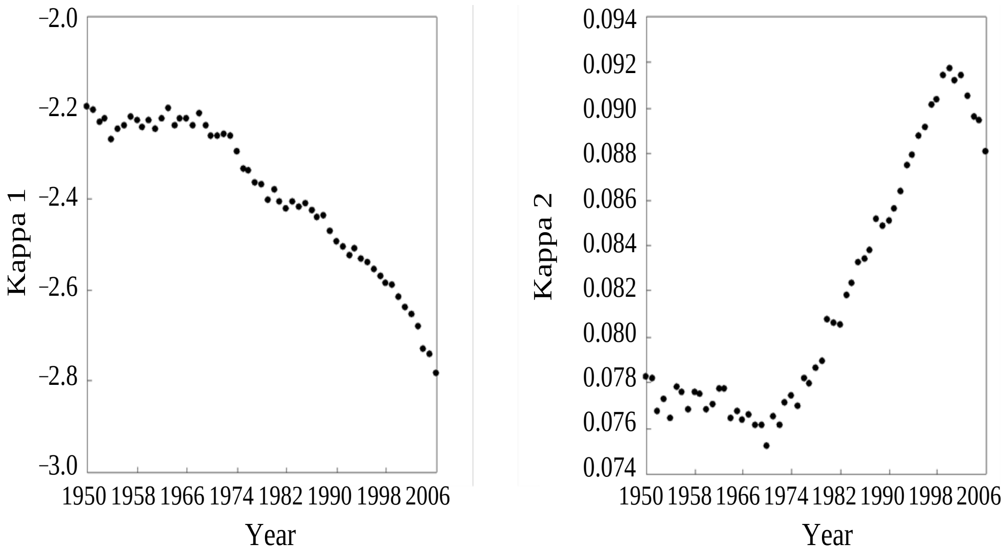

Figure 2.a Estimated kappa values for male samples

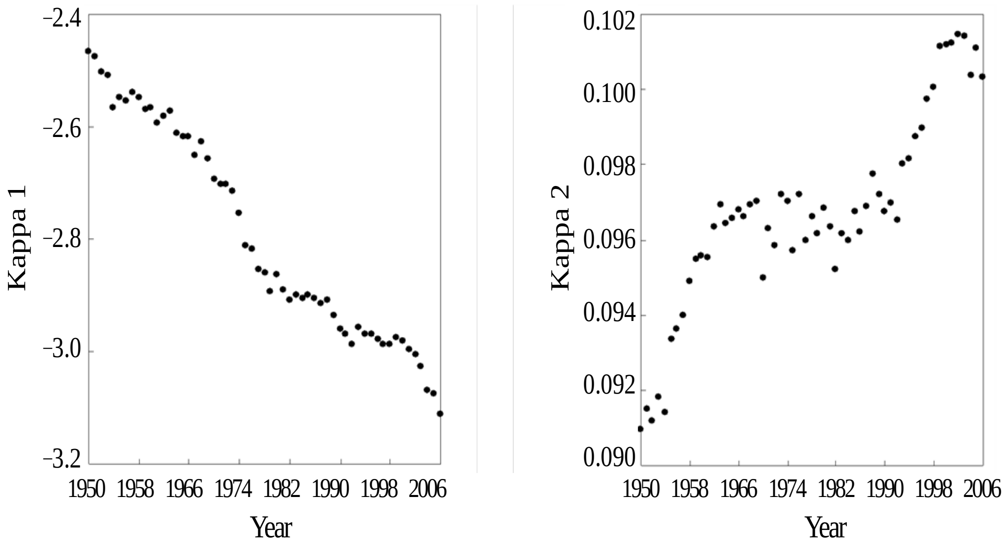

Figure 2.b Estimated kappa values for female samples

Appendix

We consider the housing price process follows an ARMA-GARCH jump model. Specifically, in a filtered probability space $\left(\Omega;\Phi;P;(\Phi_t)_{t=j\Delta t}^{T/\Delta t}\right)$ be a complete probability space, the house price return process shown in Equation (1) and (5) is given by

$$Y_t = \ln(\frac{H_t}{H_{t-\Delta t}}) = \mu_t + \varepsilon_t, \\ h_t = w + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i\Delta t}^2 + \sum_{j=1}^{p} \beta_j h_{t-j\Delta t}, \tag{A.1}$$

where $u_t = c + \sum_{i=1}^{s} \vartheta_i Y_{t-i\Delta t} + \sum_{j=1}^{m} \zeta_j \varepsilon_{t-j\Delta t}$ is the conditional mean function, given the time $(t-\Delta t)$ information $\Phi_{t-\Delta t}$; s is the order of the autocorrelation terms; m is the order of the moving average terms; ϑ_i is the i^th^-order autocorrelation coefficient; ζ_j is the j^th^-order moving average coefficient. In addition, ε_t is the total returns innovation with conditional variance h_t, given the information $\Phi_{t-\Delta t}$; p is the order of the GARCH terms; q is the order of the ARCH term; α_i is the i^{th}-order ARCH coefficient; and β_j is the j^{th}-order GARCH coefficient.

To obtain the housing price dynamic under a risk-neutral measure, we aslo employ an equivalent martingale measure using the conditional Esscher transform developed by Bühlmann et al. (1996). Due to the discount housing price under the Q measure is a martingale, we have:

$$H_{t-\Delta t} = E^Q\left(\frac{B_{t-\Delta t}}{B_t} H_t \middle| \Phi_{t-\Delta t}\right) = E^Q\left(\exp\left(-\int_{t-\Delta t}^{t} r_s ds\right) H_t \middle| \Phi_{t-\Delta t}\right) \tag{A.2}$$

To incorporate the stochastic interest rate assumption, we follow Lee et al. (2012)'s assumption that the interest rate between time t-Δt and time t is fixed at $r_{t-\Delta t}$, but that it may vary from one band to the next. Consequently, $B_t = B_{t-\Delta t}\,\mathrm{e}^{r_{t-\Delta t}\Delta t}$.

According to Lemma 5.2.2 in Shreve (2004), we obtain:

$$
\begin{aligned}
H_{t-\Delta t} &= \mathrm{e}^{-(r_{t-\Delta t}\Delta t)}\, E^Q\left(H_t \middle| \Phi_{t-\Delta t}\right) = e^{-(r_{t-\Delta t}\Delta t)} E^P\left(\frac{\Lambda_t}{\Lambda_{t-\Delta t}} H_t \middle| \Phi_{t-\Delta t}\right) \\
&= \mathrm{H}_{t-\Delta t} e^{-(r_{t-\Delta t}\Delta t)} \frac{E^P\left(\exp\left((a_t+\iota)Y_t\right) \mid \Phi_{t-\Delta t}\right)}{E^P\left(\exp\left((a_t)Y_t\right) \mid \Phi_{t-\Delta t}\right)}
\end{aligned} \tag{A.3}
$$

Or equivalently,

$$
e^{(r_{t-\Delta t}\Delta t)} = \frac{E^P\left(\exp\left((a_t+\iota)Y_t\right) \mid \Phi_{t-\Delta t}\right)}{E^P\left(\exp\left((a_t)Y_t\right) \mid \Phi_{t-\Delta t}\right)} \tag{A.4}
$$

In order for risk neutral Q to be an equivalent martingale measure, we need have

$$
E^Q\left[\exp\left(Y_t\right) \mid \Phi_{t-\Delta t}\right] = e^{(r_{t-\Delta t}\Delta t)} \tag{A.5}
$$

Because, Maheu and McCurdy (2004) has point out the conditional moments of return are

$$
\begin{aligned}
&E\left[Y_t \mid \Phi_{t-\Delta t}\right] = u_t \\
&Var\left[Y_t \mid \Phi_{t-\Delta t}\right] = h_t + \left(\phi^2 + \theta^2\right)\lambda_t = h_t^*
\end{aligned} \tag{A.6}
$$

Thus, Y_t is normally distributed with mean u_t and variance h_t^*, given the information $\Phi_{t-\Delta t}$, we obtain

$$
\begin{aligned}
E^Q\left[\exp\left(\iota Y_t\right) \mid \Phi_{t-\Delta t}\right] &= \frac{\exp\left(\left(a_t+\iota\right)u_t + \frac{1}{2}\left(a_t+\iota\right)^2 h_t^*\right)}{\exp\left(a_t u_t + \frac{1}{2} a_t^2 h_t^*\right)} \\
&= \exp\left(\left(u_t + a_t h_t^*\right)\iota + \frac{1}{2} h_t^* \iota^2\right)
\end{aligned} \tag{A.7}
$$

Therefore,

$$
E^Q\left[\exp\left(Y_t\right) \mid \Phi_{t-\Delta t}\right] = \exp\left(u_t + a_t h_t^* + \frac{1}{2} h_t^*\right)
$$

(A.8)

Through the equation (A.5) and (A.8), we have

$$u_t = r_{t-\Delta t}\Delta t - a_t h_t^* - \frac{1}{2}h_t^* \qquad \text{(A.9)}$$

Similarly, the characteristic function of ε_t under martingale measure Q is of the form:

$$\begin{aligned}
E^Q\left(\exp\left(i\varpi\varepsilon_t\right)\middle|\Phi_{t-\Delta t}\right) &= E^P\left(\frac{\Lambda_t}{\Lambda_{t-\Delta t}}e^{i\varpi\varepsilon_t}\middle|\Phi_{t-\Delta t}\right) = \frac{E^P\left(e^{a_t Y_t}e^{i\varpi\varepsilon_t} \mid \Phi_{t-\Delta t}\right)}{E^P\left(\exp\left(\left(a_t\right)Y_t\right) \mid \Phi_{t-\Delta t}\right)} \\
&= \frac{\exp\left(a_t u_t\right)E^P\left(e^{\left(a_t+i\varpi\right)\varepsilon_t} \mid \Phi_{t-\Delta t}\right)}{\exp\left(a_t u_t + \frac{1}{2}a_t^2 h_t^*\right)} = \frac{\exp\left(\frac{1}{2}\left(a_t+i\varpi\right)^2 h_t^*\right)}{\exp\left(\frac{1}{2}a_t^2 h_t^*\right)} \qquad \text{(A.10)} \\
&= \exp\left(i\varpi a_t h_t^* - \frac{1}{2}\varpi^2 h_t^*\right)
\end{aligned}$$

Consequently, ε_t under the measure Q become normally distributed, with mean $a_t h_t^*$ and variance h_t^*, given the information $\Phi_{t-\Delta t}$. That is, given the information $\Phi_{t-\Delta t}$, $\varepsilon_t^Q = \varepsilon_t - a_t h_t^*$ follow normally mean 0 and variance h_t^* under measure Q. Finally, the equation (A.1) can be rewritten as:

$$\begin{aligned}
Y_t = \ln\left(\frac{H_t}{H_{t-\Delta t}}\right) &= \mu_t + \varepsilon_t = r_{t-\Delta t}\Delta t - a_t h_t^* - \frac{1}{2}h_t^* + \varepsilon_t^Q + a_t h_t^* \\
&= r_{t-\Delta t}\Delta t - \frac{1}{2}h_t^* + \varepsilon_t^Q
\end{aligned} \qquad \text{(A.11)}$$