Two Variance Components, Variance Jumps, and the Pricing of VIX Derivatives

Chien-Ling Lo, Pai-Ta Shih, Yaw-Huei Wang, Min-Teh Yu*

Abstract

Our aim in this study is to investigate the value of two variance components and variance jumps in the pricing of VIX derivatives. In an attempt to significantly reduce the computational burden of the empirical estimation, we propose an easily implemented and efficient numerical technique for the pricing of VIX derivatives under the affine framework. Our empirical findings provide support for the use of two-variance component models as the means of capturing the fickle term structure of VIX derivatives; however, specifying a second variance component does not eliminate the need for variance jumps, since they are found to be vital for pricing short-term contracts and also effective when included in the long-run variance component. Our results are robust to the effects of the recent financial crisis period.

Key Words: VIX derivatives; Variance components; Variance jump; Affine model.
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1 Introduction

For both academics and practitioners alike, volatility is a central concept in the pricing of derivatives, with the extant literature clearly showing the variation over time in the volatility of equity returns. Hence, one of the most crucial steps in derivative pricing is that of identifying an effective way of specifying the volatility process. Balancing the tradeoff between complexity and tractability, the seminal work of Heston (1993) proposed a stochastic volatility (SV) model for capturing the stylized facts of the leverage effect and mean reversion of volatility. More importantly, Heston also provided a closed-form solution for the pricing of equity options.

However, numerous empirical studies have found strong evidence to show that the SV model is misspecified, even when a jump component is considered in the returns.[1] Thus, in an attempt to remedy the Heston SV model, subsequent studies have proposed stochastic volatility models which either incorporate variance jumps or specify the second variance component;[2] the aim of the former is to capture large volatility movements, whilst the latter allows flexibility in the volatility term structure.

Both approaches have been extensively examined in recent empirical works, with the improvements on the SV model being revealed through the use of historical returns and/or option prices of equity assets. Alizadeh, Brandt and Diebold (2002) found that short- and long-run volatility components were required to respectively explain the volatility of variance and volatility persistence. Examining two independent volatility components, Adrian and Rosenberg (2008) concluded that short- and long-run components respectively captured market skewness risk and business cycle risk.

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[1] See, for example, Bakshi, Cao and Chen (1997), Andersen, Benzoni and Lund (2002), Pan (2002) and Chernov, Gallant, Ghysels and Tauchen (2003).

[2] Bates (1996) extended the work of Heston (1993) by adding a jump component into the returns to generate the large price movements that are typical of a turbulent period. Bates (2000) subsequently extended the model by specifying an additional variance factor, whilst Duffie, Pan and Singleton (2000) introduced models with correlated jumps in the returns and variance under an affine framework.

Eraker, Johannes and Polson (2003) provided strong support for specifications with jumps in variance for a time-series analysis, such as the fitting of returns data, whilst Christoffersen, Heston and Jacobs (2009) showed that models with no second variance factor provided insufficient flexibility to explain the independent fluctuations in the level and slope of option-implied volatility. Li and Zhang (2010) also found the two-variance specification to be a promising approach for fitting option prices.[3]

Eraker (2004) noted that although the incorporation of jumps in returns and variance effectively described options and returns, the jump components added little explanatory power on the fitting options. Having identified strong evidence of the presence of jumps in variance, Broadie, Chernov and Johannes (2007) concluded that jumps in both returns and variance were crucial components of option pricing. Overall, the prior empirical works appear to have convincingly identified the value of the second variance component and variance jumps in the pricing of derivatives.

Over recent decades, the Chicago Board Options Exchange (CBOE) has been active in launching VIX futures and options in order to meet the enormous demand for volatility hedging instruments, with the open interest and trading volume in both products having grown rapidly.[4] VIX derivatives have essentially become the most prominent volatility-related product to date within the derivative markets, which raises the question of whether the most effective models in S&P 500 index option pricing also perform well in the pricing of VIX derivatives. We therefore set out in this study to provide a comprehensive discussion on the respective roles of the second variance component and variance jumps in VIX derivative pricing.[5]

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[3] Specifically, Eraker et al. (2003) support jumps in both returns and variance. The impact of jumps in returns is transient, whereas the large volatility movement generated by variance jumps is persistent.

[4] The CBOE first published the volatility index (VIX) in 1993, calculated from a series of at-the- money S&P 100 index options. In 2003, it updated the VIX definition by a model-free method using the market prices of S&P 500 index options to estimate the expected volatility of the S&P 500 index during the following 30-calendar-day period. As regards VIX derivatives, VIX futures have been traded since 2004, whilst VIX options were introduced in 2006. See Carr and Lee (2009) for an overview of the volatility derivative markets.

[5] Although some studies have attempted to jointly estimate both the S&P 500 index options and VIX derivatives in order to identify the risk premia or the joint pricing kernel (see Amengual and Xiu, 2012; Bardgett, Gouier and

According to the definition provided by the CBOE, the VIX is constructed based upon a portfolio of all out-of-the-money S&P 500 index options with a time to maturity of 30 calendar days. Given the model specifications discussed above, the VIX is equivalent to a square root function of the instantaneous variance, which increases the difficulty of pricing its derivatives. In an attempt to explain the prior empirical works, we propose an innovative approach for the valuing of VIX derivatives under the affine framework introduced by Duffie et al. (2000), with our proposed technique being both easily implemented and numerically efficient.

Our empirical results reveal that the second variance component clearly plays an important role in the pricing of VIX derivatives. The empirical term structure of VIX futures prices usually exhibits a hump-shaped pattern, with the non-parallel shift for a non-trivial proportion of trading days. We find that models with no second variance component are incapable of capturing these common features for the term structure, even when jumps in returns and variance are specified; therefore, the incorporation of the second variance component as the means of simultaneously capturing the short- and long-run fluctuations remains a promising approach to the fitting of VIX derivatives prices. Nevertheless, the addition of variance jumps in the Heston SV model with jumps in returns may not significantly improve the pricing of VIX derivatives.

Given that a VIX derivative is a European-style contract, the derivative price is dependent only upon the terminal VIX level; however, the mean-reversion characteristic of volatility may mitigate the impacts generated by variance jumps prior to the expiry date. This mitigation does not appear in the pricing of index options, essentially because a jump in variance has an immediate effect, namely, a change in the S&P 500 index level, even if the

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Leippold, 2013; Song and Xiu, 2014), joint estimation is not the focus in the present study, for two reasons. Firstly, since the trading targets for the two derivatives differ, this gives rise to the further issue of the avoidance of the impact of different types of trading behavior in both markets. Secondly, the estimation results on both markets are quite mixed, depending on their weights. Since the empirical performance of these models in the index option market has already been extensively explored, we focus on investigations into the VIX derivatives market only.

terminal volatility returns to its pre-jump level and there is no mean reversion in the index itself. The second variance component does not, however, eliminate the need for variance jumps.

Our empirical results show that the pricing performance arising from the incorporation of variance jumps is comparable to that of adding the second variance component when considering only short-term contracts, for two reasons: firstly, the impacts of volatility jumps at expiry may not be fully mitigated by the mean-reversion effects over a short period; and secondly, the term structure of VIX derivatives is less complicated over a small range, such that the additional variance component has little chance of exhibiting its strength.

As a result, a crucial issue under the two-variance framework is where to jump, and we find that specifying variance jumps in the long-run component can significantly improve the fitting of VIX option prices; given that the mean-reversion speed of the long-run variance component is slower than that of the short-run variance component, the mitigation caused by mean reversion is somewhat limited.

Whilst recent empirical works on the pricing of VIX derivatives, such as Branger and Völkert (2012) and Mencía and Sentana (2013), have attempted to determine the most suitable model for price fitting, we explore the respective values of two existing approaches adopted in the index options literature. Overall, our empirical results are consistent with those reported on variance jumps by Mencía and Sentana (2013), who found that adding jumps into the VIX logarithm yielded a minor pricing improvement, and Branger and Völkert (2012), who found that variance jumps mainly influenced short-term contracts. We go on to further explain why the addition of variance jumps may provide superior performance and analyze which variance component should incorporate the jump component.

Unlike Mencía and Sentana (2013), we adopt a consistent framework to provide a link with the extant index options literature and overcome the difficulties involved in the valuation of VIX derivatives. As compared to Branger and Völkert (2012), we add further discussion on the two-variance framework and find that the second variance component does indeed play a crucial role in the pricing of VIX derivatives.[6]

The remainder of this paper is organized as follows. The model setup and proposed closed-form approximation for VIX derivative prices are introduced in Section 2, followed in Section 3 by a description of the data and methodology used for our empirical analyses. Our empirical results and in-depth discussion are respectively presented in Sections 4 and 5, with robustness checks being provided in Section 6. Finally, the conclusions drawn from this study are presented in Section 7

2 The Model

2.1 Model Specification

We consider the class of affine SV models introduced by Duffie et al. (2000), with the risk-neutral joint process of log price and its variance being specified under this framework as an affine structure balancing the tradeoff between the complexity and tractability of the valuation of financial derivatives. Affine SV models, including the Heston (1993) SV model, the Bates (1996) SV model with price jumps (SVJ) and numerous extended SV models, have been widely used in the option pricing literature, with two particular alternative extensions being the addition of variance jumps (Duffie et al., 2000) and specifying the second variance component (Bates, 2000; Christoffersen et al., 2009); we provide comprehensive investigations of both approaches.

We assume that the level of the SPX at time t, denoted by P_t, involves the following risk-neutral dynamic:

$$\frac{dP_t}{P_t} = (r-q)dt + \sqrt{V_{1,t}}dW_{1,t} + \sqrt{V_{2,t}}dW_{2,t} + (JdN_t - \lambda\bar{\mu}dt), \qquad (1)$$

[6] Although Branger and Völkert (2012) adopted a consistent framework, their empirical estimations excluded the jump component in prices so as to reduce the number of estimated parameters. However, since many studies, such as Todorov and Tauchen (2011), have concluded that the addition of variance jumps does not eliminate the need for price jumps, in the present study we simultaneously consider jumps in both returns and variance.

$$dV_{1,t} = \kappa_1(\theta_1 - V_{1,t})dt + \sigma_1\sqrt{V_{1,t}}dW_{1,t}^v + \xi^v dN_t\,, \tag{2}$$

$$dV_{2,t} = \kappa_2(\theta_2 - V_{2,t})dt + \sigma_2\sqrt{V_{2,t}}dW_{2,t}^v\,, \tag{3}$$

$$\text{Cov}(dW_{j,t}, dW_{j,t}^v) = \rho_j dt, \qquad j = 1, 2, \tag{4}$$

and

$$\text{Cov}(dW_{i,t}, dW_{j,t}^v) = \text{Cov}(dW_{i,t}, dW_{j,t}) = \text{Cov}(dW_{i,t}^v, dW_{j,t}^v) = 0, \qquad i \neq j, \tag{5}$$

where r refers to the risk-free rate; q is the continuous dividend yield; $V_{1,t}$ and $V_{2,t}$ capture the two types of instantaneous variances at time *t*; κ_j and θ_j respectively denote the mean-reversion speed and the long-run mean level of $V_{j,t}$; σ_j refers to the volatility of variance j; $W_{j,t}$ and $W_{j,t}^v$ are Wiener processes with the correlation structure specified in Equations (4) and (5); N_t is a Poisson process with constant intensity λ; ξ^p and ξ^v are the respective random jump sizes in the SPX and its variance; and $J \equiv \mathrm{e}^{\xi^p} - 1$ is the random percentage jump in SPX with mean $\bar{\mu}$. As regards the correlated jump sizes, we follow the extant literature to assume that ξ^v is exponentially distributed with positive mean μ_v, and that $\xi^p|\xi^v$ is normally distributed with mean $\mu_p + \rho_J\,\xi^v$ and variance σ_p^2.[7]

We make no attempt in the present study to create a new valuation model, but instead, set up our model in a way which nests two alternative extensions of the Bates (1996) SVJ model from within the literature. Firstly, we can dispose of Equation (3) by setting $V_{2,t} = \theta_2 = 0$ in order to obtain the SV model with correlated jumps (SVCJ model) proposed by Duffie et al. (2000). Alternatively, by removing the variance jump component, $\xi^v dN_t$, which is achieved by setting $\mu_v = 0$, we can then obtain the two-variance SV model with price jumps (2-SVJ model) proposed by Bates (2000). If $\lambda = 0$, then the SVJ and the 2-SVJ are respectively reduced to the Heston (1993) SV model and the Christoffersen et al. (2009) two-variance SV (2-SV) model.[8]

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[7] It then follows that $\bar{\mu} = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\xi^p} - 1] = \mathbb{E}_t^{\mathbb{Q}}[\mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\xi^p}|\xi_j^v]] - 1 = \exp\left\{\mu_p + \frac{\sigma_p^2}{2}\right\}/(1 - \rho_J\mu_v) - 1$.

[8] This result is consistent with the finding of Todorov and Tauchen (2011), who noted that jumps in price and variance occur simultaneously and exhibit highly negative dependency. Our proposed model can also be reduced to the Sepp (2008) stochastic volatility with variance jumps (SVVJ) model by setting $V_{2,t} = \theta_2 = \mu_p = 0$, and

Although the central tendency model of Duffie et al. (2000) does offer an alternative way of specifying the second variance component, Branger and Völkert (2012) demonstrated its minor influence on VIX options, since the model does not provide flexibility for the volatility of variance. In addition, for a technical reason, the theoretical price of VIX derivatives under the central tendency model does not offer analytical tractability and requires further numerical calculation for an effective solution; thus, this type of model is not taken into consideration in our study.

2.2 The CBOE VIX and Its Derivatives

The squared VIX index, introduced by the CBOE in 2003, is defined as a portfolio of all out-of-the-money S&P 500 index options with time-to-maturity of 30 calendar days:

$$\left(\frac{VIX_t}{100}\right)^2 \equiv \frac{2e^{r\Delta t}}{\Delta t}\left[\int_0^{S_t e^{r\Delta t}} \frac{P_t^{SPX}(K,\Delta t,r)}{K^2}dK + \int_{S_t e^{r\Delta t}}^{\infty} \frac{C_t^{SPX}(K,\Delta t,r)}{K^2}dK\right], \quad (6)$$

where Δt =30/365, and $P_t^{SPX}(K,\tau,r)$ and $C_t^{SPX}(K,\tau,r)$ are the respective values of put and call options traded at time t with strike price K, time to maturity τ, and discount rate r.[9] Applying the spanning formula of Bakshi and Madan (2000), the VIX squared under Equations (1)–(5) can be derived as:

$$\left(\frac{VIX_t}{100}\right)^2 = \frac{2}{\Delta t}\mathbb{E}_t^{\mathbb{Q}}\left[-\ln\left(\frac{P_{t+\Delta t}}{P_t e^{r\Delta t}}\right)\right] = A + X_t, \quad (7)$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ denotes the risk-neutral expectation under the martingale measure $\mathbb{Q}$, $X_t \equiv B_1 V_{1,t} + B_2 V_{2,t}$ defines a weighted variance, $A = A_1 + A_2 + \lambda A_J$, and

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to the stochastic volatility with independent jumps (SVIJ) model of Duffie et al. (2000) by setting $V_{2,t} = \theta_2 = \rho_J = 0$. However, for models with a single variance component, for simplicity, we focus on the SVJ and SVCJ models.

[9] An alternative definition of the VIX squared is the quadratic variation of the log return process $\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\frac{dP_u}{P_u}\right)^2\right]$. See, for example, Amengual and Xiu (2012), Branger and Völkert (2012), Britten-Jones and Neuberger (2000), Carr and Wu (2009), and Wu (2011).

$$
\begin{aligned}
B_j &= \frac{1-e^{-\kappa_j \Delta t}}{\kappa_j \Delta t},\ j = 1,2, \\
A_j &= \theta_j[1 - B_j],\ j = 1,2, \\
A_J &= \frac{\mu_v}{\kappa_1}[1 - B_1] + 2\left[\frac{e^{\mu_p + \frac{\sigma_p^2}{2}}}{1-\rho_J \mu_v} - 1 - \mu_p - \rho_J \mu_v\right].
\end{aligned} \tag{8}
$$

It should be noted that the total variance, VIX_t^2, comprises of two parts. The continuous part of the j^{th} variance component, $A_j + B_j V_{j,t}$, is a weighted average of $V_{j,t}$ and its long-run mean level θ_j, with the weight, B_j, being related to its mean-reversion speed. The jumps in both price and variance also affect the total variance, with the total impact being derived as λA_J. As noted in the related literature, VIX is irrelevant to ρ_j (the correlation coefficient between index price and its variance).[10]

The linear relationship between VIX squared and instantaneous variances in Equation (7) implies that a VIX derivatives payoff is a square root function in state variables. According to risk-neutral valuation theory, the value of a VIX call option at time t, with strike price K, time to maturity τ and discount rate r, is expressed as

$$
\begin{aligned}
C_t^{VIX}(K,\tau,r) &\equiv e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}[(VIX_T - K)^+] \\
&= 100e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{A + X_T}\, I_{\{X_T \geq K_0^2 - A\}}\right] - Ke^{-r\tau}Pr_t^{\mathbb{Q}}(VIX_T \geq K),
\end{aligned} \tag{9}
$$

where $I_{\{\cdot\}}$ denotes the indicator function, $T = t + \tau$, $K_0 = K/100$, and $Pr_t^{\mathbb{Q}}(\cdot)$ denotes the risk-neutral probability measure. It should be noted that the VIX derivatives prices are difficult to solve due to the square root payoff; we therefore go on in the following sub-section to propose a closed-form approximation.

2.3 Closed-Form Approximation for VIX Derivatives

The aim of our approximation is essentially to transfer a thorny problem into a manageable subject. Specifically, we use N piecewise exponential curves to approximate the target

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[10] This formula can be reduced to Duan and Yeh (2010) by setting $V_{2,t} = \theta_2 = \mu_v = 0$, and to Cheng, Ibraimi, Leippold, and Zhang (2012) by setting $V_{2,t} = \theta_2 = 0$.

payoff in the interval $\left[X_0^{(k)}, X_N^{(k)}\right]$, which covers k standard deviations around the mean of the state variable X_T.[11] Consequently, we derive the closed-form solution for the substitute payoff. This approach follows Duffie et al. (2000) who derived a closed-form solution for option prices when the payoff is an exponential function in state variables. Although Duffie et al. (2000) also provided the closed-form option prices for linear payoffs, exponential curves provide a simpler way of solving the analytic form for the 'ordinary differential equations' (ODEs) when deriving the option prices.[12] More importantly, exponential curves fit the target payoff more effectively.

Figure 1 illustrates a concrete example of $k \in \{3,6\}$ and $N \in \{1,4\}$ under the SVCJ model with parameters based on Duffie et al. (2000): κ_1=3.5, θ_1=0.01, σ_1=0.15, λ =0.5, $\bar{\mu} = -0.1$, μ_v=0.05, σ_p=0.0001, $\rho_J = -0.4$, $V_{1,t}$=0.008, r =0.03, τ=1, and moneyness $m = 1$.[13] The solid and dashed curves respectively depict the target function and our exponential approximations, with $\left\{X_0^{(k)}, M_1^{(k)}, X_1^{(k)}, \ldots, M_N^{(k)}, X_N^{(k)}\right\}$ marking all of the fitting points; in particular, $M_n^{(k)} \equiv \frac{1}{2}\left(X_{n-1}^{(k)} + X_n^{(k)}\right)$ denotes the midpoint of each sub-interval. Appendix A provides the complete fitting scheme and all of the explicit formulae.

< Figure 1 is inserted about here >

We go on to offer a closed-form approximation for VIX option prices in Proposition 1,

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[11] In our notation, the square root payoff in Equation (9), $\sqrt{A + X_T}\, I_{\{X_T \geq K_0^2 - A\}}$, is approximated by

$$\sum_{n=1}^{N-1}(a_n + b_n e^{c_n X_T})\, I_{\left\{X_{n-1}^{(k)} \leq X_T \leq X_n^{(k)}\right\}} + (a_N + b_N e^{c_N X_T}) I_{\left\{X_{N-1}^{(k)} \leq X_T\right\}},$$

or equivalently,

$$(a_1 + b_1 e^{c_1 X_T}) I_{\left\{X_0^{(k)} \leq X_T\right\}} + \sum_{n=1}^{N-1}(a_{n+1} - a_n + b_{n+1} e^{c_{n+1} X_T} - b_n e^{c_n X_T})\, I_{\left\{X_n^{(k)} \leq X_T\right\}},$$

where (a_n, b_n, c_n) are corresponding coefficients for each exponential curve.

[12] We refer to Equations (2.5)-(2.6) and (2.15)-(2.16) in Duffie et al. (2000) for the ODEs for the respective exponential and linear payoffs. The first step in solving the ODEs of the linear payoffs is to solve the ODEs of the exponential payoffs.

[13] For simplicity, the Duffie et al. (2000) parameters are slightly adjusted in the present study. The parameters that were estimated by Duffie et al. (2000) were $\kappa = 3.46, \theta = 0.008, \sigma = 0.14, \lambda = 0.47, \bar{\mu} = -0.1, \mu_v = 0.05, \sigma_p = 0.0001, \rho_J = -0.38$, and $V_0 = 0.007569$.

where the pricing errors of the proposed approximation are caused by the fitting error multiplied by the corresponding probability. Since both the target and the substitute payoffs are continuous functions, with increases in the specified fitting range, *k*, and the number of exponential curves, *N*, the fitting error converges to zero. Our numerical analysis (Appendix B) shows that our approximation with (*k*, *N*) = (6,4) can effectively value VIX options, since it performs with similar accuracy, albeit with much less computational time being required, operating as much as eighty times faster than Lian and Zhu (2013), a less time-consuming approach within the literature.[14]

PROPOSITION 1. (CLOSED-FORM APPROXIMATION).

The VIX call option can be approximated by

$$C_t^{VIX}(K,\tau,r) = 100e^{-r\tau}\left[\frac{\Phi(0)}{2} - \frac{1}{\pi}\int_0^{\infty}\frac{1}{v}Imag\big(\Phi(v)+\widetilde{\Phi}(v)\big)dv\right], \qquad (10)$$

where

$$\Phi(v) = [a_N\psi(-iv) + b_N\psi(c_N - iv)]e^{ivX_{N-1}^{(k)}} - K_0\psi(-iv)e^{iv(K_0^2-A)},$$

$$\widetilde{\Phi}(v) = \sum_{n=1}^{N-1}\left[(a_n\psi(-iv) + b_n\psi(c_n - iv))\left(e^{ivX_{n-1}^{(k)}} - e^{ivX_n^{(k)}}\right)\right],$$

where (a_n, b_n, c_n), $X_n^{(k)}$, and $\psi(\cdot)$ are analytic terms as defined in Appendix A.

PROOF. See Appendix A.

3 Data Description and Empirical Methodology

3.1 Data Description

Our primary dataset, which is obtained from the CBOE, includes the end-of-day bid and ask quotes of VIX options for the 2007-2010 period.[15] Since our sample period covers the recent financial crisis, in Section 6.1, we examine whether our empirical results are robust to the

[14] Branger and Völkert (2012) proposed a similar formula to Lian and Zhu (2013); however, further numerical calculations were required to solve the ODEs, whilst Bardgett et al. (2013) valued VIX options using a Fourier cosine expansion approach involving numerous complex error functions. In the present study, our comparisons are confined to the Lian and Zhu (2013) study.

[15] Although VIX options were launched on 24 February 2006, our sample period runs from 2007 in order to avoid the liquidity issue at the early stage of this newly-introduced product.

crisis period. Since VIX options are European-style options traded on the CBOE, we use mid-quotes to represent the option prices. We obtain the daily closing levels of VIX along with the option data, and in order to investigate the performance of various models on pricing VIX futures, we also obtain the daily settlement prices of VIX futures from the CBOE. The risk-free rate is calculated for each derivative contract through the interpolation of zero curve surfaces obtained from the OptionMetrics database to fit the contract maturity.

Due to the liquidity concern, we adopt some filtering rules commonly used within the related literature: (i) we omit those options with fewer than seven days and more than 366 days to maturity; (ii) all observations with zero trading volume are discarded; (iii) we exclude all observations with bid prices lower than the minimum tick size, US$0.1, or with ask prices lower than the bid prices; and (iv) we eliminate observations violating the arbitrage conditions.[16] Our resultant sample provides a total of 86,149 observations, comprising of 55,965 calls and 30,184 puts. We follow Bakshi et al. (1997) to classify the VIX options into 18 categories, as described below, based upon their moneyness (m) and time to maturity (τ), where moneyness is defined as the strike price divided by the VIX futures price.

A VIX call option is classified as deep in the money (DITM) if $m \leq 0.75$, in the money (ITM) if $m \in (0.75, 0.9]$, slightly in the money (SITM) if $m \in (0.9, 1.0]$, slightly out of the money (SOTM) if $m \in (1.0, 1.1]$, out of the money (OTM) if $m \in (1.1, 1.35]$, and deep out of the money (DOTM) if $m > 1.35$.[17] However, in contrast to Bakshi et al. (1997), we adopt wider partitions, as in Mencía and Sentana (2013), essentially because the VIX is much more

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[16] First, we eliminate observations violating the arbitrage bounds:

$$\max\{0, e^{-r\tau}(F_t^{VIX}(\tau) - K)\} \leq C_t^{VIX}(K, \tau, r) \leq e^{-r\tau} F_t^{VIX}(\tau)$$

and

$$\max\{0, e^{-r\tau}(K - F_t^{VIX}(\tau))\} \leq P_t^{VIX}(K, \tau, r) \leq e^{-r\tau} K.$$

Next, for each option maturity, if the prices are not monotonic in the strike prices, we eliminate those observations with lower volumes. It should be noted that the observations that are not monotonic in time to maturity should not be eliminated due to the mean-reversion property of volatility.

[17] Similarly, a VIX put option is classified as DOTM if $m \leq 0.75$, OTM if $m \in (0.75, 0.9]$, SOTM if $m \in (0.9, 1.0]$, SITM if $m \in (1.0, 1.1]$, ITM if $m \in (1.1, 1.35]$, and DITM if $m > 1.35$.

volatile than the SPX.[18]

In terms of time to maturity, we classify VIX options into groups of short-term ($\tau <$ 60/365), mid-term (60/365$\leq \tau <$180/365), and long-term ($\tau \geq$180/365) maturities. For each category, the average mid-quote (*Price*), the total number of observations (*Obs*), the average trading volume (*Vol*), and the average open interest (*OIT*) are reported in Table 1, with Panels A and B respectively reporting the results for call and put options.

< Table 1 is inserted about here >

The average prices range from \$0.45 (short-term, DOTM) to \$12.12 (short-term, DITM) for VIX call options and from \$0.34 (short-term, DOTM) to \$19.62 (mid-term, DITM) for VIX put options. In general, a longer maturity does not imply a higher option price; the main reason for this is that long-term contracts are clustered in 2007, a period with relatively low VIX levels. We also find that for both trading volume and open interest, DITM, ITM, and SITM VIX call options are less liquid than the respective DOTM, OTM, and SOTM VIX put options. We therefore replace all categories of the in the money VIX call option by their corresponding out of the money VIX put options via the VIX put-call parity.[19]

3.2 Empirical Methodology

We follow Bakshi et al. (1997) for the estimation of the model parameters from VIX options, so as to minimize the daily sum of the squared pricing errors. We also employ the VIX formula, Equation (7), with the VIX time series as a constraint, following a widely used approach in the VIX-related literature.[20] In other words, the instantaneous variance can be identified by the VIX constraint for models with single variance component (SV, SVJ, and SVCJ), treating the second instantaneous variance as another parameter for those models with

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[18] The partitions adopted by Mencía and Sentana (2013) comprised of $\ln(m) < -0.3$, $\ln(m) \in [-0.3, -0.1)$, $\ln(m) \in [-0.1, 0.1)$, $\ln(m) \in [0.1, 0.3)$, and $\ln(m) \geq 0.3$.

[19] The value of a VIX put option can be expressed as $P_t^{VIX}(K,\tau,r) \equiv e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}[(K - VIX_T)^+]$, and VIX put–call parity is given by

$$P_t^{VIX}(K,\tau,r) = C_t^{VIX}(K,\tau,r) + e^{-r\tau}\left(K - F_t^{VIX}(\tau)\right),$$

where $F_t^{VIX}(\tau)$ denotes the level of VIX futures at time t, with time to maturity τ.

[20] Examples include Duan and Yeh (2010), Branger and Völkert (2012), and Song and Xiu (2014).

two variance components (2-SV and 2-SVJ).[21] Specifically, we solve the following minimization for each day:

$$\begin{cases} \min_{\Theta, V_{2,t}} SSE_t \equiv \sum_{j=1}^{n_t} \left(C_j\left(\Theta, V_{1,t}, V_{2,t}\right) - C_j^* \right)^2 \\ \text{s.t.} \qquad V_{1,t} = \left[\left(\frac{VIX_t}{100} \right)^2 - A - B_2 V_{2,t} \right] / B_1, \end{cases} \tag{11}$$

where $\Theta \equiv \left(\kappa_1, \theta_1, \sigma_1, \kappa_2, \theta_2, \sigma_2, \lambda, \mu_p, \sigma_p, \mu_v, \rho_J\right)$ are the structure and jump-related parameters; SSE_t denotes the sum of squared error at time t; n_t denotes the observations at time t; $C_j\left(\Theta, V_{1,t}, V_{2,t}\right)$ and C_j^* respectively refer to the theoretical and market prices of a call option; and VIX_t denotes the closing levels of the VIX at time t.

When a model with more parameters fits the in-sample data more effectively, this raises the question of whether it also has superior out-of-sample performance; we therefore explore the out-of-sample results in an attempt to determine whether the extra parameters cause over-fitting. Following Bakshi et al. (1997), we use the in-sample daily estimated parameters and variances to compute the next day's option prices for each day, with the exception of the final day.

In the objective dollar pricing error in Equation (11) more weight is assigned to the relatively expensive VIX options; alternatively, by minimizing the sum of the squared relative errors, less weight is assigned to the relatively expensive VIX options, whilst the minimization of the sum of the squared errors in Black-Scholes implied volatility leads to less weight being assigned to near the money VIX options. Hence, both of these alternative weighting schemes are examined as a check for the robustness of our results in Section 6.2.

Although the minimization in Equation (11) involves numerous parameters and extremely complex objective functions, our proposed approximation offers an effective alternative approach. Considering the trade-off between computational accuracy and

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[21] Although treating the instantaneous variance as another parameter may not the optimal approach for estimation, this approach is seen as a simpler and popular alternative within the literature (Bakshi et al., 1997; Christoffersen et al., 2009). For the monthly estimation, we adopt an iterative two-stage procedure, as in Bates (2000), Huang and Wu (2004), and Christoffersen et al. (2009), as detailed in Section 5.1.

efficiency, we select (*k, N*) = (6, 4) to implement the pricing formula of VIX derivatives in our subsequent empirical analysis, with the computation of 100 option prices taking about 1.15 seconds for the single-variance model, and about 4.30 seconds for the two-variance model.

In order to ensure that the approximation method selected in this study does not mislead the estimation, we carry out additional investigations into the approximation errors across alternative implementation methods in Section 6.3. Given that a call option reduces to a futures contract when the strike price and discount rate are set as zero, the same estimation method can be applied to VIX futures; thus, we address this issue in Section 5.1, where the theoretical and market prices of call options are replaced by those of futures.

4 Preliminary Results on VIX Options

4.1 Parameter Estimates

The summary statistics of the parameter estimates from the daily updated frequency are reported in Table 2 for the single variance component model (Panel A) and the two-variance component model (Panel B), with the penultimate rows of the two panels reporting the average daily root mean-squared errors (RMSEs). For the two-variance model, we follow a number of prior related studies to identify the two variance components by their mean reversion speed, in which the factor with faster (slower) reversion speed captures the short-run (long-run) variance.[22] In order to clarify the notations, the parameters for the short- and long-run variance components are respectively denoted by $(\kappa_s, \theta_s, \sigma_s, V_{s,t})$ and $(\kappa_l, \theta_l, \sigma_l, V_{l,t})$, with the suffixes being disposed of for the single-variance models $(\kappa, \theta, \sigma, V_t)$.

<Table 2 is inserted about here>

[22] Examples include Adrian and Rosenberg (2008), Alizadeh et al. (2002) and Christoffersen et al. (2009).

As shown in Panel A of Table 2, the respective daily average mean-reversion speeds, κ, for the SV, SVJ and SVCJ models are 3.395, 3.897 and 4.510, thereby implying that the half-life period of variance shocks is roughly three months. These estimated values are consistent with the extant literature on index options, such as Christoffersen et al. (2009), in which the estimated κ under the SV model during their 1990-2004 sample period was found to be between 1.60 and 4.43. Panel B of Table 2 shows that for two-variance models, the estimated κ_s is approximately 15, whilst the estimated κ_l is approximately 1.8, which means that the two-variance models are indeed capable of capturing the differences in the variance components, in terms of their mean reversion speed. The half-life of the short-run variance is close to one month, whilst that of the long-run variance is roughly six months.

As compared to the literature on index options (such as Eraker et al., 2003), our estimated volatilities under the single-variance models ($\sqrt{V_t}$) are largely due to the financial crisis during our sample period (resulting in a volatile VIX process). The volatilities estimated by Eraker et al. (2003) are roughly 15 per cent, whilst the $\sqrt{V_t}$ estimated in the present study are 25.8 per cent for the SV model, 18.7 per cent for the SVJ model and 20.5 per cent for the SVCJ model; however, the respective median levels are only 22.8 per cent 14.8 per cent and 16.7 per cent.[23]

Figure 2a illustrates the spot volatility estimates for single-variance models over our sample period, with all estimated volatilities being highly correlated to the VIX level and all of the coefficients being in excess of 0.90.[24] Conversely, the estimated volatility levels under the two-variance models ($\sqrt{V_{s,t}}$ and $\sqrt{V_{l,t}}$) range between 14.9 per cent and 17.4 per cent. Figure 2b

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[23] Similarly, the volatilities of variance (σ, σ_s, and σ_l) are also generally found to be quite high. The daily average σ for the SV model is 0.852, but its median is only 0.564. For the two-variance models, one of the estimated σ_j reaches a high of approximately 2.4, which is in line with the findings of Bates (2000) and Christoffersen et al. (2009). In particular, the higher σ_j estimated by Christoffersen et al. (2009) reaches an average of 3.667, and was as high as 9.43 in 1994.

[24] Although we incorporate the VIX constraint into our estimation, the estimated instantaneous volatility is not necessary highly correlated to the VIX level. Specifically, $VIX_t = 100 \times \sqrt{A + B_1 V_t}$, and the correlation coefficient may be low if the estimated A and B_1 are not stable.

(2c) illustrates the short-run (long-run) volatility estimates over our sample period for the two-variance models. The volatility estimates for these models are also found to be highly correlated to the VIX level, with coefficients in excess of 0.75. Finally, the jump parameters, comprising of the estimated jump size in returns (μ_p) and the correlation between two jumps (ρ_J), are both found to be negative, which is consistent with the extant index options literature.

<Figures 2a-2c are inserted about here>

4.2 Pricing Error

As regards the average in-sample RMSE for nested models, models with more parameters always outperform those with fewer parameters, since the latter can be regarded as constrained models. For example, the average RMSEs are 0.233 for the SV model, 0.150 for the SVJ model and 0.122 for the SVCJ model, whilst the averages are 0.089 for the 2-SV model and 0.085 for the 2-SVJ model; nevertheless, the latter group appears to generally outperform the former. Our non-tabulated results show that in both the in-sample and out-of-sample results, and across both moneyness groups and time to maturity periods, those models with a second variance component (2-SV and 2-SVJ) consistently outperform those with no second variance component (SV, SVJ and SVCJ).

Two statistical tests are adopted in Table 3 to facilitate an examination of whether the improvements made by the extended models are statistically significant. Panels A and B respectively report the mean and median of the differences in the daily RMSE, with the difference being defined by the error of the compared model specified in the first row minus that of the selected model specified in the first column. Panel A examines whether the means of the daily RMSE for any two models are equal by utilizing the two-sample *t*-test with unequal variances. Given the sensitivity to extreme mean values, Panel B also considers the Wilcoxon matched-pairs signed-rank test to investigate the null hypothesis that the median of the differences in the daily RMSEs between any two models will be zero.[25]

<Table 3 is inserted about here>

The positive means and medians in all of the results reveal better fitting performance for the selected models, with the in-sample results showing that the differences are statistically significant at the 5 per cent level, with the exception of the mean of the differences between the 2-SV and the 2-SVJ models. These findings indicate that the improvements made by adding jumps into the returns may not be statistically significant for the pricing of VIX options once the second variance component is incorporated. The out-of-sample predictions are also found to provide qualitatively similar results to the in-sample estimations.

Since the SVCJ and the 2-SVJ models represent two alternative ways of improving on the SVJ model, we focus on comparisons between the SVJ and the SVCJ models in Table 4 to observe the improvements made by variance jumps, as well as comparisons between the SVJ and the 2-SVJ models in Table 5 to observe the improvements made by the long-run variance component.

In both tables, the in-sample and out-of-sample results are reported on the mean (median) equality test in Panel A (Panel B).[26] Using the SVJ model as the benchmark, we examine the null hypothesis that the mean/median of the differences between the absolute pricing errors of the SVCJ and 2-SVJ models for VIX options, across both moneyness and time to maturity, will be greater than zero.[27]

To facilitate our examination of the fitting performance across moneyness levels and

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[25] The test statistic is

$$Stat_{x,y} = \sum_{j=1}^{\mathcal{T}} I_{\{d_j(x,y)>0\}} rank(|d_j(x,y)|),$$

where $d_j(x,y)$ is the RMSE of Model x minus that of Model y for the j^{th} trading date in our sample, and $I_{\{.\}}$ denotes the indicator function. The standardized version of this statistic is asymptotically normal.

[26] Although the 2-SV model and the Sepp (2008) SVCJ model are two alternative ways of improving on the SV model, models incorporating jumps in variance only are not popular in the options literature.

[27] It should be noted that the pricing errors defined in Tables 4 and 5 are the absolute pricing errors for each transaction. The test statistic is:

$$Stat_{x,y} = \sum_{j=1}^{\mathcal{N}} I_{\{d_j(x,y)>0\}} rank(|d_j(x,y)|),$$

where $\mathcal{N}$ is the total number of options, $d_j(x,y)$ is the difference of the absolute pricing error of Models x and y for the jth VIX option in our sample.

time to maturity periods, all of the data are divided into five moneyness categories and three time-to-maturity periods. For space-saving purposes, we combine SOTM and SITM into one group, referred to as 'near the money' (NTM).

Table 4 reveals that whilst the SVCJ model clearly does make significant improvements on the SVJ model for all VIX options data, the improvements do not hold for all categories, since the absolute pricing errors of the SVCJ model for numerous in-the-money and long-term categories are not found to be significantly lower than those of the SVJ model at the 5 per cent significance level. Indeed, for some long-term categories, the pricing errors of the SVCJ model are even found to be significantly higher than those of the SVJ model, an observation which suggests that time-to-maturity plays a vital role in variance jump performance.

<Table 4 is inserted about here>

In contrast, Table 5 shows that at the 5 per cent significance level, the absolute pricing errors of the 2-SVJ model are significantly lower than those of the SVJ model for all categories, with only one exception, the long-term, NTM sub-category in the out-of-sample results of the mean equality test. In addition, the improvements made by the 2-SVJ model are generally found to be greater than those of the SVCJ model. Although the better performance of the 2-SVJ model may arguably be attributable to the number of parameters, our non-tabulated results examining the pairs of SVJ-2-SV show the same patterns. The pricing errors of the 2-SV model are also smaller than those of the SVCJ model, despite the fact that the latter has more parameters than the former.

<Table 5 is inserted about here>

Overall, the in-sample and out-of-sample results provide consistent findings; that is, specifying the second variance component can significantly raise the precision of the evaluation of VIX options, whilst the inclusion of variance jumps can capture the secondary effect. As noted by Bates (2000), a jump component generates more flexibility for the skewness and kurtosis of the underlying distribution, whilst the second variance component allows for a more complicated volatility term structure. Thus, our preliminary findings indicate that relaxing flexibility for the term structure may be more crucial for the pricing of VIX options. Consequently, it should be of some interest to explore certain other factors, such as the reasons why the second variance component is so important, whether the two-variance structure eliminates the need for variance jumps, and what the actual values of the variance jumps are.

5 Discussion

5.1 The Value of the Second Variance Component

Despite the fact that a growing number of studies have reported the valuable role of the second variance component,[28] the term structure of VIX derivatives has not been extensively explored. Branger and Völkert (2012) briefly noted that whilst the two-variance structure allows for a more flexible VIX futures term structure, it cannot generate flexible patterns on VIX option-implied volatility. As such, they did not consider two-variance models and offered no detailed discussions on the VIX futures term structure. Conversely, Bardgett et al. (2013) suggested that specifying a stochastic central tendency might help to provide a better representation of the term structure of VIX options.

In the present study, we offer theoretical justifications for examining the valuable role of the second variance component. Firstly, we observe the patterns of the VIX futures term structure, essentially because futures contracts provide a clear avenue for investigating the impact on the term structure. The VIX futures term structure provides the market expectations on the VIX level across different maturity periods. When the current VIX level is relatively low (high), the VIX futures term structure is generally found to be upward (downward) sloping, due to the stylized fact of the mean-reversion of volatility; however, we find that the VIX futures term structure often exhibits a U-shaped or hump-shaped pattern.[29]

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[28] See, for example, Christoffersen et al. (2009) and Egloff, Leippold and Wu (2010).

[29] Let us suppose that the observed prices for each trading date, t, are $F_{t,1}, F_{t,2}, \dots, F_{t,N}$, if an $F_{t,j}$, j = 2, 3, … ,

In Appendix C, we offer theoretical explanations as to why single-variance models are incapable of generating a hump-shaped pattern, even when the variance jumps are specified, whereas the hump-shaped pattern is easily explained by two-variance components when the short-run (long-run) variance is lower (higher) than its corresponding long-run mean level.

In Figures 3a-3d, we demonstrate the fitting performances of the SV, SVCJ and 2-SV models based upon four typical patterns of the empirical term structure of VIX futures, comprising of: (i) upward-sloping (11 December 2009); (ii) downward-sloping (24 October 2008); (iii) U-shaped (6 March 2007); (iv) and hump-shaped (7 April 2009).[30] Interestingly, Figures 3a-3d reveal that the 2-SV model significantly outperforms the SV and SVCJ models in the hump-shaped pattern, whilst the SV and SVCJ models are found to perform as well as the 2-SV model in all other patterns.

<Figure 3 is inserted about here>

Since the hump-shaped pattern may simply arise from the mean-reversion expectations of investors, we offer an alternative analysis in an attempt to emphasize the importance of the second variance component. Observing the fluctuations in the VIX futures term structures for periods of two consecutive days, we find that of the 1,005 pairs, 364 exhibit crosses.[31] We provide examples of two pairs of VIX futures term structures; the first, observed on 10/11 August 2009 and illustrated in Figure 4a, has a hump-shaped pattern, whilst the second, observed on 14/15 March 2007 and illustrated in Figure 4b, has a U-shaped pattern, with the hump-shaped (U-shaped) pattern revealing parallel (non-parallel) shift. Clearly, this type of

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$N-1$ exists, which simultaneously satisfies $F_{t,j} \geq F_{t,j-1}$ and $F_{t,j} \geq F_{t,j+1}$, we can then conclude that the term structure contains at least one hump, and similar rules can be used to examine the other patterns. Of the total of 1,006 days of observations in our 2007-2010 sample period, the term structure was found to be strictly increasing (decreasing) on 266 (166) days, and exhibiting at least one smile for 389 days, and at least one hump for 533 days.

[30] Given that the number of available futures contracts is very low, we use synthetic observations of futures prices generated by fitting the available market prices of futures with the cubic spline to estimate the model parameters.

[31] Since the observed times to maturity are not constant across days, we adopt the cubic spline to generate a smooth term structure function. To avoid unnecessary errors, we do not extrapolate the term structure beyond the observed ranges.

term structure variation indicates opposing changes in short- and long-run expectations, which is precisely captured by the two-variance models, whereas the single-variance models with jumps are unable to generate this common situation.[32]

<Figures 4a and 4b are inserted about here>

Following on from our brief discussion on VIX futures term structures, we return to the empirical performance of VIX option pricing. Inspired by our observations of the changes in the VIX futures term structures over two consecutive days, we further examine the empirical performance of VIX option pricing based upon an alternative updating frequency,[33] with the monthly estimation results showing that the average RMSEs are 0.396 for the SV model, 0.324 for the SVJ model and 0.322 for the SVCJ model.

When we switch from daily to monthly updating frequencies for single-variance models with jumps, the advantage becomes smaller, with the improvements made by both the SVJ and SVCJ models being found to be statistically insignificant in the mean equality test. Our non-tabulated results also reveal that for both the SVJ and the SVCJ models, the improvements are close to zero when we take into consideration lower updating frequencies, such as quarterly or annual estimates.

By contrast, the monthly averages for the RMSE are 0.396 for the SV model, 0.193 for the 2-SV model and 0.187 for the 2-SVJ model. The improvements made by the two-variance models remain statistically significant, regardless of the frequency of the estimations. These results are consistent with the findings for the VIX futures term structures; that is, those

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[32] Although not provided here, the technical details are available from the authors on request.

[33] It should be noted that the instantaneous variance is time-varying, while the parameters in Θ are assumed to be constant over time. As discussed in Section 3.2, the instantaneous variance $V_{1,t}$ can be uniquely determined by Θ for the single-variance models after employing the VIX constraint; however, for multi-factor models, the variance swap rate cannot uniquely identify the two components of instantaneous variance in one constraint. We therefore adopt the iterative two-step procedure used by Bates (2000), Huang and Wu (2004) and Christoffersen et al. (2009), which is detailed as follows. In Step 1, given Θ, we solve Equation (11) to obtain the sequences of $V_{2,t}$ at each date, and in Step 2, given a sequence of $V_{2,t}$ we minimize the aggregate sum of the squared error $\sum_{t=1}^{\mathcal{T}} SSE_t$ subject to the VIX constraint to obtain the model parameters Θ, where $\mathcal{T}$ denotes the total number of days. This procedure iterates between Step 1 and Step 2 until the aggregate SSE_t converges, with our proposed approximation once again significantly lowering the computational burdens.

models without the second variance component are incapable of capturing the various changes in the term structure of VIX derivatives, although this failing is not apparent in the daily estimates.

In summary, in this section we have demonstrated the way in which the two-variance model provides flexibility for the VIX futures term structure and identified when this flexibility is crucial for the pricing of VIX options. Furthermore, the fickle patterns observed in the empirical term structure of the VIX derivatives explain why the two-variance specification is so successful. Not only do we explain our empirical findings, but we also offer theoretical justifications.

5.2 Short-run Variance Component and the Need for Variance Jumps

Despite the compelling evidence in support of the existence of variance jumps,[34] our preliminary results, along with the findings of Mencía and Sentana (2013), demonstrate that only modest improvements on the pricing of VIX derivatives are provided by the inclusion of variance jumps. Since variance jumps intuitively capture short-run information, whilst two-variance models capture both short- and long-run information, this raises the question of whether the short-run variance component in the two-variance model eliminates the need for variance jumps; in an attempt to answer this question, we investigate the improvements in pricing made by variance jumps after specifying the second variance component.

We extend our empirical estimations in Section 4 using the full model, Equations (1)-(5), referred to as the 2-SVCJ model, where variance jumps occur in either short- or long-run variance processes, depending on which position better minimizes the pricing error. The pricing errors between the 2-SVJ and the 2-SVCJ models, which describe the contributions of variance jumps when the short- and long-run variance components are specified, are reported in Table 6, from which we can see that the 2-SVCJ model makes significant improvements on the 2-SVJ model

[34] See, for example, Eraker (2004), Wu (2011) and Todorov and Tauchen (2011).

for all VIX options data.

<Table 6 is inserted about here>

The mean (median) in-sample improvement is found to be 0.0055 (0.0006), whilst the mean (median) out-of-sample improvement is found to be 0.0036 (0.0003). When examining the different moneyness categories and time to maturity periods, the pattern is found to be similar to that observed in Table 4; that is, the pricing improvements for long-term contracts may not be statistically significant, essentially because the jump component primarily captures short-run information.

However, variance jumps seem to be crucial for short-term contracts even if the variances are decomposed into short- and long-run components, a finding which indicates that the short-run variance component does not eliminate the need for variance jumps. In Sections 5.3 and 5.4, we will go on to identify precisely when the role of variance jumps is at its most crucial in the pricing of VIX options.

5.3 The Value of Variance Jumps under Single-variance Models

In this section, in an attempt to more clearly illustrate the value of variance jumps, we duplicate the empirical estimations presented earlier, adopting only short-term $(\tau < 60/365)$ data, for two reasons.[35] Firstly, the mean-reversion of volatility may mitigate the jump impact at expiry, with such mitigation being potentially more significant for long-term contracts. Secondly, as demonstrated in Section 5.1, short-term contracts cannot generate complicated VIX futures term structures.

The mean and median equality tests on the pricing errors for our short-term estimations are reported in Table 7, where the in-sample results show that the two-variance models still perform better than the SVCJ model; however, as compared with Table 3, the advantages become much smaller. In particular, the difference between the SVCJ and 2-SV models in the mean equality test

[35] The short-term data definition, which is consistent with Bakshi et al. (1997), is shown in Section 3.1.

is found to be insignificant. The out-of-sample results further show that of all the models, the SVCJ model exhibits superior performance. Although the differences between the SVCJ and two-variance models are not statistically significant in some cases, the short-term results differ quite markedly from our previous findings using all of the data.

<Table 7 is inserted about here>

A summary of the pricing errors under three different scenarios is provided in Table 8. Under the first scenario, we minimize the daily RMSE using all VIX options data (the preliminary results presented in Section 4); the second scenario reports the monthly estimations, as shown in Section 5.2; in the third scenario, we minimize the daily RMSEs using data on short-term VIX options only. The average daily/monthly RMSEs under the three scenarios are reported in Panel A, the transaction-weighted average RMSEs are reported in Panel B as a check for robustness, and Panel C provides a comparison of the RMSE performance of the 2-SV and 2-SVJ models. Both weighted schemes imply the same conclusion, that under the third scenario, the SVCJ model provides significant improvements on the SVJ model.[36] Although the two-variance models are still found to perform better than the SVCJ model in terms of short-run in-sample estimations, the SVCJ model is found to outperform all other models in the out-of-sample predictions.

<Table 8 is inserted about here>

Panel C of Table 8 provides further analysis of the percentages of trading days and months when the SVCJ model outperforms the 2-SV and 2-SVJ models; ideally, a superior model should have a consistently better fit with the market price. Under the first scenario, the SVCJ is found to outperform the 2-SV (2-SVJ) model in only 16.1 per cent (10.6 per cent) of the trading

[36] The differences between the SVJ and SVCJ models are also found to be insignificant for the monthly estimations, as explained in Section 5.2

days in our sample period, thereby indicating that the two-variance models generally perform better than the SVCJ model.[37]

As regards the daily estimations on short-term data, since the SVCJ model is found to outperform the 2-SV (2-SVJ) model in 49.4 per cent (40.3 per cent) of the trading days in our sample period, this provides support for the valuable role played by variance jumps in short-term contracts. More importantly, the out-of-sample ratios are all found to be higher than the corresponding in-sample ratios, thereby indicating that the SVCJ may cause less over-fitting than the two-variance models.

In order to identify whether a particular model performed better over specific periods, we illustrate the in-sample RMSEs over time for the SVJ, SVCJ and 2-SVJ models in Figures 5a-5c. For the daily and monthly results on all data (Figures 5a and 5b), the RMSEs of the 2-SVJ model are consistently found to be lower than those of the SVCJ model, and this remains so even when observing only the 2007- 2009 financial crisis period. However, for the daily results on short-term data, the SVCJ and 2-SVJ models are found to be comparable, with the 40.3 per cent of trading days captured in Panel C of Table 8 apparently being randomly distributed across the whole sample period, thereby eliminating any sample specific concerns.

<Figures 5a-5c are inserted about here>

The findings in this section provide strong support for the valuable role of variance jumps in the pricing of short-term contracts, with the impact of variance jumps in the pricing of long-term VIX derivatives being potentially mitigated by the mean reversion of volatility. Such mitigation is not addressed within the extant index options literature, essentially because there is no mean-reversion in underlying assets; conversely, variance jumps with no second variance

[37] Under scenario 2, the SVCJ model always generates larger pricing errors for the in-sample results due to its inflexibility with regard to the VIX derivatives term structure. In such cases, the two-variance models are the ideal candidates for the pricing of VIX options.

component are able to effectively capture the simple term structure of VIX options for short-term contracts.

Although Branger and Völkert (2012) also found that specifying variance jumps primarily improved the pricing of short-term VIX options, they attributed the success to the right skewness in the instantaneous variance. The focus in their study was essentially on the jump impacts across different moneyness categories, whereas we offer alternative explanations using time to maturity periods.

5.4 The Value of Variance Jumps under Two-variance Models

In addition to the length of time to maturity, another determinant of our mean-reversion mitigation argument is the actual speed of mean reversion; hence, in this section, we analyze the effects that the mean-reversion speed has on the overall impacts of the variance jumps. Although we argue that the impacts of the variance jumps are mitigated by the mean-reversion property of volatility, essentially as a result of the European-style design of the contracts, intuitively, this argument should not hold when the mean-reversion speed is extremely slow. We therefore provide an alternative examination of variance jumps in this section for models with two variance components, analyzing the position of the short- and long-run variance components in order to determine which jump component is more crucial.

Following the estimation of the 2-SVCJ model in Section 5.2, we report the percentages of trading days exhibiting jumps in the short- and long-run variance components in Table 9, with Panel A showing that 69.3 per cent of all variance jumps take place in the long-run component, with an average mean-reversion speed of 1.902. The remaining 30.7 per cent of all trading days exhibit jumps in the short-run variance component, with an average mean-reversion speed of 13.370.[38]

[38] 258 of the 1,006 trading days over our sample period exhibited no variance jumps, thereby implying that specifying two-variance components alone fits the market price well. To compare the contribution of short- and long-run jumps more clearly, "no-jump" cases are excluded from Table 9; nevertheless, the inclusion of trading days with no jumps does not distort our conclusions.

<Table 9 is inserted about here>

As regards the ratio between the two mean-reversion speeds, κ_s/κ_l, we report the results on sub-groups with κ_s/κ_l levels less than (greater than) specific quantiles in Panel B (Panel C) of Table 9, sorted by κ_s/κ_l. Panel B shows that the percentage of short-run jumps increases when the two speeds, κ_s and κ_l, begin to converge; in particular, the percentage of short-run jumps in the first decile is 48.0 per cent, with averages of κ_s = 5.389 and κ_l = 4.668. In contrast, Panel C reveals that with an increase in the difference between the two speeds, there will be a corresponding increase in the percentage of long-run jumps; indeed, the percentage of long-run jumps in the tenth decile is 78.7 per cent, with averages of κ_s = 0.027 and κ_l = 21.359.

Although we argue that the mean-reversion feature may mitigate the impacts of variance jumps, the effects are found to be quite limited under slow mean-reversion speeds. The sensitivity of a VIX call price to the impact of variance jumps and time to maturity periods under the 2-SVCJ model is illustrated in Figures 6a-6b, with parameters based upon our earlier estimation results: κ_s=12, κ_l=2, $\sqrt{\theta_s}$=$\sqrt{\theta_l}$=0.2, σ_s=2, σ_l=0.75, $V_{s,t}$=$V_{l,t}$=0.15, $\lambda = 0.25$, μ_p = –0.25, σ_p =0.15, $\mu_v \in \{0.01, 0.02 \ldots, 0.20\}$, ρ_J = –0.4, $r = 0.03$, $\tau \in \{0.05, 0.10, \ldots, 1\}$ and strike price K=28.[39]

Figure 6a (Figure 6b) illustrates the scenario for jumps in short-run (long-run) variance components, with Figure 6a showing that the VIX call prices are relatively insensitive to the variance jumps, particularly for longer time to maturity periods; however, Figure 6b shows that the VIX call prices remain sensitive to the variance jumps for long-term contracts, as a result of their slow mean-reversion speed.

<Figure 6 is inserted about here>

Finally, as shown earlier in Table 2, the standard error of the estimated long-run level

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[39] The means of the estimated parameters are κ_s=13.726, $\sqrt{\theta_s}$=0.183, σ_s=2.134, $\sqrt{V_{s,t}}$=0.150, κ_s=1.970, $\sqrt{\theta_s}$= 0.211, σ_s=0.782, $\sqrt{V_{s,t}}$=0.148, λ=0.275, μ_p=–0.248, σ_p=0.171, μ_v=0.146, and ρ_J=–0.409.

$\sqrt{\theta_l}$ for the long-run variance component was 0.021, much higher than that for the short-run component. After specifying the variance jumps in the long-run component, the standard error declines to 0.007 and the average $\sqrt{\theta_l}$ is reduced from 0.382 to 0.211. These results indicate that the 2-SVCJ model with jumps in the long-run variance component offers more stable and reasonable parameter estimates than those provided by either the 2-SV or 2-SVJ models.

In summary, the role of variance jumps in the pricing of VIX derivatives is essentially similar to that in the pricing of index options; that is, the primary aim is to capture short-run information. However, since the mean-reversion nature of volatility mitigates the value of variance jumps in a valuation sense, the poor performance of volatility jumps in the preliminary results actually stems from the features of the contract and the underlying assets, ultimately leading to conclusions that differ from those reported in the extant literature on index options.

Several recent studies have suggested that providing sufficient flexibility for the volatility of variance is particularly crucial for the pricing of VIX derivatives, whereas the stylized fact of mean reversion partly offsets the jump impact on the volatility of variance.[40] By identifying the value of the variance jumps for those models with both one and two variance components, we provide insights which go beyond those of the prior studies (such as Mencía and Sentana, 2013).

6 Tests for Robustness

6.1 Financial Crisis Period

We now reexamine the results using the specific sample period of December 2007 to May 2009 (the financial crisis period defined by the NBER Business Cycle). Table 10 essentially duplicates Table 8, but with results on the financial crisis period only. Due to objective dollar pricing errors, the RMSEs during the financial crisis period are, in general, found to be greater;

[40] See Amengual and Xiu (2012), Branger and Völkert (2012) and Mencía and Sentana (2013).

however, our prior conclusions remain robust.

<Table 10 is inserted about here>

Panels A and B of Table 10 show that the two-variance models outperform the SVCJ model with the exception of the out-of-sample results for short-term contracts, whilst Panel C shows that the two-variance models consistently perform better than the SVCJ model on the monthly in-sample results, and that the SVCJ model outperforms the two-variance models on more than 50 per cent of the trading days in the out-of-sample short-term contract results. Building on the prior discussion, most of the ratios in Panel C are found to be smaller than in Table 8, implying even greater advantages of the two-variance models during the financial crisis period.

This finding is interesting since jump components are designed to capture large movements; however, VIX derivatives are forward-looking contracts. Due to the stylized fact of volatility persistence, market participants may not expect a large upward movement after a dramatic rise in volatility;[41] thus, the addition of jumps in variance during the financial crisis period does not appear to offer any better fit for the pricing of VIX derivatives. Instead, as shown in Schwert (2011), volatility actually exhibits greater mean-reversion during the 2008-2009 financial crisis period.

6.2 Alternative Weighting Schemes

Although this study compares affine SV models by minimizing the sum of the squared error, SSE_t, we adopt two alternative objective functions as our checks for the robustness of the results. Firstly, we minimize the sum of the squared relative error ($SSRE_t$), which assigns more weight to less expensive options:

$$\min_{\Theta,V_{2,t}} SSRE_t \equiv \sum_{j=1}^{n_t} \left(\frac{C_j(\Theta,V_{1,t},V_{2,t}) - C_j^*}{C_j^*} \right)^2 . \quad (12)$$

[41] The probability of a large upward movement after a dramatic rise in volatility is intuitively smaller than under normal conditions. In particular, since the variance jump component in the SVCJ model is assumed to be exponentially distributed, the model cannot capture large downward movements.

Secondly, we minimize the sum of the squared error for Black-Scholes implied volatilities rather than that for option prices. This ‘sum of the squared implied volatility errors’ ($SSIVE_t$), which assigns more weight to NTM options, can be approximated using the Christoffersen et al. (2009) approach:

$$\min_{\Theta,V_{2,t}} SSIVE_t \equiv \sum_{j=1}^{n_t} \left(\frac{C_j(\Theta,V_{1,t},V_{2,t}) - C_j^*}{\text{Vega}_{j,t}} \right)^2 , \quad (13)$$

where $\text{Vega}_{j,\text{t}}$ denotes the Black–Scholes vega for the VIX call, C_j^*.

The RMSE, root mean-squared relative error (RMSRE) and implied volatility root mean-squared error (IVRMSE), which are respectively obtained from the minimizations, Equations (11), (12) and (13), are reported in Table 11. The sample period for this robustness test covers the first quarter of 2007 (a relatively tranquil period) and the third quarter of 2008 (a turbulent period). Consistent with our main results, the results in Table 11 show that the 2-SV model significantly outperforms the SVCJ model regardless of the error functions, thereby highlighting the merits of the second variance component.[42]

<Table 11 is inserted about here>

6.3 Approximation Error

To ensure that our approximations do not confound our empirical results, we investigate the approximation errors using two approaches. Firstly, we examine the pricing errors of our approximation using our estimated parameters and the closed-form solution detailed in Appendix A. Next, setting our estimated parameters as initial values, we redo the empirical estimation using the less efficient formula of Lian and Zhu (2013) under the SV, SVJ and SVCJ models in order to examine the convergence of the minimization; both investigations highlight the merits of our approach. As compared to the closed-form solution, the RMSEs for our approximations are less than 0.02, whilst the RMSREs for our approximations are less than 1

[42] Since the 2-SV has successfully outperformed the SVCJ model, we provide no further examination of the 2-SVJ model.

per cent for all of the models adopted in this study. Furthermore, we can find no significant improvement in the minimizations when adopting the Lian and Zhu (2013) formula.

7 Conclusions

The two main approaches for improving on the Heston SV model in the extant index options literature on volatility model specifications are identifying either the variance jumps or the second variance component (Bates, 1996, 2000; Duffie et al., 2000). We examine whether these two approaches also perform well in the pricing of VIX derivatives, which are, thus far, the most prominent volatility-related products in the derivative markets. In an attempt to significantly reduce the computational burden of the empirical estimations, we also propose an easily implemented and efficient numerical approximation for the pricing of VIX derivatives under the affine framework.

Our general findings show that in VIX derivative pricing, specifying the second variance component does have merit. Given the stylized fact of the mean-reversion of volatility, the term structure of VIX futures prices invariably exhibits hump-shaped patterns and non-parallel shift; we show that models with no second variance component provide insufficient flexibility for the term structure. Conversely, no significant improvements are discernible in the pricing of VIX derivatives by the incorporation of variance jumps in models with a single variance component.

As the price of a European-style VIX derivative is dependent only on the terminal VIX level, the mean-reversion nature of volatility mitigates the large movements generated by variance jumps. In contrast, as in stock prices, the level of the S&P 500 index exhibits no mean reversion; thus, jumps in variance have direct impacts on changes in the index level even if the terminal volatility returns to its pre-jump level. Therefore, given their distinct features, a model which shows promise in describing index options, may not be readily applied to the pricing of VIX derivatives.

The two-variance specification does not, however, eliminate the need for variance jumps in

VIX option pricing. We have found that when examining only short-term contracts, variance jump pricing performance is comparable to that of the second variance component; in this case, the impact of mean-reversion mitigation is limited and the term structure of VIX derivatives is less complex. We have found that incorporating variance jumps into the long-run variance component under the two-variance framework can further improve the pricing fit, with this improvement being particularly significant when there are differences in the speed of the two variance components.

Our findings are also found to be robust to the financial crisis period. Given the stylized fact of volatility persistence, market participants may not expect a large upward movement after a dramatic rise in volatility, and indeed, the addition of variance jumps during the financial crisis period offers no better fit for the pricing of VIX derivatives; indeed, volatility is found to exhibit more mean reversion after such a rise in volatility, with the two-variance models exhibiting consistent superior performance during the 2008-2009 crisis period.

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Table 1. Descriptive Statistics of VIX Options

This table reports the descriptive statistics of VIX options trading during the 2007 to 2010 period, with the moneyness being defined as the ratio of the strike price to the VIX futures price. Deep in the money (DITM), in the money (ITM), slightly in the money (SITM), slightly out of the money (SOTM), out of the money (OTM), and deep out of money (DOTM) are respectively defined as $m \leq 0.75$ $m \in (0.75, 0.9]$, $m \in (0.9, 1.0]$, $m \in (1.0, 1.1]$, $m \in (1.1, 1.35]$, and $m > 1.35$ for VIX call options, and as $m > 1.35$ $m \in (1.1, 1.35]$, $m \in (1.0, 1.1]$, $m \in (0.9, 1.0]$, $m \in (0.75, 0.9]$, and $m \leq 0.75$ for VIX put options. The maturity levels are defined as short-term ($\tau < 60/365$), mid-term ($60/365 \leq \tau < 180/365$), and long-term ($\tau \geq 180/365$) maturity. *Price*, *Obs*, *Vol*, and *OIT* respectively refer to the average mid-quote, the number of observations, the average trading volume and open interest.

Variables	Moneyness Level						
	DITM	ITM	SITM	SOTM	OTM	DOTM	All
Panel A: VIX Call Options							
Short-term Maturity							
Price (US$)	12.12	5.20	3.12	2.06	1.12	0.45	3.43
Obs	3,192	3,804	2,523	2,266	4,846	6,498	23,129
Vol	264.6	881.7	2652.2	4539.2	4562.9	2593.8	2600.3
OIT	8596.3	11765.1	23668.0	36986.3	41975.2	37124.4	28551.4
Mid-term Maturity							
Price (US$)	10.45	5.66	4.04	3.09	2.01	0.84	3.53
Obs	3,415	4,682	3,250	3,140	6,185	9,136	29,808
Vol	95.1	335.2	736.5	832.1	867.4	820.3	662.9
OIT	3178.1	4549.0	7331.4	9637.8	9316.4	8677.6	7486.0
Long-term Maturity							
Price (US$)	7.21	4.60	3.42	2.78	1.94	0.88	3.53
Obs	550	641	385	312	559	581	3,028
Vol	76.5	67.0	124.0	446.2	240.2	594.5	248.2
OIT	1672.9	820.4	1762.6	3779.9	2898.2	8930.0	3339.6
All Maturity Period							
Price (US$)	10.94	5.39	3.62	2.67	1.64	0.68	3.49
Obs	7,157	9,127	6,158	5,718	11,590	16,215	55,965
Vol	169.2	544.1	1483.1	2280.1	2382.3	1522.9	1441.1
OIT	5478.9	7294.7	13676.5	20156.2	22662.1	20086.4	15967.5
Panel B: VIX Put Options							
Short-term Maturity							
Price (US$)	18.07	6.80	3.39	1.80	0.73	0.34	5.48
Obs	2,464	3,522	2,148	2,508	3,407	1,116	15,165
Vol	50.8	569.0	2265.6	4111.8	3415.3	1515.1	2020.1
OIT	2056.1	14411.8	31224.5	34514.4	29841.3	20977.6	22059.8
Mid-term Maturity							
Price (US$)	19.62	7.95	4.49	2.84	1.26	0.50	4.40
Obs	1,247	1,976	1,729	2,362	4,222	2,377	13,913
Vol	34.3	152.9	642.1	972.7	1044.9	768.8	718.1
OIT	297.0	2092.3	6328.2	10132.6	10166.2	9329.4	7509.3
Long-term Maturity							
Price (US$)	14.61	6.56	4.00	2.55	1.22	0.47	4.00
Obs	138	168	88	173	340	199	1,106
Vol	16.3	20.9	364.8	1037.8	762.7	127.3	453.9
OIT	55.0	83.2	1247.9	9823.1	6094.2	1148.8	3735.5
All Maturity Period							
Price (US$)	18.45	7.19	3.88	2.31	1.03	0.45	4.93
Obs	3,849	5,666	3,965	5,043	7,969	3,692	30,184
Vol	44.2	407.6	1515.5	2536.1	2046.2	959.8	1362.6
OIT	1414.5	9690.6	19702.8	22247.6	18404.2	12409.4	14681.5

Table 2. Parameter Estimates of VIX Options

This table reports the estimated parameters for the models with a single variance component (Panel A) and two variance component (Panel B), with the estimates being generated by minimizing the sum of the squared pricing errors with the market price of all VIX options traded during the 2007-2010 period. RMSE reports the average daily root mean-squared error. NOP stands for the number of estimated parameters.

Panel A: Single Variance Component Models				Panel B: Two Variance Components Models		
	SV	SVJ	SVCJ		2-SV	2-SVJ
$\boldsymbol{\kappa}$	3.395	3.897	4.510	κ_s	15.181	14.356
	(0.113)	(0.070)	(0.063)		(0.639)	(0.553)
$\sqrt{\boldsymbol{\theta}}$	0.273	0.238	0.208	$\sqrt{\theta_s}$	0.204	0.195
	(0.003)	(0.003)	(0.003)		(0.003)	(0.003)
$\boldsymbol{\sigma}$	0.852	1.111	0.850	σ_s	2.372	2.394
	(0.024)	(0.019)	(0.018)		(0.119)	(0.097)
$\sqrt{\boldsymbol{V_t}}$	0.258	0.187	0.205	$\sqrt{V_{s,t}}$	0.163	0.149
	(0.004)	(0.004)	(0.004)		(0.003)	(0.003)
	–	–	–	κ_l	1.895	1.759
					(0.067)	(0.062)
	–	–	–	$\sqrt{\theta_l}$	0.391	0.382
					(0.021)	(0.021)
	–	–	–	σ_l	0.756	0.767
					(0.026)	(0.024)
	–	–	–	$\sqrt{V_{l,t}}$	0.174	0.158
					(0.003)	(0.003)
$\boldsymbol{\lambda}$	–	0.444	0.468	λ	–	0.107
		(0.014)	(0.019)			(0.005)
$\boldsymbol{\mu_p}$	–	–0.203	–0.158	μ_p	–	–0.297
		(0.005)	(0.005)			(0.009)
$\boldsymbol{\sigma_p}$	–	0.190	0.127	σ_p	–	0.168
		(0.004)	(0.004)			(0.004)
$\boldsymbol{\mu_v}$	–	–	0.358		–	–
			(0.013)			
$\boldsymbol{\rho_J}$	–	–	–0.385		–	–
			(0.007)			
RMSE	0.233	0.150	0.122	RMSE	0.089	0.085
NOP	3	6	8	NOP	7	10

Table 3. Mean and Median Equality Tests

This table reports the statistics of the two-sample *t*-test and the Wilcoxon signed-rank test examining whether there were any differences in the mean (Panel A) and median (Panel B) of the daily root mean-squared errors between two models of VIX options traded during the 2007-2010 period. The difference is defined by the error of the compared model specified in the first row minus that of the error of the selected model specified in the first column. The standardized test statistics are reported in the parentheses. Figures in **bold** text indicate insignificance at the 5% level.

	In-sample Results				Out-of-sample Results			
	SV	SVJ	SVCJ	2-SV	SV	SVJ	SVCJ	2-SV
Panel A:	Mean Equality Test							
SVJ	0.083				0.077			
	(21.63)				(10.59)			
SVCJ	0.111	0.028			0.099	0.022		
	(28.86)	(7.71)			(13.44)	(2.93)		
2-SV	0.144	0.061	0.033		0.120	0.043	0.021	
	(43.87)	(19.95)	(10.62)		(16.90)	(5.88)	(2.80)	
2-SVJ	0.148	0.065	0.037	**0.004**	0.124	0.046	0.024	**0.003**
	(45.35)	(21.43)	(12.04)	**(1.82)**	(17.36)	(6.34)	(3.26)	**(0.48)**
Panel B:	Median Equality Test							
SVJ	0.081				0.076			
	(–26.91)				(–25.52)			
SVCJ	0.103	0.027			0.092	0.020		
	(–27.31)	(–26.49)			(–26.35)	(–23.33)		
2-SV	0.143	0.049	0.021		0.116	0.034	0.012	
	(–27.48)	(–27.45)	(–22.72)		(–27.16)	(–25.23)	(–16.54)	
2-SVJ	0.146	0.053	0.025	0.002	0.118	0.038	0.015	0.001
	(–27.48)	(–27.48)	(–25.22)	(-22.71)	(–27.23)	(–25.80)	(–19.81)	(–12.29)

Table 4. Pricing Improvements Attributable to Variance Jumps

This table reports the two-sample *t*-test and Wilcoxon signed-rank test results examining the null hypothesis that the mean (Panel A) and median (Panel B) of the difference between the absolute pricing errors of the two models will be zero for VIX options traded during the 2007-2010 period. The pricing errors between the SVJ and the SVCJ models are investigated across moneyness levels and time to maturity periods. The standardized test statistics are reported in the parentheses. Figures in **bold** text indicate insignificance at the 5% level.

Time to Maturity Periods	Moneyness Levels					
	DITM	ITM	NTM	OTM	DOTM	All
Panel A: Mean Equality Test						
In-sample Results						
Short-term	**0.013**	0.026	0.047	0.016	0.047	0.035
	(0.85)	(7.65)	(22.50)	(8.91)	(31.29)	(24.71)
Mid-term	0.034	0.028	0.017	0.014	0.041	0.028
	(5.29)	(11.03)	(9.66)	(11.07)	(36.60)	(28.99)
Long-term	**0.023**	**0.007**	**–0.004**	–0.013	0.017	**0.003**
	(1.83)	**(0.92)**	**(–0.77)**	(–2.62)	(3.08)	**(1.12)**
All Maturity	0.027	0.026	0.029	0.016	0.043	0.029
	(4.20)	(12.94)	(22.20)	(17.28)	(47.51)	(37.37)
Out-of-sample Results						
Short-term	**0.004**	**0.013**	0.031	0.015	0.043	0.027
	(0.15)	**(1.70)**	(7.07)	(4.82)	(22.84)	(9.93)
Mid-term	**0.023**	0.019	0.012	0.011	0.037	0.022
	(1.77)	(3.97)	(3.84)	(4.97)	(26.46)	(12.35)
Long-term	**0.012**	**0.007**	**–0.009**	–0.016	**0.010**	**–0.001**
	(0.41)	**(0.46)**	**(–0.98)**	(–2.26)	**(1.60)**	**(–0.27)**
All Maturity	**0.016**	0.016	0.019	0.013	0.039	0.023
	(1.35)	(3.78)	(7.62)	(7.89)	(34.48)	(15.37)
Panel B: Median Equality Test						
In-sample Results						
Short-term	0.004	0.017	0.032	0.008	0.045	0.025
	(–4.33)	(–19.21)	(–33.06)	(–15.11)	(–48.58)	(–60.86)
Mid-term	0.021	0.015	0.011	0.013	0.032	0.018
	(–15.79)	(–21.63)	(–22.27)	(–25.12)	(–48.43)	(–63.36)
Long-term	0.010	0.003	–0.008	–0.015	0.008	**–0.002**
	(–4.63)	(–2.91)	(–3.31)	(–9.36)	(–6.20)	**(–0.21)**
All Maturity	0.015	0.015	0.016	0.009	0.036	0.019
	(–16.28)	(–29.02)	(–38.18)	(–26.47)	(–68.86)	(–86.73)
Out-of-sample Results						
Short-term	**0.003**	0.009	0.023	0.008	0.041	0.020
	(–1.42)	(–9.82)	(–21.68)	(–13.70)	(–42.62)	(–46.31)
Mid-term	0.009	0.010	0.008	0.008	0.029	0.013
	(–7.97)	(–14.23)	(–14.30)	(–17.43)	(–42.69)	(–47.51)
Long-term	**0.006**	**0.002**	–0.006	–0.015	0.009	–0.003
	(–1.86)	**(–1.37)**	(–3.92)	(–7.98)	(–4.48)	(–2.36)
All Maturity	0.007	0.009	0.011	0.007	0.033	0.015
	(–7.70)	(–17.20)	(–24.47)	(–20.00)	(–60.30)	(–64.96)

Table 5. Pricing Improvements Attributable to the Second Variance Component

This table reports the two-sample *t*-test and Wilcoxon signed-rank test results examining the null hypothesis that the mean (Panel A) and median (Panel B) of the difference between the absolute pricing errors of the two models will be zero for VIX options traded during the 2007-2010 period. The pricing errors between the SVJ and the 2-SVJ models are investigated across moneyness levels and time to maturity periods. The standardized test statistics are reported in the parentheses. Figures in **bold** text indicate insignificance at the 5% level.

Time to Maturity Periods	Moneyness Levels					
	DITM	ITM	NTM	OTM	DOTM	All
Panel A: Mean Equality Test						
In-sample Results						
Short-term	0.160	0.072	0.051	0.037	0.056	0.059
	(12.74)	(23.72)	(25.13)	(23.56)	(41.32)	(48.53)
Mid-term	0.106	0.053	0.035	0.030	0.050	0.048
	(19.83)	(24.21)	(22.18)	(25.26)	(47.81)	(57.52)
Long-term	0.069	0.059	0.032	0.037	0.055	0.047
	(6.53)	(8.61)	(6.29)	(8.88)	(11.94)	(18.88)
All Maturity	0.120	0.061	0.042	0.034	0.052	0.052
	(22.69)	(34.57)	(33.93)	(40.09)	(64.14)	(76.45)
Out-of-sample Results						
Short-term	0.059	0.025	0.030	0.023	0.046	0.034
	(2.20)	(3.36)	(6.95)	(7.55)	(25.49)	(13.09)
Mid-term	0.068	0.033	0.022	0.020	0.043	0.034
	(5.64)	(7.14)	(7.45)	(9.31)	(31.59)	(19.95)
Long-term	0.079	0.049	**0.015**	0.025	0.043	0.036
	(3.12)	(3.40)	**(1.76)**	(4.04)	(7.58)	(7.71)
All Maturity	0.066	0.030	0.025	0.022	0.044	0.034
	(5.70)	(7.42)	(10.15)	(13.68)	(41.16)	(23.79)
Panel B: Median Equality Test						
In-sample Results						
Short-term	0.080	0.051	0.035	0.025	0.047	0.040
	(−16.86)	(−28.25)	(−30.58)	(−29.55)	(−54.51)	(−73.94)
Mid-term	0.068	0.036	0.028	0.025	0.043	0.035
	(−24.03)	(−27.86)	(−28.90)	(−31.44)	(−51.54)	(−75.29)
Long-term	0.041	0.051	0.024	0.024	0.032	0.030
	(−6.84)	(−9.35)	(−8.31)	(−11.31)	(−12.99)	(−22.17)
All Maturity	0.070	0.043	0.030	0.025	0.044	0.037
	(−30.05)	(−40.77)	(−42.81)	(−44.54)	(−75.37)	(−107.68)
Out-of-sample Results						
Short-term	0.028	0.017	0.023	0.016	0.041	0.029
	(−5.91)	(−9.27)	(−17.24)	(−18.83)	(−46.90)	(−45.98)
Mid-term	0.038	0.021	0.017	0.018	0.039	0.027
	(−11.06)	(−14.09)	(−16.48)	(−19.54)	(−44.45)	(−49.74)
Long-term	0.047	0.027	0.008	0.014	0.027	0.020
	(−4.47)	(−5.27)	(−2.54)	(−5.88)	(−9.63)	(−12.65)
All Maturity	0.035	0.020	0.019	0.017	0.040	0.027
	(−13.12)	(−17.37)	(−23.88)	(−27.77)	(−64.46)	(−68.85)

Table 6. Pricing Improvements Attributable to Variance Jumps When the Second Variance Component is Specified

This table reports the two-sample *t*-test and Wilcoxon signed-rank test results examining the null hypothesis that the mean (Panel A) and median (Panel B) of the difference between the absolute pricing errors of the two models will be zero for VIX options traded during the 2007-2010 period. The pricing errors between the 2-SVJ and the 2-SVCJ models are investigated across moneyness levels and time to maturity periods. The standardized test statistics are reported in the parentheses. Figures in **bold** text indicate insignificance at the 5% level.

Time to Maturity Periods	Moneyness Levels					
	DITM	ITM	NTM	OTM	DOTM	All
Panel A: Mean Equality Test						
In-sample Results						
Short-Term	**0.0081**	0.0071	0.0091	0.0072	0.0071	0.0076
	(0.85)	(3.06)	(5.91)	(6.32)	(7.36)	(8.52)
Mid-Term	**0.0033**	0.0041	0.0050	0.0041	0.0038	0.0041
	(0.88)	(2.74)	(4.06)	(4.32)	(5.31)	(6.97)
Long-Term	**−0.0013**	**−0.0011**	**0.0022**	**0.0009**	0.0061	**0.0021**
	(−0.17)	**(−0.20)**	**(0.52)**	**(0.30)**	(2.16)	**(1.15)**
All	**0.0045**	0.0052	0.0067	0.0056	0.0052	0.0055
	(1.16)	(3.99)	(7.05)	(8.60)	(9.18)	(11.04)
Out-of-sample						
Short-Term	**0.0047**	**0.0053**	**0.0054**	**0.0042**	0.0053	0.0050
	(0.19)	**(0.75)**	**(1.27)**	**(1.42)**	(3.36)	(2.01)
Mid-Term	**0.0042**	**0.0025**	**0.0029**	**0.0025**	**0.0022**	**0.0026**
	(0.38)	**(0.58)**	**(1.01)**	**(1.25)**	**(1.83)**	**(1.64)**
Long-Term	**−0.0000**	**−0.0001**	**0.0028**	**0.0009**	**0.0052**	**0.0023**
	(−0.00)	**(−0.00)**	**(0.36)**	**(0.18)**	**(1.22)**	**(0.56)**
All	**0.0041**	**0.0036**	**0.0040**	0.0032	0.0035	0.0036
	(0.38)	**(0.93)**	**(1.65)**	(2.08)	(3.80)	(2.64)
Panel B: Median Equality Test						
In-sample Results						
Short-Term	0.0005	0.0005	0.0014	0.0009	0.0012	0.0010
	(−4.82)	(−8.35)	(−16.67)	(−16.36)	(−20.72)	(−31.97)
Mid-Term	0.0001	0.0006	0.0004	0.0003	0.0004	0.0004
	(−3.64)	(−8.97)	(−10.02)	(−10.78)	(−13.68)	(−22.03)
Long-Term	**−0.0001**	**0.0000**	0.0006	**0.0001**	0.0005	0.0002
	(−0.26)	**(−0.11)**	(−2.07)	**(−1.21)**	(−3.73)	(−3.54)
All	0.0003	0.0006	0.0007	0.0005	0.0007	0.0006
	(−5.55)	(−12.00)	(−18.80)	(−18.82)	(−24.05)	(−37.54)
Out-of-sample Results						
Short-Term	0.0002	0.0003	0.0006	0.0004	0.0006	0.0005
	(−2.54)	(−5.64)	(−8.84)	(−8.61)	(−14.42)	(−19.28)
Mid-Term	0.0004	0.0003	0.0003	0.0001	0.0002	0.0002
	(−2.97)	(−4.97)	(−6.89)	(−6.71)	(−7.89)	(−13.59)
Long-Term	**−0.0005**	**−0.0002**	**0.0003**	**0.0003**	0.0003	0.0002
	(−0.12)	**(−0.08)**	**(−1.36)**	**(−1.11)**	(−2.35)	(−2.38)
All	0.0003	0.0003	0.0004	0.0002	0.0003	0.0003
	(−3.71)	(−7.28)	(−11.11)	(−10.75)	(−15.44)	(−22.84)

Table 7. Mean and Median Equality Tests for Only Short-Term VIX Options

This table reports the two-sample *t*-test (in-sample) and Wilcoxon signed-rank test (out-of-sample) results examining whether the mean (Panel A) and median (Panel B) of the daily root mean-squared errors between the two models differ for only short-term VIX options traded during the 2007-2010 period. The difference is defined by the error of the compared model specified in the first row minus that of the selected model specified in the first column. The standardized test statistics are reported in the parentheses. Figures in **bold** text indicate insignificance at the 5% level.

	In-sample Results				Out-of-sample Results			
	SV	SVJ	SVCJ	2-SV	SV	SVJ	SVCJ	2-SV
Panel A: Mean Equality Test								
SVJ	0.056				0.050			
	(15.29)				(5.81)			
SVCJ	0.102	0.046			0.082	0.031		
	(28.53)	(15.17)			(9.30)	(3.53)		
2-SV	0.107	0.051	**0.005**		0.072	0.021	**–0.010**	
	(32.72)	(19.08)	**(1.85)**		(7.89)	(2.34)	**(–1.04)**	
2-SVJ	0.113	0.057	0.011	0.006	0.080	0.030	**–0.001**	**0.008**
	(34.85)	(21.61)	(4.26)	(2.94)	(9.07)	(3.35)	**(–0.15)**	**(0.89)**
Panel B: Median Equality Test								
SVJ	0.056				0.049			
	(–25.58)				(–23.88)			
SVCJ	0.091	0.043			0.071	0.025		
	(–27.36)	(–27.31)			(–25.75)	(–23.34)		
2-SV	0.096	0.044	0.000		0.067	0.021	–0.003	
	(–27.42)	(–25.68)	(–2.92)		(–24.20)	(–18.21)	(–8.29)	
2-SVJ	0.099	0.051	0.003	0.001	0.071	0.025	**–0.001**	0.001
	(–27.44)	(–27.36)	(–10.18)	(–15.09)	(–24.89)	(–21.23)	**(–1.41)**	(–12.48)

Table 8. Summary of Pricing Error

This table reports the in-sample (IS) and out-of-sample (OS) 'root mean-squared error' (RMSE) averages for all of the SV models examined in this study under three different scenarios: Scenario 1 examines daily data on the RMSE for VIX options in all time-to-maturity periods; Scenario 2 is based on the monthly estimation for VIX options in all time-to-maturity periods; and Scenario 3 examines daily data on short-term VIX options only. The RMSE is averaged across trading days/months in Panel A and transactions in Panel B. Panel C reports the percentages of trading days in which the RMSE of the SVCJ model outperformed that of the 2-SV and 2-SVJ models. The sample period runs from 2007 to 2010, with the short-term VIX options being defined by $\tau < 60/365$.

Models	Scenario 1 (Daily)		Scenario 2 (Monthly)		Scenario 3 (Daily)	
	IS	OS	IS	OS	IS	OS
Panel A: Average Daily/Monthly RMSE						
SV	0.233	0.298	0.396	0.651	0.152	0.250
SVJ	0.150	0.220	0.324	0.594	0.096	0.200
SVCJ	0.122	0.198	0.322	0.592	0.050	0.169
2SV	0.089	0.178	0.193	0.517	0.045	0.178
2SVJ	0.085	0.174	0.187	0.516	0.039	0.170
Panel B: Average Transaction RMSE						
SV	0.267	0.367	0.499	0.972	0.204	0.358
SVJ	0.181	0.302	0.422	0.926	0.139	0.319
SVCJ	0.155	0.287	0.421	0.925	0.103	0.299
2SV	0.111	0.265	0.247	0.816	0.078	0.317
2SVJ	0.107	0.263	0.242	0.816	0.073	0.302
Panel C: SVCJ Advantage Ratio (%) Compared to 2-SV and 2-SVJ Models						
2SV	16.1	27.6	0.0	21.3	49.4	63.6
2SVJ	10.6	22.2	0.0	19.1	40.3	52.5

Table 9. Jumps in Short- and Long-run Variance Components

This table reports the percentages of trading days exhibiting variance jumps in short-run variance component (with a rapid reversion speed) and long-run variance components (with a slower reversion speed) under the 2-SVCJ model. The mean reversion speed denotes the mean level of the estimated speed of mean reversion κ_s and κ_l. Panel A reports the results for all the trading days with variance jumps, with Panel B (Panel C) reporting the results for subgroups with κ_s/κ_l which are ≤ or ≥ specific quantiles sorted by κ_s/κ_l, the ratio between the two mean-reversion speeds.

Variables	Mean Reversion Speed		Trading Days (%)	
	κ_s	κ_l	V_s	V_l
Panel A: All Trading Days with Variance Jumps				
	13.370	1.902	30.7	69.3
Panel B: Subgroups with $\boldsymbol{\kappa_s/\kappa_l}$ ≤ Specific Quantiles				
≤ Quantile 50	8.958	3.232	33.2	66.8
≤ Quantile 25	6.312	4.053	34.8	65.2
≤ Quantile 10	5.389	4.668	48.0	52.0
Panel C: Subgroups with $\boldsymbol{\kappa_s/\kappa_l}$ ≥ Specific Quantiles				
≥ Quantile 50	17.782	0.571	28.3	71.7
≥ Quantile 75	19.030	0.119	22.5	77.5
≥ Quantile 90	21.359	0.027	21.3	78.7

Table 10. Summary of Pricing Errors During the Financial Crisis Period

This table reports the in-sample (IS) and out-of-sample (OS) 'root mean-squared error' (RMSE) averages for all of the SV models examined in this study during the financial crisis period (December 2007 to May 2009) under three different scenarios: Scenario 1 examines daily data on the RMSE for VIX options in all time-to-maturity periods; Scenario 2 is based on the monthly estimation for VIX options in all time-to-maturity periods; and Scenario 3 examines daily data on short-term VIX options only. The RMSE is averaged across trading days/months in Panel A and transactions in Panel B. Panel C reports the percentages of trading days in which the RMSE of the SVCJ model outperformed that of the 2-SV and 2-SVJ models. The sample period runs from 2007 to 2010, with the short-term VIX options being defined by $\tau < 60/365$.

Models	Scenario 1 (Daily)		Scenario 2 (Monthly)		Scenario 3 (Daily)	
	IS	OS	IS	OS	IS	OS
Panel A: Average Daily/Monthly RMSE						
SV	0.228	0.320	0.452	0.841	0.150	0.287
SVJ	0.181	0.283	0.385	0.802	0.112	0.252
SVCJ	0.159	0.266	0.385	0.803	0.066	0.229
2SV	0.104	0.230	0.219	0.702	0.052	0.238
2SVJ	0.100	0.226	0.214	0.695	0.050	0.230
Panel B: Average Transaction RMSE						
SV	0.271	0.426	0.626	1.336	0.223	0.456
SVJ	0.233	0.402	0.548	1.296	0.190	0.434
SVCJ	0.214	0.391	0.547	1.297	0.159	0.419
2SV	0.143	0.353	0.312	1.144	0.114	0.445
2SVJ	0.138	0.350	0.310	1.135	0.113	0.421
Panel C: SVCJ Advantage Ratio (%) Compared to 2-SV and 2-SVJ Models						
2SV	1.6	16.3	0.0	16.7	38.7	51.2
2SVJ	2.9	16.5	0.0	16.7	42.4	57.3

Table 11. Pricing Errors for Alternative Weighting Schemes

This table reports the pricing errors obtained from three minimizations in the SV, SVJ, SVCJ and 2-SV models during the first quarter of 2007 and the third quarter of 2008. RMSE denotes the root mean-squared error from minimization (11), RMSRE denotes the root mean-squared relative error from minimization (12) and IVRMSE denotes the implied volatility root mean-squared error from minimization (13).

Models	RMSE	RMSRE	IVRMSE
Panel A: In-sample Results			
SV	0.143	0.172	0.083
SVJ	0.096	0.103	0.053
SVCJ	0.077	0.079	0.038
2-SV	0.058	0.053	0.023
Panel B: Out-of-sample Results			
SV	0.183	0.300	0.221
SVJ	0.144	0.198	0.166
SVCJ	0.129	0.152	0.152
2-SV	0.116	0.133	0.119

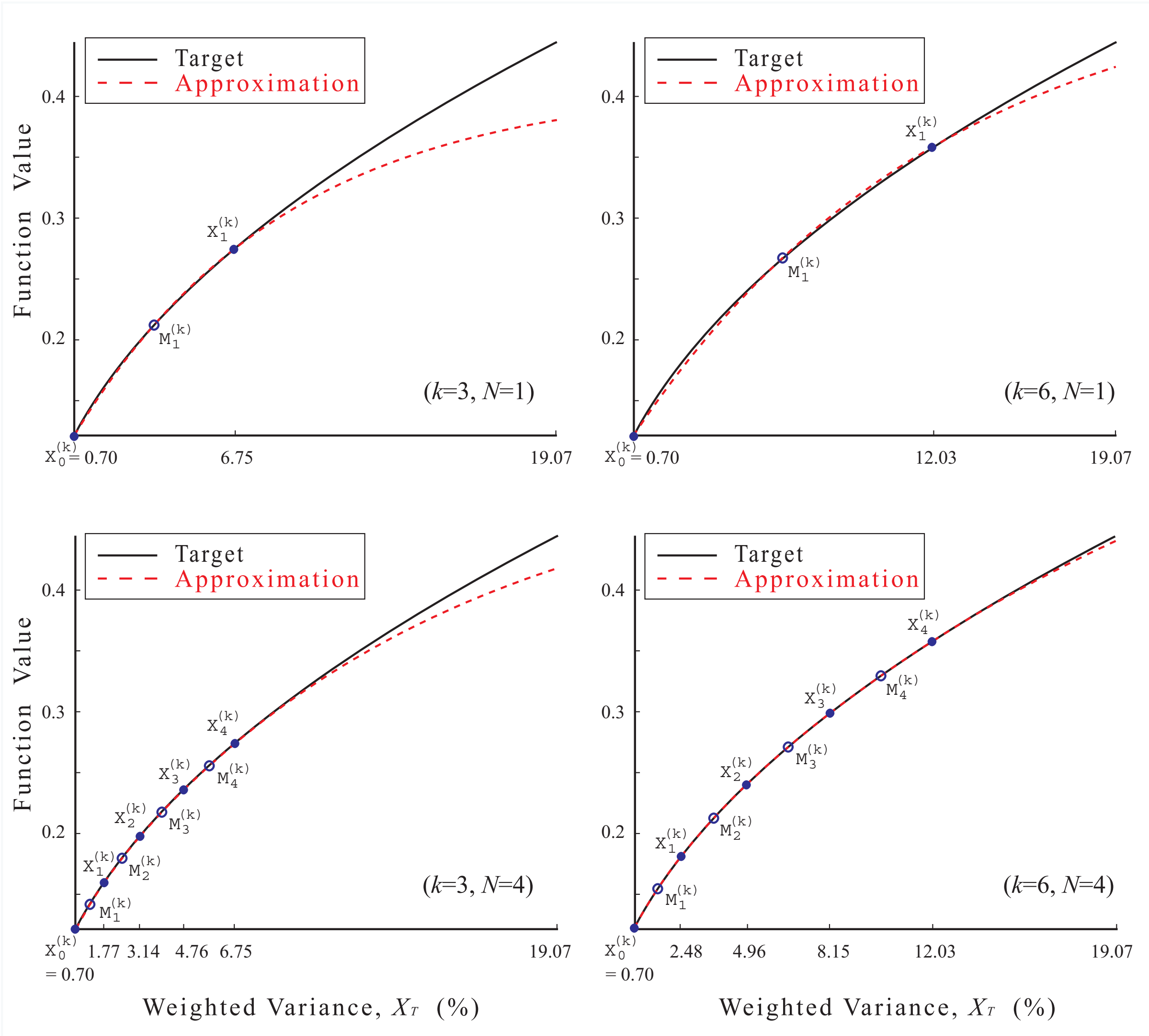

Figure 1. Fitting Points and Fitting Performance

This figure illustrates the fitting points in our approximation with different fitting area choices (k = 3 and k = 6) and different numbers of exponential curves (N = 1 and N = 4) under the SVCJ model with the following parameters (based on Duffie et al., 2000): κ_1=3.5, θ_1=0.01, σ_1=0.15, λ=0.5, $\bar{\mu} = -0.1$, μ_v=0.05, σ_p=0.0001, $\rho_J = -0.4$, $V_{1,t}$=0.008, r=0.03, τ=1, and moneyness, m=1. The solid curves depict the target function $\sqrt{A + X_T}\, I_{\{X_T \geq K^2 - A\}}$ and the dashed curves depict the approximations (10). The fitting points $X_n^{(k)}$ and $M_n^{(k)} \equiv \left(X_n^{(k)} + X_{n-1}^{(k)}\right)/2, n = 1,2, \ldots, N$, are detailed in Appendix A.

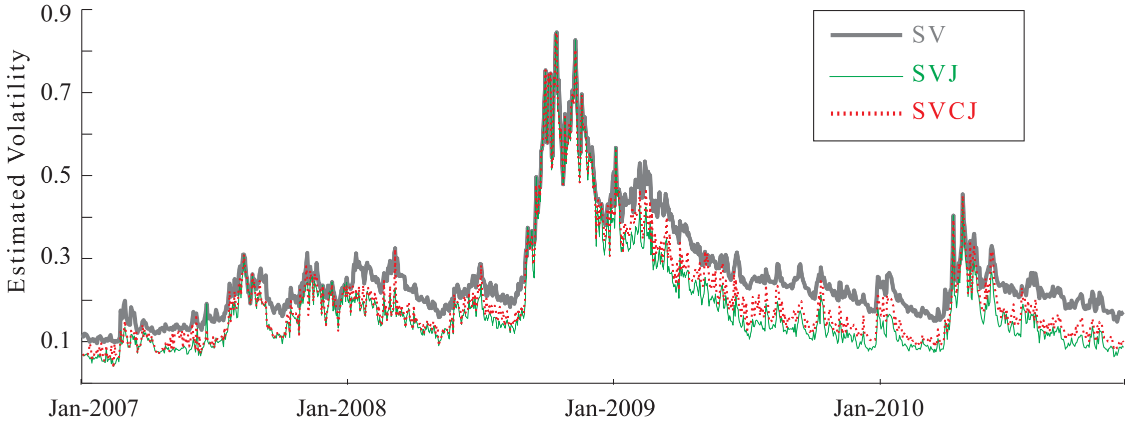

Fig.2a Estimated volatility for single-variance models

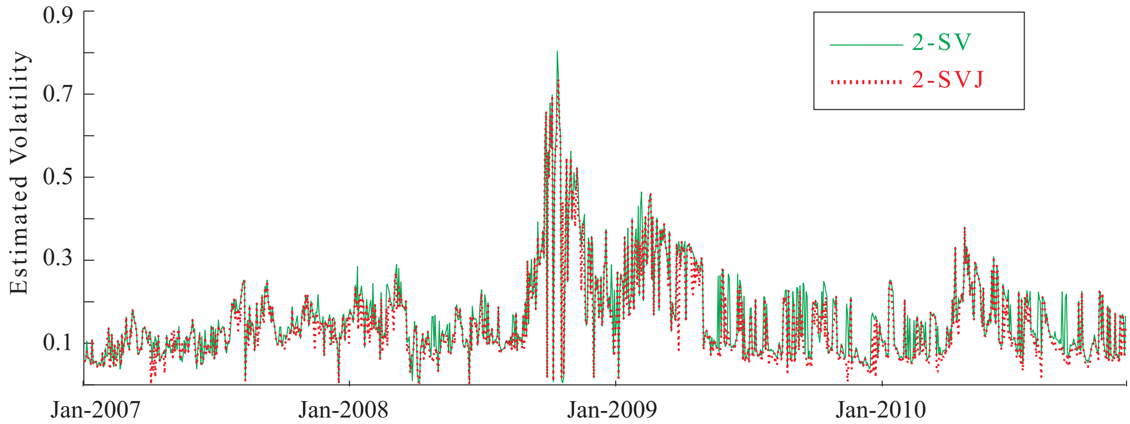

Fig.2b Estimated short-run volatility for two-variance models

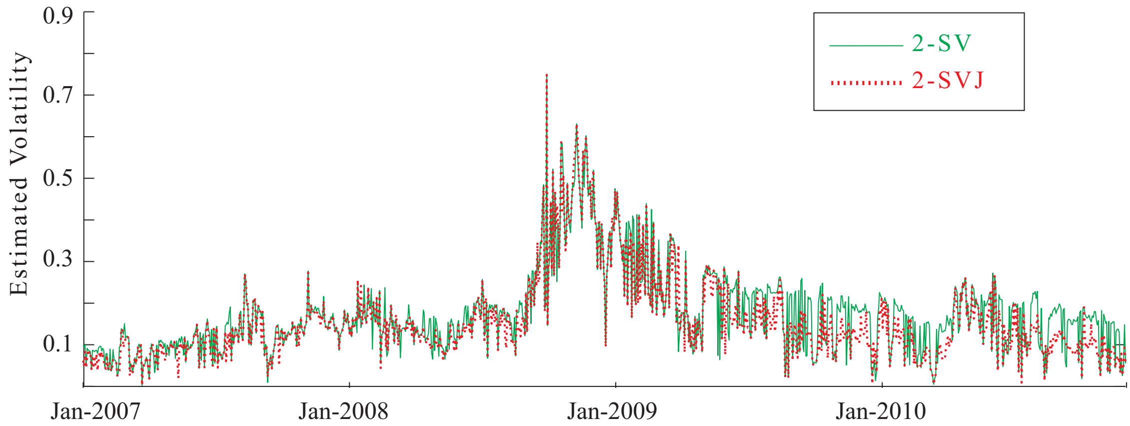

Fig.2c Estimated long-run volatility for two-variance models

Figure 2. Estimated Volatility for Single- and Two-variance Models

Fig.2a illustrates the estimated volatility $\sqrt{V_t}$ for the SV, SVJ, and SVCJ models, whilst Fig.2b and 2c respectively illustrate the estimated short-run volatility $\sqrt{V_{s,t}}$ and long-run volatility $\sqrt{V_{l,t}}$ for the 2-SV and the 2-SVJ models. The estimates are generated by minimizing the sum of the squared pricing errors with the market prices of VIX options traded during 2007-2010 sample period.

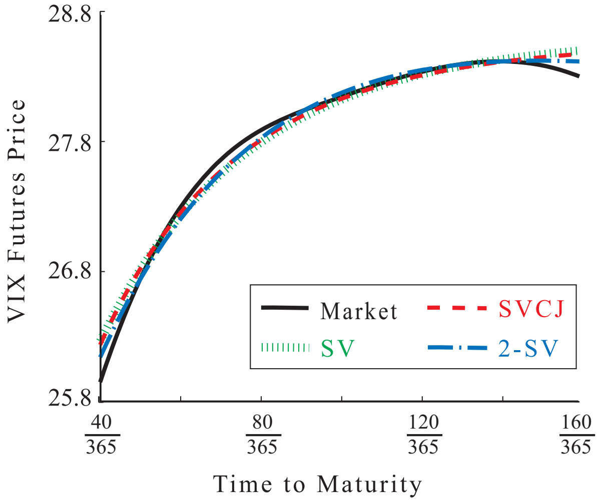

Fig.3a Upward-sloping, 11 Dec 2009

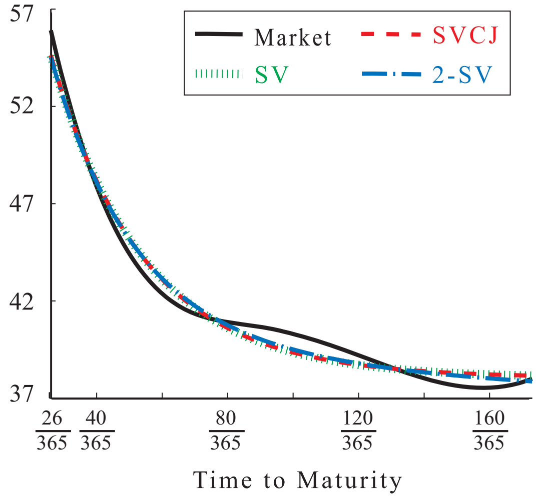

Fig.3b Downward-sloping, 24 Oct 2008

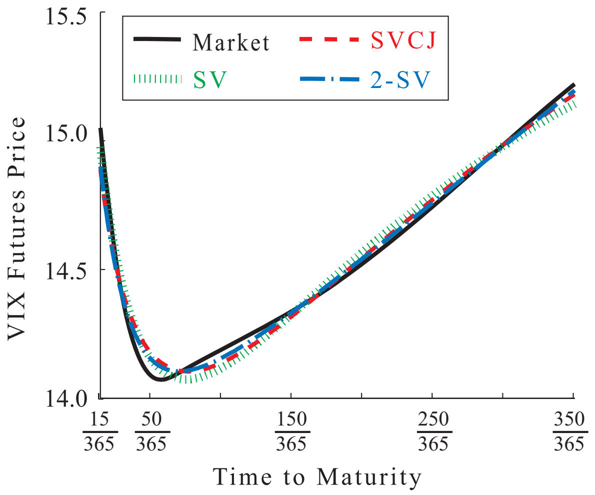

Fig.3c U-shaped, 6 Mar 2007

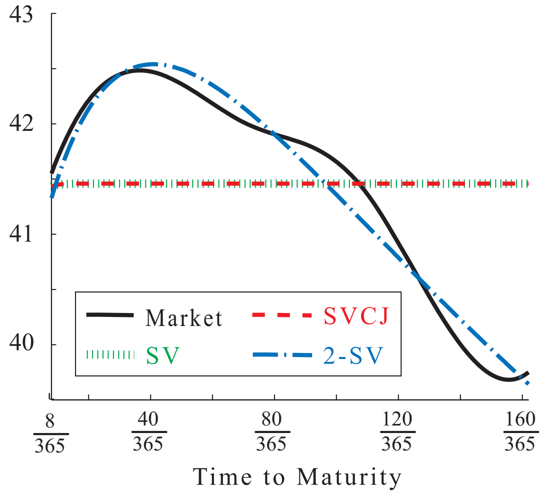

Fig.3d Hump-shaped, 7 Apr 2009

Figure 3. VIX Futures Term Structure

This figure illustrates four typical patterns of term structure drawn from the market prices of VIX futures observed on 11 December 2009 (upward-sloping), 24 October 2008 (downward-sloping), 6 March 2007 (U-shaped) and 7 April 2009 (hump-shaped). The solid curve denotes the term structure constructed by interpolating the market prices of VIX futures using cubic splines, whilst the remaining curves denote the theoretical term structures with calibrated parameters of the SV, SVCJ and 2-SV models using our approximation with $(k, N) = (6,4)$.

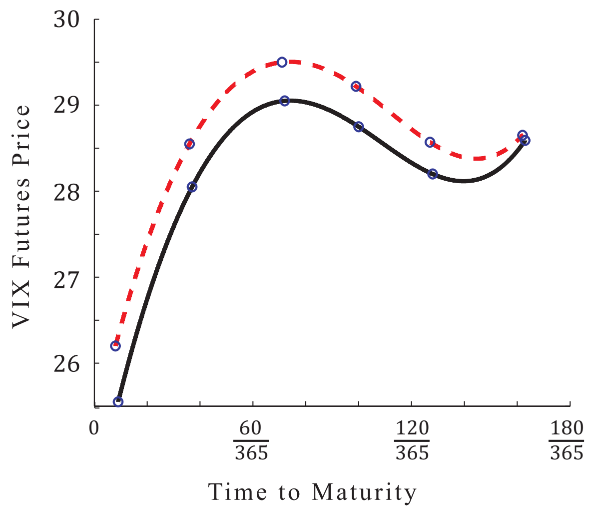

Fig.4a Parallel shift, 10-11 Aug 2009

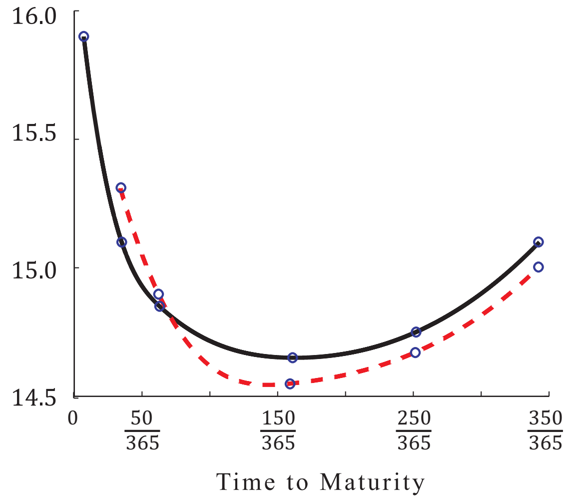

Fig.4b Non-parallel shift, 14-15 Mar 2007

Figure 4. Changes in VIX Futures Term Structure

This figure illustrates changes in VIX futures term structures on two consecutive trading days. The curves are obtained using cubic splines, with the marked points around the curves being the observed market prices of VIX futures. Fig.4a reveals the parallel shift observed on 10 August 2009 (solid line) and 11 August 2009 (dashed line), whilst Fig.4b reveals the non-parallel shift observed on 14 March 2007 (solid line) and 15 March 2007 (dashed line).

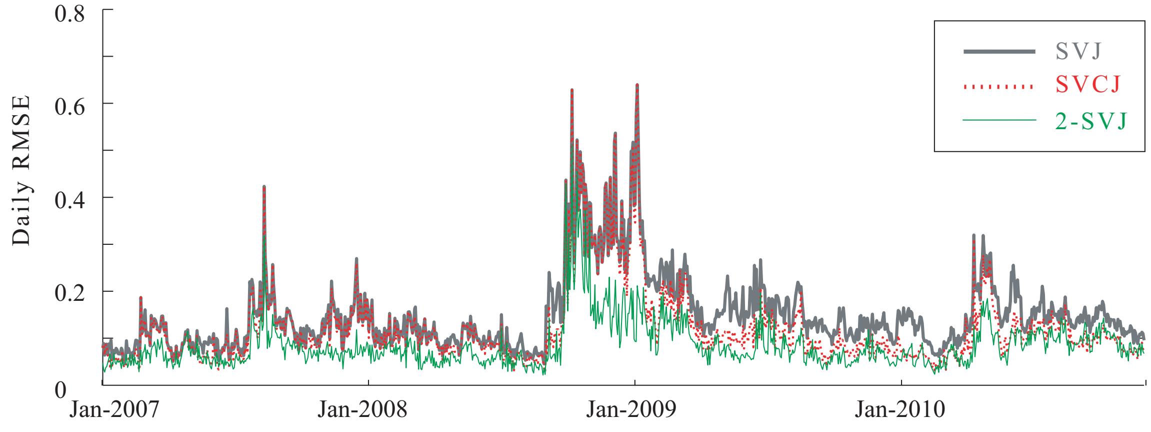

Fig.5a Daily estimates on the full sample data

Fig.5b Monthly estimates on the full sample data

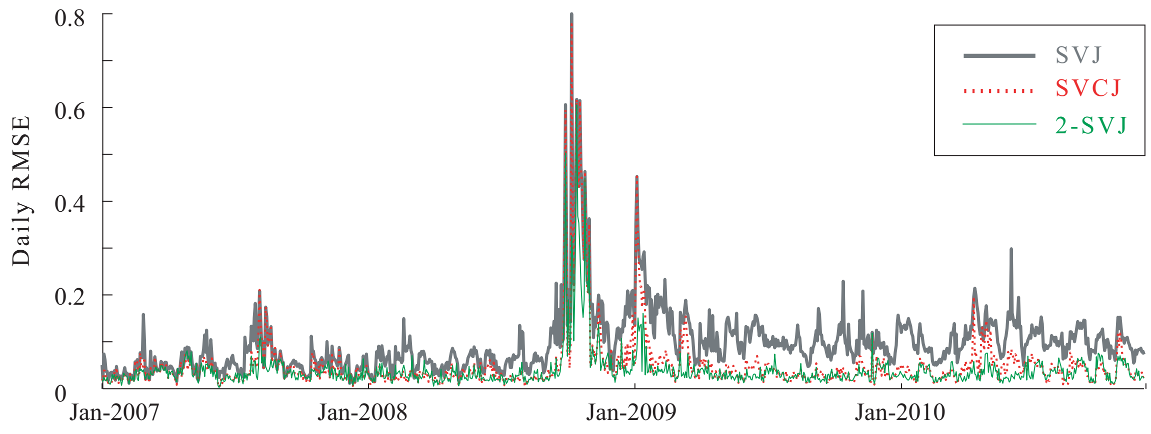

Fig.5c Daily estimates on short-term data only

Figure 5. In-sample Root Mean-Squared Errors (RMSE)
This figure illustrates the root mean-squared errors (RMSE) over time for the SVJ, SVCJ and 2-SVJ models. Fig.5a (Fig.5b) shows the daily (monthly) estimates on the full sample data, whilst Fig.5c shows the daily estimates based upon the short- term data only.

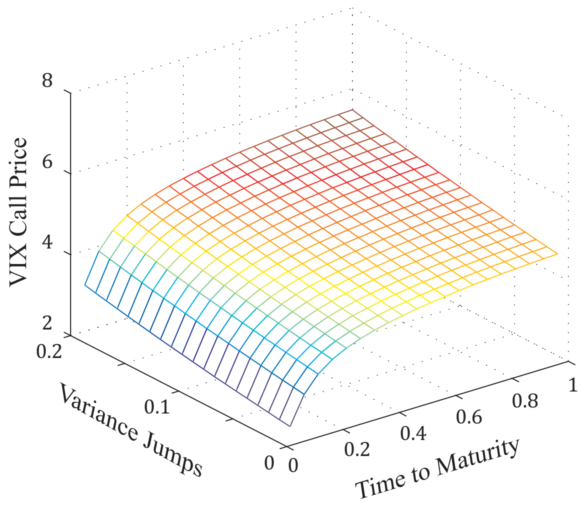

Fig.6a Short-run variance jumps

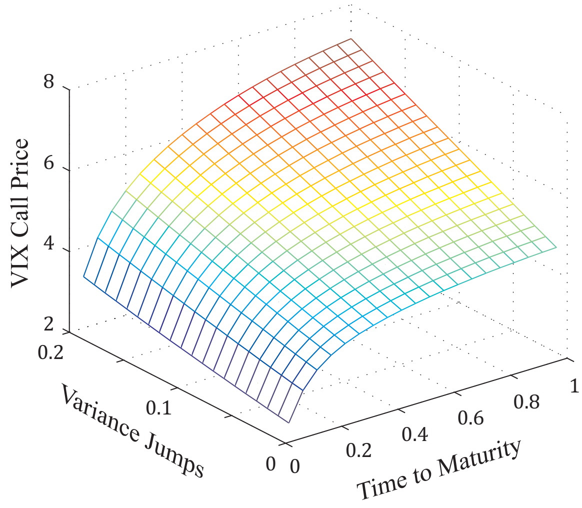

Fig.6b Long-run variance jumps

Figure 6. Jumps in Short- and Long-run Variance Components

This figure illustrates the sensitivity of VIX call prices to variance jumps (μ_v) and time to maturity (τ) under the 2-SVCJ model with the parameters based upon our estimation results, κ_s=12, κ_l=2, $\sqrt{\theta_s}$=$\sqrt{\theta_l}$=0.2, σ_s=2, σ_l=0.75, $\sqrt{V_{s,t}}$=$\sqrt{V_{l,t}}$=0.15, λ=0.25, $\mu_p = -0.25$, σ_p=0.15, $\mu_v \in \{0.01, 0.02, \ldots, 0.20\}$, $\rho_J = -0.4$, r=0.03, $\tau \in \{0.05, 0.10, \ldots, 1.0\}$, and strike price K=28.

Appendix A. Closed-form Approximation for VIX Derivatives

The Fitting Scheme

As noted earlier in Section 2.3, *N* exponential curves are used to approximate the target payoff in the interval $\left[X_0^{(k)}, X_N^{(k)}\right]$, which covers *k* standard deviations around the mean of the state variable, X_T. This approach, which identifies how much information is incorporated, is expressed as:

$$X_0^{(k)} = \max\{K^2 - A, \mu - k\delta\} \quad \text{and} \quad X_N^{(k)} = \max\{K^2 - A, \mu\} + k\delta,$$

where μ and δ, which respectively denote the mean and the standard deviation of X_T, are derived from the Lemma (below).

As regards the choice of the other fitting points, $\{X_1^{(k)}, X_2^{(k)}, \dots, X_{N-1}^{(k)}\}$, the equally-divided partition is the simplest candidate; however, in order to fit the target more effectively, we follow Liu (2010) to determine the partitions according to their curvature levels, as follows:

$$X_n^{(k)} = \left[\frac{(N-n)\sqrt{A+X_0^{(k)}} + n\sqrt{A+X_N^{(k)}}}{N}\right]^2 - A, \tag{A1}$$

where $n = 1, 2, \dots, N-1$.

Liu (2010) demonstrated that the linear approximation with partition points which satisfies the non-linear equations $f'(X_n) = \frac{f(X_{n+1}) - f(X_{n-1})}{X_{n+1} - X_{n-1}}$, fits the target payoff more effectively than that with an equally-divided partition, where *f* denotes the target function.

We can derive Equation (A1) by applying mathematical induction and then setting $f(x) = \sqrt{A + x}$. Although Liu (2010) considered linear curve fitting, our non-tabulated results indicate that the performance of exponential curve fitting using the curvature partition also dominates the curve fitting approach using the equally-divided partition $X_n = \frac{(N-n)X_0 + nX_N}{N}$.

LEMMA. (MEAN AND VARIANCE) *The mean* $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ *and the variance* $\mathbb{V}_t^{\mathbb{Q}}[\cdot]$ *of the instantaneous variance* $V_{j,t+\tau}$, $j = 1,2$, *can be derived from*

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[V_{j,t+\tau}\right] &= \left[e^{-\kappa_j\tau}V_{j,t} + \theta_j(1-e^{-\kappa_j\tau})\right] + \lambda\left[\frac{\mu_v}{\kappa_1}(1-e^{-\kappa_1\tau})\right] I_{\{j=1\}}, \\
\mathbb{V}_t^{\mathbb{Q}}\left[V_{j,t+\tau}\right] &= \frac{\sigma_j^2}{\kappa_j}(1-e^{-\kappa_j\tau})\left(e^{-\kappa_j\tau}V_{j,t} + \frac{\theta_j}{2}(1-e^{-\kappa_j\tau})\right) \\
&\quad + \lambda\left[\frac{\sigma_1^2\mu_v}{2\kappa_1^2}(1-e^{-\kappa_1\tau})^2 + \frac{\mu_v^2}{\kappa_1}(1-e^{-2\kappa_1\tau})\right] I_{\{j=1\}},
\end{aligned}$$

where $I_{\{\cdot\}}$ *denotes the indicator function, and the mean and variance of* X_T *can be derived from* $\mu = B_1\mathbb{E}_t^{\mathbb{Q}}\left[V_{1,t+\tau}\right] + B_2\mathbb{E}_t^{\mathbb{Q}}\left[V_{2,t+\tau}\right]$ *and* $\delta^2 = B_1^2\mathbb{V}_t^{\mathbb{Q}}\left[V_{1,t+\tau}\right] + B_2^2\mathbb{V}_t^{\mathbb{Q}}\left[V_{2,t+\tau}\right]$.[43]

For each exponential curve, $\widetilde{f_n} = a_n + b_n e^{c_n x}$, $n = 1,2,\dots,N$, we fit the two endpoints and the midpoint of each sub-interval $\left[X_{n-1}^{(k)}, X_n^{(k)}\right]$ to solve the corresponding parameters (a_n, b_n, c_n) as follows:

$$\begin{aligned}
c_n &= \frac{2}{X_n^{(k)} - X_{n-1}^{(k)}} ln\left[\left(\sqrt{A + X_n^{(k)}} - \sqrt{A + M_n^{(k)}}\right) / \left(\sqrt{A + M_n^{(k)}} - \sqrt{A + X_{n-1}^{(k)}}\right)\right], \\
b_n &= \left(\sqrt{A + X_n^{(k)}} - \sqrt{A + X_{n-1}^{(k)}}\right) / \left(e^{c_n X_n^{(k)}} - e^{c_n X_{n-1}^{(k)}}\right), \qquad \text{(A2)} \\
a_n &= \sqrt{A + X_{n-1}^{(k)}} - b_n e^{c_n X_{n-1}^{(k)}},
\end{aligned}$$

where $M_n^{(k)} \equiv \frac{1}{2}\left(X_{n-1}^{(k)} + X_n^{(k)}\right)$ denotes the midpoint of $\left[X_{n-1}^{(k)}, X_n^{(k)}\right]$. Thus, all of the fitting points $\left\{X_0^{(k)}, M_1^{(k)}, X_1^{(k)}, M_2^{(k)}, X_2^{(k)}, \dots, M_N^{(k)}, X_N^{(k)}\right\}$ are uniquely determined for given levels of N and k.[44]

The payoff for the remaining range of $X_T \geq X_N^{(k)}$ is approximated by the terminal function $\widetilde{f_N}$, an approach which can reduce the fitting error, particularly when the tail probabilities are non-trivial.

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[43] The technical details are available on request from the authors.

[44] Although selecting the midpoint is the simplest way to solve the parameters, it is not the only way. If we choose $M_n^{(k)}$ according to the curvature rule, the corresponding parameters (a_n, b_n, c_n) are difficult to solve. However, as the pricing error rapidly converges to zero, the choice of $M_n^{(k)}$ is not a major issue here; nevertheless, the linear approximation has only two parameters that can be uniquely determined by the endpoints X_{n-1} and X_n.

Closed-Form Approximation

Following Duffie et al. (2000) and Bates (2000) to define the instrumental function, $G_{a,b}(y) \equiv \mathbb{E}_t^{\mathbb{Q}}\left[e^{a \cdot X_T} I_{\{b \cdot X_T \leq y\}}\right]$. Under the affine model, Equations (1)–(5), the $G_{a,b}$ function can be derived by the Fourier transform as:

$$G_{a,b}(y) = \frac{\psi(a; V_{1,t}, V_{2,t}, \tau)}{2} - \int_0^\infty \frac{1}{\pi v} Imag\left[\psi(a + ivb; V_{1,t}, V_{2,t}, \tau)e^{-ivy}\right] dv \,, \text{(A3)}$$

where $Imag[\cdot]$ denotes the imaginary part of a complex number, $i = \sqrt{-1}$, $\psi(z; V_{1,t}, V_{2,t}, \tau) \equiv \psi_1(z; V_{1,t}, \tau)\psi_2(z; V_{2,t}, \tau)$, and

$$\psi_j(z; V_t, \tau) = e^{\alpha_j(B_j z, \tau) + \beta_j(B_j z, \tau)V_t + \lambda \gamma_j(B_j z, \tau)}, \; j = 1, 2, \qquad \text{(A4)}$$

with

$$\begin{aligned} \alpha_j(z, \tau) &= -\frac{2\kappa_j \theta_j}{\sigma_j^2} ln\left[1 - \frac{\sigma_j^2}{2\kappa_j}(1 - e^{-\kappa_j \tau})z\right], \; j = 1, 2, \\ \beta_j(z, \tau) &= \frac{2\kappa_j z}{\sigma_j^2 z + \left(2\kappa_j - \sigma_j^2 z\right)e^{\kappa_j \tau}}, \; j = 1, 2, \\ \gamma_1(z, \tau) &= \frac{2\mu_v}{2\kappa_1 \mu_v - \sigma_1^2} ln\left[1 + \frac{(2\kappa_1 \mu_v - \sigma_1^2)z}{2\kappa_1(1 - \mu_v z)}(1 - e^{-\kappa_1 \tau})\right], \end{aligned} \qquad \text{(A5)}$$

and $\gamma_2(z, \tau) = 0$. It should be noted that when $V_{2,t} = \theta_2 = 0$ (that is, the SVCJ model), the analytic solutions, Equations (A4) and (A5), reduce to those of Lian and Zhu (2013).

The risk-neutral probability of a VIX option which is expiring 'in the money' $Pr_t^{\mathbb{Q}}(VIX_T \geq K)$ can subsequently be exactly solved by $G_{0,-1}(A - K_0^2)$, and the risk- neutral expectation of the substitute payoff can be expressed as:

$$a_1 G_{0,-1}\left(-X_0^{(k)}\right) + b_1 G_{c_1,-1}\left(-X_0^{(k)}\right)$$

$$+ \sum_{n=1}^{N-1}\left[(a_{n+1} - a_n)G_{0,-1}\left(-X_n^{(k)}\right) + b_{n+1}G_{c_{n+1},-1}\left(-X_n^{(k)}\right) - b_n G_{c_n,-1}\left(-X_n^{(k)}\right)\right]. \qquad \text{(A6)}$$

We obtain the closed-form approximation by expanding the $G_{a,b}$ functions in Equation (A6) by Equation (A3) and rewriting Equation (9). It should be noted that each $G_{a,b}$ includes one integral that can be efficiently computed, whilst the approximation for VIX options requires 3N integrals.[45] As a result, in Proposition 1, we rearrange the formula to become a single integral formula only, which yields a more computationally efficient result.

Appendix B. Numerical Analysis

In order to examine the accuracy of our approximations, we first of all construct the exact solution (Exact) using a time-consuming double-integral method which is similar to the benchmark of Cheng et al. (2012). By definition, the VIX option can be expressed as an integral containing the conditional probability density function (p.d.f.) of the instantaneous variance $f(X_T|X_t)$, expressed as:

$$C_t(K,\tau,r) \equiv 100e^{-r\tau}\int_{K_0^2-A}^{+\infty}\left(\sqrt{A+X_T}-K_0\right)f(X_T|X_t)dX_T. \tag{B1}$$

In particular, the conditional p.d.f. of X_T can be solved by:

$$f(X_T|X_t) \equiv \frac{d}{dy}Pr_t^{\mathbb{Q}}(X_T \le y) = \frac{1}{\pi}\int_0^{\infty} Real\left[\psi(iv)e^{-ivy}\right]dv, \tag{B2}$$

where $Real[\cdot]$ is the real part of a complex number. For simplicity, $\psi(z; V_{1,t}, V_{2,t}, \tau)$ is denoted by $\psi(z)$ in the present study. The benchmark value can then be constructed via numerical methods, such as the Simpson method.[46]

For the approximation error, we consider two measurements, the root mean- squared error (RMSE) and root mean-squared relative error (RMSRE) defined as:

$$\text{RMSE} = \sqrt{\tfrac{1}{n}\textstyle\sum_{j=1}^{n}\left(C_j - C_j^*\right)^2}, \tag{B3}$$

$$\text{RMSRE} = \sqrt{\tfrac{1}{n}\textstyle\sum_{j=1}^{n}\left(\frac{C_j - C_j^*}{C_j^*}\right)^2}, \tag{B4}$$

where n is the number of the observations, C_j denotes the approximated values and C_j^* denotes the benchmark values. To analyze the computational efficiency, we report the total

[45] Specifically, the exponential approximation, Equation (A6), requires $3N-1$ integrals, with the last term in Equation (9), $Pr_t^{\mathbb{Q}}(VIX_T \ge K)$, requiring one integral.

[46] To calculate the improper integral in (B1), we take the upper limit of the integral as $\mu + 20\delta$, which is sufficiently large, and adopt the Simpson method with 20,000 subintervals of equal length. To calculate the improper integrals in $f(X_T|X_t)$ and equation (B2), we choose the upper limit of the integral as 10^4.

computing time (Time) obtained by the Matlab package on Microsoft Windows 7 based on Intel® Core™ i7-2600M CPU @ 3.40GHz and 8.00GB RAM.

Table B1 reports the VIX option prices under the SVCJ model with parameters based on Duffie et al. (2000): κ_1=3.5, θ_1=0.01, $\sigma_1 \in$ {0.10, 0.15, 0.20}, $\lambda \in$ {0.4, 0.5, 0.6}, $\bar{\mu} = -0.1$, μ_v=0.05, σ_p=0.0001, $\rho_J = -0.4$, $V_{1,t}$=0.008. The other parameters are set as $r = 0.03$, $\tau = 1$, and moneyness, $m \in \{0.85, 1.00, 1.15\}$. For comparison, Table B1 also uses the Lian and Zhu (2013) pricing formula, which provides a closed-form expression for VIX option prices under the SVCJ model; however, this formula involves a complex error function which still requires numerical approximation.[47]

<Table B1 is inserted about here>

The results in Table B1 show that the RMSREs for our approximations of (k, N) = (3,1), (3,2), (3,4) and (3,8) are respectively 4.42%, 0.97%, 0.39%, and 0.29%, and the RMSREs for approximations of (k, N) = (6,1), (6,2), (6,4) and (6,8) are respectively 8.67%, 1.76%, 0.19% and 0.02%. As regards the computational time for calculating 27 VIX option prices, our respective (k, N) = (3,1), (3,2), (3,4) and (3,8) approximations take 0.04, 0.08, 0.18 and 0.35 seconds, whilst the (k, N) = (6,1), (6,2), (6,4) and (6,8) approximations take 0.04, 0.12, 0.31 and 0.72 seconds.

For a given k, the role of N is a trade-off between accuracy and computational burdens. As the components in the VIX option formula, Equation (13), are proportional to the number of exponential curves, the growth in computing time is in the order of N; however, the pricing error decays rapidly. As compared to Lian and Zhu (2013), Table B1 shows that our approximation with (k, N) = (6,4) is a more efficient method of valuing VIX options since our approximation performs with similar accuracy, but requires much less computational time,

[47] We use the Matlab function "mfun('erf',-)" to calculate the complex error function, selecting $\phi_R = 1$. It should be noted that in the Lian and Zhu (2013) formula, ϕ_R is a positive number, as defined in their Equation (A7), and is theoretically independent of the VIX option price.

operating as much as eighty times faster.

The VIX option prices under two-variance models require more computing time due to the more complex integrands. Our non-tabulated results show that to calculate 27 VIX option prices under the 2-SVJ model with parameters based on Bates (2000), our approximations with (k, N) = (3,1), (3,2), (3,4) and (3,8) take 0.09, 0.28, 0.62, and 1.21 seconds and our approximations with (k, N) = (6,1), (6,2), (6,4) and (6,8) take 0.09, 0.41, 1.16, and 2.54 seconds, respectively.[48] Since the empirical estimation in this study involves a large number of data observations and minimizes highly complicated error functions, our proposed method offers an ideal approach for this type of empirical analysis.

Appendix C. VIX Futures Single-variance Models Term Structure

In an attempt to theoretically explain why single-variance models cannot generate the hump-shaped pattern for the term structure of VIX futures, we follow Bates (2006) to use the Taylor expansion to observe the theoretical VIX futures:

$$\mathbb{E}_t^{\mathbb{Q}}[VIX_{t+\tau}] \approx 100\sqrt{\mathbb{E}_t^{\mathbb{Q}}[A + B_1 V_{1,t+\tau}]}\left[1 - \frac{\mathbb{V}_t^{\mathbb{Q}}[A+B_1V_{1,t+\tau}]}{8\left(\mathbb{E}_t^{\mathbb{Q}}[A+B_1V_{1,t+\tau}]\right)^2}\right]. \quad \text{(C1)}$$

By the Lemma shown in Appendix A, we have:

$$\mathbb{E}_t^{\mathbb{Q}}[V_{1,t+\tau}] = \left[\theta_1 + \lambda\frac{\mu_v}{\kappa_1} + e^{-\kappa_1\tau}\left(V_{1,t} - \theta_1 - \lambda\frac{\mu_v}{\kappa_1}\right)\right]. \quad \text{(C2)}$$

The expectation $\mathbb{E}_t^{\mathbb{Q}}[A + B_1 V_{1,t+\tau}]$ is strictly increasing (decreasing) in τ when $V_{1,t}$ is less (greater) than $\theta_1 + \lambda\mu_v/\kappa_1$, with the variance $\mathbb{V}_t^{\mathbb{Q}}[A + B_1 V_{1,t+\tau}]$ increasing in the short run, thus forcing VIX futures to decrease;[49] however, the variance effect disappears in the long-run due to the rapidly growing denominator. This explains why single-variance models

[48] The RMSREs for our (k, N) = (3,1), (3,2), (3,4), and (3,8) approximations are 3.38%, 0.98%, 0.56% and 0.44%, whilst those for our (k, N) = (6,1), (6,2), (6,4) and (6,8) approximations are 6.59%, 1.11%, 0.15% and 0.06%, respectively. For simplicity, we slightly adjust the parameters of Bates (2000) to: κ_1=1, θ_1=0.01, $\sigma_1 \in \{0.50,\ 0.55,\ 0.60\}$, κ_2=2, θ_2=0.02, $\sigma_2 \in \{0.30,\ 0.35,\ 0.40\}$, λ=1, $\bar{\mu} = -0.05$, σ_p=0.1, $V_{1,t}$=0.010 $V_{2,t}$=0.015. The parameters estimated by Bates (2000) were κ_1=0.91, θ_1=0.010989, σ_1=0.582, κ_2=1.76, θ_2=0.022727, σ_2=0.346, λ_0=0.9035, $\bar{\mu} = -0.057$, σ_p=0.102, $V_{1,t}$=0.00963, and $V_{2,t}$=0.01352.

[49] When τ is close to 0, the variance $\mathbb{V}_t^{\mathbb{Q}}[A + B_1 V_{1,t+\tau}]$ can be simplified as $B_1^2(\sigma_1^2 V_{1,t} + 2\lambda\mu_v^2)\tau + O(\tau)$.

with jumps cannot capture the hump-shaped pattern, even if the SVCJ model uses more parameters than the 2-SV model, although the single- variance models with jumps can generate the U-shaped pattern when $V_{1,t}$ is smaller than, but close to, $\theta_1 + \lambda\frac{\mu_v}{\kappa_1}$. In this case, the variance effect dominates the short-run mean effect, but is dominated by the long-run mean effect.

	$\mathbb{E}_t^{\mathbb{Q}}[A + B_1 V_{1,t+\tau}]$	$\mathbb{V}_t^{\mathbb{Q}}[A + B_1 V_{1,t+\tau}]$	$\mathbb{E}_t^{\mathbb{Q}}[VIX_{t+\tau}]$
$V_{1,t} < \theta_1 + \lambda\mu_v/\kappa_1$	Increasing in τ	Increasing in τ	Upward-sloping/U-shaped
$V_{1,t} > \theta_1 + \lambda\mu_v/\kappa_1$	Decreasing in τ	Increasing in τ	Downward-sloping

Table B1. VIX Option Prices under the SVCJ Model

This table reports the VIX option prices generated by the exact solution (Exact, as detailed in Appendix B), our approximations under the SVCJ model (which cover k standard deviations using N exponential curves) and Lian and Zhu (2013). The parameters $(\kappa_1, \theta_1; \bar{\mu}, \sigma_p, \mu_v, \rho_J; V_{1,t}, r, \tau)$ which are based on Duffie et al. (2000), are set as (3.5, 0.01; –0.1, 0.0001, 0.05, –0.4; 0.008, 0.03, 1), with m denoting the moneyness K/F_t. RMSE and RMSRE respectively refer to the root mean-squared error and the root mean-squared relative error. The total computing time (Time) is obtained by the Matlab package on Microsoft Windows 7 and based on Intel® Core™ i7-2600 CPU @ 3.40GHz and 8.00GB RAM.

(σ_1, λ, m)	Exact	Our Model with k = 3				Our Model with k = 6				LZ
		N = 1	N = 2	N = 4	N = 8	N = 1	N = 2	N = 4	N = 8	
(0.10,0.4,0.85)	3.6470	3.4182	3.6053	3.6373	3.6401	3.2068	3.5485	3.6389	3.6468	3.6503
(0.10,0.4,1.00)	2.1320	2.0346	2.1072	2.1224	2.1252	1.9518	2.0915	2.1264	2.1315	2.1367
(0.10,0.4,1.15)	1.2178	1.1787	1.2034	1.2094	1.2112	1.1594	1.2069	1.2161	1.2175	1.2239
(0.10,0.5,0.85)	3.9868	3.7426	3.9425	3.9773	3.9801	3.5123	3.8811	3.9776	3.9864	3.9876
(0.10,0.5,1.00)	2.3735	2.2707	2.3480	2.3641	2.3670	2.1773	2.3304	2.3673	2.3728	2.3763
(0.10,0.5,1.15)	1.3953	1.3547	1.3812	1.3873	1.3891	1.3293	1.3837	1.3935	1.3949	1.3999
(0.10,0.6,0.85)	4.3112	4.0555	4.2656	4.3022	4.3049	3.8077	4.2007	4.3014	4.3108	4.3112
(0.10,0.6,1.00)	2.6050	2.4980	2.5795	2.5960	2.5989	2.3943	2.5601	2.5985	2.6043	2.6073
(0.10,0.6,1.15)	1.5625	1.5207	1.5488	1.5549	1.5566	1.4890	1.5502	1.5606	1.5620	1.5667
(0.15,0.4,0.85)	3.5990	3.3884	3.5618	3.5889	3.5922	3.1813	3.5122	3.5912	3.5984	3.6014
(0.15,0.4,1.00)	2.2428	2.1474	2.2207	2.2339	2.2364	2.0564	2.2052	2.2386	2.2424	2.2473
(0.15,0.4,1.15)	1.3729	1.3294	1.3585	1.3648	1.3666	1.2996	1.3594	1.3711	1.3726	1.3781
(0.15,0.5,0.85)	3.9320	3.7065	3.8936	3.9236	3.9269	3.4807	3.8388	3.9247	3.9327	3.9338
(0.15,0.5,1.00)	2.4722	2.3709	2.4492	2.4634	2.4659	2.2700	2.4316	2.4673	2.4715	2.4749
(0.15,0.5,1.15)	1.5365	1.4914	1.5222	1.5286	1.5304	1.4562	1.5222	1.5344	1.5360	1.5404
(0.15,0.6,0.85)	4.2554	4.0156	4.2139	4.2459	4.2493	3.7720	4.1549	4.2459	4.2546	4.2554
(0.15,0.6,1.00)	2.6928	2.5872	2.6696	2.6844	2.6869	2.4766	2.6502	2.6876	2.6921	2.6950
(0.15,0.6,1.15)	1.6915	1.6455	1.6776	1.6841	1.6858	1.6047	1.6767	1.6894	1.6910	1.6950
(0.20,0.4,0.85)	3.5656	3.3705	3.5315	3.5557	3.5590	3.1666	3.4880	3.5580	3.5648	3.5678
(0.20,0.4,1.00)	2.3463	2.2530	2.3257	2.3377	2.3400	2.1547	2.3110	2.3425	2.3458	2.3498
(0.20,0.4,1.15)	1.5235	1.4775	1.5096	1.5158	1.5175	1.4380	1.5089	1.5217	1.5231	1.5278
(0.20,0.5,0.85)	3.8884	3.6773	3.8520	3.8787	3.8822	3.4553	3.8033	3.8801	3.8876	3.8887
(0.20,0.5,1.00)	2.5649	2.4659	2.5438	2.5566	2.5590	2.3582	2.5269	2.5607	2.5643	2.5670
(0.20,0.5,1.15)	1.6772	1.6294	1.6633	1.6697	1.6714	1.5849	1.6617	1.6752	1.6767	1.6808
(0.20,0.6,0.85)	4.2016	3.9775	4.1637	4.1924	4.1960	3.7379	4.1106	4.1928	4.2009	4.2009
(0.20,0.6,1.00)	2.7760	2.6726	2.7547	2.7681	2.7705	2.5556	2.7359	2.7716	2.7755	2.7776
(0.20,0.6,1.15)	1.8234	1.7746	1.8098	1.8163	1.8180	1.7250	1.8073	1.8214	1.8230	1.8265
RMSE		0.1455	0.0278	0.0087	0.0062	0.2899	0.0601	0.0058	0.0006	0.0032
RMSRE		0.0442	0.0097	0.0039	0.0029	0.0867	0.0176	0.0019	0.0002	0.0019
Time (sec.)		0.04	0.08	0.18	0.35	0.04	0.12	0.31	0.72	25.18