The Closed-form Pricing Formula for a Risky Asset When Its Risky Factors Follow Gamma Distributions

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Abstract:

This paper constructed a pricing model for the asset with multi-risks by specifying the risky factors (i.e., interest rate and termination hazard rates) to follow gamma distributions. The model not only avoids the possibility of the termination hazard rate taking an irrational (i.e., negative) value, but it also makes it easier to derive a closed-form valuation formula for a risky asset. Our model can also effortless apply because the parameters of the gamma distribution can easily be estimated from market data. An example using Taiwanese bond data illustrates how the model can be utilized for practical applications. To facilitate understanding of how accurately the different models price risky assets, we compare their pricing errors for different hazard rate specifications assuming normal and gamma distributions. The results show the valuation with the gamma distribution is better than with normal distribution, and also reveal our closed-form formula is realistic and accurate in its applications. Therefore, it should help market participants more accurately price risky securities and manage complicated portfolios effectively.

Keywords: Hazard Rate, Gamma Distribution, Pricing, Closed-form Formula

1. Introduction:

Because of the recent crises involving subprime mortgages in the U.S. and the government bond markets in Europe, researchers and market practitioners have been paying a great deal of attention to studies of the specific termination risks (e.g., default and prepayment) associated with risky assets (e.g., bonds and mortgages). Managing these is an important but difficult assignment for financial institutions. Since a closed-form pricing formula can adequately appraise asset values from correlated multiple risk sources, it can greatly reduce the hard works of managing risks. By utilizing a suitable closed-form pricing formula, portfolio managers and financial institutions can not only accurately price risky assets but also perform efficient hedging analyses.[1] Therefore, it is important and even necessary for market practitioners and financial institutions to derive a closed-form pricing formula for risky assets. The main goal of this paper is to provide a general and accurate valuation model to accomplish this task.

To assess these probabilities of risky events (e.g., default, prepayment, liquidity risk) occurring prior to contract maturity, researchers usually utilize two models: the structural-form model (see, Merton 1974, Kau, Keenan, Muller III and Epperson, 1993; Yang, Buist and Megbolugbe, 1998; Ambrose and Buttimer, 2000; Azevedo-Pereira, Newton and Paxson, 2003) and the reduced-form model (see, Jarrow and Turnbull, 1995; Jarrow, 2001). Because it is easier to derive a closed-form formula by using the reduced-form model than the structural-form model, researchers have increasingly chosen the reduced-form model for pricing risky securities and

[1] Some of the other advantages of a closed-form formula are as follows: (1) it helps us better appreciate how sensitive asset values are to changes in relevant factors; (2) it improves calculation efficiency; (3) it provides basic building blocks that financial institutions can use to price complicated financial products (Liao, Tsai and Chiang, 2008).

determining the probability of termination (Liao, Tsai, and Chiang, 2008; Tsai, Liao, and Chiang, 2009; Tsai and Chiang, 2012). For our study, we also chose the reduced-form model for this purpose.

The reduced-form model usually specifies the unpredictability of risky events as exogenous random variables that follow a Poisson distribution (Bielecki and Rutkowski, 2001). The analyses of default risk depend mainly on how the termination hazard rate is appropriately specified. Nowadays, the termination hazard rate for a risky asset is usually specified as following one of the following three distributions: the normal distribution, the log-normal distribution, and the non-central chi-square distribution. Each specification has its specific advantages and problems when pricing a risky asset.

The termination hazard rate follows a normal distribution if it is specified as a Vasciek-Form (Vasicek, 1977). As shown by Jarrow (2001) and by Liao, Tsai and Chiang (2008), it is easiest to derive a closed-form formula for a risky asset with multi-risks under this construction. To depict reasonably the risks that an asset incurs, the hazard rates associated with the various risks have to be modeled as the function of state variables (i.e., interest rate and stock return). It is also necessary to consider the correlations among the hazard rates in the pricing model. For example, the decisions of prepayment and default for mortgage obligations are viewed as the correlated competing risks and thus are jointly estimated when investigating mortgages and mortgage-based securities (Han and Hausman, 1990; Sueyoshi, 1992; McCall, 1996; Deng, Quigley and Van Order, 2000; Kuznetsovski and Hwang, 2010). Nowadays, among the above mentioned three distributions, this is only possible to more easily deal with this situation if hazard rates are assumed to follow a normal distribution. However, some scholars have argued that such a specification yields hazard rates with unrealistic values (e.g., negative values), which in turn leads to inaccurate pricing of the securities. To avoid such problems, additional constraints are needed that restrict the range of values the hazard rate can assume (see Liao et al., 2008).

In what is usually called the Cox hazard rate model for analyzing termination probabilities for risky securities, the termination hazard rates are specified as following a log-normal distribution (see Schwartz and Torous, 1989; 1993). The advantage of this specification is that it ensures the hazard rate will always take a positive value, thereby avoiding incorrect prices for the risky securities. However, the adoption of Cox's model in a pricing framework results in a "double exponential" form,[2] which induces an infinite expectations for the accumulated factors (see, Miltersen, Sandmann and Sondermann, 1997). Therefore, a closed-form formula for a risky asset cannot be obtained by specifying a lognormal distribution.

Many scholars, such as Duffie (2005), have used the non-central chi-square distribution (also defined as the CIR-Form, see Cox, Ingersol and Ross, 1985) to describe the distribution of hazard rates. Compared with the two distributions quoted above, the advantages of the chi-square specification are: (1) the hazard rate always takes a positive value and (2) a closed-form formula for the risky asset can be obtained. However, incorporating this specification in the valuation model seems to make the derivation procedure more complicated; especially, it is quite difficult to derive the closed-form formula when the multi-risks and their correlations are included in the valuation model. Also, the parameters for the non-central chi-square distribution are difficult to estimate from historical market data. These disadvantages greatly restrict the application of this specification for market participants.

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[2] The term "double exponential" is an exponential function of exponential function.

In short, market participants will be dissatisfied with the assumption of a normal distribution because the specification of the hazard rates may cause incorrect pricing, with the assumption of a log-normal distribution because it is difficult to obtain the closed-form solution, and with the assumption of a non-central chi-square distribution because the model is too complicated to be used for empirical application studies. Thus, a more appropriate model for valuating a risky asset must achieve the following goals: (1) It must avoid the generation of unreasonable values for the hazard rate, (2) it must be easy to obtain a closed-form formula from it, and (3) it must be convenient for practical applications. For simultaneously satisfying all these purposes, the main purpose of this paper is to support a model that can achieve all these goals for market participants. We derived a general closed-form formula for a risky asset based on the assumption of that hazard rates follow a gamma distribution, as shown by Hall (2000).

Our model has the following advantages: (1) Because the hazard rate of a risk event is always a positive value assuming a gamma distribution, the pricing is always reasonable; thus, our specification is better than specifying a normal distribution. (2) The pricing model is stable and more rational in practical applications, because the gamma distribution is approximately stationary for a population that fluctuates around a stable equilibrium (Dennis and Patil, 1984). (3) The gamma distribution is quite general; in theory, it can approximate to the normal and chi-square distributions (Johnson, Kotz and Balakrishnan, 1994), meaning that our valuation model is more general than the traditional models. (4) A moment-generating function is useful when deriving a closed-form pricing formula for securities (Chiang and Tsai, 2010). The gamma distribution always has a moment-generating function that can be used to derive a closed-form pricing formula for risky securities, making our model better than Cox's hazard rate model and the CIR model. (5) It is very easy to estimate the parameters of the gamma distribution, which therefore makes it very easy for market participants to apply the model in practice. (6) Our model is quite flexible, because it can be extended to apply to cases with correlated multi-risk sources, such as correlated and simultaneously considered prepayment and default risks for pricing a mortgage.

Because of the above advantages, we recommend that when evaluating a risky asset, market participants and financial scholars assume a gamma distribution for hazard rates rather than the specifications used in traditional research. To the best of our knowledge, our model is the first that derives a closed-form formula for assets with multiple correlated risks while specifying that hazard rates follow such distribution.

To demonstrate how this valuation framework can be used for practical applications, we used data obtained from Taiwanese bond market to illustrate how we estimate the parameters of our model. We also show how our pricing formula can be used to valuate these risky securities. To clearly illustrate which hazard-rate distribution prices the risky asset most accurately, we compare the pricing errors for a normal distribution and a gamma distribution.

The remainder of this paper is organized as follows. Section 2 illustrates our valuation framework, including the identification of the risky asset components and our treatment of the asset's multiple risks. This section also explains the specifications for the hazard-rate functions. Finally, it presents a closed-form solution for the risky asset. The third section describes the empirical methods for estimating the necessary parameters from historical data and presents the sensitivity analyses. The final section summarizes our findings.

2. The Model

We construct a model for the pricing of a risky asset that includes multi-risks. In the definition of the termination risk, a risky asset holder would incur a loss if the asset is terminated early. For example, that liquidity risk is the risk that a given security or asset cannot be traded quickly enough in the market to prevent a loss (or obtain the required profit). In such a situation, if investors need to quickly sell their holdings, they should sell at a discount with respect to the current market price. Another example is the default risk, that the amount that the risky asset holder may lose in an investment if the asset is defaulted prior to the maturity. The termination risks may include the default risk, liquidity risk[3] and prepayment risk, and so on.

We assume the risky asset promise to pay $c(u)$ dollar at time u and M dollar at maturity date T, where $t < u \leq T$. We assume the payment for the risky asset can be terminated by K kinds of risks. At each time point, two situations appear for the risky asset: terminated by one kind of risk, or to maintain. When the risky asset is early terminated, the risky asset holder may get a loss from risk.

For analyzing the expected loss, it must to identify the loss rate from risks and termination probabilities for risks. Let us denote the random variables τ_i as the terminated time of risk i during the period from t to T, for $i = 1, 2, \cdots, K$. We use τ to represent the time when the loan payment is first terminated, where $\tau = \min(\tau_1, \tau_2, \cdots, \tau_K)$. If $\tau = \tau_i$, the loan is terminated by the risk i, then the cash flow is $M\pi_i(\tau)$, where $\pi_i(\tau)$ is the fractional recovery rate.[4] Otherwise, if the loan doesn't be terminated until maturity date, the cash flow is equal to the $c(u)$ at each time u and M at maturity date. The value of such risky asset is therefore denoted

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[3] If one wants to analyze an expected liquidity risk, one must identify its expected loss rate and the probability of an early sell. A similar idea has been suggested by Duffie and Singleton (1999).

[4] We let M dollar to be the denominator of recovery rate is.

as (Jarrow and Turnbull, 1992; Bielecki and Rutkowski, 2001; Tsai, Liao and Chiang, 2009; Tsai and Chiang, 2012):

$$P_1(t,T) = E[\sum_{i=1}^{K} M\pi_i(\tau)\exp(-\int_t^{\tau_i} r(s)ds)I_{\{\tau=\tau_i,\tau\le T\}}$$

$$+\int_t^T c(u)\exp(-\int_t^u r(s)ds)I_{\{\tau>u\}}du + M\exp(-\int_t^T r(s)ds)I_{\{\tau>T\}}], \qquad (1)$$

where

$P_1(t,T)$ is the value of risky asset at time t,

$E[\cdot]$ is an expected operator,

$r(s)$ is the instantaneous default-free short-term interest rate at time s, and

$I_{\{\}}$ is the indicator function.

The first term on the right side of Equation (1) is the expected current value of the risky asset if risk event i occurs. The second term is the expected current survival value of the risky asset which is not terminated during contract period. The third term is the expected current survival value of the risky asset which is not terminated until maturity time T. We can also express the valuation formula as follows (Bielecki and Rutkowski, 2001):

$$P_1(t,T) = \int_t^T ME[\exp(-\int_t^u r(s)ds)(\sum_{i=1}^{K}\pi_i(u)f_i(t,u))]du$$

$$+\int_t^T c(u)E[\exp(-\int_t^u r(s)ds)G(t,u)du] + ME[\exp(-\int_t^T r(s)ds)G(t,T)], \quad (2)$$

$G(t,u)$ is the survival probability until time u, and

$f_i(t,u)$ is termination probability density function for risk i at time u.

Under the reduced-form model, we assume a stochastic hazard rate $h_i(u)$ for risk i at time u. Then, the survival function and the probability density function for risk i can be obtained. Given the stochastic hazard rates $h_i(u)$, the survival function can generally be defined as follows:

$$G(t,u) = \exp(-\int_t^u (\sum_{i=1}^{K} h_i(u))du). \tag{3}$$

The termination probability density function for risk i can be described as follows:

$$f_i(t,u) = h_i(u)G(t,u). \tag{4}$$

Under the reduced-form model, the value of the risky asset at time t can be rewritten as follows (cf. Bielecki and Rutkowski, 2001; Liao et al., 2008; Tsai and Chiang, 2012):

$$\begin{aligned} P_1(t,T) = &\int_t^T ME[\sum_{i=1}^{K} \pi_i(u)h_i(u)\exp(-\int_t^u (r(s)+\sum_{i=1}^{K} h_i(s))ds)]du \\ &+ \int_t^T c(u)\ E[\exp(-\int_t^u (r(s)+\sum_{i=1}^{K} h_i(s))ds)]du \\ &+ ME[\exp(-\int_t^T (r(s)+\sum_{i=1}^{K} h_i(s))ds)]. \end{aligned} \tag{5}$$

According to Equation (5), to derive a closed-form formula for the risky asset first must derive the closed-form solutions for the following two terms:

$$E[\int_t^T \pi_i(u)h_i(u)\exp(-\int_t^u (r(s)+\sum_{i=1}^{K} h(s))ds)du], \text{ and} \tag{6}$$

$$E[\exp(-\int_t^u (r(s)+\sum_{i=1}^{K} h_i(s))ds)]. \tag{7}$$

These two expected values should be solved under the assumption that recovery rates, the termination hazard rates and interest rate follow stochastic processes. However, in traditional research, the recovery rate, which is estimated from market data, has usually been treated as constant or a deterministic variable (Jarrow and Turnbull, 1995). Nevertheless, there is evidence that whether the recovery rate is assumed to be stochastic or constant has no significant impact on the pricing of risky products such as mortgages (Jokivuolle and Peura, 2003). In focusing our investigation on the hazard rate, we assume that the recovery rate is a constant (i.e., $\pi_i(u) = \pi_i$). Based on this assumption, we need to derive the following solutions:

$$\Omega^G(t,u) = E[\exp(-\int_t^u (r(s) + \sum_{i=1}^K h_i(s))ds)], \text{ and} \tag{8}$$

$$\Omega_i^f(t,u) = E[h_i(u)\exp(-\int_t^u (r(s) + \sum_{i=1}^K h(s)ds)]. \tag{9}$$

If we know the distribution of $h_i(u)$, we can get solutions for Equations (8) and (9). Then, we can derive a closed-form formula for the price of the risky asset with multi-risks. In traditional studies, the default hazard rates $h_i(u)$ are generally specified as following one of three distributions: normal, lognormal, or non-central chi-square.

To provide for easy calculation of the closed-form formula, accuracy of pricing, and ease of practical applications, we model hazard rates to follow a gamma distribution as follows:[5]

$$\log(h_i(u)) = E[\log(h_i(u))] + \log(W_i(u)), \tag{10}$$

where

$W_i(u)$ is a random variable for risk i at time u, which follows a gamma distribution.

In traditional studies, the hazard rate is always assumed to be influenced by

[5] The specification for a stochastic process following gamma distribution can be found in Dennis and Patil (1984).

interrelated state variables such as the risk-free interest rate, and the stock price index. In empirical procedures, when using Cox's hazard model, the estimation of $E[h_i(u)]$ can be described as follows:

$$E[h_i(u)] = h_{0,i}(u)\exp(A_i' X(u)); \qquad (11)$$

where

$h_{0,i}(u)$, which is deterministic, denotes the baseline hazard rate for risk i at time u;

$X(u)$ stands for a set of the state variables in the regression;

A_i is a vector of the coefficients of the hazard rate function for risk i, with each coefficient denoting the relative magnitude of the effect of each state variable on the hazard rate.

The termination risks may be influenced by common factors such as interest rates and the stock price index. To accurately price the risky asset with multi-risks, we also have to consider the influenced effects from common factors and correlations among the hazard rates. We let

$$W_i(u) = \Lambda'_i V^0(u) + V_i(u), \text{ for } i = 1,\cdots,K, \qquad (12)$$

where

$V^0(u) = [V_1^0 \quad V_2^0 \quad \cdots \quad V_N^0]'$ is a vector for N kinds of common factors that affects all $W_i(u)$;

$\Lambda_i = [\xi_1^i \quad \xi_2^i \quad \cdots \quad \xi_N^i]'$ is a vector of influential effects;

ξ_n^i represents the influential effect of n -th common factor, $n = 1, \cdots, N$, on the i -th hazard rate; and

$V_i(u)$ is the specific factor for risk i .

We let V_n^0 and $V_i(u)$ follow a gamma distribution with parameters (α_n^0 , β_n^0) and (α_i , β_i), respectively. In addition, all elements in $V^0(u)$, $V_i(u)$ and $V_k(u)$, for $i \neq k$, are mutually independent random variables. For a gamma distribution with parameters (α , β), the expected value and variance are $\alpha\beta$ and $\alpha\beta^2$, respectively. We let $Var(V^0)$ be the variance-covariance matrix of V^0 . According to Equation (12), the variance of $W_i(u)$ is $\Lambda'_i Var(V^0)\Lambda_i + \alpha_i\beta_i^2$. Moreover, $\Lambda'_i Var(V^0)\Lambda_k$ is the covariance between $W_i(u)$ and $W_k(u)$. Therefore, one can calculate the correlation between $W_i(u)$ and $W_k(u)$ by their covariance and variances. In terms of these properties, we can discuss the influence of the correlation among the hazard rates on the price of the risky asset. If we assume that the risk i is independent with other risks, then let $\Lambda_i = 0$.

In general, the interest rate is usually treated as a common factor that affects the termination hazard rates. Since the interest rate is always a positive value, we can also model it to follow a gamma distribution.[6] Moreover, we let the interest rate be the first common factor in our model. We express the interest rate as follows:

$$\log(r(u)) = E[\log(r(u))] + \log(V_1^0(u)) , \tag{13}$$

For simplicity, we let the expected value of interest $E[r(u)]$ and the expected value of termination hazard rate $E[h_i(u)]$ be denoted as $\Phi_1^0(u)$ and $\Phi_i(u)$ in the later

[6] The interest rate has been assumed to follow a gamma distribution in traditional researches (Heston 2007).

expression, respectively.

If $V(u)$ (i.e., $V^0(u)$ and $V_i(u)$) follows a gamma distribution with parameters α and β, the probability density function of $V(u)$ can be described as $p(v|\alpha,\beta)=\dfrac{\beta^{\alpha}v^{\alpha-1}\exp(-\beta v)}{\Gamma(\alpha)}$, and the moment-generating function of $\Phi(u)V(u)$ can be expressed as:

$$E[\exp(\Phi(u)V(u))]=\int_0^{\infty}\exp(-(\Phi(u))v)\times p(v)dv,=(1+\frac{\Phi(u)}{\beta})^{-\alpha}. \quad (14)$$

In light of the above results, we can easily derive the closed-form formulas for $\Omega^G(t,T)$ and $\Omega_i^f(t,u)$. They are (see Appendix A):

$$\Omega^G(t,T)=\exp(-\int_t^T\delta(s)ds), \text{ and} \quad (15)$$

$$\Omega_i^f(t,u)=g_i(u)du\exp(-\int_t^u\delta(s)ds), \quad (16)$$

where

$$\delta(s)=\sum_{n=1}^{N}\alpha_n^0\ln(1+\frac{C_n(s)}{\beta_n^0})+\sum_{i=1}^{K}\alpha_i\ln(1+\frac{\Phi_i(s)}{\beta_i}),$$

$$g_i(u)=\sum_{n=1}^{N}\frac{\alpha_n^0C_n(u)}{\beta_n^0+C_n(u)}+\frac{\alpha_i\Phi_i(u)}{\beta_i+\Phi_i(u)},$$

$$C_1(s)=\Phi_1^0(s)+\sum_{i=1}^{K}\xi_1^i\Phi_i(s) \text{ and } C_l(s)=\sum_{i=1}^{K}\xi_l^i\Phi_i(s), \text{ for } l=2,\cdots,N.$$

Finally, the closed-form formula for the risky asset with K kinds of risks can be solved as follows:

$$P_1(t,T)=M\int_t^T\sum_{i=1}^{K}\pi_ig_i(u)\exp(-\int_t^u\delta(s)ds)du$$

$$+\int_t^Tc(u)\exp(-\int_t^T\delta(s)ds)du+M\exp(-\int_t^T\delta(s)ds). \quad (17)$$

If we can estimate the expected values of termination hazard rates and the parameters of the gamma distributions, the price of the risky asset can be obtained for practical applications. In the following section, we show how one can use historical data to estimate the necessary parameters from our model.

3. Empirical Method and Results

In this example, we let the risky asset be a zero coupon bond (ZCB) with one dollar face value. We use Taiwanese market data to show how one can use historical data to estimate the necessary parameters for application. For simplicity, we assume that the only risk associated with the ZCB is default. Yield data are used to calculate the default hazard rate of the defaultable ZCB. The daily data for the four ranks of the Taiwanese corporate bond yields are based on a 10-year Taiwanese corporate bond taken from the TEJ Databank.[7] We choice the risk-free interest rate and the daily returns of the stock price index to be the state variables. For empirical purpose, we allow these state variables to be deterministic. The risk-free interest rate was obtained from the yield of a 10-year Taiwanese government bond. The daily returns of the stock price index were obtained from the Taiwan Stock Exchange. The sample period, Nov. 1, 2005 to Oct. 7, 2011, yielded 1478 observations for each variable. The variables are the four ranks (i.e. AAA, AA, A and BBB) of the Taiwanese corporate bond yields, the risk-free interest rates, and the returns of the stock price index.

Table 1 presents descriptive statistics for the sample. Specifically, it gives the mean, standard deviation, maximum value, median, minimum value, skewness, and kurtosis for each variable. The results indicate that the standard deviation of the yield for a higher credit rank is always larger than that for a lower credit rank of the Taiwanese corporate bond. One can thus infer that the higher the credit rank of a

7 TEJ is a famous databank with financial data from Taiwan.

Taiwanese corporate bond**,** the more volatile its yield.

The default hazard rate for each rank of the corporate bond is calculated by the following equation (Jarrow and Turnbull, 1995; Duffie and Singleton, 1999):

$$h(u) = (y(u) - r(u))/(1-\pi), \tag{18}$$

where

$y(u)$ is the yield of the corporate bond at time u, and

π is the recovery rate given default.

For simplicity, we assume that the recovery rate is a constant and we let $\pi = 0.7$.[8] We also let the baseline default hazard rate (denoted as $h_0(s)$) is a constant during contract period; that is $h_0(s) = h_0$. If the default hazard rate is specified as following a normal distribution, they can be described as follows:

$$h(u) = h_0 + a_1 r(u) + a_2 r_S(u) + \varepsilon(u), \tag{19}$$

where $r_S(u)$ is the return of the stock price index and $\varepsilon(u)$ is the residual term.

In the case of gamma distribution, one can use Cox's hazard model to estimate the expected hazard rate; we have

$$\log(h(u)) = \log(h_0) + a_1 r(u) + a_2 r_S(u) + \varepsilon(u), \tag{20}$$

The coefficients in the linear function expressed in Equations (19) and (20) are estimated by panel regression. It can be seen from Table 2 that these estimated parameters are as follows: a_1 = -0.6847 and a_2 = -0.0080 under the normal distribution, and a_1 =-64.3450 and a_2 =-1.1797 under the gamma distribution. All are significant at the 1% level. In terms of results, we find that the default hazard rate

[8] We have used a different value of π to perform the empirical estimation. The main conclusion is similar with present version.

for the Taiwanese corporate bond is negatively correlated with both the risk-free interest rate and the returns of the stock price index.

After the estimates are calculated using panel regression, the residuals are employed to estimate the parameters for the normal and gamma distributions. The mean and standard deviation for distributions were calculated from the residuals in Equations (19) and (20). The parameters from the gamma distribution can easily be estimated by applying the following formulas: $\hat{\alpha} = \frac{\bar{x}^2}{s^2}$ and $\hat{\beta} = \frac{\bar{x}}{s^2}$, where $\bar{x}$ is the mean of the residual and s is the standard deviation of the residual.

The resulting estimates for each distribution are shown in Table 3. The values of μ in the normal distribution for the four credit ranks of the bonds (AAA, AA, A, and BBB) are 1.5947×10^{-17}, 2.2801×10^{-17}, -1.1647×10^{-17}, and 1.0413×10^{-17} respectively. The values of σ in the normal distribution for the four credit ranks of the bonds are 2.8662×10^{-3}, 2.6294×10^{-3}, 2.7747×10^{-3}, and 3.7040×10^{-3} respectively. With the gamma distribution, we have $\hat{\alpha} = 7.4908$ and $\hat{\beta} = 6.9163$ for the AAA bond, $\hat{\alpha} = 12.292$ and $\hat{\beta} = 11.77$ for the AA bond, $\hat{\alpha} = 18.202$ and $\hat{\beta} = 17.677$ for the A bond, and $\hat{\alpha} = 40.37$ and $\hat{\beta} = 39.845$ for the BBB bond.

After obtaining the necessary parameters above, we can calculate the credit ranks of the bond values, given that default hazard rates follow each of the two distributions. Because we use the ZCB with only one default risk, we have $c(u) = 0$, $M = 1$ and $K = 1$ in the evaluation. We let the maturity date of the defaultable ZCB be 10 years and Δt be 1 year. For each rank, its expected hazard rate until the maturity date is a constant, e.g., $\Phi(u) = \Phi(t)$, for $u \ge t$. In addition, we let each expected future default hazard rate be based on the current values of the state vector.

We have the following:

$$\Phi(t) = \begin{cases} h_0 + a_1 r(t) + a_2 r_S(t), & \text{under the normal distribution} \\ h_0 \exp(a_1 r(t) + a_2 r_S(t)), & \text{under the gamma distribution} \end{cases}.$$

Under a discrete time framework, the closed-form formula for the valuation model with default hazard rate following a normal distribution can be described as follows:

$$P_1(t,T) = \pi \sum_{j=1}^{9} P_0(t,t_j)(1-\omega)\omega^j + P_0(t,T)\omega^{10}, \tag{21}$$

where

$P_0(t,t_j)$ is the value of a default-free ZCB or discount factor, and

$\omega = \exp(-(\Phi(t) + \mu - \frac{1}{2}\sigma^2))$ denotes the survival probability for each period.

With the hazard rate following the gamma distribution, we have (see also Appendix A):

$$G^E(t,T) = (1 + \frac{\Phi(t)}{\beta})^{-\alpha(T-t)}, \text{ and } g(u) = \frac{\alpha\Phi(t)}{\beta + \Phi(t)}. \tag{22}$$

In addition, according to Equation (16), the closed-form formula of a defaultable ZCB can then be solved as follows:

$$P_1(t,T) = \pi\alpha(\beta + \Phi(t))^{-1} \sum_{j=1}^{9} P_0(t,t_j)(\frac{\beta + \Phi(t)}{\beta})^{-\alpha(t_j - t)} + P_0(t,T)(\frac{\beta + \Phi(t)}{\beta})^{-\alpha(T-t)}. \tag{23}$$

After obtaining the expected price of the bonds for each of these specifications, we can measure the pricing error and compare their accuracy. The pricing error can be calculated using the following equation:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}\frac{\sqrt{(P_i^R - P_i^T)^2}}{P_i^R}, \tag{24}$$

where

MSE is the mean square error for each distribution,

N is the sample size, and

P^R and P^T are the real and expected ZCB prices, respectively.

Table 4 shows that the mean square errors are 0.0148, 0.0145, 0.0177, and 0.0253 respectively for the four credit ranks (i.e. AAA, AA, A, and BBB) of the bonds when specifying the hazard rates as normally distributed. When the hazard rates follow a gamma distribution, the mean square errors are 0.0123, 0.0135, 0.0170, and 0.0215. The overall average pricing error for the four credit ranks of the bonds for the valuation with the gamma distribution (0.0161) is smaller than that with the normal distribution (0.0181). Therefore, our model is more accurate than the model with the normal distribution. Furthermore, our model is not only creates a closed-form formula, but it also does not generate irrational values for the hazard rate, which is the main pricing problem when hazard rates are assumed to follow a normal distribution. Because of these reasons, it can help market participants manage complicated portfolios more effectively and perform more accurate hedging analyses.

4. Conclusion

It has recently become clear that it is important for market participants to consider correlated multi-risks (e.g., default, liquidity, prepayment) when valuating a risky asset. A closed-form formula for pricing such securities not only can help these persons better understand how sensitive these prices are to changes in relevant variables, but it also provides a useful tool for portfolio managers who need to optimize complicated portfolios and perform accurate hedging analyses. Therefore, the ability to solve a closed-form formula for risky securities is a major objective of pricing theory. Our objective was to use the reduced-form model to derive such a formula for assets with correlated multi-risks.

In previous studies using the reduced-form model for this purpose, the hazard rates associated with risky events were usually specified as following one of three distributions: normal (e.g., Vasciek model), log-normal, (e.g., Cox's hazard model), or non-central chi-square (e.g., CIR model). However, there are disadvantages to specifying hazard rates as following these three distributions. If they follow the normal distribution, they may take a negative value. If they follow the log-normal distribution, the closed-form formula for valuating risky securities cannot be obtained. If they follow the non-central chi-square distribution, the pricing formula for model with multi-risks becomes hard to be derived and the parameters of the distribution difficult to estimate with real data. These problems greatly restrict the application of the model for market participants.

To satisfy market participants' needs and overcome the above problems, the main objective of our study was to create an appropriate model and derive a closed-form formula for valuating risky securities with multi-risks. We based our formula on the assumption that hazard rates and common factors (e.g., interest rate) follow the gamma distribution. Our model has several advantages. To begin with, it can improve pricing accuracy because it doesn't generate unreasonable hazard values, and it is stable and rational in practical applications. Second, it is more flexible than traditional models because the gamma distribution under some specific parameters can approximate to the normal and chi-square distributions. Most important of all, our model can easily be used to obtain a closed-form formula for pricing risky securities with correlated multi-risks, and the model's parameters can very easily be estimated from historical market data. In short, we believe that our model is more useful for market participants and financial scholars than traditional models that assume hazard rates follow distributions other than gamma.

Our model has many theoretical and practical advantages. We proposed an empirical analysis for the implementation of our valuation model. We sampled Taiwanese bond data to show how one can use our model to price risky securities and estimate the essential parameters. The empirical results from this exercise reveal that risk-free interest rates and the returns of the stock price index have a statistically significant negative effect on default hazard rates. Furthermore, our study compared pricing errors when hazard rates follow the normal and gamma distributions. The results show that the pricing error is smaller for the valuation with the gamma distribution than the valuation with the normal distribution. Our model is better than its competitors because it is more accurate and yields a closed-form formula, thereby helping market participants more efficiently optimize complicated portfolios and undertake hedging analyses. As for future research, one can use our model to investigate financial products with other risks (e.g., mortgage contracts). Also, one can extend our model to incorporate stochastic loss rates in models that appraise risky securities.

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Appendix A:

This appendix shows how to derive Equations (15), (16) and (22). We firstly assume the possible termination event belong to a finite set. Let $d = \frac{T-t}{\Delta t}$, with $t = t_0 < t_1 < \cdots < t_d = T$. Let $p_j(v_1^0, \cdots, v_N^0, v_1, \cdots, v_K)$ denote the joint probability density function of $V_1^0(t_j)$, $\cdots, V_N^0(t_j)$, $V_1(t_j)$, $\cdots$, and $V_K(t_j)$, with $j = 0,1,\cdots,d$. Because $V_n^0(t_j)$ and $V_i(t_j)$ are mutually independent random variables, the joint probability density function of $V_1^0(t_j)$, $\cdots, V_N^0(t_j)$, $V_1(t_j)$, $\cdots$, and $V_K(t_j)$ can be defined as follows (see Johnson and Kotz, 1972):

$$p_{t_j}(v_0, v_1, \cdots, v_K) = \prod_{n=1}^{N} p(v_n^0 \mid \alpha_n^0, \beta_n^0) \prod_{i=1}^{K} p(v_i \mid \alpha_i, \beta_i),$$

where

$p(v \mid \alpha, \beta) = \dfrac{\beta^{\alpha} v^{\alpha-1} \exp(-\beta v)}{\Gamma(\alpha)}$ is the probability density function of $V(t_j)$ given the parameters α and β.

Therefore, $\Omega^G(t_0, t_s)$ can be expressed as follows:

$$\Omega^G(t_0, t_s) = E[\exp(-\sum_{j=1}^{s}(r(t_j) + \sum_{i=1}^{K} h_i(t_j)\Delta t))]$$

$$= E[\exp(-(\Phi_1^0(t_1)V_1^0(t_1) + \Phi_1^0(t_2)V_1^0(t_2) + \cdots + \Phi_1^0(t_d)V_1^0(t_d))\Delta t)]$$

$$\times E[\exp(-\sum_{i=0}^{K}(\Phi_i(t_1)W_i(t_1) + \Phi_i(t_2)W_i(t_2) + \cdots + \Phi_i(t_d)W_i(t_d))\Delta t)]$$

$$= E[\exp(-\sum_{j=1}^{d} C'(t_j)V^0(t_j)\Delta t)] \times \prod_{i=1}^{K} E[\exp(-\sum_{j=1}^{d} \Phi_i(t_j)V_i(t_j)\Delta t)]$$

$$= \prod_{n=1}^{N} \Xi_n^0(t_0, t_d)] \prod_{i=1}^{K} \Xi_i(t_0, t_d)],$$

where

$C(t_j) = [C_1(t_j) \quad C_2(t_j) \quad \cdots \quad C_N(t_j)]'$, we have $C_1(t_j) = \Phi_1^0(t_j) + \sum_{i=1}^{K} \xi_1^i \Phi_i(t_j)$

and $C_l(t_j) = \sum_{i=1}^{K} \xi_l^i \Phi_i(t_j)$, for $l = 2, \cdots, N$,

$$\Xi_n^0(t_0, t_d)] = E[\exp(-\sum_{j=1}^{d} C'(t_j) V^0(t_j) \Delta t)]$$

$$= \prod_{j=1}^{d} (1 + \frac{C_n(t_j)}{\beta_n^0})^{-\alpha_n^0 \Delta t} = \exp(-\sum_{j=1}^{d} \alpha_n^0 \ln(1 + \frac{C_n(t_j)}{\beta_n^0}) \Delta t) \text{, and}$$

$$\Xi_i(t_0, t_d) = E[\exp(-\sum_{j=1}^{d} \Phi_i(t_j) V_i(t_j))]$$

$$= \prod_{j=1}^{d} (1 + \frac{\Phi_i(t_j)}{\beta_i})^{-\alpha_i \Delta t} = \exp(-\sum_{j=1}^{d} \alpha_i \ln(1 + \frac{\Phi_i(t_j)}{\beta_i}) \Delta t) .$$

According to the previous formula, we derive the following:

$$E[\exp(-a \sum_{i=1}^{K} (\sum_{j=1}^{s} (r(t_j) + h_i(t_j)) \Delta t) + b h_k(t_s)) \Delta t)] = \prod_{n=1}^{N} \Gamma_n^0(t_0, t_s) \prod_{i=1}^{K} \Gamma_i(t_0, t_s) ,$$

where

$$\Gamma_n^0(t_0, t_s) = \prod_{j=1}^{s-1} (1 + \frac{a C_n(t_j)}{\beta_n^0})^{-\alpha_n^0 \Delta t} (1 + \frac{(a + b I_{\{k\}}) C_n(t_s)}{\beta_n^0})^{-\alpha_n^0 \Delta t} ,$$

$$\Gamma_i(t_0, t_s)] = \prod_{j=1}^{s-1} (1 + \frac{a \Phi_i(t_j)}{\beta_i})^{-\alpha_i \Delta t} (1 + \frac{(a + b I_{\{k\}}) \Phi_i(t_s)}{\beta_i})^{-\alpha_i \Delta t} \text{, and}$$

$I_{\{k\}}$ is an indicator function, where $I_{\{k\}} = 1$ if $i = k$; and $I_k = 0$ if $i \neq k$.

We let $\prod_{n=1}^{N} \Gamma_n^0(t_0, t_s) \prod_{i=1}^{K} \Gamma_i(t_0, t_s)$ be differentiated with respect to b. The expected termination probability with risk i can be expressed by the following:

$$\Omega_i^f(t_0,t_s)] = E[\frac{\partial \exp(-a\sum_{i=1}^{K}\sum_{j=1}^{s}(r(t_j)+h_i(t_j)) + bh_k(t_s))}{\partial b}]\Big|_{a=1,b=0}$$

$$= g_i(t_s)\prod_{n=1}^{N}\Xi_n^0(t_0,t_s)\prod_{i=1}^{K}\Xi_i(t_0,t_s),$$

where

$$g_i(t_s) = (\sum_{n=1}^{N}\frac{\alpha_n^0 C_n(t_s)}{\beta_n^0 + C_n(t_s)} + \frac{\alpha_i \Phi_i(t_s)}{\beta_i + \Phi_i(t_s)})\Delta t .$$

If we let time interval Δt approach 0, the discrete time series is transformed into a continuous time process. We have

$$\lim_{\Delta t\to 0}\Xi_n^0(t,T)] = \lim_{\Delta t\to 0}\exp(-\sum_{j=1}^{d}\alpha_n^0\ln(1+\frac{C_n(t_j)}{\beta_n^0})\Delta t)$$

$$= \lim_{\Delta t\to 0}\exp(-\sum_{j=1}^{d}\alpha_n^0\ln(1+\frac{C_n(t_j)}{\beta_n^0})\Delta t) \quad = \exp(-\int_t^T \alpha_n^0\ln(1+\frac{C_n(s)}{\beta_n^0})ds) \quad ,$$

and $\lim_{\Delta t\to 0}\Xi_i(t,T)] = \exp(-\int_t^T \alpha_i\ln(1+\frac{\Phi_i(s)}{\beta_i})ds)$.

Therefore, we can be express $\Omega^G(t,u)$ and $\Omega_i^f(t,u)$ as follows:

$$\Omega^G(t,u) = \exp(-\int_t^u \delta(s)ds), \text{ and } \Omega_i^f(t,u) = g_i(u)du\,\Omega^G(t,u),$$

where

$$\delta(s) = \sum_{n=1}^{N}\alpha_n^0\ln(1+\frac{C_n(s)}{\beta_n^0}) + \sum_{i=1}^{K}\alpha_i\ln(1+\frac{\Phi_i(s)}{\beta_i}), \text{ and}$$

$$g_i(u) = \sum_{n=1}^{N}\frac{\alpha_n^0 C_n(u)}{\beta_n^0 + C_n(u)} + \frac{\alpha_i\Phi_i(u)}{\beta_i + \Phi_i(u)} .$$

They are Equations (15) and (16). In the empirical section, we let $\Phi(u) = \Phi(t)$, $N = 0$, $K = 1$ and $\Delta t = 1$. Therefore, we have:

$$\Omega^G(t,T) = (1+\frac{\Phi(t)}{\beta})^{-\alpha(T-t)}, \text{ and } g(u) = \frac{\alpha\Phi(t)}{\beta + \Phi(t)} .$$

That is Equation (22).

Table 1: Summary statistics for corporate bond yields and state variables

	Corporate bond yield				Interest Rate	The Return of Stock Price Index
	AAA	AA	A	BBB		
Mean	2.2790	2.4178	2.6656	3.1347	1.8334	0.0152
Standard Deviation	0.3867	0.3512	0.2828	0.2135	0.4498	1.4425
Maximum	3.0753	3.1747	3.3237	3.5993	2.8247	6.5246
Median	2.2634	2.3850	2.6342	3.1548	1.7435	0.1082
Minimum	1.6571	1.8243	2.1023	2.7385	1.1379	-6.7351
Skew	0.2708	0.2918	0.41537	0.0444	0.4939	-0.4214
Kurtosis	2.0480	2.2035	2.3510	1.9054	1.9088	5.5691
Sample Number	1478	1478	1478	1478	1478	1478

Note: This table shows the mean, standard deviation, maximum value, median, minimum value, skewness, and kurtosis for each variable, including the four daily ranks (AAA, AA, A, and BBB) of Taiwanese corporate bond yields, risk-free interest rates, and returns of the stock price index. The yields were used to calculate the value of the defaultable ZCB. The ranks of the bond yields are based on the 10-year Taiwanese corporate bond taken from the TEJ Database. The risk-free interest rate was obtained from the yield of the 10-year Taiwanese government bond. The daily returns of the stock price index were obtained from the Taiwan Stock Exchange. The sample period, from Jan.11, 2005 to Oct. 10, 2011, yielded 1478 observations for each variable.

Table 2: Estimates of the coefficients in the linear regressions using panel regression

		the coefficient	P-value
Normal distribution	a_1	-0.6847***	0.0000
	a_2	-0.0080***	0.0016
Gamma distribution	a_1	-64.3450***	0.0000
	a_2	-1.1797***	0.0000

Note: We used panel regression to estimate the coefficients in Equations (19) and (20). a_1 and a_2 represent the relative magnitudes of the effects of the interest rate and returns of the stock price index respectively on the default hazard rate. "***" denotes significance at the 1% level.

Table 3: Estimates of the parameters in each distribution for the four ranks of corporate bond yields

		AAA	AA	A	BBB
Normal distribution	μ	1.5947×10^{-17}	2.2801×10^{-17}	-1.1647×10^{-17}	1.0413×10^{-17}
	σ	2.8662×10^{-3}	2.6294×10^{-3}	2.7747×10^{-3}	3.7040×10^{-3}
Gamma distribution	α	7.4908	12.2920	18.2020	40.3700
	β	6.9163	11.7700	17.6770	39.8450

Note: μ and σ are the mean and standard deviation of each distribution. For the normal distribution, these parameters were calculated from the residuals in Equation (19). The parameters α and β for the gamma distribution, obtained by the method of moments, are $\hat{\alpha} = \frac{\bar{x}^2}{s^2}$ and $\hat{\beta} = \frac{\bar{x}}{s^2}$, where $\bar{x}$ and s are the sample mean and standard deviation for the residuals in Equation (20).

Table 4: The average expected values and pricing errors for the corporate bond

Distribution Types	Rank	Average Real Values	Average Expected Values	Average MSE	Average MSE for All Ranks
Normal	AAA	0.7989	0.8012	0.0148	0.0181
	AA	0.7880	0.7915	0.0145	
	A	0.7690	0.7746	0.0177	
	BBB	0.7346	0.7444	0.0253	
Gamma	AAA	0.7989	0.8012	0.0123	0.0161
	AA	0.7880	0.7916	0.0135	
	A	0.7690	0.7755	0.0170	
	BBB	0.7346	0.7461	0.0215	

Note: The third column gives the means of real values for the four corporate bond yield ranks. The fourth and fifth columns give the means of expected prices and pricing errors for the four ranks assuming that hazard rates follow the normal and gamma distributions respectively. The means of real values were calculated from the yields from the four ranks of the 10-year Taiwanese corporate bond. The means of expected values, assuming normal and gamma distributions for hazard rates, were calculated from Equations (21) and (23). The MSE were calculated from Equation (24).