Family-Run Firms: Growth and Financing Choices

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Abstract

I present a model of growth and financing choices by the head of the family of a family-run firm. I assume that letting family members participate in decision making provides the head of the family with satisfaction but is neither compatible with maximizing cash-flows nor with allowing efficient collaboration from a partner that could bring funds and also advice (or complementary assets). I first show how the head of the family’s decision to pursue high growth versus low growth depends on family size, composition, and on cultural norms. Then, I relate this choice to the decision to sell the firm out, approach a VC, or approach a bank.

Key words: Family-run firms, growth strategy, bank, venture capital, sale-out, debt, equity, convertibles. JEL classification: G24, G3, G32
1 Introduction

Family firms represent the vast majority of privately held corporations and a large fraction of listed corporations around the world. A major issue is whether they perform better than non-family firms (e.g., Anderson and Reeb, 2003; Benedsson et al., 2007; Claessens et al., 2002; Cronqvist and Nilsson, 2003; McConaughy et al., 1998; Miller et al., 2007, Sraer and Thesmar, 2007; Villalonga and Amit, 2008). Indeed, family firms are quite specific because of the interplay of the family system and the business system. The bright side of home life overlapping work life is family members’ deeper sense of obligation towards the firm, that translates into employees’ higher loyalty and commitment. The desire to pass the business onto the next generation generates “patient capital” that allows family firms to pursue investment opportunities that more “myopic” widely held firms would not. Less bureaucratic decision making renders them more responsive to business opportunities. Besides, family stands for quality on the product market. And the family name helps the firm secure public transfers when family members are politically connected. The dark side is that family firms may suffer from capital restrictions (because family fears losing control), risk-aversion (due to the lack of diversification of human and financial capital) and reluctance to change (to not betray the values and culture of the family). Also, family firms may be prone to inter-generational squabbles or sibling rivalry, entrenchment by the older generation, and nepotism (with the direct consequence of lowering financial returns, and the indirect consequence of making it difficult to attract and retain competent and motivated non-family managers).

Yet, it should not be overlooked that there also exist important systematic differences among family firms. Family firms range from small to large, listed or unlisted companies, operate in sectors as diverse as the high-tech industries, agriculture, manufacturing or construction. In this paper, I determine how family characteristics (e.g., composition, culture) affect the decision to privilege either low growth in order to maintain harmony of the family or high growth, and relate this choice to the financing policy.

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1 According to the Family Firm Institute, they make up more than 60% of all European companies, and an even greater percentage in the U.S., Latin America, Asia, Africa, and the Middle-East. In fact, whatever the definition of the family firms used, all studies acknowledge their economic prevalence.
of the firm.

There exist several definitions of family firms that largely explain the opposing results obtained by empirical literature as to whether these firms outperform their counterparts (Miller et al., 2007). For instance, McConaughy et al. (1998) count as a family firm any company run by a founder or member of the founder family. Gomez-Mejia et al. (2007) insist on multiple family members being involved in owning and operating the business. Villalonga and Amit (2006) examine different levels of family ownership or management. In this paper, I focus on family-managed firms, that is, firms in which a member of the family (to be short, the “head of the family”) runs the company while several family members are on the payroll. It just amounts to saying that the family as an institution is essential to run the firm. Family members may possess unique competencies or privileged connections with suppliers, stakeholders, or politicians. Thus, the issue raised in this paper is not whether family members on the payroll should be replaced by outsiders but rather what strategy should be chosen, i.e., high versus low growth.

Family members are emotionally interdependent, affecting one another’s thoughts, feelings and actions, soliciting each other’s attention, approval, and support, and reacting to one another’s needs, expectations, and troubles (Kerr and Bowen, 1988). In particular, what characterizes the head of the family (hereafter, HOF), just like parents in the households, is a strong sense of duty towards other family members. Translated into economic wording, it amounts to saying that the HOF’s utility increases when other family members’ utility increases (see Becker, 1974, 1976, 1981 for an economic theory of the family). According to psychologists, this includes (but of course in not restricted to) developing family members’ self esteem, including their sense of usefulness, which was identified by Maslow (1943) as a major need. Self esteem emerges naturally in the course of development (Rogers, 1951), is partially determined by genetics (Neiss et al., 2006), but also depends on parental style

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2Maslow’s (1943) hierarchy of needs has been criticized for many reasons. In particular, it may not hold in all cultural contexts (e.g., Hofstede, 1984; Cianci and Gambrel, 2003). Also, the postulate that lower level needs must be satisfied before an individual seeks to satisfy higher level needs is severely questioned. However, the needs identified by Maslow, though later refined by other academics, remain broadly accepted.
(Coopersmith, 1967), especially on the parents’ willingness to discuss matters. I acknowledge that the HOF derives some non-monetary satisfaction from letting family members on the payroll take part in decision making, which is consistent with the casual observation that business affairs are discussed at home (James, 1999). In general, consensus building is the objective of such discussions that take place either informally or formally through family councils. While this indisputably reinforces the cohesion of the family, a factor that certainly has some good points from an economic (and non-economic) perspective, this also leads to poor decision making, especially when other family members on the payroll are less skilled than the HOF, when the desire to treat each family member with equal respect makes the HOF weight equally the points of view of the informed (or skilled) ones and the uninformed (or less skilled) ones, or when the multiplicity of players with differing agenda causes confusion or inertia. As long as the project is profitable, this does not prevent the firm from obtaining outside funds. However, it may prevent the firm from obtaining funds from “partners” that also have the potential for providing the firm with strategic advice, connections with professionals, key personnel (what venture capitalists usually do) or complementary assets (what another firm operating in the same industry can do). Indeed, letting family members interfere is unlikely to be compatible with receiving efficient support (e.g., Strike, 2012). Since the existence of discussions among family members on the payroll about how the firm is/should be run is generally not observable (i.e., squabbles relayed by the business press being the exception), supportive outside investors fear that the HOF indulge in privileging what recent research in the management literature (e.g., Gomez-Mejia et al., 2007) has termed socio-emotional wealth at the expense of profitability, and may refuse to exert the effort expected of them to increase the firm’s cash-flows. Then two related questions emerge: What financial contract (i.e., cash-flow sharing rule), if any, disciplines the HOF, allows active participation from a

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3 A growing body of the (academic and non-academic) management literature investigates how family firms could/should be advised by a variety of actors (e.g., consultants, private bankers, family therapists, psychologists or psychiatrists). See the special 2013 issue of the Family Firm Review.

4 A fortiori, this is certainly not verifiable by a court of law.

5 There is a burgeoning literature that assesses empirically the magnitude of socio-emotional wealth. For instance, Dressler and Tauer (2012) estimate that, around New York, a farm manager loses $22,000 annually when working on his family’s farm rather than with a non-family employer.
partner, and thus leads to higher growth? Is the HOF better off signing such a contract or choosing the lower-growth strategy?6

To answer these questions, I present the model of a cash-poor family firm. The HOF has to decide whether the firm embraces high growth or low growth. Whatever the strategy chosen, the firm is run by the family and must find external resources. The low-growth strategy gives non-monetary satisfaction to the HOF by letting family members on the payroll take part in decision making. This non-monetary satisfaction increases with the number of family members involved in decision making and comes at the expense of cash-flows. Under the high-growth strategy cash-flows are higher, for two reasons. First, because the HOF maximizes financial returns; second, because the HOF receives support from the provider of funds (e.g., strategic advice, access to qualified personnel, complementary assets, etc.). Such support is useless when family members other than the HOF interfere. Both the HOF’s strategic decision and support from the partner are unobservable, resulting in double-sided moral hazard. Cash flows are verifiable. The HOF maximizes the sum of the value of his shares and the non-pecuniary satisfaction from involving family members on the payroll in decision making.

I identify three regimes, depending on the number of family members on the payroll (i.e., family size to be short). In the first regime, family is small. The HOF is better off maximizing cash-flows and seeking support from the partner, that is, embracing high growth. To provide the partner with high-powered incentives to support the firm, the HOF makes the partner residual claimant of the cash-flows, that is, sells the firm out but stays as a wage-earner. A simple example of an acquirer is that of a deep-pocketed, external-growth inclined firm operating in the same industry. The HOF’s incentives to maximize cash-flows derive from a combination of a fixed salary and a bonus if the firm does well. In the second regime, the size of the family is limited enough to make it worth embracing high growth, again. The HOF keeps a suitably chosen fraction of the firm’s common stock. The partner receives

6Croce et al. (2013) analyze empirically how the will to protect socioemotional wealth affects the decision of family firms to approach venture capitalists. They show that reluctance to approach venture capitalists is higher for first generation managers.
convertible shares, that is, typically what venture capitalists hold. In the third regime, family is large. Accordingly, the HOF privileges non-monetary satisfaction rather than cash-flows, which leads to lower growth. Issuing pure debt to the provider of funds, which is typical of bank financing, allows the HOF to implement this strategy.

For given family size, family composition matters. Indeed, it is likely that the HOF values more highly letting the HOF’s children rather than other family members take part in decision making. This naturally influences the decision to opt for low versus high growth, and the related choice to approach banks rather than venture capitalists or sell the firm. Also, cultural norms influence these decisions since the attachment to family (hence the satisfaction derived from letting family interfere) varies across countries, succession rules differ, etc. This is discussed in detail in the text.

Just a few theory papers (James, 1999; Burkart et al., 2003) study family firms. Succession is an important issue. Burkart et al. (2003) examine how monetary private benefits (and amenities) derived from controlling the firm influence the decision by the founder to hire a better-qualified professional as a successor rather than a family member. The present paper differs in three respects. First, I consider the choice of a growth strategy by a founder who continues to run the firm (alongside with the family). In chronological order, the problem studied here thus precedes the one studied by Burkart et al. (2003). Second, the firm needs outside financing which poses the question of the optimal financial claims to issue. Finally, the present paper focuses on non-monetary benefits of control (i.e., socio-emotional wealth in the words of Gomez-Mejia et al., 2007) that come at the expense of profits, rather than on the extraction of monetary private benefits (i.e., tunneling), or on amenities that do not reduce the cash-flow potential of the firm. The issue studied by Burkart et al. (2003) requires, as the authors note, that outside managers (or firms) do not have the capital to buy the firm out. Were it the case, a maximizing-utility founder would better off sell the business to the best possible manager (or firm) and distribute the proceeds among family members. Thus, by itself, the founder’s desire to maximize the financial well-being of his family does not necessarily lead to any inefficiency for the business. Burkart
et al.’s (2003) assumption makes sense in a variety of contexts. I study here the complementary case in which a stock of skilled capital (venture capital or other firms in the same industry) is available. Then, it is essential that non-monetary factors enter into the HOF’s decision.

The present paper is also related to optimal contracting with venture capitalists (Casamatta, 2003; Inderst and Müller, 2004; Repullo and Suarez, 2004) in that convertibles claims represent the solution to double-sided moral hazard in the second regime identified here. A difference with that literature is that other financial arrangements prove necessary in the other two regimes.

Finally, the paper is connected to previous research that analyzed the tightness of relationships between investors and entrepreneurs concerned with too much intervention of the former. However, these papers have not considered this issue in a context in which investors can support entrepreneurs (e.g., Aghion and Bolton, 1992; Pagano and Röell, 1994; Burkart et al., 1997; Holmström and Tirole, 1997). Besides, they did not consider the optimal claims to be issued (e.g., Rajan, 1992; Pagano and Röell, 1994; Burkart et al., 1997). The only exception I am aware of is Marx (1998). However, she considers benevolent entrepreneurs. This (partially) explains why convertibles (or a mix of debt and equity) is always optimal in Marx’s setting while this solution is only optimal in one regime in the present framework.

The rest of the paper is organized as follows. Section 2 presents the model and a full information benchmark. Section 3 examines the HOF’s choice of a high-growth versus low-growth strategy, depending on family characteristics. Section 4 details how the optimal sharing rule of cash-flows that maximizes the HOF’s utility can be implemented with financial claims observed in practice. Concluding remarks follow. Proofs are in the appendix.
2 The Model

2.1 Assumptions

Consider a firm run by the HOF. \(N\) family members are on the payroll. Currently, they take part in decision making, which generates satisfaction \(f(N)\) to the HOF. For simplicity, \(f(N) = N\). I discuss in sections 3.4, 3.5 and 3.6 alternative assumptions about \(f\) to consider the impact of family composition and cultural norms. In the future, if the status quo prevails, the firm is expected to yield cash-flows that are normalized to zero with no loss of generality.

The HOF considers a growth opportunity that requires a verifiable financial investment \(I_f\). If growth materializes, cash-flows are \(R\). If growth fails to materialize, cash-flows are \(r\), with \(r < I_f < R\). Once \(I_f\) is incurred, the HOF is faced with a dilemma. Still allowing family members on the payroll (hereafter referred to as family for the sake of brevity) to take part in decision making does not allow the firm to benefit from the full potential of the growth opportunity. Specifically, if family remains involved in decision making, growth materializes with probability \(p_l\), whereas keeping family on the payroll, but at arm’s length, increases this probability up to \(p_h\), with \(p_l < p_h < 1\). The HOF’s choice is unobservable.

For simplicity, the (head of the) family is supposed to be wealthless and thus needs to raise \(I_f\) externally\(^7\). The HOF issues financial claims to a wealthy “partner”, based on the firm’s verifiable cash-flows. The partner has the potential for improving the firm’s profitability provided that the HOF excludes family from decision making. Then the probability that growth materializes increases from \(p_h\) to \(p_h + I_s \alpha\) (with \(0 < \alpha \leq 1 - p_h\)), where \(I_s \in [0, 1]\) is the partner’s unobservable supportive investment. This investment can be thought as, e.g., effort to understand the firm’s business and make valuable suggestions, a contribution in the form of intangible assets, etc. This investment costs costs \(k \frac{I_s}{2}\) to the partner.

\(^7\)The HOF is allowed to raise more than \(I_f\), that is, \(I_f + t\) with \(t > 0\). For the sake of concision, I will mention this up-front transfer \(t\) only to the extent that it increases the firm’s value by allowing a better design of the incentive scheme.
Finally, let us introduce some notation and technical assumptions. Denote by $v_l \overset{d}{=} p_l R + (1-p_l) r - I_f$ the firm’s market value when the HOF associates family to decision making (making support from the partner useless). In the following, $v_l > 0$ for simplicity, which implies that there is no credit rationing. Denote by $v_h \overset{d}{=} p_h R + (1-p_h) r - I_f$ the firm’s market value when the HOF prevents family from making decisions and the partner does not provide support. Denote by $V \overset{d}{=} (p_h + \alpha I_s) R + [1 - (p_h + \alpha I_s)] r - k \frac{I_s}{2} - I_f$ the firm’s market value when the HOF prevents family from making decisions and the partner provides support. Let $\Delta R \overset{d}{=} R - r$ and $\delta p \overset{d}{=} p_h - p_l$. The constant $k$, parametrizing the partner’s supportive investment, is such that $k \geq \max \left\{ \alpha \Delta R; \left. \frac{\alpha^2 \Delta R^2}{2 \delta p} \right\} \right.$. This assumption ensures that in equilibrium the supportive investment belongs to the interval $[0, 1]$ which is consistent with $I_s$ also being a probability. The riskless interest rate is normalized to zero. Partners are competitive. All parties are risk-neutral and protected by limited liability.

2.2 First-best Case

As a benchmark for the analysis, let actions be contractible. First suppose that the HOF keeps family on the payroll but at arm’s length and requires support from the partner. This high-growth strategy yields $V^{FB} \overset{d}{=} v_h + \frac{\alpha^2 \Delta R^2}{2 k}$. Next suppose that the HOF allows family to still take part in decision making and accordingly does not require the partner’s support. This low-growth strategy yields $v_l + N$. Thus, the HOF opts for the high-growth strategy if $N \leq N^{FB} \overset{d}{=} \delta p \Delta R + \frac{\alpha^2 \Delta R^2}{2 k}$. If otherwise, the HOF opts for the low-growth strategy. The next section considers the impact of moral hazard on these results.

3 High Growth Versus Low Growth

I first examine the case in which the HOF opts for high growth and the case in which the HOF opts for low growth. Then, I study the HOF’s decision depending on family size, composition, and cultural norms.
Suppose the partner receives $r_p$ when growth fails to materialize and $R_p$ when growth materializes. Limited liability on the HOF’s and the partner’s sides imposes that $0 \leq r_p \leq r$ and $0 \leq R_p \leq R$.

### 3.1 High Growth

Suppose the HOF excludes family from decision making. Then, the partner chooses $I_s$ so that

$$I_s \in \arg \max_{\hat{I}_s} \left( p_h + \alpha \hat{I}_s \right) R_p + \left[ 1 - (p_h + \alpha \hat{I}_s) \right] r_p - \frac{k \hat{I}_s^2}{2} - I_f. \quad (1)$$

I will refer to (1) as to the partner’s incentive compatibility constraint. The first-order condition leads to

$$I_s = \frac{\alpha (R_p - r_p)}{k}. \quad (2)$$

The HOF indeed excludes family from decision making if and only if the HOF’s utility when doing so is greater than when family is involved in decision making. This requires

$$(R - R_p) - (r - r_p) \geq \frac{N}{\delta p + I_s \alpha}. \quad (3)$$

I will refer to (3) as to the HOF’s incentive compatibility constraint. Observe that since the partner’s support rises the probability that growth materializes by $I_s \alpha$ in expectation, it renders the high-growth strategy more attractive to the HOF.

The partner’s participation constraint is given by

$$(p_h + \alpha I_s) R_p + \left[ 1 - (p_h + \alpha I_s) \right] r_p - \frac{k I_s^2}{2} - I_f \geq 0. \quad (4)$$

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8 Any contract that satisfies the HOF’s limited liability constraints also implies that the HOF’s expected revenue is positive, which in turn ensures that the HOF breaks even since the HOF’s financial input is zero.

9 For simplicity, assume that the HOF maximizes the market value of the company when indifferent between doing so and enjoying $N$. 
Let us rewrite the firm’s value as

\[ V = v_h + \left[ \alpha \Delta R I_s - \frac{k}{2} I_s \right]. \]  \tag{5}

The term into brackets reflects the partner’s support to increase the expected cash-flows (\(\alpha \Delta R\) represents the expected gain when the partner’s supportive investment is relevant, while \(I_s\) is the probability that support is relevant) net of the cost of the supportive investment. Eq. (5) shows that maximizing \(V\) requires maximizing \(I_s\)- observe that \(V\) increases in \(I_s\) up to \(I_{FB} = \frac{\alpha \Delta R}{k}\). According to (2), it amounts to setting \(R_p - r_p\) as high as possible under the set of constraints given by (3), (4), and the limited liability conditions. The main problem is to satisfy simultaneously the HOF’s and the partner’s incentive compatibility constraints. Indeed, for incentive reasons, each party involved should be deprived from all cash-flows when growth fails to materialize and obtain all cash-flows when growth materializes\(^{10}\).

Thus, fostering the HOF’s incentives automatically diminishes the partner’s incentives, and vice versa. Combining (2) and (3) and solving for the optimal level of supportive investment in the absence of any other constraint leads to

\[ R_p - r_p \leq \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2}. \]  \tag{6}

For intermediate values of \(N\), the optimal \((r_p, R_p)\) simply results from the combination of the two incentive compatibility constraints and the partner’s participation constraint (the latter being binding since partners are competitive). Observe that because the partner’s support is useless if the HOF involves family into decision making, the first objective of the contract is to induce the HOF to avoid

\(^{10}\)Introducing a third party (e.g., a pure financier) would be useful if this third party were (i) passive and (ii) allocated all the cash-flows when growth fails to materialize and no cash-flow when growth materializes (for example): both the partner and the HOF would be severely punished when growth fails to materialize, which fosters incentives. It would allow to break the budget constraint in the spirit of Holmström (1982). Nevertheless, such a third party’s reward scheme (live or die) is difficult to implement (see Innes, 1990) in the sense that the partner and the HOF are induced to collude when growth fails to materialize: the wealthy partner provides the amount \((R_p - r_p)\), and together with the HOF, they claim that growth materialized in order not to pay back the third party. Hence, the third party’s reward has to be non-decreasing in the outcome, which eventually does not facilitate the design of incentives.
doing so. Then, *residual* incentives are given to the partner. Hence, \( r_p \) increases in \( N \), whereas \( R_p \) decreases in \( N \).

For low enough and high enough values of \( N \), the presence of limited liability constraints on both sides renders more difficult the design of incentives. Specifically, when \( N = \bar{N} \), the HOF’s incentive problem is serious enough to make it necessary to deprive the HOF from all the cash-flows when growth fails to materialize, i.e. \( r_p = r \). Simultaneously, the partner’s participation constraint is just binding. Hence, when \( N > \bar{N} \), encouraging the HOF would command either to increase \( r_p \), which is impossible without violating the HOF’s limited liability constraint, or to decrease \( R_p \), which is impossible without violating the partner’s participation constraint.

When \( N = \bar{N} \) and \( I_f \) is low, i.e., \( I_f < p_h R + \frac{\alpha^2 \Delta R^2}{2k} \), making the partner just break even implies that \( r_p = 0 \). Thus, when \( N < \bar{N} \), \( r_p \) would be strictly negative in the absence of limited liability constraint since \( r_p \) decreases in \( N \). Because the partner is actually protected by limited liability, keeping \((R_p - r_p)\) as high as possible for incentive purposes imposes both to set \( r_p \) equal to zero and to increase \( R_p \). Hence, solving the maximization program implies that the partner receives a rent: the partner’s expected revenue more than offsets the sum of the partner’s financial and supportive investments. However, as partners are competitive, the HOF recoups \( V \), provided that the partner makes an up-front transfer \( t \) equal to the rent.

Whatever \( N \), substituting \((r_p, R_p)\) given by (6) into (2) implies that \( I_s \) strictly decreases in \( N \). Indeed, as family size increases, the difference between the HOF’s reward when growth materializes and the reward when growth fails to materialize must widen in order to satisfy (3). It lowers \((R_p - r_p)\) and in turn lessens the partner’s incentives to support the firm. The next proposition summarizes these results.

**Proposition 1** The high-growth strategy is possible when \( N \leq \bar{N} \), where \( \bar{N} \) is defined in the Appendix.

The partner’s supportive investment and the firm’s value \( V \) strictly decrease in \( N \). The HOF earns \( V \).

\[11 I_f \geq p_h R + \frac{\alpha^2 \Delta R^2}{2k}, \bar{N} < 0.\]
3.2 Low Growth

Assume now that the objective of the HOF is to let family continue to take part in decision making, i.e., the HOF opts for the low-growth strategy. Then, the partner’s advice is useless. Thus, the partner optimally undertakes no supportive investment. The HOF indeed opts for low growth if

\[
\frac{N}{\delta p} \geq (R - R_p) - (r - r_p),
\]

while the partner breaks even if

\[
p_l R_p + (1 - p_l) r_p - I_f \geq 0.
\]

A quick observation of (7) and (8) leads to the conclusion that, whatever \( N \), these constraints (as well as limited liability conditions) are compatible. The reason is that \( v_l > 0 \), i.e., the market value of the firm is (strictly) positive even when family takes part in decision making so that there always exists a contract that makes the two parties break even. Note that a sufficient condition to satisfy all the constraints is to set \( r_p = r \) and \( R_p = R \). Coupled with an up-front transfer from the partner to the HOF, it would ensure that the latter obtains the full value of the low-growth strategy, \( v_l \). However, this condition is not necessary. Accordingly, another sharing rule of cash-flows that avoids a transfer is proposed in the next section. These results are summarized in the next proposition.

**Proposition 2** The low-growth strategy is possible whatever \( N \). The HOF earns \( v_l \), that strictly increases in \( N \).

3.3 The Growth Decision

Simple computation shows that:

**Proposition 3** The HOF opts for high growth if \( N \leq \bar{N} \). If otherwise, the HOF opts for low growth.
Observe that since $N < \delta p \Delta R < N^{FB}$, the HOF less often opts for the high-growth strategy than in the first-best case. The reason is that $I_s < I_s^{FB}$. Thus, any mechanism that would limit the extent of moral hazard on the VC’s or the HOF’s side would have positive effects on economic growth. The next sections examine how the composition of the family, cultural norms, and succession rules influence the HOF’s decision.

### 3.4 Composition of the Family

I have assumed so far that $N$ represented the number of family members involved in the firm. However, all family members are presumably not equal in the HOF’s heart, so that satisfaction from letting them take part in decision making should not be equal. Assume that the HOF’s satisfaction is given by

$$f(N) = n \times 1 + (N - n) \times (1 - \beta),$$

where $n$ is the number of the HOF’s children on the payroll, $(N - n)$ is the number of family relatives on the payroll, and $\beta > 0$ reflects preference for children. Then, holding $N$ constant, Proposition 3 implies that firms are more likely to exhibit lower growth when the number of children on the payroll is larger.

### 3.5 Cultural Norms Across Countries

The role and status of families widely differ across countries (Bertrand and Schoar, 2007). Thus, taking for granted the preference for children over other family members, the HOF’s satisfaction can be rewritten as

$$f(N) = \rho [n \times 1 + (N - n) \times (1 - \beta)],$$

where $\rho$ reflects the value attributed to family. Holding family composition constant, Proposition 3 implies that in countries in which family is more highly valued, firms are more likely to exhibit lower growth than in countries where family values are less prevalent. This eventually impedes economic
growth. This implication is consistent with empirical evidence. Using the results obtained from several waves of the World Values Survey, Bertrand and Schoar (2007) show a negative correlation between the strength of family values on the one hand and firm size and number of listed firms as a fraction of total firm population on the other hand. At the macro level, Bertrand and Schoar (2007) show that G.D.P. per capita is negatively correlated with family strength.

3.6 Succession Rules

Cultural differences across countries and across families also pertain to succession rules. At the one end of the spectrum is primogeniture, or the fact that the eldest son inherits from the family’s whole wealth. At the other end of the spectrum is equal sharing between children. Succession rules quite likely have consequences on the role of children prior to the succession. In turn, the satisfaction from involving children in decision making would differ. When equal sharing is the rule, (10) still holds. When primogeniture is the rule, the HOF’ satisfaction can be written as

\[
f(N) = \rho \left[ 1 \times \sigma + (n - 1) \times (1 - \lambda) + (N - n) \times (1 - \beta) \right], \tag{11}\]

where \( \sigma > 1 > \beta > \lambda \). To the extent that the satisfaction derived from favoring the eldest son outweighs the dissatisfaction from neglecting the other children (i.e., \( \sigma > 1 + \lambda (n + 1) \)), then, holding \( n \) and \( N \) constant, Proposition 3 implies that, in cases in which primogeniture is the rule, firms are more likely to exhibit higher growth.\textsuperscript{12} The next section examines the implementation of the optimal sharing rule of cash-flows.

\textsuperscript{12}This holds true to the extent that the satisfaction derived from favoring the eldest son does not outweigh the dissatisfaction from neglecting the other children.
4 How to Implement the Growth Strategy

The purpose of this section is to present how the contracts obtained above in an abstract way can be implemented with contractual tools observed in the real world. This is detailed in the next proposition.

**Proposition 4** The optimal sharing rule of cash-flows entails:

- If \( N \in [0, N] \), selling out the firm. The HOF is compensated for continuing running the firm through a base salary and a bonus.
- If \( N \in [N, N] \), issuing convertible preferred equity to a venture capitalist (or a mix of debt and equity to a bank).
- If \( N \in [N; \infty] \), issuing straight debt to a bank.

To get a better intuition of these results, I start with intermediate values of \( N \).

4.1 Issuing Convertibles to a Venture Capitalist or a Mix of Debt and Equity to a Bank

When family size is intermediate, the HOF must face adequate incentives to exclude family from decision making while the partner must face adequate incentives to support the firm as much as possible.

Selling convertible preferred equity to the partner achieves this twofold objective. Note that convertible preferred equity is the typical claim held by venture capitalists, a class of investors that provide entrepreneurs with funds and advice (e.g. Sahlman, 1990; Gompers, 1996; Kaplan and Strömberg, 2003). Let \( r_p \) be the minimum pay-off guarantied by convertible preferred equity and \( \delta \) the fraction of equity obtained upon conversion into common stocks. To foster incentives, the claim is structured such that conversion is worth if and only if growth materializes (i.e., \( \delta r < r_p < \delta R \)). It implies \( I_s = \frac{\delta}{\delta} (\delta R - r_p) \). Hence, the higher the venture capitalist’s support should be, the larger the equity stake in case of conversion and the smaller the minimum guarantied pay-off \( r_p \). Besides, to motivate the HOF, \( r_p \) and \( \delta \) are such that the percentage of cash-flows the HOF receives rises as growth materializes. This is in line with Kaplan and Strömberg’s (2003) observation: “The VC stake is a median 7.9\%
lower (average 11.5%) under full vesting and good performance compared to the minimum vesting, bad performance state”.

In the framework adopted in this paper, one can replicate convertibles by a mix of debt and equity. Contrary to the U.S. law (since the Glass-Steagall Act, and although it has been modified to some extent), the law in Continental Europe and Japan allows banks to hold equity stakes. For instance, Berglöf and Perotti (1994) note that “[T]he keiretsu main bank holds from 2 to 5 percent [of the group’s companies]”. This is consistent with the supportive role banks play in these countries and thus deserves some comments. Let $d$ denote the face value of debt and $\theta$ denote the dilution. Debt and dilution have the following features: $\theta$ decreases in $N$, whereas $d$ increases in $N$. In words, the larger the family size, the lower the fraction of equity granted to the bank and the higher the level of debt. This is necessary to induce the HOF to exclude family from decision making. Naturally, this leads to lower support from the bank. It is worth noting that selling both debt and equity prove necessary for incentive purposes, except in the following two cases: when $N = \bar{N}$, pure debt is feasible; when $N = \bar{N}$, common equity is feasible.

### 4.2 Selling Out the Firm

When family size is low, that is, $N < \bar{N}$, just issuing common equity to the partner as when $N = \bar{N}$ would not induce the partner to perform the supportive investment expected of him. Indeed, equity leaves too much of a stake when growth fails to materialize. By contrast, making the partner residual claimant of the cash-flows (i.e., making the partner buy the firm) proves efficient. As a residual claimant of the cash-flows, the partner holds common equity and incurs the cost of compensating the HOF who now acts as an employee. To induce the HOF to exclude family from decision making, the HOF receives a suitably chosen package of a base salary $(r - r_p)$ and a bonus $[\{(R - R_p) - (r - r_p)\}$ when growth materializes. Note that since $N$ is low, the bonus required is small.13 Business life shows

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13 For the lowest values of $N$, the acquiring firm pays a premium $t$ over $I_f$ before the parties undertake actions. As discussed in section 3, it allows to optimize the partner’s incentives to undertake the supportive investment while making the HOF earn the whole NPV.
examples of families that sold out their businesses and thereafter still worked therein. Think of the €4.3 billion ($6 billion) sale of Bulgari, a jeweller, to LVMH, a French luxury-goods giant, in 2011. The Bulgari group got €1.9 billion in LVMH shares for their controlling 50.4% stake in the business, which is run by a great-grandson of the Greek craftsman who founded it in Rome in 1884.

### 4.3 Issuing Straight Debt to a Bank

When \( N \) is high, that is, \( N > \overline{N} \), the HOF prefers involving family in the decision-making process, even if it leads to lower growth. Issuing straight debt to a bank satisfies the properties of the optimal sharing rule of cash-flows provided that the face value of debt, \( R_p \), given by the combination of (7) and (8) (binding) is set high enough to deprive the HOF from a sufficient fraction of the cash-flows when growth materializes. Intuition says that this is easier when \( N \) takes high enough values. Simple computation shows that this indeed is possible when \( N > \overline{N} \).

The three regimes obtained above are consistent with empirical evidence that family firms follow differing strategies in terms of growth. For instance, Poutziouris (2000) categorizes U.K. small and medium-sized family firms into four generic groups. “Exiters” (less than 4%) consider exit options through trade sale (or even flotation). “Open-growth stars” (21%) are interested in increasing the size or scale of the business and do not abide dogmatically to introverted family business traditions. “Traditionalists” (61%) are interested in maintaining the status quo and retaining control across generations. Finally, “strugglers” (15%) have no clear strategic orientation and try to survive even if it comes at the expense of retaining control. The first three groups match the three arrangements detailed in Proposition 4 and commented just above.

Note that it is also possible to interpret the results obtained in Proposition 4 according to family composition and cultural norms. Proposition 4 implies that, all else equal, the higher the number of children on the payroll, the lower the probability to sell out the firm and to approach venture capitalists rather than issue pure debt to banks. Proposition 4 implies that in countries in which family
is more highly valued, firms should resort more frequently to pure debt contracts with banks and less frequently to convertible claims sold to venture capitalists. They also should be less prone to selling out their business.

5 Conclusions

Although much of the financial economics literature on family firms focuses on whether these firms perform better or worse than their non-family counterparts, it should not be overlooked that there also exist important systematic differences among family firms. In this paper, I determine how family characteristics affect the decision to privilege either low growth (in order to maintain harmony of the family) or high growth, and relate this choice to the financing policy of the firm. I assume that letting family members participate in decision making provides the head of the family with satisfaction but is neither compatible with maximizing cash-flows nor with allowing efficient collaboration from a partner that could bring funds and also advice or complementary assets. I first show how the head of the family’s decision to pursue high growth versus low growth depends on family size, composition, and on cultural norms. Then, I relate this choice to the decision to sell the firm out, approach a VC, or approach a bank.

Appendix

For the sake of brevity, \((IC)_p\) refers to the incentive compatibility constraint of the partner and \((PC)_p\) to his participation constraint. The same remark applies to the constraints pertaining to the HOF. \((LL)_r\) refers to the limited liability constraint when growth fails to materialize.

Proof of Proposition 1

First, let us rearrange the maximization program. Consider \((IC)_p\). The partner’s supportive investment is given by the FOC: \(I_s = \frac{\alpha(R_p - r_p)}{k}\). The partner’s utility function is concave since \(k > 0\) so that \(I_s\) is the maximum. Since \(I_f > r, R_p > r_p\). Hence, \(I_s > 0\). Furthermore, (12) will imply that \(I_s \leq 1\). This is consistent with \(I_s\) being a probability. Replacing \(I_s\) into \((IC)_{HOF}\) leads to
\[ \alpha^2 (R_p - r_p)^2 - (\alpha^2 \Delta R - k\delta p) (R_p - r_p) - k (\delta p \Delta R - N) \leq 0. \] Since \( \alpha^2 \Delta R - k\delta p \leq 0 \), it reduces to

\[ R_p - r_p \leq \frac{\alpha^2 \Delta R - k\delta p + \sqrt{(\alpha^2 \Delta R + k\delta p)^2 - 4\alpha^2 kN}}{2\alpha^2} \quad \text{and} \quad (12) \]

\[ N < \delta p \Delta R. \quad (13) \]

Substituting \( I_s \) into \((PC)_p\) leads to \( p_h R_p + (1 - p_h)r_p + \frac{\alpha^2 (R_p - r_p)^2}{2k} - I_f \geq 0 \).

Similarly, substituting \( I_s \) into the objective function leads to

\[ V = v_h + \frac{\alpha^2 (R_p - r_p) \Delta R}{k} - \frac{\alpha^2 (R_p - r_p)^2}{2k}. \quad (14) \]

Maximizing (14) with respect to \( R_p \) and \( r_p \) is equivalent to maximizing \((R_p - r_p)\) since \( V \) is strictly increasing in \((R_p - r_p)\) on the interval \([0, \Delta R]\): \( v_h \) depends neither on \( R_p \) nor on \( r_p \). \((R_p - r_p) < \Delta R\), while \( k > 0 \) and \( \alpha > 0 \). Note that given \( R_p - r_p < \Delta R \) when \( N > 0 \) (see (12)):

\[ r_p \leq r \Rightarrow R_p < R. \quad (15) \]

Furthermore,

\[ r_p \geq 0 \Rightarrow R_p > 0 \quad (16) \]

because \( R_p > r_p \). Given (15) and (16), . To summarize, the maximization program can be rewritten as

\[ \max_{R_p, r_p} (R_p - r_p) \]

\[ \text{s.t.} \quad (IC)_{HOF} : R_p - r_p \leq \frac{\alpha^2 \Delta R - k\delta p + \sqrt{(\alpha^2 \Delta R + k\delta p)^2 - 4\alpha^2 kN}}{2\alpha^2} \]

\[ (PC)_p : p_h R_p + (1 - p_h)r_p + \frac{\alpha^2 (R_p - r_p)^2}{2k} - I_f \geq 0 \]

\[ (LL)_r : 0 \leq r_p \leq r. \]

Next, let us solve the above program. Making \((IC)_{HOF}\) bind is the best potential solution.
Let us check that \((PC)_p\) is satisfied.

**Case 1: the partner’s participation constraint is binding**

Combining \((IC)_{HOF}\) and \((PC)_p\) both satisfied with equality leads to a system of two equations and two unknowns, the solutions of which are

\[
\begin{align*}
    r_p &= I_f - p_h \left[ \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2} \right] \\
    R_p &= I_f + (1 - p_h) \left[ \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2} \right] - \frac{\alpha^2}{2k} \left[ \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2} \right]^2.
\end{align*}
\]

(17)

(18)

It is straightforward that \(r_p\) strictly increases in \(N\). Conversely, \(R_p\) strictly decreases in \(N\). For

\[
\frac{\partial R_p}{\partial N} (N) = \frac{1}{2} \left[ 1 - \frac{2k - \left( \alpha^2 \Delta R + k (p_h + p_l) \right)}{\sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}} \right] < 0
\]

which is verified. Indeed, on the one hand, (i) \(k \geq \alpha \Delta R\), (ii) \(1 - p_h \geq \alpha\) and (iii) \((1 - p_l) > 0\) imply that \((1 - p_l) \left[ \Delta R - \frac{k (1 - p_h)}{\alpha} \right] \leq 0\). On the other hand, \(N > 0\).

When growth fails to materialize, the HOF is protected by limited liability if \(r_p \leq r\) which is equivalent to

\[
N \leq \bar{N} \equiv \frac{d}{\delta p} \Delta R - 2 (I_f - r) + \left[ \alpha^2 \Delta R + k (p_h + p_l) \right] \left[ \frac{-p_h k + \sqrt{(p_h k)^2 + 2 \alpha^2 k(I_f - r)}}{\alpha^2 k} \right].
\]

(19)

Note that \(k \geq \frac{\alpha^2 \Delta R}{\delta p}\) and \(I - r > 0\) imply that \(\bar{N} < \delta p \Delta R\). Besides, \(\bar{N} > 0 \Leftrightarrow I_f < p_h R + (1 - p_h) r + \frac{\alpha^2 \Delta R^2}{2k}\), which is a necessary condition to be financed. When the project fails, the partner is protected
by limited liability if \( r_p \geq 0 \) which is equivalent to

\[
\Leftrightarrow \quad N \geq N \frac{d}{\alpha^2 \Delta R - 2 I_f} + \left[ \frac{\alpha^2 \Delta R + k (p_h + p_l)}{\alpha^2 k} \right]^2 \frac{-p_h k + \sqrt{(p_h k)^2 + 2 \alpha^2 k I_f}}{\alpha^2 k}.
\] (20)

Consequently, \((LL)_r\) is compatible with \((PC)_p\) and \((IC)_{HOF}\) both binding on \([N, \bar{N}]\). Observe that \(\bar{N} > 0 \Leftrightarrow I_f < p_h R + \frac{\alpha^2 \Delta R^2}{2k}\).

**Case 2: the partner’s participation constraint is not binding**

Now, consider the interval \([0, N]\). Suppose \((IC)_{HOF}\) is kept binding so that \((R_p - r_p)\) is equal to the best potential solution. Then, as shown above, \(r_p \geq 0\) is not compatible with having the partner just break even. Suppose instead that

\[
p_h R_p + (1 - p_h) r_p + \frac{\alpha^2 (R_p - r_p)^2}{2k} - I_f = t \quad \text{with } t > 0.
\] (21)

Then, setting

\[
r_p = 0
\] (22)

is consistent with \((LL)_r\) and implies, because \((IC)_{HOF}\) is binding, that

\[
R_p = \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2}.
\] (23)

Combining (21), (22) and (23) leads to

\[
t(N) = \frac{p_h}{2 \alpha^2} \left[ \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2} \right] + \frac{1}{8 \alpha^2 k} \left[ \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4 \alpha^2 k N}}{2 \alpha^2} \right]^2 - I_f.
\] (24)

Observe that \(t(N) > 0\) and \(\frac{\delta t}{\delta N}(N) < 0\). As investors are competitive, one can set up a mechanism that
allows the entrepreneur to capture \( V \), even on \([0, \overline{N}]\). This takes the form of an up-front transfer equal to \( t \) from the partner to the entrepreneur.

Combining (17) and (18), or (22) and (23), with \( I_s = \frac{\alpha (R_p - r_p)}{k} \) gives the level of supportive investment by the partner on \([0, \overline{N}]\):

\[
I_s = \frac{\alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 kN}}{2\alpha k}.
\] (25)

Accordingly,

\[
V = v + \frac{\Delta R}{2k} \left[ \alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 kN} \right]
- \frac{1}{8\alpha^2 k} \left[ \alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 kN} \right]^2.
\] (26)

That \( \frac{\delta E}{\delta N} (N) < 0 \) while \( \frac{\delta V}{\delta I_s} (I_s) > 0 \) implies that \( \frac{\delta V}{\delta I_s} (N) < 0 \). Since \((LL)_r\) is verified, \((PC)_{HOF}\) which was not mentioned in the text for the sake of conciseness - is satisfied. Besides, the HOF captures \( V \).

**Proof of Proposition 4**

Let \( N \) be such that \( r_p \times R - r \times R_p = 0 \), or

\[
N = \frac{d}{\delta p \Delta R - 2I_f + \left[ \Delta R \left( \alpha^2 \Delta R + k (p_h + p_l) \right) \right] + 2kr} \times \left[ \frac{-k [p_h R + (1 - p_h) r] + \sqrt{[(p_h R + (1 - p_h) r) k]^2 + 2k \alpha^2 \Delta R^2 I_f]}{k \alpha^2 \Delta R^2} \right] .
\] (27)

The implementation of the optimal financial contract corresponding to the case where \( N \in [0, \overline{N}] \) is analyzed in the text. Consider the complementary case where \( N \in [\underline{N}, \overline{N}] \). The optimal contract can be implemented in the following two ways.

**Mix of debt and equity**

Let \( \theta \) be the percentage of equity the partner gets and \( d \) the level of debt the partner is entitled to.
Since $N \in \overline{N, N}$ $\Rightarrow r_p < r$, then $d < r$. Thus, a mix $(d, \theta)$ of debt and equity is characterized by

\begin{align*}
    d + \theta(r - d) &= r_p \quad (28) \\
    d + \theta(R - d) &= R_p \quad (29) \\
    0 < d < r \quad (30) \\
    0 < \theta < 1. \quad (31)
\end{align*}

Solving the system ((28), (29)) of two equations and two unknowns gives

\begin{align*}
    d &= \frac{r_p \times R - r \times R_p}{\Delta R - (R_p - r_p)} \quad (32) \\
    \theta &= \frac{R_p - r_p}{\Delta R}. \quad (33)
\end{align*}

According to (32), $d < r \Leftrightarrow N < \overline{N}$ which is verified. Besides, $r_p \times R - r \times R_p > 0$ (since $N > \overline{N}$) implies that $d > 0$. Thus (30) is verified. It is easy to check (31) from (33) as $0 < R_p - r_p < \Delta R$.

Replacing $r_p$ and $R_p$ by their values into (32) and (33), and re-arranging leads to

\begin{align*}
    \theta &= \left[ \frac{\alpha^2 \Delta R - k\delta p + \sqrt{(\alpha^2 \Delta R + k\delta p)^2 - 4\alpha^2 kN}}{2\alpha^2 \Delta R} \right] \quad (34) \\
    d &= r - \left[ \frac{2\alpha^2 \Delta R}{\alpha^2 \Delta R + k\delta p - \sqrt{(\alpha^2 \Delta R + k\delta p)^2 - 4\alpha^2 kN}} \right] \\
    &\times \left[ r - I - \frac{P_p}{2\alpha^2k} \left[ \frac{\alpha^2 \Delta R - k\delta p + \sqrt{(\alpha^2 \Delta R + k\delta p)^2 - 4\alpha^2 kN}}{\alpha^2 \Delta R - k\delta p + \sqrt{(\alpha^2 \Delta R + k\delta p)^2 - 4\alpha^2 kN}} \right] \right]. \quad (35)
\end{align*}

Note that $\theta$ is decreasing in $N$, while $d$ is increasing in $N$ (recall that $r_p$ is increasing in $N$).

**Convertible preferred equity or convertible debt for the partner**

Convertible preferred equity is characterized by a minimum amount $r_p$ (with $r_p \leq r$) that the holder of the claim gets, which guaranties him a superior return, as compared to the common stock-holders’
one when the project fails. Thus, if $\delta$ is the fraction of preferred stock the partner is entitled to, it must verify $\delta r_p < \frac{r_p}{R} < \delta R$ in order to have conversion occur if and only if the project succeeds. Assume for simplicity that conversion does not trigger any issuance of new claim: the financial contract specifies that the entrepreneur releases some of her claims to the partner (such contracts actually exist). Convertible preferred equity is feasible if one can find $r$ and $\delta$ such that

$$\frac{r_p}{R} = r_p$$  \tag{36}$$

$$\delta R = R_p$$  \tag{37}$$

$$\delta r < \frac{r_p}{R} < \delta R$$  \tag{38}$$

$$0 < \frac{r_p}{R} < r$$  \tag{39}$$

$$0 < \delta \leq 1.$$  \tag{40}$$

Note that $\frac{r_p}{R} = r_p$ ensures that (39) is verified since $(LL)_r$ holds. It follows from (i) $r_p = r_p$, (ii) $r_p < R_p$ and (iii) $R_p = \delta R$ that $\frac{r_p}{R} < \delta R$ is verified. Furthermore, since $\delta r < r_p \Leftrightarrow r_p \times R - R_p \times r > 0$ is verified (since $N > N$) and since $\frac{r_p}{R} = r_p$, it follows that $r_p > \delta r$ is also verified. Thus, (38) is satisfied. Then, one can rewrite (37) as

$$\delta = \frac{R_p}{R}.$$  \tag{41}$$

From (37) and given that $0 < R_p < R$ (implied by $(LL)_r$), it is straightforward that (40) holds. Finally,
replacing $r_p$ and $R_p$ by their values into (36) and (41) leads to\footnote{In the simple one-period model without tax considered here, convertible preferred equity is equivalent to convertible debt. Convertible debt is characterized by a level of debt $d'$ (with $d' < r$) and a fraction $\delta$ of equity which is granted to the partner if the partner converts debt into equity. Note that $\delta R^T < d' < \delta R^{Ih}$ implies that the partner converts debt into equity if and only if growth materializes. As a consequence, $d'(= r)$ is given by (42), while $\delta$ is such as in (43).}

\[
\begin{align*}
    r_p &= I - \frac{p_h}{2\alpha^2} \left[ \alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 k N} \right] \\
    &\quad - \frac{1}{8k\alpha^2} \left[ \alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 k N} \right]^2 \tag{42}
\end{align*}
\]

\[
\begin{align*}
    \delta &= \frac{I + \frac{(1-p_h)}{2\alpha^2} \left[ \alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 k N} \right]}{R} \\
    &\quad - \frac{1}{8k\alpha^2} \left[ \alpha^2 \Delta R - k \delta p + \sqrt{(\alpha^2 \Delta R + k \delta p)^2 - 4\alpha^2 k N} \right]^2 \tag{43}
\end{align*}
\]

References


