EQUITY OPTIONS, CREDIT DEFAULT SWAPS AND LEVERAGE: A SIMPLE STOCHASTIC-VOLATILITY MODEL FOR EQUITY AND CREDIT DERIVATIVES

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Abstract

The aim of this paper is to define a model which allows traders to assess the value of equity and credit derivatives in a unified framework. We propose closed-form formulas which traders could use to evaluate equity, equity options and credit default swaps (CDSs) in a consistent way. The model can also be used to solve the *inverse problem*, that is to extract credit-risk sensitive information from market quotes of equity/credit derivatives. In particular, we wish to estimate the firm's *leverage*, as it is perceived by traders. This goal is achieved within a model *à la* Leland (1994), where stockholders have a perpetual American option to default. After making the case for modeling debt in terms of a single perpetual-bond equivalent issue, we define leverage, show the stochastic nature of equity volatility and derive the term structures of default probabilities and credit spreads by making use of the first-passage time distribution function. Then, we give new formulas for call and put options written on stockholders' equity. The formulas, which depend on the leverage parameter L and make use of the *univariate* normal distribution function, are consistent with the volatility skew observed in the equity options market and converge to the Black-Scholes-Merton (BSM) equations for $L \rightarrow 1$. All the Greeks are simple functions of the standard corresponding letters of the BSM model. The paper concludes with an application of the model to the case of Lehman Brothers and General Motors.

Journal of Economic Literature classification: G13 (Financial Economics, General Financial Markets, Contingent Pricing; Options Pricing).

Keywords: equity options, credit default swaps, leverage, stochastic volatility, perpetual options, first-touch digitals, Greeks, default probabilities, put-call parity.

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CONTENTS

1. INTRODUCTION

The aim of this paper is to define a model which allows traders to assess the value of equity and equity/credit derivatives in a unified framework. We propose closed-form formulas which traders could use to evaluate equity, equity options and credit default swaps (CDSs) in a consistent manner. The model can also be used to solve the *inverse problem*, that is to extract credit-risk sensitive information from market quotes. In particular, the model allows an estimate of the firm's leverage, as it is perceived by traders.

In *equity option markets*, traders often quote the implied volatilities of European options – calculated according to the Black-Scholes-Merton model – rather than prices. The quotes reflect the shape of the probability distribution of future returns of the underlying asset. The implied volatilities generally decrease as the strike price increases, showing a typical downward-sloping *volatility skew*. This is consistent with the hypothesis that the actual distribution of future returns used by option traders has extra weight on the left tail with respect to the normal distribution of Black-Scholes-Merton model.

One possible explanation for the volatility skew in equity options concerns leverage:[1]

> As a company's equity declines in value, the company's leverage increases. This means that the equity becomes more risky and its volatility increases. As a company's equity increases in value, leverage decreases. The equity then becomes less risky and its volatility decreases. This argument suggest that we can expect the volatility of a stock to be a decreasing function of the stock price ...

To define leverage, we need first to define debt. A useful approach is to assume that the mixture of bonds with different coupons, bank loans and leases of a firm can be approximated by a single issue of a *perpetual* fixed-rate bond with the same "stochastic duration" of the actual debt.[2] The assumption of a infinitely-lived security is not only mathematically convenient, but also a good proxy for a short-term debt rolled over again and again as with perpetual floaters. This assumption has been extensively used by Leland (1994, 1995, 2006, 2009).[3] In his model, stockholders have a perpetual American option to default. Our changes with respect to the original Leland model (1994) are in the spirit of Goldstein-Ju-Leland (2001), where equity depends on the tax rate.

In the markets for *credit default swaps*, the term structure of CDS spreads reflects the expectations of market participants about the firm's default risk for various time horizons (typically 1, 3, 5, 7, 10 years) and the requested (credit / liquidity) premia. The main advantage of our structural model is that the default barrier is *endogenously* given as solution of a stockholders' maximization problem. This allows to derive closed-form formula for default probabilities and credit spreads which make use of the first-passage time distribution function. We do not need to separately estimate *ad hoc* values of recovery rates.

The structure of the paper is as follows. First, we characterize the different claims hold by the firm's main *stakeholders*, then we argue that the actual debt can be approximated by a single perpetual-bond equivalent issue and highlight the model's differences with respect to Leland original article. After defining leverage, we show the stochastic nature of equity volatility and derive the term structures of default probabilities and credit default swaps (CDSs) spreads. Then, we derive new formulas for equity options and show how the model's parameters can be estimated by using quotes of equity and equity options. In particular, we show how to use market data to extract the traders' perceptions of a firm's leverage, measured in terms of a single perpetual-bond equivalent issue.

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[1] HULL, John C., *Options, Futures, and Other Derivatives*, 8th ed., Pearson, p. 415, 2011.

[2] The stochastic duration of a bond is defined as the time to maturity of a zero-coupon bond with the same sensitivity to changes of interest rates, i.e. with the same basis risk. "If we wish stochastic duration, D_3, to be a proxy for basis risk of coupon bonds with the units of time, then it is natural to define it as the maturity of a discount bond with the same risk." (p. 56). See COX, John C., INGERSOLL, Jonathan E., and ROSS, Stephen A., "Duration and the Measurement of Basis Risk", *Journal of Business*, vol. 52, no. 1, pp. 51-61, January 1979.

[3] LELAND, Hayne, "Corporate Debt Value, Bond Covenants, and Optimal Capital Structure", *Journal of Finance*, 49 (4), pp. 1213-52, September 1994. LELAND, Hayne, "Bond Prices, Yield Spreads, and Optimal Capital Structure with Default Risk", Finance Working Paper no. 240, Haas School of Business, University of California at Berkeley, January 1995. LELAND, Hayne, "Princeton Lectures" [*Lecture 1* - Pros and Cons of Structural Models - An Introduction, *Lecture 2* - A New Structural Model, *Lecture 3* - Financial Synergies and the Optimal Scope of the Firm - Implications for Mergers and Structured Finance], 2006. LELAND, Hayne, "Structural Models and the Credit Crisis", China International Conference in Finance, July 8, 2009.

TABLE 1 Contracts between stakeholders.

Contracts	*Stakeholders*			
	Stockholders	Bondholders	Third parties	Tax Authority
Firm's assets	V_0	-	-	-
Risk-free bond	$-Z$	Z	-	-
Option to default	$P \equiv (Z - V_b)\, p_b$	$-P \equiv -(Z - V_b)\, p_b$	-	-
Bankruptcy security	-	$-A \equiv -\alpha\, V_b\, p_b$	$A \equiv \alpha\, V_b\, p_b$	-
Tax claims	$-G_S \equiv -\theta\,(V_0 - Z + P)$	$-G_B = -\theta\,(Z - P - A)$	$-G_U \equiv -\theta\, A$	$G_0 \equiv G_S + G_B + G_U$
Total	$S_0 \equiv (1 - \theta)\,(V_0 - Z + P)$	$B_0 \equiv (1 - \theta)\,(Z - P - A)$	$U_0 \equiv (1 - \theta)\, A$	$G_0 \equiv \theta\, V_0$

Note: p_b is the value of a *perpetual* first-touch digital option which pays \$1 when $V = V_b$ at default time τ.

2. CAPITAL STRUCTURE

The contractual relationships among the various firm's "stakeholders", considered in the Leland model, can be synthesized as in Table 1. The firm's ownership is shared between stockholders, bondholders, third parties (lawyers, accountants, courts, etc.) and the tax authority, which has the right to receive the share θ of the firm's earnings (θ is the tax rate). The assets' value, V, is divided in two parts: θV to the tax authority and $(1 - \theta)V$ to the other stakeholders.

Stockholders issue a perpetual bond with nominal value Z, coupon $C = r\,Z$, and market value B. Because of firm's limited liability, they have an option to default, that is the (perpetual) right to sell the assets at price Z to the bondholders. In other terms, they have a perpetual American put option, with strike Z and market value P, written on V.

When $V = V_b$, stockholders exercise their option to default. This prevents equity's value to become negative. When the (put) option is exercised, stockholders sell the firm's assets, whose value is V_b, and receive Z from bondholders. Such a bankruptcy triggers the execution of another contract. When the firm defaults, third parties claim a share α of the firm.

Therefore, the contracts "negotiated" among the various parties are as follows:

1. *stockholders* use their own capital to buy the firm's assets. Assets, whose current value is V_0, can be tangible and intangible (including human capital);
2. *Tax Authority* claims a share θ of the firm's assets as soon as the firm is created (the *Tax Authority* is a "special partner" of *stockholders*);
3. *bondholders* buy a perpetual fixed-rate bond from stockholders. The face value of the *corporate bond*, with coupon C, is Z. The bond contains two embedded options: a short perpetual option to default in favor of *stockholders*, and a short perpetual digital option (or *bankruptcy security*) in favor of *third parties*. The option to default, with strike Z, is optimally exercised at the default time τ, when $V = V_b$. The perpetual digital option, with barrier V_b, offers a rebate $\alpha\, V_b$ ($0 < \alpha < 1$) at τ. As soon as the bond is issued, the tax burden θV_0 is redistributed among the three firm's claimants, to include the newcomers (bondholders and third parties).

The current values of the four securities (stock, corporate bond, bankruptcy security, tax claim) are, respectively, S_0, B_0, U_0, G_0.

Here we assume that individuals (stockholders, bondholders, third parties) are taxed at the same effective rate, θ. Besides, we assume that stockholders, bondholders, third parties are taxed, respectively, only when dividends, interests and fees are paid. This means, in particular, that – in order to avoid double taxation – retained earnings are not taxed.[4]

[4] In the notation used by Goldstein-Ju-Leland (2001), interest payments to investors are taxed at a personal rate τ_i, "effective" dividends are taxed at τ_d, and corporate profits are taxed at τ_c. We assume that $\tau_d = \tau_i$ and $\tau_c = 0$. Therefore, the effective tax rate, τ_{eff}, defined by $(1 - \tau_{\text{eff}}) = (1 - \tau_c)(1 - \tau_d)$ is simply equal to $\tau_d = \tau_i$ (and to θ, in our notation).

Stakeholders

The *Tax Authority* is long on a simple linear contract, with current value

$$G_0 \equiv \theta V_0 \tag{1}$$

This is the result of three claims:

1. a claim toward stockholders, with current value $G_S = \theta\,(V_0 - Z + P)$;
2. a claim toward bondholders, with current value $G_B = \theta\,(Z - P - A)$;
3. a claim toward third parties, with current value $G_U = \theta\,A$.

Third parties (lawyers, etc.) have a portfolio with current value

$$U_0 \equiv (1-\theta)\,A \tag{2}$$

They are:

1. long on a perpetual *bankruptcy security*, with current value A, bought from bondholders;
2. short on the tax claim $G_U \equiv \theta\,A$.

Bondholders have a portfolio with current value

$$B_0 \equiv (1-\theta)(Z - P - A) \tag{3}$$

They are:

1. long on a perpetual *risk-free bond*, with constant value Z, bought from stockholders;
2. short on a perpetual *option to default*, with current value P, sold to stockholders;
3. short on a perpetual *bankruptcy security*, with current value A, sold to third parties;
4. short on the tax claim $G_B = \theta\,(Z - P - A)$.

Finally, *stockholders* have a portfolio with current value

$$S_0 \equiv (1-\theta)(V_0 - Z + P) \tag{4}$$

They are:

1. long on the firm's assets, with current value V_0;
2. short on a perpetual *risk-free bond*, with constant value Z, sold to bondholders;
3. long on a perpetual *option to default*, with current value P, bought from bondholders;
4. short on the tax claim $G_S = \theta\,(V_0 - Z + P)$.

Note that the total value of liabilities, $S_0 + B_0 + U_0 + G_0$, should be equal to the value of the firm's assets, V_0:

$$\begin{aligned} S_0 + B_0 + U_0 + G_0 &= (1-\theta)(V_0 - Z + P) + (1-\theta)(Z - P - A) + (1-\theta)\,A + \theta V_0 \\ &= V_0 \end{aligned} \tag{5}$$

Dynamics of Assets' Value

The dynamics for the value, V, of the firm's assets is described by a diffusion-type stochastic process with stochastic differential equation

$$dV = (\mu_V - q_V)\,V\,dt + \sigma_V\,V\,dz$$

where

μ_V is the instantaneous expected rate of return on the firm per unit time;
q_V is the payout rate (to shareholders, bondholders and Tax Authority);
σ_V is the asset volatility (i.e. the standard deviation of the assets' rate of return per unit of time);
dz is a standard Wiener process

Because of Merton's hedging argument, the price, f, of any time-insensitive derivative with no intermediate payments (as the *perpetual option to default* or the *bankruptcy security*) should satisfy the following differential equation

$$\frac{1}{2}\frac{d^2 f}{dV^2}\sigma_V^2 V^2 + (r - q_V)V\frac{df}{dV} - rf = 0$$

where r is the risk-free interest rate.

In this context, it is appropriate to note that Merton's hedging argument *does* hold even if all of the firm's assets are not tradable nor observable. Although V may not be the value of a traded asset, trading of equity (or other contingent claims) allows use of V as the state variable:[5]

> To understand the intuition of the replication argument, consider an analogy with an ordinary stock option model. Fundamentally, the option can be priced precisely because we can replicate its payoff using the stock and risk free bonds. However, we can just as well value the stock by replicating its payoff using the (traded) option. In the same fashion, we can value the firm's assets using stocks and risk free bonds. [Ericsson & Reneby (2002), p. 5]

Under the assumption of a geometric Brownian motion for V, we can price the contracts negotiated by the various stakeholders.

Contracts

The current value, A, of the *bankruptcy security* is equal to:

$$A = \alpha V_b\, p_b \tag{6}$$

where

- α is the ratio between (direct and indirect) bankruptcy costs and the market value of assets prior to bankruptcy;[6]
- V_b is the optimal default trigger, chosen by stockholders to maximize the value of equity:[7]

$$V_b = Z\frac{\gamma_2}{\gamma_2 - 1} \tag{7}$$

- p_b is the value of a *perpetual* first-touch digital option which pays \$1 when $V = V_b$:[8]

$$p_b = (V_0 / V_b)^{\gamma_2} \tag{8}$$

- γ_2 is the elasticity of the perpetual first-touch digital option with respect to V:

$$\gamma_2 \equiv \frac{-(r - q_V - \sigma_V^2/2) - \sqrt{(r - q_V - \sigma_V^2/2)^2 + 2\sigma_V^2 r}}{\sigma_V^2}$$

- Z is the constant value of a *perpetual risk-free bond*:

$$Z = \frac{C}{r} \tag{9}$$

- C is the instantaneous coupon per year of a *perpetual risk-free bond*.

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[5] ERICSSON, Jan, and RENEBY, Joel, "A Note on Contingent Claims Pricing with Non-Traded Assets", SSE/EFI Working Paper Series in Economics and Finance No. 314, June 2002.

[6] Goldstein-Ju-Leland (2001, footnote 20, p. 497) report that direct bankruptcy costs, estimated by Warner (1977), are about 1% of assets' value: "As measured here, the cost of bankruptcy is on average about one percent of the market value of the firm prior to bankruptcy" [Warner (1977), p. 377]. See WARNER, Jerold B., "Bankruptcy Costs: Some Evidence", *Journal of Finance*, vol. 32, no. 2, pp. 337–47, May 1977. However, taking account for indirect costs, Goldstein-Ju-Leland choose α = 5% in their base case.

[7] As we will see later, this is different from the default trigger originally derived by Leland [1994, Equation (14), p. 1222].

[8] For a proof of p_b see, for instance, BARONE, Gaia, "European Compound Options Written on Perpetual American Options", *Journal of Derivatives*, Spring 2013.

The current value, P, of the *option to default* is equal to:[9]

$$P = (Z - V_b)p_b \tag{10}$$

Therefore, substituting (10) into (4) gives the current value, S_0, of equity as

$$S_0 \equiv (1-\theta)[V_0 - Z + (Z - V_b)(V_0 / V_b)^{\gamma_2}] \tag{11}$$

Payouts

The model does not allow the firm to change its business risk, measured by σ_V, which is a constant, but allows for the liquidation of assets to make interest, dividend and tax payments. The firm's payout policy is defined by q_V

$$\begin{aligned} q_V V &= \text{net interests} + \text{net dividends} + \text{taxes} \\ &= \text{gross interests} + \text{gross dividends} \\ &= q_B \frac{B}{1-\theta} + q_S \frac{S}{1-\theta} \end{aligned} \tag{12}$$

where

q_B is the before-tax bond yield

$$q_B = r\frac{(1-\theta)Z}{B} \tag{13}$$

q_S is the before-tax dividend yield.

The payout rate, q_V, determines the cash flow $q_V V$ which is taken out of the assets of the firm. What is left out of this cash flow (after paying interest on debt) is paid out to shareholders as dividends. If $q_V V$ is insufficient to cover coupons on the bonds, shareholders receive a negative dividend (i.e., contribute additional cash to the firm). A negative dividend (a cash-flow crisis) does not mean that that it is optimal to default: Expected future cash flows could be sufficiently high to induce stockholders to keep the firm alive.

Default Point and Renegotiation

Stockholders have to determine the optimal default point, V_b. We will suppose that V_b is not affected by renegotiation. In other terms, stockholders of distressed firms will not try to persuade bondholders to renegotiate the contractual terms, even if they have a common interest in avoiding the losses associated with bankruptcy.

The sub-optimality of renegotiation, for both stockholders and bondholders, has been argued by Ingersoll (1987, p. 419):

> A natural question at this point is, if the firm is bankrupt at time *T*, why should the bondholders not renegotiate the contract in hopes that subsequent good fortune would allow the firm to pay them in full? The answer should be clear. Under the current contract they have the right to receive all the assets of the firm. Why should they settle for less? If the firm is fortunate, they can have all of the profits rather than sharing them. Of course, they would renegotiate if the shareholders made the right concession – add more money to the firm. However, they would have to add enough to make up the bankruptcy shortfall plus an amount equal to whatever claim on the refinanced firm they would like to own. But again, why should they do that? It would cost more than the value they would receive in return.

Therefore we will assume that, if bankruptcy occurs, bondholders receive all assets (*after costs*) and stockholders none.

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[9] Note that our $Z - P$ is equal to $F(V)$ in Black & Cox [(1976), Equation (16), p. 364], where $c = r Z$, $\overline{V} = V_b$, $V = V_0$ and $\alpha = -\gamma_2$:

$$F(V) = \frac{c}{r} + \left(\overline{V}^{\alpha+1} - \frac{c}{r}\overline{V}^{\alpha}\right)V^{-\alpha}$$

The optimal default point, V_b, is chosen by stockholders as the level of V that maximizes the value of equity. By (11), maximizing S_0 with respect to V_b gives

$$V_b = Z \frac{\gamma_2}{\gamma_2 - 1} \tag{14}$$

Greeks

By (4), the value of equity is

$$S_0 \equiv (1-\theta)(V_0 - Z + P) \tag{15}$$

and, by (10) and (8), the value of the perpetual option to default is

$$P = (Z - V_b)(V_0 / V_b)^{\gamma_2}$$

Therefore, the delta, Δ_P, and gamma, Γ_P, of the perpetual option to default are

$$\Delta_P \equiv \frac{\partial P}{\partial V} = \gamma_2 \frac{P}{V} \tag{16}$$

$$\Gamma_P \equiv \frac{d^2 P}{d V^2} = \frac{\gamma_2 (\gamma_2 - 1) P}{V^2}$$

and the delta, Δ_S, and gamma, Γ_S, of equity are

$$\Delta_S \equiv \frac{d S}{d V} = (1-\theta)(1 + \Delta_P) \tag{17}$$

$$\Gamma_S \equiv \frac{d^2 S}{d V^2} = (1-\theta)\Gamma_P$$

By taking the limits of Δ_S for $V \to V_b$ and $V \to \infty$, we can see that $0 \leq \Delta_S < (1 - \theta)$. Besides, $\Gamma_S > 0$.

Therefore, S is a convex function of V, consistently with the "option-like" nature of equity. Figure 1 reports equity, S, as a function of asset's value, V, for different levels of debt's notional value, Z

Leverage

We define leverage, L, as the ratio between the non-Government value, $(1 - \theta)V$, of firm's assets and the value, S, of equity:

$$L = \frac{(1-\theta)V}{S} \tag{18}$$

Substituting (15) into (18) and taking the limits of L for $V \to \infty$ and $V \to V_b$ shows that $1 < L < +\infty$.

Equity Volatility

By Itô's lemma, the equity volatility, σ_S, is

$$\sigma_S = \frac{\Delta_S V \sigma_V}{S} \tag{19}$$

Substituting (16)-(18) into (19) gives

$$\sigma_S = \left(1 + \gamma_2 \frac{P}{V}\right) L \sigma_V \tag{20}$$

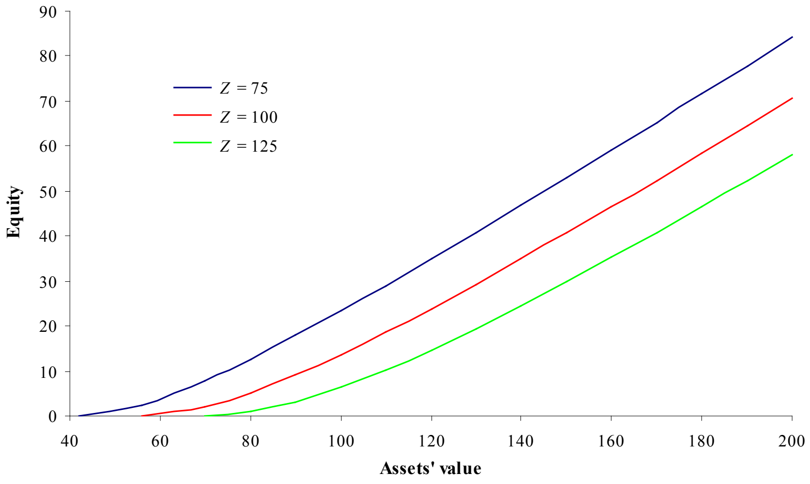


Figure 1 Equity, S, as a function of asset's value, V, for different levels of debt's notional value, Z. ($r = 4\%$, $q_V = 6\%$, $\sigma_V = 10\%$, $\theta = 35\%$, $\alpha = 5\%$).

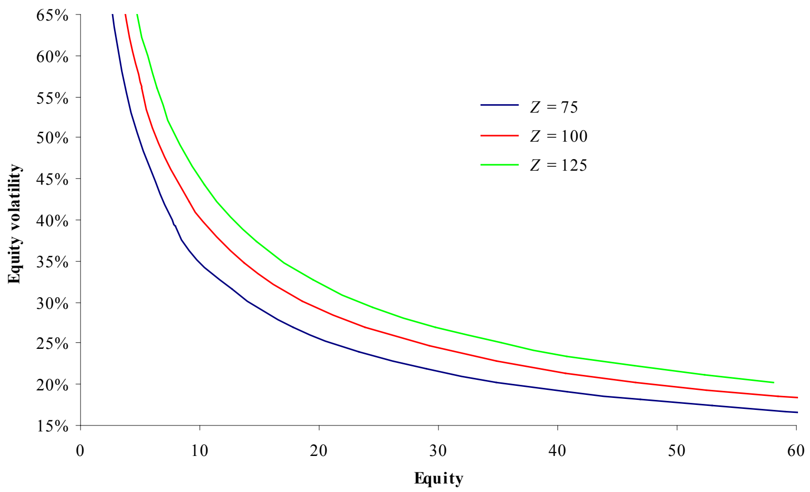


Figure 2 Equity volatility, σ_S, as a function of equity's value, S, for different levels of debt's notional value, Z. ($r = 4\%$, $q_V = 6\%$, $\sigma_V = 10\%$, $\theta = 35\%$, $\alpha = 5\%$).

Equation (20) reveals the stochastic nature of equity volatility, which is a complex function of V.[10] Taking the limits of σ_S for $V \to \infty$ and $V \to V_b$ shows that $\sigma_V < \sigma_S < +\infty$.

Figure 2 reports equity volatility, σ_S, as a function of equity's value, S, for different levels of debt's nominal value, Z.

Some values of σ_S, as a function of Z, q_V and σ_V, are shown in Table 2, together with dividend yield (q_S), leverage (L), default trigger (V_b), and values of the firm's stakes (S_0, B_0, U_0, G_0).

The Case for Debt with Infinite-Maturity

A fundamental property of the Leland model is that debt is approximated by a single perpetual bond. While equity is simply a perpetual, residual claim on firm's assets, debt is much more difficult to define in general terms. It can be devised in various ways.

[10] Equation (20) shows that σ_S is a function of the random asset value, V. Strictly speaking, our model is a *local-volatility* model, which belongs to the more general class of *stochastic-volatility* models. The term *fully stochastic-volatility* has been often used to describe models where the asset volatility has a randomness of its own and is driven by a different Wiener process.

TABLE 2 The firm's stakes (S_0, B_0, U_0, G_0), default trigger (V_b), leverage (L), dividend yield (q_S) and equity volatility (σ_S).

					Z						Z		
			0	25	50	75	100		0	25	50	75	100
					q_V						q_V		
			1.3%	2.0%	2.7%	3.5%	4.3%		1.3%	2.0%	2.7%	3.5%	4.3%
	5%		65.00	48.75	32.50	16.53	5.09		0.00	16.25	32.50	48.37	58.84
	10%		65.00	48.75	32.71	18.65	9.11		0.00	16.25	32.25	46.01	54.91
σ_V	15%	S_0	65.00	48.84	33.75	21.57	13.05	B_0	0.00	16.15	31.13	42.99	51.07
	20%		65.00	49.19	35.41	24.65	16.81		0.00	15.77	29.41	39.89	47.39
	25%		65.00	49.84	37.34	27.66	20.38		0.00	15.11	27.45	36.89	43.90
	5%		0.00	0.00	0.00	0.10	1.07		35.00	35.00	35.00	35.00	35.00
	10%		0.00	0.00	0.04	0.34	0.97		35.00	35.00	35.00	35.00	35.00
σ_V	15%	U_0	0.00	0.01	0.12	0.43	0.88	G_0	35.00	35.00	35.00	35.00	35.00
	20%		0.00	0.03	0.18	0.46	0.80		35.00	35.00	35.00	35.00	35.00
	25%		0.00	0.05	0.21	0.45	0.72		35.00	35.00	35.00	35.00	35.00
	5%		0.00	23.60	46.17	66.02	80.79		1.00	1.33	2.00	3.93	12.77
	10%		0.00	20.62	39.45	55.41	68.15		1.00	1.33	1.99	3.49	7.13
σ_V	15%	V_b	0.00	17.46	33.14	46.46	57.55	L	1.00	1.33	1.93	3.01	4.98
	20%		0.00	14.64	27.78	39.08	48.76		1.00	1.32	1.84	2.64	3.87
	25%		0.00	12.28	23.35	33.03	41.49		1.00	1.30	1.74	2.35	3.19
	5%		2.00%	2.05%	2.15%	3.03%	5.89%		5.00%	6.67%	10.00%	19.05%	42.82%
	10%		2.00%	2.05%	2.14%	2.68%	3.29%		10.00%	13.33%	19.63%	31.22%	49.92%
σ_V	15%	q_S	2.00%	2.05%	2.07%	2.32%	2.30%	σ_S	15.00%	19.90%	27.80%	39.16%	54.40%
	20%		2.00%	2.03%	1.98%	2.03%	1.78%		20.00%	26.17%	34.65%	45.32%	58.28%
	25%		2.00%	2.01%	1.87%	1.81%	1.47%		25.00%	32.07%	40.68%	50.63%	62.00%

Note: $V_0 = 100$, $r = 4\%$, $\theta = 35\%$, $\alpha = 5\%$.

Often debt has a finite maturity, but this is not always the case. Leland (1994) assumes that it has an infinite maturity:

> Very long time horizons for fixed obligations are not new, either in theory or in practice. The original Modigliani and Miller (1958) argument assumes debt with infinite maturity. Merton (1974) and Black and Cox (1976) look at infinite maturity debt in an explicitly dynamic model. Since 1752 the Bank of England has, on occasion, issued Consols, bonds promising a fixed coupon with no final maturity date. And preferred stock typically pays a fixed dividend without time limit. [Leland (1994, p. 1215)]

Debt can be coupon-bearing or have no coupon, can be repaid by amortization or by a balloon payment at maturity, can pay a fixed, variable or step (mixed) interest, can be assisted by option-like covenants (in favor of the issuer or the holder), can include not-paid salaries and delayed payments to suppliers, can have different priority and subordination in the event of dissolution of the firm. Last, but not least, debt is always embedded in derivatives contracts which the firm could have entered into. The debt embedded in derivatives contracts can be extremely relevant:[11]

> Long-Term Capital Management used its $2.2 billion in capital from investors as collateral to buy $125 billion in securities, and then used those securities as collateral to enter into exotic financial transactions valued at $1.25 trillion. [Kahn and Truell (1998)]

In the well-known Moody's KMV approach, the reference debt – used to define the "default point" – is simply equal to the short-term liabilities plus half of long-term liabilities, both measured in nominal, accounting, book units.[12]

Over time, the firm's financial structure (leverage, maturity, etc.) changes. Debt can be rolled-over into other longer-maturity loans. In particular, firms can use revolving credit lines or loan commitments – made by a bank or other financial institution – to increase the flexibility of their financial structure.

Theoretically, firms can roll-over their debt an infinite number of times. This makes reasonable to approximate the actual debt structure with a unique perpetual *fixed-rate* bond, as in Merton (1974), Black-Cox (1976), Leland (1994), Goldstein-Ju-Leland (2001) and many other papers.[13]

From a theoretical standpoint, the time independency of perpetual securities assures that the firm's capital structure does not change abruptly over time, thus simplifying the analysis.[14]

From an empirical standpoint, the assumption of a single perpetual issue is an important advantage with respect to models which assume a finite-maturity debt, because there is no need to estimate the debt's maturity parameter, T.[15]

Differences with Respect to Leland's Original Model

The firm's capital structure we defined is different from that used by Leland (1994) in his original article. As in Leland, the value of equity, S, does not depend on the parameter α, which measures bankruptcy costs. However, differently from Leland, equity *does* depend on the effective tax rate, θ.

The definitions of debt, D, and equity, E, used by Leland [1994, Equations (7) and (13)] can be obtained, respectively, by merging the claims of bondholders with the claims of the Tax Authority towards bondholders and by merging the claims of stockholders with the claim of the Tax Authority towards stockholders plus "the value of the tax deduction of coupon payments", $TB(V) = \theta\, Z(1 - p_b)$.

---

[11] KAHN, Joseph, and TRUELL, Peter, "Troubled Investment Fund's Bets Now Estimated at $1.25 Trillion", *Wall Street Journal*, September 26, 1998.

[12] Moody's KMV defines the default point as "The point to which a firm's asset value must fall before the firm is unable to raise capital to meet either a principal or interest payment. It is approximately equal to the total amount of short-term liabilities, plus half of the long-term liabilities (precise definition varies by industry)." See Moody's KMV "Credit Monitor Quick Reference", 2004.

[13] BLACK, Fischer S., and COX, John C., "Valuing Corporate Securities - Some Effects of Bond Indenture Provisions", *Journal of Finance* 31 (2), 351-367, 1976. GOLDSTEIN, Robert, JU, Nengjiu, and LELAND, Hayne, "An EBIT-Based Model of Dynamic Capital Structure", *Journal of Business*, vol. 74, no. 4, 2001.

Leland (1995) has also proposed an "exponential model" where the firm retires the perpetual debt at a proportional rate m through time. This roll-over debt structure, with regular repayments and renewals of principal and of coupon, should guarantee a stationary debt structure. However Décamps and Villeneuve (2008) show that this extension of Leland's original model does not allow close-form formulas because the default point is not constant anymore, but depends on time. See LELAND, Hayne, "Bond Prices, Yield Spreads, and Optimal Capital Structure with Default Risk", Finance Working Paper no. 240, Haas School of Business, University of California at Berkeley, January 1995; DÉCAMPS, Jean-Paul, and VILLENEUVE, Stéphane, "On the modeling of Debt Maturity and Endogenous Default - A Caveat", Working Paper, May 2008.

[14] Note that, because the debt is fixed in nominal terms, the actual leverage tends to decrease in real terms over time. An alternative assumption would be to model debt as a perpetual *floating-rate* bond. However, if coupons were continuously paid at the rate r, the basis risk, i.e. the bond's sensitivity to interest rates, would be null (and this would contradict the empirical evidence). See COX, John C., INGERSOLL, Jonathan E., and ROSS, Stephen A., "An Analysis of Variable Rate Loan Contracts", *Journal of Finance*, vol. 35, no. 2, pp. 389-403, May 1980.

[15] MERTON, Robert C., "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates," *Journal of Finance* 29, No. 2 (May 1974), pp. 449–470, reprinted in Robert C. Merton, *Continuous-Time Finance*, Chapter 12 (Malden, MA: Blackwell, 1990), pp. 388–412.

There is some evidence that the longer is T the better is the Merton model fit: "Experimenting with different choices, we find that choosing a longer maturity, and hence giving a larger weight to volatility, generates predictions more correlated with market observations. We present our results based on $T = 10$". See BAI, Jennie, and WU, Liuren, "Anchoring Corporate Credit Spreads to Firm Fundamentals", Working Paper, June 2010 (p. 9).

By Table 1 and Equations (8) and (10):

$$
\begin{aligned}
D &\equiv B_0 + G_B \\
&= (1-\theta)(Z-P-A) + \theta(Z-P-A) \\
&= Z-P-A \\
&= Z-(Z-V_b)\,p_b - \alpha V_b\, p_b \\
&= Z+[(1-\alpha)V_b - Z]\,p_b \\
&= Z+[(1-\alpha)V_b - Z](V_0/V_b)^{\gamma_2}
\end{aligned}
$$

and

$$
\begin{aligned}
E &\equiv S_0 + G_S + \theta Z(1-p_b) \\
&= (1-\theta)(V_0 - Z + P) + \theta(V_0 - Z + P) + \theta Z(1-p_b) \\
&= V_0 - Z + P + \theta Z(1-p_b) \\
&= V_0 - Z + (Z-V_b)\,p_b + \theta Z(1-p_b) \\
&= V_0 - (1-\theta)Z + [(1-\theta)Z - V_b]\,p_b \\
&= V_0 - (1-\theta)Z + [(1-\theta)Z - V_b](V_0/V_b)^{\gamma_2}
\end{aligned}
$$

Our stricter definition of equity, S_0, is equal to the definition of equity, E_{solv}, used by Goldstein-Ju-Leland [2001, Equation (19)].[16]

Some of the consequences of our different approach, with respect to Leland (1994), are:

1. the value of equity *does* depend on the tax rate θ;[17]
2. the default trigger, V_b, is different from (higher than) the default trigger obtained by Leland;
3. it is not possible to derive an optimal capital structure by balancing tax advantages with potential default costs.

Another consequence is that – for evaluating equity options – we cannot use the formulas derived *within* the original Leland model.

3. TERM STRUCTURE OF DEFAULT PROBABILITIES AND CREDIT DEFAULT SWAPS

Default Probabilities

Let $Q(T)$ denote the probability of default between time 0 and time T (included). This is equal to the probability that V reaches V_b before T (or in T). Therefore, $Q(T)$ is equal to the first-passage time distribution function:[18]

$$
Q(T) = N(-z_1) + \left(\frac{V_0}{V_b}\right)^{2(1-\lambda)} N(-z_2) \tag{21}
$$

where

$$
z_1 = \frac{\ln(V_0/V_b) + (r - q_V - \sigma_V^2/2)T}{\sigma\sqrt{T}}
$$

[16] According to Equation (19) of Goldstein-Ju-Leland (2001), $E_{\text{solv}} = (1-\tau_{\text{eff}})(V_{\text{solv}} - V_{\text{int}})$ where $V_{\text{int}} = (C/r)(1-p_B)$. In our notation, $S_0 = E_{\text{solv}}$, $\theta = \tau_{\text{eff}}$, $V_0 - V_b\,p_b = V_{\text{solv}}$, $Z(1-p_b) = V_{\text{int}}$, $p_b = p_B$. Therefore, $S_0 = (1-\theta)[V_0 - V_b\,p_b - Z(1-p_b)] = (1-\theta)[V_0 + (Z-V_b)\,p_b - Z] = (1-\theta)(V_0 + P - Z)$. In the notation used by Goldstein-Ju-Leland (2001), $BC_{def} = \alpha V_{def} = \alpha V_b p_b$ is the value of the bankruptcy claim. We assume that also third parties have to pay taxes. Therefore, our expression for the value, U_0, of the bankruptcy claim is equal to $\alpha V_b p_b$ pre-multiplied by $(1-\theta)$.

[17] "In contrast to Leland (1994), equity is a decreasing function of τ_{eff}." Goldstein *et al*, p. 497.

[18] See Equation (34b), p. 353, in INGERSOLL, Jonathan E., *Theory of Financial Decision Making,* Rowman & Littlefield, 1987. Transformed into our notation, $x_0 = \ln(V_0/V_b)$, $\mu = r - q_V - \sigma_V^2/2$, $2\mu/\sigma^2 = 2(\lambda - 1)$, $t_0 = 0$.

TABLE 3 Default probabilities.

	ACTUAL DEFAULT PROBABILITIES (%) [source: Moody's (1970-2009)]								
	Maturity (years)								
Rating	1	2	3	4	5	7	10	15	20
Aaa	0.000	0.012	0.012	0.037	0.105	0.245	0.497	0.927	1.102
Aa	0.022	0.059	0.091	0.159	0.234	0.384	0.542	1.150	2.465
A	0.051	0.165	0.341	0.520	0.717	1.179	2.046	3.572	5.934
Baa	0.176	0.494	0.912	1.404	1.926	2.996	4.851	8.751	12.327
Ba	1.166	3.186	5.583	8.123	10.397	14.318	19.964	29.703	37.173
B	4.546	10.426	16.188	21.256	25.895	34.473	44.377	56.098	62.478
Caa	17.723	29.384	38.682	46.094	52.286	59.771	71.376	77.545	80.211

	THEORETICAL DEFAULT PROBABILITIES (%)										
	Maturity (years)										
Rating	1	2	3	4	5	7	10	15	20	*Z*	σ_V (%)
Aaa	0.000	0.001	0.015	0.057	0.127	0.316	0.614	0.992	1.220	60	11.5
Aa	0.000	0.005	0.051	0.159	0.314	0.683	1.208	1.829	2.196	65	12.0
A	0.001	0.106	0.461	0.973	1.530	2.567	3.766	4.991	5.658	70	12.5
Baa	0.210	2.036	4.528	6.858	8.860	11.970	15.092	18.097	19.768	80	15.0
Ba	1.933	8.363	14.220	18.859	22.533	27.943	33.240	38.468	41.594	90	20.0
B	4.708	16.491	26.138	33.488	39.216	47.592	55.841	64.196	69.418	110	35.0
Caa	13.644	30.656	41.560	49.063	54.588	62.289	69.526	76.567	80.829	140	40.0

Note: The theoretical default probabilities are based on Equation (21), where the risk-free interest rate *r* has been replaced with the *actual* drift rate μ_V. The model's parameters are: $V = 100$, $\mu_V = 5\%$, $q_V = 0\%$, $\theta = 35\%$, $\alpha = 5\%$ (the values of *Z* and σ_V are a function of the rating class).

$$z_2 = \frac{\ln(V_0 / V_b) - (r - q_V - \sigma_V^2 / 2)T}{\sigma\sqrt{T}}$$

$$\lambda = 1 + \frac{r - q_V - \sigma_V^2 / 2}{\sigma_V^2}$$

Equation (21) gives the term structure of unconditional (cumulative) default probabilities and allows an easy calculation of the term structure of conditional default probabilities (or *hazard rates / default intensities*).

Table 3 shows that, by using the *actual* drift rate μ_V instead of *r*, the model can be calibrated to fit the term structures of *actual* default probabilities.[19]

In the table, historical default frequencies calculated by Moody's for different rating classes have been reported together with theoretical default probabilities obtained for different values of the model's parameters *Z* and σ_V.

[19] It should be noted that *actual* default probabilities shown in the table *do not* include the (high) risk premiums asked by bond traders as a compensation for the (high) systematic risk of bond portfolios.

CDS Spreads

Standard single-name credit default swaps (CDSs), where the protection buyer makes periodic payments and has the right to sell at par the bonds issued by the reference entity, can be priced in the framework given by the model.[20]

To explain the calculations, let us consider a n-year CDS whose payment dates are at times t_i ($1 \leq i \leq m \times n$), where m is the number of payments per year. The value, A_1, of an annuity which pays $1/m$ at each payment date until default, τ, or maturity, $T = t_{m \times n}$, whichever comes first, is

$$A_1 = \frac{1}{m} \sum_{i=1}^{m \times n} e^{-y_i t_i} [1 - Q(t_i)] \tag{22}$$

where y_i is the risk-free interest rate for maturity t_i and $Q(t_i)$ is the default probability given by (21).

We assume "instant recovery". In other terms, we assume that – at default time, τ – the protection buyer receives an instant payment equal to $1 - R$, where the recovery rate R is *endogenously* given by the following formula:

$$R = \frac{(1-\alpha) V_b}{Z} \tag{23}$$

Substituting (14) into (23), the instant payoff $1 - R$ to the protection buyer is equal to

$$1 - R = \frac{1 - \alpha \gamma_2}{1 - \gamma_2}$$

The present value of the payoff is

$$(1-R) p_b(T) \tag{24}$$

where $p_b(T)$ is the value of a *T-maturity* first-touch digital option, which pays a unit *at time* τ ($\tau \leq T$) if the firm defaults at τ.

Generally, credit default swaps specify that the protection buyer must, at default, pay the CDS spread that has accrued since the last coupon date. In order to take this accrual payment into account, we subtract one half of the regular payment from the instant payoff $1 - R$:

> For reasonably small default probabilities and intercoupon periods, the expected difference in time between the credit event and the previous coupon date is just slightly less than one-half, in expectation, of the length of an intercoupon period, assuming that the default risk is not concentrated at a coupon date. Thus, for purposes of pricing in all but extreme cases, one can think of the credit swap as equivalent to payment at default of face value less recovery value less one-half of the regular default-swap premium payment [Duffie-Singleton (2003), pp. 183-4].

Therefore, the present value, A_2, of the *net* CDS payoff is

$$A_2 = \left[(1-R) - \frac{1}{2} \frac{s}{m} \right] p_b(T) \tag{25}$$

where s is the CDS spread per year.

The CDS contract is fair when the present value, sA_1, of the payments equals the present value, A_2, of the net payoff, or

$$sA_1 = A_2 \tag{26}$$

[20] For "Model-Based CDS Rates", see Section 8.4 in DUFFIE, Darrell, and SINGLETON, Ken J., *Credit Risk*, Princeton University Press, 2003.

By substituting (22) and (25) into (26), the CDS breakeven spread is

$$s = \frac{(1-R)\,p_b(T)}{\dfrac{1}{m}\left\{\dfrac{1}{2}\,p_b(T) + \sum_{i=1}^{m\times n} e^{-y_i\,t_i}[1-Q(t_i)]\right\}} \tag{27}$$

where the value, $p_b(T)$, of a finite-maturity first-touch digital option, with barrier $V_b < V_0$, has been derived by Rubinstein and Reiner (1991) as:[21]

$$p_b(T) = (V_b / V_0)^{a+b}\, N(-z) + (V_b / V_0)^{a-b}\, N(-z + 2\,b\,\sigma_V \sqrt{T}) \tag{28}$$

with

$$z \equiv \frac{\ln(V_b / V_0)}{\sigma_V \sqrt{T}} + b\,\sigma_V \sqrt{T} \qquad a \equiv \frac{r - q_V - \sigma_V^2 / 2}{\sigma_V^2} \qquad b \equiv \frac{\sqrt{(r - q_V - \sigma_V^2 / 2)^2 + 2\,\sigma_V^2\, r}}{\sigma_V^2}$$

It can be shown that (28) converges to (8) when $T \to \infty$.

4. EQUITY OPTIONS

Similarly to Toft (1994), Toft & Prucyk (1996) and Barone (2011), we can derive the value, c, of a European call, with strike K and maturity T, written on S as:[22]

$$\begin{aligned} c &= e^{-rT}\, E(S_T - K \mid V_T > V_T^* \cap V_t > V_b) \times \text{Prob}(V_T > V_T^* \cap V_t > V_b) \\ &= (1-\theta)(V_{doc} + P_{uop} - Z\,R_{doc}) - K\,R_{doc} \end{aligned} \tag{29}$$

where

V_{doc} is the value of a down-and-out asset-or-nothing call, with strike V_T^* and barrier V_b, which pays V_T at time T if $V_T > V_T^*$ and $V_t > V_b$ $(0 < t \le T)$:

$$V_{doc} = V_0 e^{-q_V T}[N(x) - (V_b / V_0)^{2\lambda} N(y)]$$

P_{uop} is the value of an up-and-out asset-or-nothing put, with strike P_T^* and barrier P_b, which pays P_T at time T if $P_T < P_T^*$ and $P_t < P_b$:

$$P_{uop} = P_0[N(-x_P) - (P_b / P_0)^{2\lambda_P} N(-y_P)]$$

R_{doc} is the value of a down-and-out cash-or-nothing call, with strike V_T^*, barrier V_b and unit rebate R, which pays R = \$1 in T if $V_T > V_T^*$ and $V_t > V_b$:

$$R_{doc} = e^{-rT}[N(x - \sigma_V \sqrt{T}) - (V_b / V_0)^{2\lambda - 2} N(y - \sigma_V \sqrt{T})]$$

V_T^* is the critical asset value that makes the call finish at the money;

$$V_T^* = V_0\, e^{(r - q_V - \sigma_V^2 / 2)T + \varepsilon^* \sigma_V \sqrt{T}}$$

P_T^* is the value of the perpetual first-touch digital when $V = V_T^*$;

$$P_T^* = P_0\, e^{(r - \sigma_P^2 / 2)T - \varepsilon^* \sigma_P \sqrt{T}}$$

ε^* is the standardized normal shock that makes the equity call finish at the money;

$N(\cdot)$ is the standard normal distribution function;

and

$$x \equiv \frac{\ln(V_0 / V_T^*)}{\sigma_V \sqrt{T}} + \lambda\,\sigma_V \sqrt{T} \qquad x_P \equiv \frac{\ln(P_0 / P_T^*)}{\sigma_P \sqrt{T}} + \lambda_P\, \sigma_P \sqrt{T}$$

---

[21] RUBINSTEIN, Mark e REINER, Eric, “Unscrambling the Binary Code”, *Risk*, vol. 4, no. 9, pp. 75-83, October 1991.

[22] TOFT, Klaus Bjerre Toft, “Options on Leveraged Equity with Default Risk”, Walter A. Haas School of Business, University of California at Berkeley, July 1994. TOFT, Klaus Bjerre Toft, and PRUCYK, Bryan, “Options on Leveraged Equity: Theory and Empirical Tests”, *Journal of Finance*, Vol. 52, No. 3, pp. 1151-1180, July 1997. BARONE, Gaia, “Equity Options and Bond Options in the Leland Model”, “Tor Vergata” University of Rome, Working Paper, March 2011. Downloadable at http://ssrn.com/author=1004723.

$$y \equiv \frac{\ln[V_b^2/(V_0\,V_T^*)]}{\sigma_V\sqrt{T}} + \lambda\,\sigma_V\sqrt{T} \qquad y_P \equiv \frac{\ln[P_b^2/(P_0\,P_T^*)]}{\sigma_P\sqrt{T}} + \lambda_P\,\sigma_P\sqrt{T}$$

$$\lambda \equiv 1 + \frac{r - q_V - \sigma_V^2/2}{\sigma_V^2} \qquad \lambda_P \equiv 1 + \frac{r - \sigma_P^2/2}{\sigma_P^2} \qquad \sigma_P = -\gamma_2\,\sigma_V$$

Under the hypothesis that – if the firm defaults at τ – the buyer receives K at T (and not at τ), the value, p, of a European put, with strike K and maturity T, written on S is

$$p = e^{-rT} E[K - S_T \mid V_T < V_T^* \cap V_t > V_b] \times \mathrm{Prob}(V_T < V_T^* \cap V_t > V_b) + E[e^{-r\tau} K \mid V_t \le V_b] \times \mathrm{Prob}(V_t \le V_b)$$

$$= K\,(R_{dop} + R_{di}) + (1-\theta)(Z\,R_{dop} - V_{dop} - P_{uoc}) \tag{30}$$

where

R_{dop} is the value of a down-and-out cash-or-nothing put, with strike V_T^*, barrier V_b and unit rebate R, written on V, which pays R = \$1 at time T if $V_T < V_T^*$ and $V_t > V_b$ ($0 < t \le T$):

$$R_{dop} = e^{-rT}\{N(-x + \sigma_V\sqrt{T}) - N(-x_1 + \sigma_V\sqrt{T}) + (V_b/V_0)^{2\lambda-2}[N(y - \sigma_V\sqrt{T}) - N(y_1 - \sigma_V\sqrt{T})]\}$$

V_{dop} is the value of a down-and-out asset-or-nothing put, with strike V_T^* and barrier V_b, written on V, which pays V_T at time T if $V_T < V_T^*$ and $V_t > V_b$:

$$V_{dop} = V_0 e^{-q_V T}\{N(-x) - N(-x_1) + (V_b/V_0)^{2\lambda}[N(y) - N(y_1)]\}$$

P_{uoc} is the value of an up-and-out asset-or-nothing call, with strike P_T^* and barrier P_b, written on P, which pays P_T at time T if $P_T > P_T^*$ and $P_t < P_b$:

$$P_{uoc} = P_0\{N(x_P) - N(x_{1P}) + (P_b/P_0)^{2\lambda_P}[N(-y_P) - N(-y_{1P})]\}$$

R_{di} is the value of a down-and-in (*at expiry*) cash-or-nothing, with barrier V_b and unit rebate R, written on V, which pays R = \$1 at time T if $V_t \le V_b$ ($0 < t \le T$):

$$R_{di} = e^{-rT}[N(-x_1 + \sigma_V\sqrt{T}) + (V_b/V_0)^{2\lambda-2} N(y_1 - \sigma_V\sqrt{T})]$$

and

$$x_1 \equiv \frac{\ln(V_0/V_b)}{\sigma_V\sqrt{T}} + \lambda\,\sigma_V\sqrt{T} \qquad x_{1P} \equiv \frac{\ln(P_0/P_b)}{\sigma_P\sqrt{T}} + \lambda_P\,\sigma_P\sqrt{T}$$

$$y_1 \equiv \frac{\ln(V_b/V_0)}{\sigma_V\sqrt{T}} + \lambda\,\sigma_V\sqrt{T} \qquad y_{1P} \equiv \frac{\ln(P_b/P_0)}{\sigma_P\sqrt{T}} + \lambda_P\,\sigma_P\sqrt{T}$$

Table 4 shows an application of formulas (29) and (30) for call and put equity options.

Put-Call Parity

Formulas (29) and (30) satisfy the following put-call parity

$$c - p = (1-\theta)\,S_{do} - Ke^{-rT}$$

where

S_{do} is the value of a down-and-out asset-or-nothing, with barrier 0, written on S, which pays S_T at time T if $S_t > 0$ ($0 < t \le T$):

$$S_{do} = V_{do} + P_{uo} - Z\,R_{do}$$

TABLE 4 European equity options.

	A	B	C	D
1	**Equity Options**			
2				
3	Asset value (V_0)	100		
4	Bond face value (Z)	50		
5	Strike price (K)	30		
6	Time to maturity (T)	1		
7	Tax rate (θ)	35%		
8	Bankruptcy costs rate (α)	5.00%		
9	Risk-free rate (r)	5.50%		
10	Payout rate (q_V)	3.50%		
11	Asset volatility (σ_V)	20.00%		
12	Bankruptcy trigger (V_b)	31.19		
13	Option to default (P)	2.72		
14	Option to default volatility (σ_P)	33.17%		
15	Tax claim (G_0)	35.00		
16	Third parties claim (U_0)	0.15		
17	Bond (B_0)	30.58		
18	Equity (S_0)	34.27		
19	Dividend yield (q_S)	2.19%		
20	Equity volatility (σ_S)	36.22%		
21	Leverage (L)	1.90		
22	Critical asset value (V_T^*)	93.09		
23		call	put	
24	Down-and-out asset-or-nothing (V_{doc} or V_{dop})	44.67	18.10	
25	Up-and-out asset-or-nothing (P_{uop} or P_{uoc})	0.90	0.87	
26	Down-and-out cash-or-nothing [$(Z+K)R_{doc}$ or $(Z+K)R_{dop}$]	37.86	21.30	
27	Down-and-in cash-or-nothing ($K R_{di}$)	0.00	0.00	
28	Equity option (c or p)	7.72	2.34	
29		$c-p$	$(1-\theta)S_{do}-Ke^{-rT}$	
30	Put-call parity	5.38	5.38	

and

$$V_{do} = V_0 e^{-q_V T}[N(x_1) - (V_b / V_0)^{2\lambda} N(y_1)]$$

$$P_{uo} = P_0[N(-x_{1P}) - (P_b / P_0)^{2\lambda_P} N(-y_{1P})]$$

$$R_{do} = e^{-rT}[N(x_1 - \sigma_V \sqrt{T}) - (V_b / V_0)^{2\lambda - 2} N(y_1 - \sigma_V \sqrt{T})]$$

and the other variables have already been defined.

Figure 3 shows the values of call and put options, as a function of the equity value, for some model's parameters. The chart also shows the put-call parity relationship, measured by the difference $c - p$.

Implementing the Model

The required inputs for estimating the value of equity options by the Black-Scholes-Merton include S_0, q_S and σ_S. If we add leverage, L, to the list, we can estimate V_0, q_V, σ_V and Z by solving a system of four simultaneous equations.

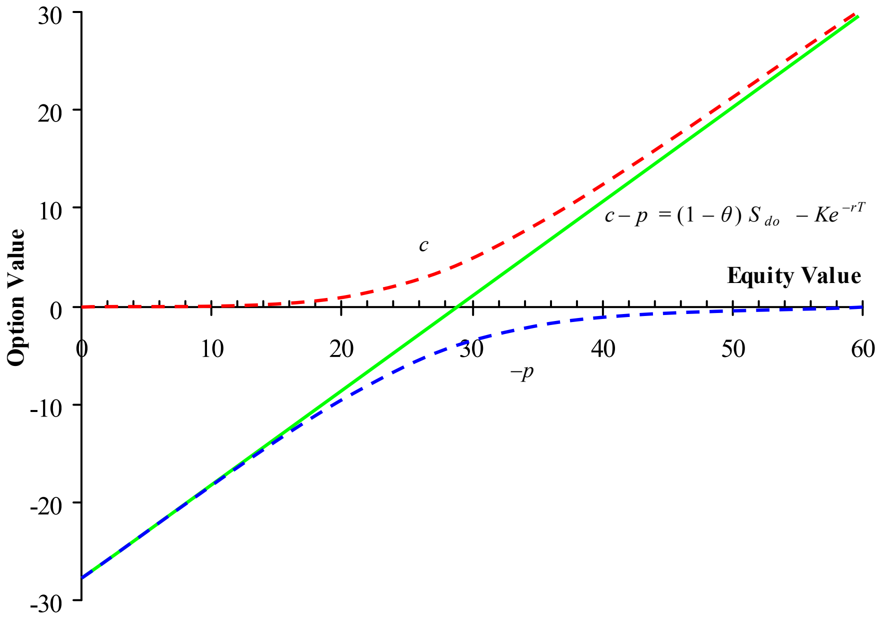


Figure 3 Equity put-call parity ($r = 5.5\%$, $q_V = 3.5\%$, $\sigma_V = 20\%$, $Z = 50$, $\theta = 35\%$, $V_b = 31.19$, $K = 30$, $T = 1$).

The first equation sets to zero the difference between the theoretical value and the market value of equity:

$$(1-\theta)[V_0 - Z + (Z - V_b)(V_0 / V_b)^{\gamma_2}] - S_0 = 0$$

The second equation, derived by applying Itô's lemma, sets to zero the difference between the theoretical level and the market estimate of equity volatility:

$$\frac{\Delta_S \sigma_V V_0}{S_0} - \sigma_S = 0$$

The third equation sets to zero the difference between the theoretical level and the market estimate of dividend yield:

$$\left(q_V \frac{V}{S} - r\frac{Z}{S}\right) - q_S = 0$$

Finally, the fourth equation sets to zero the difference between the theoretical and actual level of leverage:

$$\frac{(1-\theta)V_0}{S_0} - L = 0$$

These equations can be solved numerically. Table 5 shows a worksheet where the Excel's Solver has been used to estimate the model's parameters V_0, q_V, σ_V, Z on the basis of the input list S_0, q_S, σ_S, L (in addition to r, θ, α, K and T).

5. TWO APPLICATIONS: LEHMAN BROTHERS AND GENERAL MOTORS

In Section 3, Table 3 showed the solution of an *inverse problem*: the calculation of *implied* default probabilities based on *actual* default probabilities reported by Moody's for some rating classes. Instead in Section 4, Table 4 and Table 5 showed the solution of a *direct problem*: the value of equity options based on two different input lists, which include (V_0, q_V, σ_V, Z) or (S_0, q_S, σ_S, L).

TABLE 5 European equity options.

	A	B	C	D
1	**Equity Options**			
2		*Actual*	*Theoretical*	*Sq. errors*
3	Equity (S_0)	**34.27**	**34.27**	0.00000
4	Leverage (L)	**1.90**	**1.90**	0.00000
5	Dividend yield (q_S)	**2.19%**	**2.19%**	0.00000
6	Equity volatility (σ_S)	**36.22%**	**36.22%**	0.00000
7	Risk-free rate (r)	**5.50%**	SSE	0.00000
8	Tax rate (θ)	**35%**		
9	Bankruptcy costs rate (α)	**5%**		
10	Strike price (K)	**30.00**		
11	Time to maturity (T)	**1**		
12	Asset value (V_0)	**100.00**		
13	Payout rate (q_V)	**3.50%**		
14	Asset volatility (σ_V)	**20.00%**		
15	Bond face value (Z)	**50.00**		
16	Bankruptcy trigger (V_b)	**31.19**		
17	Option to default (P)	**2.72**		
18	Option to default volatility (σ_P)	**33.17%**		
19	Tax claim (G_0)	**35.00**		
20	Third parties claim (U_0)	**0.15**		
21	Bond (B_0)	**30.58**		
22	Critical asset value (V_T^*)	**93.09**		
23		**call**	**put**	
24	Down-and-out asset-or-nothing (V_{doc} or V_{dop})	**44.67**	**18.10**	
25	Up-and-out asset-or-nothing (P_{uop} or P_{uoc})	**0.90**	**0.87**	
26	Down-and-out cash-or-nothing [$(Z+K)\,R_{doc}$ or $(Z+K)\,R_{dop}$]	**37.86**	**21.30**	
27	Down-and-in cash-or-nothing ($K\,R_{di}$)	**0.00**	**0.00**	
28	Equity option (c or p)	**7.72**	**2.34**	
29		$c-p$	$(1-\theta)S_{do}-Ke^{-rT}$	
30	Put-call parity	**5.38**	**5.38**	

Here we show how to solve another inverse problem: the model's calibration based on the spreads of credit default swaps and the quotes of equity and equity options. We will consider market daily data on Lehman Brothers and General Motors.

Lehman Brothers

As in Brigo-Morini-Tarenghi (2009), we focus on three different dates: July 10th 2007, June 12th 2008, September 12th 2008.[23] The last date immediately precedes September 14th 2008, when Lehman filed for Chapter 11 bankruptcy protection.

Table 6 reports the (actual and theoretical) CDS spreads of Lehman together with the (actual and theoretical) quotes of Lehman's common stock.

After fixing θ, α, and r, we estimated the other model's parameters (V, Z, q_V, σ_V), by minimizing a loss function defined as the sum of weighted squared *log* differences between actual and theoretical values of both CDS spreads and equity quotes.[24]

[23] See BRIGO, Damiano, MORINI, Massimo, and TARENGHI, Marco, "Credit Calibration with Structural Models - The Lehman Case and Equity Swaps under Counterparty Risk", Working Paper, 2009.

[24] The Solver routine in Excel has been used to search for the values of the parameters that minimize the loss function. The routine works well provided that the spreadsheet is structured so that the parameters being searched for have roughly equal values. Since occasionally Solver gives a local minimum, a number of different starting values for the model's parameters has been tested.

TABLE 6 Lehman Brothers: model's calibration based on CDS spreads and equity.

	CDSs								Equity		
Date	*Maturity*	*Zero Rate*	*Default Prob.*	*Survival Prob.*	*Default Intensity*	*Actual Spread*	*Theoretical Spread*	*Squared Error*	*Market Value*	*Theoretical Value*	*Squared Error*
t	$T-t$	r	Q	$1-Q$	λ	s_{mkt}	s	$[\ln(s_{mkt}/s)]^2$	S_{mkt}	S	$[\ln(S_{mkt}/S)]^2$
7/10/2007	1	5.417%	0.68%	99.32%	0.68%	16	14	2.11	69.67	69.67	0.000
	3	5.322%	6.95%	93.05%	2.40%	29	48	25.55			**Sum of Sq. Errors**
	5	5.437%	11.58%	88.42%	2.46%	45	50	1.02			
	7	5.540%	14.53%	85.47%	2.24%	50	46	0.55			SSE
	10	5.656%	17.25%	82.75%	1.89%	58	41	11.85			41.08
6/12/2008	1	3.490%	13.69%	86.31%	14.72%	397	380	0.18	22.51	22.51	0.000
	3	4.289%	32.67%	67.33%	13.19%	315	354	1.34			**Sum of Sq. Errors**
	5	4.608%	40.37%	59.63%	10.34%	277	294	0.34			
	7	4.772%	44.63%	55.37%	8.44%	258	254	0.02			SSE
	10	4.925%	48.40%	51.60%	6.62%	240	216	1.11			3.01
9/12/2008	1	3.122%	35.83%	64.17%	44.36%	1,437	1,393	0.10	3.65	3.65	0.000
	3	3.465%	55.40%	44.60%	26.91%	902	949	0.26			**Sum of Sq. Errors**
	5	3.853%	62.08%	37.92%	19.39%	710	752	0.32			
	7	4.123%	65.67%	34.33%	15.27%	636	641	0.01			SSE
	10	4.388%	68.85%	31.15%	11.66%	588	543	0.63			1.31

	Model's Parameters				Model's Output						
Date	*Asset Value*	*Bond's Face Value*	*Payout Rate*	*Asset Volatility*	*Leverage*	*Bankruptcy Trigger*	*Option to Default*	*Option to Default Vol.*	*Bond*	*Bond Yield*	*Recovery Rate*
t	V	Z	q_V	σ_V	L	V_b	P	σ_P	B_0	y	R
7/10/2007	564.5	469.6	0,01%	14.94%	5.269	392.1	12.2	75.63%	295.3	5.85%	79.35%
6/12/2008	450.1	464.1	0,01%	16.99%	13.022	358.8	48.7	57.89%	264.7	5.61%	73.47%
9/12/2008	168.6	200.5	0,01%	18.36%	30.164	144.8	37.5	47.71%	102.8	5.56%	68.63%

Note: -The model's exogenous parameters are $\theta = 35\%$, $\alpha = 5\%$, $r = 5.66\%$ (10 Jul 2007), 4.92% (12 Jun 2008), 4.39% (12 Sep 2008). All the weights have been set to 1, except for equity weights, which are 30 (10 Jul 2007), 20 (12 Jun 2008), 10 (12 Sep 2008).

Log differences have been used because CDS spreads and equity quotes are measured in different units (and have different levels). Weights allow for the *perfect fit* of the equity quotes.

The cross-section estimation of the model parameters has been carried out separately for each of the three above mentioned dates. Results conform to expectations:

1. the assets' value, V, diminishes as time passes by;
2. leverage, L, increases throughout the observation period (from 5.269 to 30.164). As in (18), L is the ratio between the after-tax value, $(1-\theta)\,V$, of firm's assets and the value, S, of equity;
3. simultaneously, also the business risk, measured by σ_V, progressively increases (from 14.94% to 18.36%);
4. the term structures of (unconditional, *risk neutral*) default probabilities and (conditional, *risk neutral*) default intensities shift upward. However, on September 12th 2008 the immediate future of Lehman Brothers was not at all clear: the 1-year default and survival probabilities were 35.83% and 64.17%, respectively;[25]
5. the recovery rate, R, slightly goes down (from 79.35% to 68.63%).

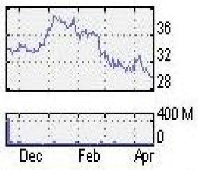
[25] The implied default intensities differ from those calculated in Brigo *et al.*, *op. cit.*, because our values are based on different recovery rates.

Symbol: GM | ALL | Go! **General Motors Co**

Options Root: GM

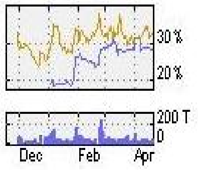
Price	Change (%)	52 wk High	52 wk Low	Stock volume	Avg. options volume	Avg. options open interest
29.97	-0.27 (-0.89%)	38.98 01/07/2011	29.97 04/18/2011	12,734,181	39,320	1,007,450

Price Chart



	Current	1 WK AGO	1 MO AGO	52 wk Hi/Date	52 wk Low/Date
HISTORICAL VOLATILITY					
10 days	21.35%	34.07%	32.90%	37.02% - 04-Feb	10.44% - 21-Jan
20 days	27.07%	29.47%	31.67%	32.85% - 24-Feb	16.08% - 17-Dec
30 days	28.38%	29.15%	28.39%	31.90% - 01-Apr	18.34% - 06-Jan
IMPLIED VOLATILITY					
IV Index call	32.55%	34.38%	33.35%	39.28% - 16-Mar	23.82% - 21-Dec
IV Index put	32.53%	34.25%	34.18%	38.63% - 23-Feb	23.41% - 20-Dec
IV Index mean	32.54%	34.31%	33.76%	38.92% - 23-Feb	23.82% - 21-Dec
HISTORICAL 30-DAYS CORRELATION AGAINST S&P 500 Index (SPX)					
30 days	34.76%	45.19%	52.98%	69.81% - 11-Mar	-10.43% - 13-Jan

Volatility Chart



Implied Vola (%)	Call Delta	Change (%)	Call Bid/Ask Mean	Expiry	Strike	Days	Put Bid/Ask Mean	Change (%)	Put Delta	Implied Vola (%)
34.63%	76.31%	-0.23 (-8.49)	2.425	May 21, 2011	28.0	33	0.450	0.02 (5.88)	-23.67%	34.60%
32.95%	65.10%	-0.25 (-12.82)	1.700	May 21, 2011	29.0	33	0.725	0.03 (3.57)	-34.89%	32.92%
32.12%	51.56%	-0.20 (-15.09)	1.125	May 21, 2011	30.0	33	1.150	0.05 (4.55)	-48.46%	32.12%
30.92%	37.40%	-0.15 (-18.18)	0.675	May 21, 2011	31.0	33	1.700	0.10 (6.25)	-62.64%	30.91%
34.61%	71.14%	-0.25 (-8.20)	2.800	Jun 18, 2011	28.0	61	0.825	0.00 (0.00)	-28.90%	34.73%
33.83%	62.23%	-0.20 (-8.51)	2.150	Jun 18, 2011	29.0	61	1.175	0.05 (4.44)	-37.80%	33.93%
32.69%	52.46%	-0.22 (-12.50)	1.575	Jun 18, 2011	30.0	61	1.600	0.05 (3.23)	-47.55%	32.81%
31.59%	42.21%	-0.20 (-15.38)	1.100	Jun 18, 2011	31.0	61	2.150	0.10 (4.88)	-57.58%	32.22%

Figure 4 CBOE options quotes of General Motors: implied volatilities (April 18th, 2011). Source: IVolatility.com

General Motors

As a second application of the model, we used market data on General Motors (GM). On April 18th 2011, the latest available date, we observed CDS spreads, options quotes, equity and equity volatility.

The implied volatilities of CBOE options quotes of GM exhibit the usual downward-sloping relationship with respect to strike prices (Figure 4).

The calibration's results are shown in Table 7. The model fits less well than for Lehman. The valuations made in the market for CDSs do not seem to be aligned with the market quotes of equity and equity options.[26] In order to explain the high CDS spreads, it would be necessary to assume a very high business risk, which would determine unreasonably high levels of equity volatility.

The conclusion, based on the model, is that the market for CDSs seems to have been much more pessimistic on the future of GM than the market for equity and equity options. If one trusts the model, there should have been room for convergence trading.

[26] Some new empirical evidence documents a "slow information diffusion between the CDS market and the stock market". See HAN, Bing, and ZHOU, Yi, "Term Structure of Credit Default Swap Spreads and Cross-Section of Stock Returns", Working Paper, March 2011.

TABLE 7 General Motors: model's calibration based on CDS spreads, equity options, equity, and equity volatility.

CDSs

Current Date	*Maturity*	*Zero Rate*	*Default Probability*	*Survival Probability*	*Av. Default Intensity*	*Actual Spread*	*Theoretical Spread*	*Squared Error*
t	$T-t$	r	Q	$1-Q$	λ	s_{mkt}	s	$[\ln(s_{mkt}/s)]^2$
4/18/2011	1	0.772%	1.51%	98.49%	1.52%	94	68	10.23%
	3	1.353%	19.37%	80.63%	7.18%	214	304	12.34%
	5	2.284%	34.58%	65.42%	8.49%	331	355	0.50%
	7	2.953%	45.27%	54.73%	8.61%	377	364	0.12%
	10	3.568%	56.24%	43.76%	8.26%	408	361	1.50%
								24.68%

Equity Options

Maturity	*Time to Maturity*	*Zero Rate*	*Strike*	*Implied Volatility*	*Critical Value*	*Market Value*	*Theoretical Value*	*Squared Error*
T	$T-t$	r	K	σ_{imp}	V_T^*	c_{mkt}	c	$[\ln(c_{mkt}/c)]^2$
5/21/2011	0.09	0.212%	28	34.63%	468.2	2.425	1.682	13.40%
		0.212%	29	32.95%	471.0	1.700	1.347	5.42%
		0.212%	30	32.12%	473.8	1.125	1.066	0.29%
		0.212%	31	30.92%	476.5	0.675	0.834	4.49%
								23.60%
6/18/2011	0.17	0.247%	28	34.61%	468.2	2.800	2.572	0.72%
		0.247%	29	33.83%	471.0	2.150	2.205	0.06%
		0.247%	30	32.69%	473.8	1.575	1.881	3.15%
		0.247%	31	31.59%	476.5	1.100	1.595	13.81%
								17.75%

Equity

Current Date	*Market Value*	*Theoretical Value*	*Squared Error*
t	S_{mkt}	S	$[\ln(S_{mkt}/S)]^2$
4/18/2011	29.97	26.08	1.93%

Equity Volatility

Market Value	*Theoretical Value*	*Squared Error*
σ_{mkt}	σ_S	$[\ln(\sigma_{mkt}/\sigma_S)]^2$
32.54%	78.24%	76.96%

Sum of Squared Errors
SSE
144.92%

Model's Parameters

Asset Value	*Bond's Face Value*	*Payout Rate*	*Asset Volatility*
V	Z	q_V	σ_V
462.6	588.7	4.39%	12.82%

Model's Output

Leverage	*Bankruptcy Trigger*	*Option to Default*	*Option to Default Vol.*	*Bond*	*Bond Yield*	*Recovery Rate*
L	V_b	P	σ_P	B_0	y	R
11.5	333.9	166.2	16.80%	267.6	5.10%	53.89%

Note: The model's exogenous parameters are $\theta = 35\%$, $\alpha = 5\%$, $r = 3.57\%$. All the weights have been set to 1.

6. CONCLUSIONS

We have presented a structural model *à la* Leland (1994), which can be used to value credit and equity derivatives in a unified framework. In the model, stockholders issue a perpetual fixed-rate bond and hold a perpetual American option to default written on the firm's assets. Stockholders determine the (constant) optimal default point which maximizes the value of equity. As a consequence, the T-year default probabilities are equal to the probabilities that the asset value reaches the default point by time T. Default probabilities can be calculated by using the first-passage time distribution function. Similarly, the *T*-year CDS spreads can be calculated by using the value of *finite-maturity* first-touch digital options.

Equity is equal to a portfolio which is long on the firm's assets, short on a perpetual risk-free bond, long on a perpetual American option to default and short on a tax claim. Equity volatility is stochastic, being a complex function of both value and volatility of firm's assets. By simply adding an extra parameter, leverage, to the standard input list of the Black-Scholes-Merton formulas, traders can evaluate equity options in a way which is consistent with the downward-sloping *volatility skew* observed in equity options markets. Given the "option-like" nature of equity, equity options are compound options written on the firm's assets. They can be valuated by the Rubinstein-Reiner (1991) formulas for binary barrier options, which only require the calculation of the *univariate* normal distribution function.

The model can be used to solve *direct* and *inverse problems*. As an example of a direct problem, we have determined the value of equity options based on two different input lists and, as an example of an inverse problem, we showed the calculation of *implied* default probabilities based on *actual* default probabilities reported by Moody's for some rating classes. Finally, we have considered market data on Lehman Brothers and General Motors and show how to calibrate the model by using the CDS spreads and the quotes of equity and equity options.

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