Valuation of Digital Options In a LIBOR Market Model Under the Merton Jump Diffusion Processes

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Abstract

The LIBOR market model features the ease for calibration procedure, and the resulting pricing formula is more tractable. This paper derives an approximate pricing formula of digital options within the multifactor LIBOR market model framework when the dynamics of the LIBOR rate with different maturity dates are effected by the distinct rare events. The digital options include delay digital options, delay interest-or-nothing digital option and delay interest-or-nothing range digital option.

1 Introduction

The LIBOR market model (LMM) is developed by Brace, Gatarek and Musiela (1997, BGM), Miltersen, Sandmann and Sondermann (1997), and Musiela and Rutkowski (1997) and has become an important model for interest rate markets. This is because that the dynamics of interest rate index do not deduced from the unobservable factors, as is HJM term structure model. Rogers (1996) examined that the rate can attain negative values with positive probability which may cause some pricing error in many cases for the Gaussian HJM term structure model. However, the underlying LIBOR rates are positive. The LMM

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is widely used by practitioners due to the advantage that the cap pricing formula in the LMM framework is the Black's formula which is consistent with market practice and makes the calibration procedure easier. Recently, several new results are proposed for pricing derivatives in the lognormal LMM, such as floating range notes (Wu and Chen, 2008).

However, the lognormal LMM cannot be calibrated adequately to the observed market data. A considerable of extensions for the LMM is proposed by using jump diffusion, Levy processes, or incorporating stochastic volatility effects. The importance of jumps has been widely documented in equity markets and in interest rate markets. Jumps provide a flexibility to matching derivative prices. Therefore, adding jumps to the lognormal LMM provides a more adequate model to calibrate the observed market data. In the line of HJM market model, Bjork et al. (1997) proposed a very general model in which the forward rate is driven by a finite number of Wiener processes plus a jump random measure. Glasserman and Kou (2003) extended the lognormal LMM to include jumps in interest rates governed market point processes. The extension from Glasserman and Kou (2003) allows for finding a no-arbitrage condition and a risk-neutral probability measure as well. Glasserman and Merener (2003) developed computational procedures for the numerical solution of LMM with jumps. Recently, several researches are concentrated on finding a formula for pricing exotic derivatives in the Levy-driven LMM.

The presented research ha two objectives. First, we derive a formula for pricing the European-style derivatives in the LMM with jumps when the dynamics of the LIBOR rate with different maturity dates are effected by the distinct rare events. And then, we use the formula to pricing digital options in the LMM with jumps. The digital options include delay digital options, delay interest-or-nothing digital option and delay interest-or-nothing range digital option. In the line of Raible (2000), a Fourier transform method is provided for Delay Digital Options to compute the value of our formula. In future studies, we derive an approximate pricing formula of floating range notes within the multifactor LMM framework with jumps.

This paper proceeds as follows. The approximate LMM with jumps are introduced in Section 2. The pricing formula for a European-style option is obtained in this LMM. Section 3 applies the formula to pricing digital options. To compute the formula numerically, we provide a Fourier transform method for digital options in Section 4.

2 The approximate LIBOR market model

On the basis of the results of HJM (1992) and Glasserman and Kou (2003), they modelled the jump diffusion model interest rate behavior in term of the forward LIBOR rates. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P)$ be a filter probability space in which is defined on a m-dimensional Brownian motion w and r random measures of jump of semimartingale with process μ_l. Let $T \in [0, T^*]$, $T^* \leq \infty$ denote the time horizon. The LIBOR rate is formulated in a discrete-tensor setting in which the maturity T is restricted to a finite set of dates $0 = T_0 < T_1 < \cdots < T_M < T_{M+1} < T^*$ and the intervals $T_{i+1} - T_i$ are equally spaced with a common spacing of δ. We assume that μ_l, $l = 1, 2, ..., r$ and w_t are independent to each other and that the processes jump μ_l have the compensator $\nu_l(t, dx) = \lambda_l F_l(dx)dt$, $l = 1, 2, ..., r$, where the expected number of jumps is λ_l and the jump size is distributed according to F_l. With each forward rate we associate jump size function $\delta_l(x, T_n)$, $l = 1, 2, ..., r$ and define

$$J(t) = \sum_{l=1}^{r} \int_{\mathbb{R}} \delta_l(x, T_n)\mu_l(dx, t),$$

where $\delta_l : \mathbb{R} \times [0, T^*] \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}([0, T^*])$-measurable.

Suppose that a bond price is specified through forward rate, ie. for $\mathrm{T}_n \in \mathbb{R}^+$.

$$\mathrm{P}(t, T_n) = \exp\left\{-\int_t^{T_n} f(t, s)ds\right\}, \quad t \leq T_n,$$

where the forward rates dynamics $f(t, T_n)$ are given by

$$df(t, T_n) = \alpha(T_n)dt + \sigma^T(T_n) \cdot dw_t + \sum_{l=1}^{r} \int_{\mathbb{R}} \delta_l(x, T_n)\mu_l(dt, dx). \tag{1}$$

The coefficients $\alpha : [0, T^*] \to \mathbb{R}$ and $\sigma : [0, T^*] \to \mathbb{R}$ are $\mathcal{B}([0, T^*])$-measurable. The coefficients satisfy the following conditions. For all finite t and $t \leq T_n$, $n = 1, 2, \ldots, M + 1$,

$$\begin{aligned}
&\textstyle\int_t^{T_n} |\alpha(s)|\, ds < \infty, \\
&\textstyle\int_t^{T_n} |\sigma_i(s)|^2\, ds < \infty, \quad i = 1, 2, \ldots, m
\end{aligned}$$

and

$$\int_{\mathbb{R}} \int_t^{T_n} |\delta_l(x, s)|^2\, \nu_i(ds, dx) < \infty, \ l = 1, 2, ..., r$$

It is convenient to extend the definitions of the coefficients by putting them equal to zero for $t > T$.

Put $A(t, T) = -\int_t^T \alpha(s)ds$, $\sigma_i^*(t, T) = -\int_t^T \sigma_i(s)ds$, $i = 1, 2, ..., m$ and $\sigma^*(t, T) =$

$(\sigma_1^*(T), ..., \sigma_m^*(T))$ and

$$D_l(t,x,T) = -\int_t^T \delta_l(x,s)ds, \; l = 1,2,...,r$$

According to Glasserman and Kou (2003), the LIBOR rate dynamics under the martingale measure is follows:

$$\frac{dL(t,T_n)}{L(t,T_n)} = \left[\begin{array}{l} \gamma_n(t)\varphi_0(t) + \gamma_n(t)\sum_{k=\lceil t\rceil}^{n} \frac{\delta\gamma_k L(t,T_k)}{1+\delta L(t,T_k)} \\ -\sum_{l=1}^{r}\int_{\mathbb{R}} H_{nl}(x)\prod_{k=\lceil t\rceil}^{n} \frac{1+\delta L(t,T_k)}{1+\delta L(t,T_k)(1+H_{kl}(x))}\varphi_l(t,x)\nu_l(t,dx) \end{array}\right] dt$$
$$+\gamma_n dw_t + \sum_{l=1}^{r}\int_{\mathbb{R}} H_{nl}(x)\mu_l(dt,dx),$$

where $\gamma_n(\cdot)$ is a bounded, adapted, $\mathbb{R}^m$-valued process and H_{nl}, $l = 1,...,r$ is deterministic functions from $\mathbb{R}$ to $\mathbb{R}$ for $n = 1,2,...,M$. For all $n = 1,2,\ldots,M$,γ_n and H_{nl} satisfy the following relation:

$$\sigma^*(t,T_{n+1}) - \sigma^*(t,T_n) = \frac{\delta\gamma_n(t)L(t,T_n)}{1+\delta L(t,T_n)}$$

and

$$\int_{T_n}^{T_{n+1}} \delta_i(t,x,s)ds = \log\left(\frac{1+\delta L_n(t^-,T_n)(1+H_{nl}(x))}{1+\delta L(t^-,T_n)}\right).$$

Here, $\lceil t\rceil$ denotes the smallest integer that is greater than t so that $\lceil t\rceil$ is the index of the next maturity as of time t.

Under forward measure $\mathrm{P_{M+1}}$ for each $n = 1,2,...,M$, Glasserman and Kou (2003) presented that

$$\frac{dL(t,T_n)}{L(t,T_n)} = \left[\begin{array}{l} \gamma_n(t)\varphi_0(t) - \gamma_n(t)\sum_{k=n+1}^{M} \frac{\delta\gamma_k L(t,T_k)}{1+\delta L(t,T_k)} \\ -\sum_{l=1}^{r}\int_{\mathbb{R}} H_{nl}(x)\prod_{k=n+1}^{M} \frac{1+\delta L(t,\mathrm{T}_k)(1+H_{kl}(x))}{1+\delta L(t,\mathrm{T}_k)}\nu_l^{M+1}(t,dx) \end{array}\right] dt$$
$$+\gamma_n(t)d\mathrm{W}_t^{M+1} + \sum_{l=1}^{r}\int_{\mathbb{R}} H_{nl}(x)\mu_l(dt,dx),$$

where $\nu_l^{M+1} = \psi_l(t,x)\nu_l(t,dx)$. There exist infinite suitable $\varphi_0(t)$ and $\varphi_l(t,x)$, $l = 1,2,\ldots,r$ such that the forward rate is arbitrage-free. Glasserman and Kou (2003) worked directly under the risk-neutral measure (Brace et al., 1997; Jamshidian, 1997; Miltersen et al., 1997) and so implicitly have $\varphi_l \equiv 1$; that is $\nu_l^{M+1} = \nu_l(t,dx) = \lambda_l F_l(dx)$.

In this paper, we assume that dynamics of the LIBOR rate with different maturity dates are effected by the distinct rare events; To model this situation, the jump size functions are selected as

$$H_{nn} = e^x - 1, \quad H_{nl} \equiv 0, \ n \neq l.$$

Then the dynamics of LIBOR rates changes to

$$\frac{dL(t,T_n)}{L(t,T_n)} = \begin{bmatrix} \gamma_n(t)\varphi_0(t) - \gamma_n(t) \displaystyle\sum_{k=n+1}^{M} \frac{\delta\gamma_k L(t,T_k)}{1+\delta L(t,T_k)} \\ -\displaystyle\int_{\mathbb{R}} (e^x - 1)(x) \prod_{k=n+1}^{M} \frac{1+\delta L(t,\mathrm{T}_k)e^x}{1+\delta L(t,\mathrm{T}_k)} \lambda_n F_n(dx) \end{bmatrix} dt$$
$$+\gamma_n(t) d\mathrm{W}_t^{M+1} + \int_{\mathbb{R}} (e^x - 1)\mu_n(dt,dx).$$

When the market price of risk $\varphi_l(x,t)$, $l = 1,2,\ldots,r$ associated with component of the jump processes are given, the market price of risk $\varphi_0(t)$ associated with the corresponding component of the Brownian motion can be obtained by solving the following equation system:

For any given $T_1, T_2, \ldots, T_M \in [0,T^*]$ with $T_1 < T_2 < \cdots < T_M$, assume that there exist maturity invariant solutions

$$\varphi_0(\cdot, T_1, T_2, \ldots, T_M) = \varphi_{0i}(\cdot) : \Omega \times [0,T_i] \to \mathbb{R} \ , i = 1,2,\ldots,m$$

to the equations of market price of risk:

$$\begin{bmatrix} m(t,T_1) \\ m(t,T_2) \\ \vdots \\ m(t,T_M) \end{bmatrix} + H \begin{bmatrix} \varphi_{01}(t) \\ \varphi_{02}(t) \\ \vdots \\ \varphi_{0m}(t) \end{bmatrix} = 0, \tag{2}$$

where

$$H = \begin{bmatrix} \sigma_1^*(t,T_1) & \cdots & \sigma_m^*(t,T_1) \\ \vdots & \ddots & \vdots \\ \sigma_1^*(t,T_M) & \cdots & \sigma_m^*(t,T_M) \end{bmatrix},$$

and

$$m(t,T_n) = A(t,T_n) + \frac{1}{2}\|\sigma^*(t,T_n)\|^2 + \int_{\mathbb{R}} (e^{D_l(t,x,T_n)} - 1)\varphi_n(t,x)\nu_n(t,dx),$$

for all $n = 1,2,\ldots,M$.

Let

$$\alpha(t,T_n,T_M) = \gamma_n(t)\varphi_0(t) - \gamma_n(t) \sum_{k=n+1}^{M} \frac{\delta\gamma_k L(t,T_k)}{1+\delta L(t,T_k)}$$

for $n = 1, 2, \ldots, M$ and $l = 1, 2, \ldots M$. The calendar time of the process $L(t, T_n)$ is frozen as its initial time τ and thus the parameters' process become deterministic. Under the initial time τ, $\alpha(t, T_n, T_M)$ and ν_n^{M+1} can be approximated by $\bar{\alpha}(t, T_n, T_M)$ and $\bar{\nu}_n^{M+1}$, which are defined by

$$\bar{\alpha}(t, T_n, T_M) = \gamma_n(t)\varphi_0(t) - \sum_{k=n+1}^{M} \frac{\delta \gamma_k L(\tau, T_k)}{1 + \delta L(\tau, T_k)}.$$

and

$$\bar{\nu}_n^{M+1}(dx) = \prod_{k=n+1}^{M} \frac{1 + \delta L(\tau, T_k) e^x}{1 + \delta L(\tau, T_k)} F_n(dx),$$

respectively.

Proposition 2.1 *Under the forward measure P_{M+1}, the dynamics of LIBOR rates are represented as*

$$\frac{dL(t, T_n)}{L(t, T_n)} = \bar{\alpha}(t, T_n, T_M)dt + \gamma_n(t)dw_t^{M+1} + \int_{\mathbb{R}} (e^x - 1)(\mu_n(dt, dx) - \lambda_n \bar{\nu}_n^{M+1}(dx)dt) \tag{3}$$

The solution for (3) is obtained as

$$L(t, T_n) = L(\tau, T_n)\dot{\exp}(Y(t, T_n)),$$

where

$$dY(t, T_n) = \bar{\alpha}^Y(t, T_n, T_M)dt + \gamma_n^Y(t)dw_t^{M+1} + \int_{\mathbb{R}} x\mu_n^Y(dt, dx).$$

The parameters of $Y(t, T_n)$ satisfy the following relations

$$\gamma_n^Y = \gamma_n, \quad \nu_n^Y(dx) = \nu_n(dx), \quad \lambda_n^Y = \lambda_n,$$

and

$$\bar{\alpha}^Y(t, T_n, T_M) = \int_\tau^t \alpha(s, T_n, T_M)ds - \int_\tau^t \frac{1}{2}||\gamma_n(s)||^2 ds - \int_{\mathbb{R}} (e^x - 1)\lambda_n \bar{\nu}_{nn}^{M+1}(dx).$$

Since $\bar{\nu}_n^{M+1}(dx)$ is a time-t independent measure, we assume that $\bar{\nu}_n^{M+1}(dx) = f_n^Q(x)dx$, where the probability density function $f_n^Q(x)$ of the jump size satisfies a normal distribution with mean η and variance ξ^2, ie.

$$f_n^Q(x) = \frac{1}{\sqrt{2\pi}\xi_n} \exp(\frac{(x - \eta_n)^2}{2\xi^2}).$$

The probability density of log return $Y(t, T_n) = \log \frac{L(t,T_n)}{L(\tau,T_n)}$ is obtained as a quickly converging series of the following form:

$$\mathbb{P}(Y(t,T_n)\in A)=\sum_{a=0}^{\infty}\mathbb{P}(N_t=a)\mathbb{P}(Y(t,T_n)\in A|N_t=a).$$

Precisely, it can be represented as the form

$$\mathbb{P}(Y(t,T_n))=\sum_{a=0}^{\infty}\frac{e^{-\lambda_n t}(\lambda_n t)^a}{a!}N(Y(t,T_n)|N_i^n=a),$$

where $k_n = e^{\eta_n+\xi_n^2/2}-1$ and $N(Y(t,T_n))$ is a normal density with mean $\int_0^t \alpha^Y(s,T_n,T_M)ds-\frac{1}{2}\int_0^t\|\gamma_n^Y(s)\|^2ds-\lambda_n k_n+i\eta_n$ and variance $\int_0^t\|\gamma_n(s)\|^2ds+i\xi_n^2$.

Theorem 2.2 *Under the forward measure P^{M+1} the characteristic function of Y_n is obtained as follows*

$$E^{P^{M+1}}[e^{izY(t,T_n)}]=e^{\psi(z,t)}\equiv\chi(z),$$

where

$$\begin{aligned}\psi(z,t)=iz&\left(\textstyle\int_0^t\alpha^Y(s,T_n,T_M)ds-\frac{1}{2}\int_0^t\|\gamma_n(s)\|^2ds-\lambda k_n\right)\\&-\tfrac{z^2}{2}\textstyle\int_0^t\|\gamma_n(s)\|^2ds+\lambda_n k_n(z)\end{aligned}$$

and $k_n(z)=\left(e^{i\eta_n z-\frac{\xi_n^2 z^2}{2}}-1\right)$.

Proof. Since w_t^{M+1} and $\mu_l^{Y_n}$ are independent for all l, the distribution of $\log Y(t,T_n)$ can be regarded as a time dependent normal distribution added with a time independent jump measure. We have

$$\begin{aligned}E^{P^{M+1}}[e^{izY(t,T_n)}]&=E^{P^{M+1}}[e^{iz\int_0^t\bar{\alpha}^Y(s,T_n,T_M)ds}]E^{P^{M+1}}[e^{iz\int_0^t\gamma_n^Y(s)dw_s^{M+1}}]\\&\quad\cdot\Pi_{i=1}^r E^{P^{M+1}}[e^{iz\int_0^t\int_{\mathbb{R}}x\mu_i(ds,dx)}]\\&=e^{\psi(z,t)}.\end{aligned}$$

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In the following theorem, we obtain a European option pricing formula under the normal jump size assumption.

Theorem 2.3 *Let $L(t,T_n)$, $n=1,2,\ldots,M$ be the dynamics of the LIBOR rate with*

$$\frac{dL(t,T_n)}{L(t,T_n)}=\bar{\alpha}(t,T_n,T_M)dt+\gamma_n(t)dw_t^{M+1}+\int_{\mathbb{R}}(e^x-1)(\mu_n(dt,dx)-\lambda_n\bar{\nu}_n^{M+1}(dx)dt),$$

where the jump process μ_n has the compensator $\nu_n(dx) = \lambda_n f_n(x)dx$, When $f_n(x)$ is a probability density of normal distribution with mean η_n and variance ξ_n^2, for all $n = 1, 2, \ldots, M$, the European option price $V(\tau, L(t, T_n))$ with payoff $F(L(T_n, T_n))$ under the P^{M+1}-measure is obtained as

$$\begin{aligned}&V(\tau, L(\tau, T_n))\\&= \textstyle\sum_{i=0}^{\infty} \mathbb{P}(N_t^n = i) V^{BS}(L_i(\tau, T_n), \int_\tau^{T_n} \alpha(s)ds, \sigma_i^2(\tau, T_n)),\end{aligned}$$

where $V^{BS}(L_i(\tau, T_n), \int_\tau^{T_n} \alpha(s)ds, \sigma_i^2(\tau, T_n))$ is a pricing formula of the European option uder the Black-Schole likely assumption; that is the forward LIBOR under the Black-Schole likely assumption is given as

$$dL(t, T_n) = L(t, T_n)(\alpha(t)dt + \sigma_{ni}(t)dz_t)$$

with initial value $L_i(\tau, T_n)$.

Proof. A European option price $V(\tau, L(t, T_n))$ with payoff $F(L(T_n, T_n))$ under the P^{M+1}-measure is calculated as

$$\begin{aligned}&V(\tau, L(\tau, T_n))\\&= E^{P^{M+1}}\left[F\left(L(\tau, T_n)e^{Y(T_n, T_n)}\right)\right]\\&= E^{P^{M+1}}\left[F\left(L(\tau, T_n)\exp\left(\textstyle\int_\tau^{T_n} \bar{\alpha}^Y(t, T_n, T_M)ds + \int_\tau^{T_n} \gamma_n^Y(s)dw_s^{n+1} + \int_{\mathbb{R}} x\mu_n^Y(dt, dx)\right)\right)\right]\\&= \textstyle\sum_{a=0}^{\infty} \mathbb{P}(N_t^n = a) E^{P^{M+1}}\left[F\left(L(\tau, T_n)\exp\left(c(\tau, T_n) + \int_\tau^{T_n} \gamma_n^Y(s)dw_s^{n+1} + \sum_{k=1}^{a} Y_k\right)\right)\right],\end{aligned}$$

where

$$c(\tau, T_n) = \int_\tau^{T_n} \alpha(s, T_n, T_M)ds - \frac{1}{2}\int_\tau^{T_n} \|\gamma_n(s)\|^2 ds - \lambda_n k_n.$$

Since

$$c(\tau, T_n) + \int_\tau^{T_n} \gamma_n^Y(s)dw_s^{n+1} + \sum_{k=1}^{i} Y_k \sim N(c(\tau, T_n) + i\eta_n), \int_\tau^{T_n} \|\gamma_n(s)\|^2 ds + i\eta_n^2), \tag{4}$$

the random variable

$$c(\tau, T_n) + i\eta_n + \sqrt{\frac{\int_\tau^{T_n} \|\gamma_n(s)\|^2 ds + i\eta_n^2}{T_n - \tau}}\, z_{T_n - \tau} \tag{5}$$

is equal to (4), where z_t is a one-dimensional Brownian motion.

Set

$$\sigma_i^2(\tau, T_n) = \frac{\int_\tau^{T_n} \|\gamma_n(s)\|^2 ds + i\eta_n^2}{T_n - \tau} \tag{6}$$

and rewrite (5) as the form

$$\int_{\tau}^{T_n} \alpha(s)ds - \frac{1}{2}\sigma_{ni}^2(t)(T_n-\tau) + i\xi_n^2(T_n-\tau) + (\lambda_n k_n)(T_n-t) + i\eta_n + \sigma_{ni}(\tau)z_{T_n-\tau}.$$

This implies that

$$V(\tau, L(\tau, T_n))$$
$$= \sum_{i=0}^{\infty} \mathbb{P}(N_t^n = i) E^z \left[F\left(L(\tau, T_n) e^{i\xi_n^2(T_n-\tau)+(\lambda_n k_n)(T_n-t)+i\eta_n} \dot{e}^{\int_{\tau}^{T_n} \alpha(s)ds - \frac{1}{2}\sigma_{ni}^2(t)(T_n-\tau)+\sigma_{ni}(\tau)z_{T_n-\tau}} \right) \right].$$

Setting

$$L_i(\tau, T_n) = L(\tau, T_n) e^{i\xi_n^2(T_n-\tau)+(\lambda_n k_n)(T_n-t)+i\eta_n},$$

we get

$$V(\tau, L(\tau, T_n))$$
$$= \sum_{i=0}^{\infty} \mathbb{P}(N_t^n = i) E^z \left[F\left(L_i(\tau, T_n) \dot{e}^{\int_{\tau}^{T_n} \alpha(s)ds - \frac{1}{2}\sigma_i^2(\tau, T_n)(T_n-\tau)+\sigma_{ni}(\tau)z_{T_n-\tau}} \right) \right]$$
$$= \sum_{i=0}^{\infty} \mathbb{P}(N_t^n = i) V^{BS}(L_i(\tau, T_n), \textstyle\int_{\tau}^{T_n} \alpha(s)ds, \sigma_i^2(\tau, T_n)).$$

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3 Delay digital option pricing formula

3.1 Delay Digital Options

An interest rate delayed digital call (put) option (DC (DP)) pays one currency unit at maturity T_{i+1} if the reference interest rate $L(T_{ij}, T_{ij})$ that matured previously at time T_{ij} with the compounding period $[T_{ij}, T_{ij}^*]$ lies above (below) the strike rate K_{ij}. The final payoff of this option at time T_{i+1} is precisely given as follows: [1]

$$DO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = 1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}}.$$

where $L(T_{ij}, T_{ij})$ is a matured LIBOR rate for the period $[T_{ij}, T_{ij}^*]$, θ set to 1 stands for a digital call and -1 for a digital put.

[1] $1_{\{\cdot\}}$ is an indicator function, defined as follows:

$$1_{\{A\}}(\omega) = \begin{cases} 1 \text{ if } \omega \in A, \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 3.1 *The value of the DO at time τ is given as follows*

$$DO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1}) \sum_{a=0}^{\infty} \frac{e^{-\lambda_{i+1}(T_{i+1}-\tau)}(\lambda_{i+1}(T_{i+1}-\tau))}{a!} N(\theta d(T_{ij}); a)$$

with

$$d(T_{ij}; a) = \frac{\log\left(\frac{L_a(\tau, T_{ij})}{K_{ij}}\right) + \rho(\tau, T_{ij}; T_{i+1}) - \frac{1}{2}\sigma_a(\tau, T_{ij})}{\sqrt{\sigma_a(\tau, T_{ij})}},$$

where

$$\rho(\tau, T_{ij}; T_{i+1}) = \int_{\tau}^{T_{ij}} \alpha(s, T_{ij}; T_{i+!})ds$$

and $\sigma_a(t, T)$ is defined in (6).

3.2 Delay Digital Range Option

A delayed range digital option (DRO) is similar to an DO except that the payment occurs as the reference rate lies inside a pre-specified range $[K_{ij}^L, K_{ij}^U]$. The final payoff of a general DRO at time T_{i+1} is defined as follows:

$$DRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = 1_{\{K_{ij}^L \leq L(T_{ij}, T_{ij}) \leq K_{ij}^U\}}. \tag{7}$$

Based on the property in probability measure theory, the DRO payoff can be expressed in terms of two DC payoffs. It means that (15) can be rewritten as follows:

$$DRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = DC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^L) - DC(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}^U).$$

Theorem 3.2 *The time τ value of the DRO is equal to*

$$DRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = DC(\tau, T_{i+1}; T_{ij}; K_{ij}^L) - DC(\tau, T_{i+1}; T_{ij}; K_{ij}^U). \tag{8}$$

Remark 3.3 *For an DO, if the maturity date T_{ij} of its reference rate equals T_{i+1} which is also the maturity date of the DO, then the DO becomes an ordinary digital option without delaying its payoff. Similarly, as $T_{ij} = T_{i+1}$, an DRO also becomes an ordinary digital range option.*

3.3 Delay Interest-or-Nothing Digital Option

A delayed interest-or-nothing digital call (put) option (DIC (DIP)) pays a floating interest payment $L(T_i, T_i)$ at maturity date T_{i+1} if the reference interest rate $L(T_{ij}, T_{ij})$ is above (below) a pre-specified strike rate K_{ij}. We state the contract formally by specifying its final payoff at time T_{i+1} as follows:

$$DIO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = L(T_i, T_i)1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}},$$

$\theta = 1$ stands for a digital call option, and -1 for a digital put option.

Theorem 3.4 *Let*

$$\frac{dR^{T_{i+1}}}{dQ^{T_{i+1}}} = e^{\gamma_i dw_t^{T_{i+1}} + \sum_{l=1}^{r} \int_{\mathbb{R}} x\mu_l(dt,dx) - \frac{1}{2}||\gamma_i||^2 - \int_{\mathbb{R}} (e^x - 1 - x)\lambda_{ij}\nu_{ij}^{T_i}(dx)dt}.$$

The time τ vale of the DIO is given as follows:

$$DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1})L(\tau, T_i)e^{\bar{\alpha}_i} E^{R^{T_{i+1}}}[1_{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}}],$$

where

$$\bar{\beta}_i = \alpha(t, T_{ij}, T_{i+1}) - \int_{\mathbb{R}} x\lambda_{ij}\nu_{ij}^{T_i}(dx).$$

Proof. The DIO under the forward measure $Q^{T_{i+1}}$ is priced as follows:

$$DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(t, T_{i+1})E^{Q^{T_{i+1}}}(L(T_i, T_i)1_{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}}).$$

$DIO(\tau, T_{i+1}; T_{ij}; K_{ij})$ can be written as follows

$$P(\tau, T_{i+1})L(\tau, T_i)e^{\bar{\beta}_i} E^{Q^{T_{i+1}}}[\frac{dR^{T_{i+1}}}{dQ^{T_{i+1}}} 1_{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}}],$$

where

$$\frac{dR^{T_{i+1}}}{dQ^{T_{i+1}}} = e^{\gamma_i dw_t^{T_{i+1}} + \sum_{l=1}^{r} \int_{\mathbb{R}} x\mu_l(dt,dx) - \frac{1}{2}||\gamma_i||^2 - \int_{\mathbb{R}} (e^x - 1 - x)\lambda_{ij}\nu_{ij}^{T_i}(dx)dt},$$

and

$$\bar{\beta}_i = \alpha(t, T_{ij}, T_{i+1}) - \int_{\mathbb{R}} x\nu_{ij}^{T_i}(dx).$$

Applying Levy-type Girsanov theorem, the dynamic of $Y(\tau, T_{ij})$ under the measure $R^{T_{i+1}}$ is

$$dY(\tau, T_{ij}) = \tilde{\alpha_{ij}}dt + \gamma_{ij}dw^{R^{T_{i+1}}} + \int_{\mathbb{R}} x\mu_{ij}^{R^{T_{i+1}}}(dx, dt),$$

where

$$\tilde{\alpha_{ij}} = \bar{\alpha}_{ij} + \frac{1}{2}||\gamma_{ij}||^2 + \int_{\mathbb{R}} xe^x\nu_{ij}^{T_i}(dx);$$

that is the Levy measure is $\nu_{ij}^{R_{ij}} = e^x\nu_{ij}^{T_i}$ under the measure $R^{T_{i+1}}$. Thus one obtains

$$DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(\tau, T_{i+1})L(\tau, T_i)e^{\bar{\alpha}_i} E^{R^{T_{i+1}}}[1_{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}}].$$

❑

Theorem 3.5 *The time τ value of the DIO is given as follows:*

$$
\begin{aligned}
&DIO(\tau, T_{i+1}; T_{ij}; K_{ij}) \\
&= P(\tau, T_{i+1}) L(\tau, T_i) e^{\bar{\alpha}_i} \textstyle\sum_{a=0}^{\infty} \frac{e^{-\lambda_{i+1}(T_{i+1}-\tau)}(\lambda_{i+1}(T_{i+1}-\tau))}{a!} N(\theta e(T_{ij}); a)
\end{aligned}
$$

with

$$
e(T_{ij}) = \frac{\ln\left(\frac{L(\tau,T_{ij})}{K_{ij}}\right) + \eta(\tau, T_i; T_{ij}; T_{i+1}) - \frac{1}{2}V_a(\tau, T_{ij})}{\sqrt{V_a(\tau, T_{ij})}},
$$

where

$$
\eta(\tau, T_i; T_{ij}; T_{i+1}) = \int_{\tau}^{T_i} \gamma(t, T_{ij}) \cdot \Big(\bar{\sigma}^{\tau}(t, T_{ij}^*) - \bar{\sigma}^{\tau}(t, T_{i+1}) + \gamma(t, T_i)\Big) dt \qquad (9)
$$

$\rho(\cdot, T_{ij}; T_{i+1})$ and $V_a(t, T)$ is defined in (6).

Proof. When the probability density function $f(x; \mu, \sigma)$ is given as a normal density with mean μ and variance σ^2, we have

$$
e^x f(x; \mu, \sigma) = e^{\mu + \frac{\sigma^2}{2}} f(x; \bar{\mu}, \sigma),
$$

where $\bar{\mu} = \mu + \sigma^2$. This implies that $\nu_{ij}^{R_{ij}}(dx) = e^x \nu_{ij}^{T_i}(dx) = e^x f^{T_i}(x; \mu, \sigma) dx = e^{\mu + \frac{\sigma^2}{2}} f(x; \bar{\mu}, \sigma)$ when the pdf $f^{T_i}(x; \mu, \sigma)$ is a normal density. ❑

Remark 3.6 *The characteristic function of Y under the forward measure $R^{T_{i+1}}$ is written as*

$$
E^{R^{T_{i+1}}}[e^{izY}] = e^{t\psi^{R^{T_{i+1}}}(z)} \equiv \chi^{R^{T_{i+1}}}(z),
$$

where

$$
\psi(z) = iz\tilde{\alpha}_{ij} - \frac{1}{2} z^2 \bar{\sigma}_n^2 + \sum_{l=1}^{r} \int_{\mathbb{R}} (e^{izx} - 1 - izx) \lambda_l \bar{\nu}_l^R(dx).
$$

The expected value $E^{R^{T_{i+1}}}[1_{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}}]$ can also be optioned by the Fourier transform method (Raible, 2000). The details of the Fourier transform method is given in Appendix B.

3.4 Delay Interest-or-Nothing Range Digital Option

A delayed interest-or-nothing range digital option (DIRO) pays a floating interest payment at maturity T_{i+1} if the reference interest rate $L(T_{ij}, T_{ij})$ lies within a pre-specified range $[K_{ij}^L, K_{ij}^U]$. The final payoff of the DIRO is defined as follows:

$$
DIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = L(T_{ij}, T_{ij}) 1_{\{K_{ij}^L \le L(T_{ij}, T_{ij}) \le K_{ij}^U\}}.
$$

Similar to DROs, DIROs can also be expressed in terms of two DICs, i.e.

$$DIRO(T_{i+1}, T_{i+1}; T_{ij}; K_{ij}) = DIC(T_{i+1}, T_{i+1}; T_{ij}; K^L_{ij}) - DIC(T_{i+1}, T_{i+1}; T_{ij}; K^U_{ij}).$$

Thus, the pricing formula of the DIRO can be expressed in terms of the pricing formulas of DICs, and the result is presented in the following theorem.

Theorem 3.7 *The time τ value of the DIRO is equal to*

$$DIRO(\tau, T_{i+1}; T_{ij}; K_{ij}) = DIC(\tau, T_{i+1}; T_{ij}; K^L_{ij}) - DIC(\tau, T_{i+1}; T_{ij}; K^U_{ij}). \tag{10}$$

Remark 3.8 *If the maturity date T_{ij} of its reference rate equals T_{i+1} that is also the maturity date of the DIO and DIRO, then the DIO and DIRO become, respectively, an ordinary interest-or-nothing digital option and an ordinary interest-or-nothing range option without delaying its payoff.*

4 Fourier transform method for Delay Digital Options

According to Theorem 3.2 in Raible (2000), the initial price of an option can be obtained as follows.

Theorem 4.1 *Consider the DO with payoff $w(L(T_{ij}, T_{ij})) = 1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}}$ at time T_{i+1}. Let $v(x) := w(e^{-x})$ denote the modified payoff function. Assume that $x \to e^{-Rx}|v(x)|$ is bounded and integrable for some $R \in \mathbb{R}$ such that the moment generating funtion $mgf(u)$ of Y satisfies $mgf(-R) < \infty$. The value of the DO at time τ is given as follow:*

$$DO(\tau, T_{i+1}; T_{ij}; K_{ij}) = \frac{e^{Y(\tau, T_{ij})R}}{2\pi} P(t, T_{i+1}) \int_{\mathbb{R}} e^{iuY(\tau, T_{ij})} L[v](R+iu)\chi(iR-u)du.$$

Proof. The DO under the forward measure $Q^{T_{i+1}}$ priced as follows

$$DO(\tau, T_{i+1}; T_{ij}; K_{ij}) = P(t, T_{i+1}) E^{Q^{T_{i+1}}}(1_{\{\theta L(T_{ij}, T_{ij}) > \theta K_{ij}\}}).$$

Applying Theorem 1, $L(T_{ij}, T_{ij})$ under the measure $Q^{T_{i+1}}$ is given by

$$L(T_{ij}, T_{ij}) = L(\tau, T_{ij}) e^{Y(T_{ij}, T_{ij})},$$

where

$$dY(t, T_{ij}) = \alpha_{ij} dt + \gamma_{ij} dw^{T_{i+1}}_t + \sum_{i=1}^{r} \int_{\mathbb{R}} x\mu_i(dt, dx)$$

with the Levy measure $\nu_i^{T_i}(dx)$. As $\theta = 1$, the pricing formula of the DC is defined as follows.

$$
\begin{aligned}
& DC(\tau, T_{i+1}; T_{ij;K_{ij}}) \\
&= P(t, T_{i+1}) E^{Q^{T_{i+1}}}[1_{e^{-Y(\tau,T_{ij})+Y(T_{ij},T_{ij})}>K_{ij}}] \\
&= P(t, T_{i+1}) E^{Q^{T_{i+1}}}[v(Y(\tau, T_{ij}) - Y(T_{ij}, T_{ij}))] \\
&= P(t, T_{i+1}) \int_{\mathbb{R}} v(Y(\tau, T_{ij}) - y)\rho(y)dy \\
&= \tfrac{e^{Y(\tau,T_{ij})R}}{2\pi} P(t, T_{i+1}) \int_{\mathbb{R}} e^{iuY(\tau,T_{ij})} L[DC](R + iu)du \\
&= \tfrac{e^{Y(\tau,T_{ij})R}}{2\pi} P(t, T_{i+1}) \int_{\mathbb{R}} e^{iuY(\tau,T_{ij})} L[v](R + iu)\chi(iR - u)du.
\end{aligned}
$$

The bilateral Laplace transform $L[\rho]$ of the density ρ is given by

$$L[\rho](z) = \int_{\mathbb{R}} e^{-zx}\rho(x)dx.$$

Hence, we have the identity $L[\rho](z) = \chi(iR - u)$. ❑

Given $\theta = 1$ in the payoff function $w(L(T_{ij}, T_{ij}))$. For $R > 0$, the transform of $v(x)$ can be deduced as

$$L[v](R + iu) = \frac{K_{ij}^{iu-R}}{R - iu}.$$

Hence the pricing formula of the DC is written as

$$DO(\tau, T_{i+1}; T_{ij}; K_{ij}) = \frac{e^{Y(\tau,T_{ij})R}}{2\pi} P(t, T_{i+1}) \int_{\mathbb{R}} e^{iuY(\tau,T_{ij})} \frac{K_{ij}^{iu-R}}{R - iu} \chi(iR - u)du.$$

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