How to apply GARCH model in risk management?
Model diagnosis on GARCH innovations

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Abstract
Risk management requires accurate estimates of the downside risk of financial investment. For financial return series, specific features such as heavy-tailedness and volatility clustering impose extra difficulties in downside risk evaluation. The Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model captures these features regardless of the distributional assumptions on the innovation process. Nevertheless, distribution of the innovations plays an important role when analyzing conditional and unconditional downside risk. We show that the diagnosis method on the heavy-tailedness of GARCH innovations in [8] is not reliable for GARCH processes that are close to non-stationarity. With comparing different tail index estimates, we provide an alternative approach which leads to a formal test on the distribution of GARCH innovations. Empirical analysis on real data confirms similar finding as in [8] that when modeling financial returns with the GARCH model, the downside distribution of innovations possesses heavier tail than the usual normality assumption.

Key words: Dynamic Risk Management, GARCH(1,1), Extreme Value Theory, Hill Estimator.

EFM Classification: 450

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1 Introduction

Downside risk of financial investment is of the major concern for investors. Risk managers often assess downside risk measures such as the Value-at-Risk (VaR) to evaluate the potential large loss of their investment portfolios. Regulators also consider downside risk of financial institutions and impose regulation to prevent severe systemic crisis based on quantitative risk measures. Having an accurate estimate of the downside risk measures is not an easy task, due to specific features of financial return series. First, the time series of financial returns exhibit volatility clustering. Second, the distribution of financial returns exhibit heavy-tails: the downside tail of the return distribution decays in a power law speed instead of the exponential speed as in the normal distribution. The two specific features impose a great challenge in producing accurate and time-varying estimates of downside risk measures.

The Generalized Autoregressive Conditional Heteroscedasticity (GARCH) model introduced by [1] attempts to capture the volatility clustering feature of financial returns by modeling the dynamic of volatility. Thus, it leads to time-varying estimate on downside risk measures such as the VaR. Another surprising fact of the GARCH-type models is that they capture the heavy-tailedness at the same time. Following the result in [7], the stationary solution of GARCH (1,1) process follows a heavy-tailed distribution, see [9], [3], etc. Therefore, the GARCH-type models turn to be an effective instrument in risk management.

This paper provides a diagnosis framework on the distributional assump-
tion of GARCH innovations. The distribution of GARCH innovations plays an important role for both conditional and unconditional risk measurement. When analyzing conditional risk, it is obvious that the conditional downside risk depends on the distribution of the innovations. When analyzing unconditional risk, we show that GARCH models with different innovations lead to different shape of the downside tail, hence different estimates of the tail risk. Therefore, an inappropriate distributional assumption of innovations may lead to either underestimate or overestimate the downside risk. It is thus important to clarify which type of GARCH innovations should be applied for modeling financial returns and how this influences the heavy-tailedness of the original GARCH process.

The most often applied GARCH (1,1) model with normal distributed innovation assumes that the conditional distribution of financial return given the return level and volatility of the previous period follows a normal distribution. In risk management, recent literature shows that the conditional normality assumption does not perform well in estimating the downside risk with a low probability, see [2]. [9] show that the GARCH process with normal innovation generates much thinner tail than that obtained from the real data. [8] show that the GARCH models with heavy-tailed innovation is more efficient in estimating and forecasting the downside risk of financial returns, whereas the estimates from GARCH models with normal innovation underestimate the potential downside risk.

[8] detect the heavy-tailedness of innovations based on estimated innovations. They find that the conditional normality assumption is not valid. Based on this fact, the estimation procedure of the GARCH model can be
divided into two steps: first, they use Quasi-Maximum-Likelihood-Estimates (QMLE) to back out the innovations; second, they use the Generalized Pareto Distribution to model the estimated innovations and consequently estimate the tail shape of the distribution of the innovations. This method has been followed by other studies, see e.g. [6].

We start by revisiting their method and show that the estimated innovations obtained from data simulated from the GARCH model with normal innovations may violate the normality assumption. This phenomenon turns apparent when the GARCH process is close to non-stationarity. Thus the [8] method is not robust in diagnosing the heavy-tailedness of GARCH innovations. We develop an alternative approach based on analyzing the tail index of the GARCH process. Our method yields a formal test on the distributional assumption of the innovations. Taking normal and Student-t distributed innovations as examples, simulations show that our method is robust in the case of near non-stationarity. Moreover, this method leads to a robust estimate of the tail index of a GARCH process.

We apply our method to the S&P 500 Composite Index and 12 S&P equity sector indices. The estimated GARCH coefficients indicate that the fitted GARCH models are close to non-stationarity. Therefore, the [8] method is not valid in this case. With our formal test, we reject the normal innovation in most cases, while can not always reject the Student-t innovation hypothesis.

The rest of the paper is organized as follows. In Section 2, we show that using the [8] approach based on estimated innovations is not robust for detecting heavy-tailedness of innovations. Instead, we develop a test via
analyzing the tail index of the GARCH (1,1) model. A discussion on the Hill estimator used in our test is given in Section 3. Simulations and empirical results are presented in Section 4. Section 5 concludes.

2 Theory

2.1 Heavy-tailedness of the GARCH Series

We consider the GARCH (1,1) model in modeling the financial returns \( \{X_t\} \) as follows,

\[
X_t = \varepsilon_t \sigma_t, \tag{1}
\]

\[
\sigma_t^2 = \lambda_0 + \lambda_1 X_{t-1}^2 + \lambda_2 \sigma_{t-1}^2, \tag{2}
\]

where \( \{\varepsilon_t\} \) are independent and identically distributed (i.i.d.) innovations with zero mean and unit variance, the parameters \( \lambda_0, \lambda_1, \lambda_2 \) are positive. Moreover, in order to have a stationary solution of the GARCH model, we assume the stationary condition \( \lambda_1 + \lambda_2 < 1 \).

The heavy-tailedness of the stationary solution of a GARCH model follows from the result of [7]. Consider a process \( \{Y_t, t \geq 1\} \) satisfying the stochastic difference equation

\[
Y_t = Q_t Y_{t-1} + M_t, \tag{3}
\]

where \( \{(Q_t, M_t)\} \) are i.i.d. \( \mathbb{R}^2 \)-valued random pairs. [7] shows that the stationary solution of the stochastic difference equation follows a heavy-
tailed distribution as follows. Suppose there exists a positive real number \( \kappa \), such that

\[
E(Q_1^\kappa) = 1, \ E(Q_1^\kappa \log Q_1) < \infty, \ 0 < E(M_1^\kappa) < \infty.
\]

Moreover, assume that \( \frac{M_1}{1-Q_1} \) is non-degenerate and the conditional distribution of \( \log Q_1 \) given \( Q_1 \neq 0 \) is nonlattice. Then the stationary solution of \( Y_t \) follows a heavy-tailed distribution as

\[
P(Y_t > x) = Ax^{-\kappa}[1 + o(1)], \ \text{as} \ x \to \infty, \quad (4)
\]

where \( \kappa \) is the so-called tail index and \( A \) is the tail scale.

The GARCH model is associated to a specific stochastic difference equation. By combining the equations (1) and (2), we derive the following stochastic difference equation on the stochastic variance series \( \{\sigma^2_t\} \) as

\[
\sigma^2_t = \lambda_0 + (\lambda_1 \varepsilon^2_t + \lambda_2)\sigma^2_{t-1}, \quad (5)
\]

which satisfies equation (3) with \( Q_t = \lambda_1 \varepsilon^2_t + \lambda_2 \) and \( M_t = \lambda_0 \). Denote \( \varphi_t = \varepsilon^2_t \). Suppose \( \kappa \) is the solution to the equation

\[
E[(\lambda_1 \varphi_t + \lambda_2)^\kappa] = 1. \quad (6)
\]

The stationary solution of \( \sigma^2_t \) follows a heavy-tailed distribution with tail index \( \kappa \). Hence, \( \sigma_t \) follows a heavy-tailed distribution with tail index \( 2\kappa \). The relation (6) implies that \( E[\varepsilon^{2\kappa}_t] < \infty \). Therefore, the tail of \( \sigma_t \) is heavier
than that of $\varepsilon_t$. If $\varepsilon_t$ follows a thin-tailed distribution such as the normal distribution, $\sigma_t$ follows a heavy-tailed distribution. If $\varepsilon_t$ follows a heavy-tailed distribution, then its tail index is higher than that of $\sigma_t$.

Furthermore, from [9], we can derive the following relation

$$
P\{|X_t| > x\} = P\{|\sigma_t \varepsilon_t| > x\} \sim E[|\varepsilon_t|]^\kappa P\{\sigma_t > x\}, \text{ as } x \to \infty. $$

Thus $|X_t|$ has a similar tail behavior as $\sigma_t$, in the sense that the tail index of $|X_t|$ is $2\kappa$.

From the discussion, we observe that the general heavy-tailed feature of the GARCH model is irrelevant to the distribution of the innovation. No matter the innovation is thin or heavy-tailed, the stationary solution of the GARCH model is always heavy-tailed. However, the shape of the tail distribution of a stationary GARCH series does depend on the distribution of innovations: the solution to equation (6) differs for different distributions of $\varphi_t$. Since different distributional assumptions on the innovations may lead to different risk analysis on a GARCH series, it is necessary to verify which innovation model fits the actual financial returns.

2.2 Diagnosing the Distribution of GARCH Innovations: the McNeil and Frey Approach

[8] (MF2000) confirm the GARCH innovation of financial returns is heavy-tailed. However, they test the heavy-tailedness of the innovations by making QQ plot on the estimated innovations against a normal distribution. Since the innovations are backed out from an estimation procedure, its heavy-
tailedness may potentially be imposed by the estimation procedure. We demonstrate this phenomenon by the following simulation.

Given the Maximum-Likelihood-Estimates (MLE) $\hat{\lambda}_0, \hat{\lambda}_1, \hat{\lambda}_2$ (see [5]), one can estimate the innovations $\hat{\varepsilon}_t$ by

$$\hat{\varepsilon}_t = X_t / \hat{\sigma}_t, \quad (7)$$

$$\hat{\sigma}^2_t = \hat{\lambda}_0 + \hat{\lambda}_1 X_{t-1}^2 + \hat{\lambda}_2 \hat{\sigma}^2_{t-1}. \quad (8)$$

Using the same data as in MF2000, we reproduce their QQ plot in Figure 1 (left panel). In addition, we simulate a series of observations from the GARCH (1,1)-normal process with the model coefficients equivalent to the MLE obtained from the real data. Then we re-estimate the innovations from (7) & (8) and display the corresponding QQ plot in Figure 1 (right panel). We find that the estimated innovations still violate the normality assumption, while exhibiting a heavy-tailed feature.

From the simulation exercise, we observe that using estimated innovation backed out from a finite sample is not a robust method in testing the heavy-tailedness of the innovations. The intuition of the non-robustness is given as follows.

By combining equation (2) and (8), we have that

$$\hat{\sigma}^2_t - \sigma^2_t = \lambda_2 (\hat{\sigma}^2_{t-1} - \sigma^2_{t-1}) = \cdots = \lambda_2^t (\hat{\sigma}^2_0 - \sigma^2_0)$$

Hence, $\sigma^2_t$ is not accurately estimated for a finite $t$. This may lead to a misestimation in the innovations. The difference between $\hat{\sigma}^2_t$ and $\sigma^2_t$ stems
Note: The QQ-plots show the estimated innovations against the sample quantiles from a normal distribution. Straight line corresponds to that the estimated innovations follow a normal distribution. The left panel demonstrates the estimated innovations from fitting the S&P 500 returns (from June 1985 to May 1989) to a GARCH (1,1) model by MLE. The right panel demonstrates the estimate innovations from fitting a simulated sample, with the same sample size, while the simulation is based on a GARCH (1,1)-normal model, with the coefficients equivalent to the MLE from S&P 500 returns.

from that between $\hat{\sigma}_0^2$ and $\sigma_0^2$. Moreover, the difference between $\hat{\sigma}_0^2$ and $\sigma_0^2$ comes from the fact that $\hat{\sigma}_0^2$ is simply some initial value chosen in estimation, while $\sigma_0^2$ follows the stationary solution of the stochastic difference equation (3), i.e. a heavy-tailed distribution. When $\lambda_1 + \lambda_2$ is close to 1, the parameter $\kappa$ from equation (6) is close to 1. In the case $\kappa = 1$, we get that $E(\hat{\sigma}_t^2) = +\infty$. Hence any initial value $\hat{\sigma}_0^2$ may underestimates the potential $\sigma_0^2$, which implies that $\hat{\sigma}_t^2$ underestimates $\sigma_t^2$. Hence $\hat{\varepsilon}_t$ may demonstrate a heavier tail than $\varepsilon_t$. This is the main intuition why given that $\varepsilon_t$ follows a normal distribution, it is still possible to obtain heavy-tailedness of $\hat{\varepsilon}_t$.

The following lemma shows theoretically that based on finite observations generated from a GARCH model with normal innovations, the estimated innovations follow a heavy-tailed distribution.

Lemma 1 Consider a GARCH (1,1) model in (1) and (2) with normal dis-
tributed innovations \{\varepsilon_t\}. Suppose \hat{\lambda}_0, \hat{\lambda}_1 and \hat{\lambda}_2 perfectly estimate \lambda_0, \lambda_1, \lambda_2, i.e. \hat{\lambda}_0 = \lambda_0, \hat{\lambda}_1 = \lambda_1, \hat{\lambda}_2 = \lambda_2. The estimated innovations from (7) and (8) with the form
\[
\hat{\varepsilon}_t = \frac{X_t}{\sigma_t} = \frac{\sigma_t}{\hat{\sigma}_t} \varepsilon_t
\]
follow a heavy-tailed distribution for finite t.

2.3 Testing Distributional Assumption of GARCH Innovations

The different distributional assumptions on the GARCH innovations lead to different level of heavy-tailedness of the GARCH series, measured by its tail index. This provides an alternative way to test which innovation fits the real data.

We investigate the difference between Student-t and normal innovations as an example. The GARCH models with these two types of innovations lead to different tail behavior. In a GARCH(1,1)-normal model, the innovation \varepsilon_t follows a standard normal distribution. The parameter \kappa solved from equation (6) is denoted as \kappa_n(\lambda_1, \lambda_2). Then the tail index of the GARCH series is \alpha_n(\lambda_1, \lambda_2) = 2\kappa_n(\lambda_1, \lambda_2). Differently, in a GARCH(1,1)-Student model, the innovation \varepsilon_t follows a Student-t distribution with degree of freedom \nu, normalized to unit variance. The parameter \kappa solved from equation (6) is then denoted as \kappa_s(\lambda_1, \lambda_2, \nu). Similarly, the tail index of the GARCH series is \alpha_s(\lambda_1, \lambda_2, \nu) = 2\kappa_s(\lambda_1, \lambda_2, \nu). Notice that \kappa_n = \kappa_s = 1 for \lambda_1 + \lambda_2 = 1.

The GARCH coefficients \lambda_1, \lambda_2 are connected to the tail index according to \alpha_n(\lambda_1, \lambda_2) or \alpha_s(\lambda_1, \lambda_2, \nu). With assuming the innovations follow a nor-
mal or Student-t distribution, we obtain the implied tail indices $\alpha_n(\hat{\lambda}_1, \hat{\lambda}_2)$ or $\alpha_s(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu})$ with the estimated GARCH coefficients $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\nu}$.

Alternatively, the tail index of a GARCH series can be estimated from the so-called Hill estimator in extreme value analysis. Let $X_1, ..., X_n$ be the observations from a heavy-tailed distribution as in equation (4). The Hill estimator is defined as

$$\hat{\alpha}_H := \left( \frac{1}{k} \sum_{i=1}^{n} 1_{\{X_i > s\}} [\log(X_i) - \log(s)] \right)^{-1},$$  \hspace{1cm} (9)

where $k = k(n)$ is the number of observations that exceed the threshold $s$, satisfying $k(n)/n \to 0$ as $n \to \infty$. The Hill estimator is usually applied to i.i.d. sample. Nevertheless, [10] show that the consistency of Hill estimator for the solutions of stochastic difference equations of the form in equation (3) holds. Hence for GARCH (1,1) process, we can apply the Hill estimator.

By testing whether the implied tail indices from the GARCH (1,1)-normal and GARCH (1,1)-Student models differ from the estimated tail index, we can formally test the two models. The following lemmas show the relations between $\alpha_n(\hat{\lambda}_1, \hat{\lambda}_2)$, $\alpha_s(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu})$ and the real tail index $\alpha$.

**Lemma 2** Suppose $X_1, ..., X_n$ follow a GARCH (1,1)-normal process, $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\nu}$ are consistent estimates of $\lambda_1$, $\lambda_2$, $\nu$ when fitting either a GARCH-normal or GARCH-Student model, $\alpha$ is the real tail index of the series. Then as $n \to \infty$,

1. $\alpha_n(\hat{\lambda}_1, \hat{\lambda}_2) \to^P \alpha$,

2. $\alpha_s(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu}) \to^P \alpha$.  

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Lemma 3 Suppose \( X_1, \ldots, X_n \) follow a GARCH (1,1)-Student process, \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu} \) are the consistent estimates of \( \lambda_1, \lambda_2, \nu \) when fitting either a GARCH-normal or GARCH-Student model, \( \alpha \) is the real tail index of the series. Then as \( n \to \infty \),

1. \( \alpha_s(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu}) \to^P \alpha \),

2. \( P(\alpha_n(\hat{\lambda}_1, \hat{\lambda}_2) > \alpha) \to 1 \).

Here the sign \( \to^P \) in the two lemmas refers to the convergence in probability. The relation \( P(\alpha_n(\hat{\lambda}_1, \hat{\lambda}_2) > \alpha) \to 1 \) indicates that \( \alpha_n(\hat{\lambda}_1, \hat{\lambda}_2) \) does not converge to the real tail index \( \alpha \) in probability if a GARCH(1,1)-Student process is mis-estimated as a GARCH-normal process. A detailed proof is given in Appendix.

From the above two lemmas, \( \alpha_s(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu}) \) is always a consistent estimate of the tail index \( \alpha \) in both models, while \( \alpha_n(\hat{\lambda}_1, \hat{\lambda}_2) \) overestimates the tail index when the observations follow a GARCH (1,1)-Student model. With considering the asymptotic behavior of the estimates of the implied tail index and the Hill estimator, we provide a formal test on distributional assumption of the innovation in GARCH model. The following theorem gives the asymptotic properties of the test statistics. With the fact that \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu} \) converges to the parameters \( \lambda_1, \lambda_2, \nu \) with a faster speed of convergence than \( \sqrt{k} \). The proof follows from the asymptotic normality of the Hill estimator, thus it is omitted here.

Theorem 4 Denote \( \hat{\alpha}_n = \alpha_n(\hat{\lambda}_1, \hat{\lambda}_2) \), \( \hat{\alpha}_s = \alpha_s(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu}) \), where \( \hat{\lambda}_1, \hat{\lambda}_2, \hat{\nu} \) are consistent estimates of \( \lambda_1, \lambda_2, \nu \), \( \alpha \) is the real tail index of the series.
$\hat{\alpha}_H$ is the Hill estimator with sample fraction $k$, such that $\sqrt{k}(\hat{\alpha}_H - 1) \to^d N(0,1)$, as $n \to +\infty$.

1. If $X_1, ..., X_n$ follow a GARCH (1,1)-normal process, then

$$\sqrt{k}(\hat{\alpha}_n - 1) \to^d N(0,1), \text{ as } n \to +\infty.$$  

2. If $X_1, ..., X_n$ follow a GARCH (1,1)-Student process, then

$$\sqrt{k}(\hat{\alpha}_s - 1) \to^d N(0,1), \text{ as } n \to +\infty.$$  

3 The Asymptotic Bias of the Hill Estimator in GARCH Series

Although the Hill estimator is a valid method in estimating the tail index, it bears potential asymptotic bias due to the fact that the tail distribution is not an exact Pareto distribution. The following lemma shows how the approximation influences the asymptotic bias of the Hill estimator.

**Lemma 5** Suppose observations are obtained from a distribution possessing a density and satisfying the following equation

$$F(x) = 1 - Ax^{-\alpha} \left[ 1 + Bx^{-\beta} + o \left( x^{-\beta} \right) \right], \quad \beta > 0, \text{ as } x \to \infty, \quad (10)$$

where $\beta$ and $B$ are the second order tail index and tail scale respectively. Let the threshold $s$ satisfy $s^\alpha/n \to 0$, $s \to \infty$ as $n \to \infty$. The asymptotic bias of the $\frac{1}{\hat{\alpha}_H}$ is

$$E \left[ \frac{1}{\hat{\alpha}_H} - \frac{1}{\alpha} \right] = -\frac{B\beta}{\alpha (\alpha + \beta)} s^{-\beta} + o \left( s^{-\beta} \right), \quad (11)$$
and variance is

$$\text{Var} \left[ \frac{1}{\hat{\alpha}_H} \right] = \frac{s^\alpha}{an\alpha^2} + o \left( \frac{s^\alpha}{n} \right). \quad (12)$$

Combining the asymptotic bias and variance, we obtain the Asymptotic Mean Squared Error (AMSE) of the estimator $\frac{1}{\hat{\alpha}_H}$ as follows,

$$\text{AMSE} \left( \frac{1}{\hat{\alpha}_H} \right) = \frac{s^\alpha}{an\alpha^2} + \frac{B^2\beta^2}{\alpha^2 (\alpha + \beta)^2} s^{-2\beta}. \quad (13)$$

One can choose the optimal threshold $s$ by minimizing (13). With the optimal threshold, the squared bias and variance vanish at the same rate.

The asymptotic bias problem turns severe for finite sample applications with serial dependence such as the GARCH series. This may potentially contaminate the test procedure introduced in Theorem 4. Therefore, we investigate the asymptotic bias of the Hill estimator applying to a GARCH series.

We start by clarifying the second order approximation for the tail distribution of the stationary solution of a GARCH model.

**Lemma 6** For the GARCH (1,1)-normal model, suppose the tail expansion of $\sigma_t^2$ satisfies equation (10), where $\alpha = \kappa$ which is the solution to the equation $E[(\lambda_1\varphi_t + \lambda_2)^\kappa] = 1$. Then, $\beta = 1$ and

$$B = \frac{\kappa \lambda_0 E[(\lambda_1\varphi + \lambda_2)^{\kappa+1}]}{1 - E[(\lambda_1\varphi + \lambda_2)^{\kappa+1}]} < 0. \quad (14)$$

**Remark 7** For the GARCH (1,1)-Student model, the second order index $\beta = 1$ and $B$ has the same form as in (14) if $E[(\lambda_1\varphi + \lambda_2)^{\kappa+1}] < \infty$. 

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Since the constant $B$ is negative for the stationary solution of the GARCH model with normal innovations, the Hill estimator applied to such a GARCH series is downward biased. We demonstrate the Hill bias for the GARCH(1,1)-normal model by simulation against various values of $\lambda_1$ and $\lambda_2$. We first fix $\lambda_1$ at 0.08, and vary $\lambda_2$ from 0.87 to 0.91, then fix $\lambda_2$ at 0.88, and vary $\lambda_1$ from 0.07 to 0.11. The simulation procedure is as follows.

1. For any fixed $\lambda_1$ and $\lambda_2$, calculate the implied tail index $\alpha = 2\kappa$, where $\kappa$ is solved from equation (6).

2. Simulate 2500 samples with sample size 5000 from the GARCH (1,1)-normal model, then calculate the Hill estimator $\hat{\alpha}_i(k)$ of the downside tail for each sample fraction $k$, where $i = 1, \ldots, 2500$, $k = 1, \ldots, 500$.

3. For each $k$, calculate the corresponding Mean Squared Error (MSE) as $MSE(\hat{\alpha}(k)) = \frac{1}{2500} \sum_{i=1}^{2500} \left( \frac{1}{\hat{\alpha}_i(k)} - \frac{1}{\alpha} \right)^2$. 

Figure 2: Hill Bias of GARCH (1,1) - normal model

Note: The plots show the Hill bias of the GARCH(1,1)-normal models at the simulated optimal threshold level. In the left panel, we fix $\lambda_1 = 0.08$ and plot the bias against $\lambda_2$, while in the right panel, we fix $\lambda_2 = 0.88$ and plot the bias against $\lambda_1$. 

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4. Find the optimal sample fraction $k^*$ through minimizing the MSE,

5. Plot the bias $\frac{1}{2500} \sum_{i=1}^{2500} (\hat{\alpha}_i(k^*) - \frac{1}{\alpha})$ against $\lambda_1$ or $\lambda_2$.

We observe that the bias problem turns severe for $\lambda_1 + \lambda_2 \ll 1$, while it is of a lower importance when $\lambda_1 + \lambda_2$ is close to 1. Hence our test via comparing implied tail indices with the estimated tail index by the Hill estimation is particularly effective for the case $\lambda_1 + \lambda_2$ close to 1.

4 Simulation and Empirical Studies

4.1 Simulation

We use simulation to validate our new diagnosis method on the distribution of GARCH innovations. We consider two types of distributional assumptions, normal and Student-t distributed innovations.

We simulate observations from each model, while fitting the generated data to both models. The GARCH coefficients are estimated by the MLE outlined in [5]. With the estimated GARCH coefficients, we calculate the implied tail indices $\hat{\alpha}_n$ and $\hat{\alpha}_s$ according to the fitted models. Moreover, we obtain the Hill estimates $\hat{\alpha}_H$ by choosing an optimal sample fraction $k$.$^1$

Each simulated sample consists of 4,000 observations. This is close to the sample size we use later for real data analysis. For each model and fitting procedure, we run 100 simulations to obtain an average estimate for each tail index estimator.

$^1$The optimal $k$ is chosen from the first stable region of the tail index estimates in the Hill plots, which is the plot of the estimates against various potential $k$ levels, see [4].
Table 1: Tail index estimates of GARCH (1,1)-normal Series

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\alpha$</th>
<th>$\hat{\alpha}_n$</th>
<th>$\hat{\alpha}_s$</th>
<th>$\hat{\alpha}_H$</th>
<th>$\lambda_1$</th>
<th>$\alpha$</th>
<th>$\hat{\alpha}_n$</th>
<th>$\hat{\alpha}_s$</th>
<th>$\hat{\alpha}_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>11.88</td>
<td>12.02</td>
<td>11.55</td>
<td>6.15</td>
<td>0.07</td>
<td>13.98</td>
<td>14.44</td>
<td>13.80</td>
<td>6.47</td>
</tr>
<tr>
<td>0.88</td>
<td>10.62</td>
<td>11.34</td>
<td>10.88</td>
<td>5.94</td>
<td>0.08</td>
<td>10.62</td>
<td>11.63</td>
<td>11.16</td>
<td>5.86</td>
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<tr>
<td>0.89</td>
<td>9.12</td>
<td>8.82</td>
<td>8.51</td>
<td>5.30</td>
<td>0.09</td>
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<td>7.89</td>
<td>7.63</td>
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<tr>
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<td>7.17</td>
<td>4.86</td>
<td>0.10</td>
<td>5.62</td>
<td>5.88</td>
<td>5.71</td>
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<tr>
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<td>5.07</td>
<td>4.31</td>
<td>0.11</td>
<td>3.70</td>
<td>3.96</td>
<td>3.87</td>
<td>3.65</td>
</tr>
</tbody>
</table>

**Note:** This table presents the tail index estimation results of the data simulated from a GARCH (1,1)-normal model. For simulating the data, in the left part of the table we fix the GARCH coefficient $\lambda_1 = 0.08$ and vary $\lambda_2$ from 0.87 to 0.91, while in the right part, we fix $\lambda_2 = 0.88$ and vary $\lambda_2$ from 0.7 to 0.11. For each model, 100 samples are simulated with each sample consisting of 4,000 observations. The simulated data are fitted to a GARCH (1,1)-normal model and a GARCH (1,1)-Student-t model. With the coefficients estimated from the Maximum Likelihood method, we calculate the implied tail indices $\hat{\alpha}_n$, $\hat{\alpha}_s$. $\hat{\alpha}_H$ is the Hill estimate from the simulated data. Numbers reported are the average levels across 100 samples.

Tables 1 presents the simulation results of GARCH (1,1)-normal model. For the GARCH coefficients, we first fix $\lambda_1 = 0.08$ with varying $\lambda_2$ from 0.87 to 0.91, and then fix $\lambda_2 = 0.88$ with varying $\lambda_1$ from 0.07 to 0.11, $\lambda_0 = 0.5$.

From the results, we observe that both $\hat{\alpha}_n$ and $\hat{\alpha}_s$ robustly estimate the tail index $\alpha$. However, the Hill estimates $\hat{\alpha}_H$ considerably underestimate the tail index $\alpha$, i.e. overestimates the heavy-tailedness. The underestimation is severe particularly for the case that $\lambda_1 + \lambda_2$ is relatively low. This can be explained by the downward bias of the estimate, see Lemma 6. Nevertheless, the Hill estimator performs well in the case that $\lambda_1 + \lambda_2$ is close to 1, in other words, when the GARCH series is close to non-stationarity.

The simulation results of GARCH (1,1)-Student model are presented in Table 2. We use the same parameters $\lambda_0$, $\lambda_1$ and $\lambda_2$ as in simulations for the GARCH (1,1)-normal model, while the degree of freedom for Student-t innovation is set to $\nu = 6$. Different from the normal case, only $\hat{\alpha}_s$ robustly
Table 2: Tail index estimates of GARCH (1,1)-Student series

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>$\alpha$</th>
<th>$\hat{\alpha}_n$</th>
<th>$\hat{\alpha}_s$</th>
<th>$\hat{\alpha}_H$</th>
<th>$\lambda_1$</th>
<th>$\alpha$</th>
<th>$\hat{\alpha}_n$</th>
<th>$\hat{\alpha}_s$</th>
<th>$\hat{\alpha}_H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.87</td>
<td>5.94</td>
<td>12.54</td>
<td>6.00</td>
<td>3.94</td>
<td>0.07</td>
<td>6.48</td>
<td>15.25</td>
<td>6.68</td>
<td>4.03</td>
</tr>
<tr>
<td>0.88</td>
<td>5.56</td>
<td>11.50</td>
<td>5.81</td>
<td>3.90</td>
<td>0.08</td>
<td>5.56</td>
<td>11.28</td>
<td>5.74</td>
<td>3.78</td>
</tr>
<tr>
<td>0.89</td>
<td>5.08</td>
<td>9.54</td>
<td>5.10</td>
<td>3.55</td>
<td>0.09</td>
<td>4.66</td>
<td>8.21</td>
<td>4.79</td>
<td>3.64</td>
</tr>
<tr>
<td>0.90</td>
<td>4.42</td>
<td>8.29</td>
<td>4.78</td>
<td>3.51</td>
<td>0.10</td>
<td>3.78</td>
<td>6.06</td>
<td>3.95</td>
<td>3.26</td>
</tr>
<tr>
<td>0.91</td>
<td>3.46</td>
<td>5.42</td>
<td>3.46</td>
<td>3.07</td>
<td>0.11</td>
<td>2.88</td>
<td>3.93</td>
<td>2.91</td>
<td>2.95</td>
</tr>
</tbody>
</table>

Note: This table presents the tail index estimation results of the data simulated from a GARCH (1,1)-Student model. For simulating the data, in the left part of the table we fix the GARCH coefficient $\lambda_1 = 0.08$ and vary $\lambda_2$ from 0.87 to 0.91, while in the right part, we fix $\lambda_2 = 0.88$ and vary $\lambda_2$ from 0.7 to 0.11. The degree of freedom for the Student-t innovation $\nu = 6$. For each model, 100 samples are simulated with each sample consisting of 4,000 observations. The simulated data are fitted to a GARCH (1,1)-normal model and a GARCH (1,1) - Student-t model. With the coefficients estimated from the Maximum Likelihood method, we calculate the implied tail indices $\hat{\alpha}_n$, $\hat{\alpha}_s$. $\hat{\alpha}_H$ is the Hill estimator from the simulated data. Numbers reported are the average levels across 100 samples.

estimates the tail index of the GARCH series in this case. The normal innovation fitted estimate $\hat{\alpha}_n$ generally overestimates, while the Hill estimator $\hat{\alpha}_H$ underestimates the tail index $\alpha$ for the case that $\lambda_1 + \lambda_2$ is relatively low. Similar to the GARCH (1,1)-normal model, the Hill estimator $\hat{\alpha}_H$ performs well for the case that the GARCH series is close to non-stationarity.

As a conclusion, comparing the implied tail index with the Hill estimate yield a valid test on the null hypotheses that the GARCH innovations follow a particular distribution, under which the implied tail index is calculated. This method is efficient particularly when the GARCH series is close to non-stationarity.

4.2 Empirical Study on Equity Indices

In this section, we apply our method to detect the distribution of the GARCH innovations once fitting the GARCH model to real data. The data
Table 3: Tail index estimates of Real Data

<table>
<thead>
<tr>
<th>Sector</th>
<th>GARCH-normal</th>
<th>GARCH-Student</th>
<th>Tail Index</th>
<th>$\hat{\alpha}_n$</th>
<th>$\hat{\alpha}_s$</th>
<th>$\hat{\alpha}_H$</th>
<th>$p_n$</th>
<th>$p_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P</td>
<td>0.0770 0.9174</td>
<td>0.0766 0.9225</td>
<td>6.3377</td>
<td>3.92</td>
<td>2.18</td>
<td>2.95</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>Auto</td>
<td>0.0616 0.9290</td>
<td>0.0655 0.9284</td>
<td>5.9823</td>
<td>6.48</td>
<td>3.34</td>
<td>2.93</td>
<td>0.00</td>
<td>0.42</td>
</tr>
<tr>
<td>BioTech</td>
<td>0.0608 0.9346</td>
<td>0.0536 0.9455</td>
<td>5.9460</td>
<td>4.48</td>
<td>2.34</td>
<td>3.29</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>Media</td>
<td>0.0933 0.9257</td>
<td>0.0675 0.9285</td>
<td>7.8791</td>
<td>4.10</td>
<td>3.10</td>
<td>3.18</td>
<td>0.01</td>
<td>0.82</td>
</tr>
<tr>
<td>Chemical</td>
<td>0.0818 0.9182</td>
<td>0.0747 0.9238</td>
<td>6.6023</td>
<td>2.02</td>
<td>2.32</td>
<td>3.18</td>
<td>0.05</td>
<td>0.14</td>
</tr>
<tr>
<td>Pharm</td>
<td>0.0755 0.9152</td>
<td>0.0798 0.9152</td>
<td>5.9035</td>
<td>5.10</td>
<td>2.80</td>
<td>3.58</td>
<td>0.00</td>
<td>0.13</td>
</tr>
<tr>
<td>Retail</td>
<td>0.0649 0.9298</td>
<td>0.0594 0.9373</td>
<td>7.5192</td>
<td>4.48</td>
<td>3.12</td>
<td>3.73</td>
<td>0.22</td>
<td>0.32</td>
</tr>
<tr>
<td>SoftWare</td>
<td>0.0596 0.9356</td>
<td>0.0562 0.9438</td>
<td>6.3752</td>
<td>4.66</td>
<td>2.00</td>
<td>3.24</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Transport</td>
<td>0.0587 0.9365</td>
<td>0.0664 0.9260</td>
<td>6.7542</td>
<td>4.76</td>
<td>3.74</td>
<td>3.16</td>
<td>0.00</td>
<td>0.04</td>
</tr>
<tr>
<td>Aero</td>
<td>0.0739 0.9197</td>
<td>0.0727 0.9198</td>
<td>6.7959</td>
<td>4.30</td>
<td>3.50</td>
<td>3.57</td>
<td>0.07</td>
<td>0.87</td>
</tr>
<tr>
<td>Steel</td>
<td>0.0525 0.9418</td>
<td>0.0541 0.9433</td>
<td>7.0443</td>
<td>5.84</td>
<td>3.00</td>
<td>3.22</td>
<td>0.00</td>
<td>0.71</td>
</tr>
<tr>
<td>Insurance</td>
<td>0.0925 0.9041</td>
<td>0.0891 0.9109</td>
<td>5.3377</td>
<td>2.86</td>
<td>2.02</td>
<td>2.53</td>
<td>0.52</td>
<td>0.31</td>
</tr>
<tr>
<td>Bank</td>
<td>0.0870 0.9130</td>
<td>0.0863 0.9137</td>
<td>7.1837</td>
<td>2.02</td>
<td>2.00</td>
<td>2.12</td>
<td>0.75</td>
<td>0.71</td>
</tr>
</tbody>
</table>

**Note:** This table presents the results of S&P 500 index and 12 equity indices from US market (01.01.1995-31.12.2010). With the coefficients estimated from the Maximum Likelihood method, we calculate the implied tail indices $\hat{\alpha}_n$, $\hat{\alpha}_s$. $\hat{\alpha}_H$ is the Hill estimate. $p_n$, $p_s$ are the corresponding $p$-values based on the test statistics in theorem 4 under 95% confidence level.

set consists of S&P 500 Composite Index and 12 S&P equity sector indices. The time series of daily data runs from 1 January 1995 to 31 December 2010, with a sample of size 4174. All data are collected from Datastream.

We fit the data by both GARCH models with normal and Student-t innovations. With the estimated GARCH coefficients $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\nu}$, we report the implied tail indices $\hat{\alpha}_n$, $\hat{\alpha}_s$. We also estimate the tail index by the Hill estimator $\hat{\alpha}_H$. By applying the test in Theorem 4, we report the corresponding $p$-values of the hypothesis tests that either normal innovations or the Student-t innovations is regarded as the null hypothesis. The results are reported in Table 3.

Firstly, it is notable that for all 13 series, $\hat{\lambda}_1 + \hat{\lambda}_2$ under any model is close to 1. Hence, the GARCH series, no matter which type of innovations,
are close to non-stationarity. This is exactly the case in which the MF2000 method based on estimated innovations fails, while our method based on comparing the implied tail indices with the Hill estimates produces robust testing result. Next we observe that $\hat{\alpha}_n$ is generally higher than $\hat{\alpha}_s$ in all indices except Chemical and Banking sectors. Furthermore, in the $p$-value columns, we observe that for 9 out of the 13 indices the null hypothesis of Student-t innovations are not rejected under 95% confidence level, whereas for 9 out of 13 indices the null hypothesis of normal innovations are rejected under the same confidence level. For those indices that both null hypothesis are rejected, the corresponding $p$-values for Student-t innovations are much higher than those for the normal innovations.

Therefore, we conclude that the GARCH innovations of the equity indices are more likely to follow the Student-t distribution. One potential reason of such a preference is the fact that the Student-t distribution exhibits heavy-tails while the normal-distribution does not. Hence, we draw the same conclusion as in MF2000 that assuming conditional heavy-tailedness is necessary when applying GARCH models to financial returns, albeit from a different approach.

5 Conclusion

This paper investigates the diagnosing method on the distribution of GARCH innovations. We first show that the method in [8] based on estimated innovations is not reliable to detect the heavy-tailedness of the GARCH innovations, particularly for the case that the GARCH process is close to non-
stationarity. Even though the observations are generated from a GARCH model with normal innovations, the estimated innovations from a finite sample follow a heavy-tailed distribution.

We provide an alternative approach on diagnosing the distribution of GARCH innovations by comparing implied tail indices with the estimate form the Hill estimator. A formal test is established from such an approach. In our method, the asymptotic bias of the Hill estimator may potentially contaminate the test. Nevertheless, the potential contamination is of less problematic once the GARCH process is close to non-stationarity. The simulation results support our proposed approach.

We apply our method to 13 stock indices. The estimated GARCH coefficients in both models indicates near non-stationarity. Hence, the MF2000 method is not valid for real data. Nevertheless, the results of our method confirm that a GARCH model with Student-t innovation fits the data better than a GARCH-normal model.

Our findings lead to important implication in dynamic risk management: when applying a GARCH model to evaluate conditional risk measures, such as VaR, conditional heavy-tailed distribution should be considered. Using a conditional normal model may underestimate the potential dynamic risk. Moreover, it is not proper to evaluate the heavy-tailedness based on the estimated innovations. That may overestimate the conditional risk measures. Lastly, to accurately evaluate the unconditional tail risk, it is better to consider the implied tail index instead of using Hill estimator. The economic significance on the difference between the two types of GARCH model is left for future research.
References


