Optimizing Bounds on Security Prices in Incomplete Markets.  
Does Stochastic Volatility Specification Matter?  

Naroa Marroquín-Martínez\(^a\)  
University of the Basque Country  

Manuel Moreno\(^b\) *  
University of Castilla La-Mancha  

January 15, 2012

Abstract

We extend and generalize some results on bounding security prices under two stochastic volatility models that provide closed-form expressions for option prices. In detail, we compute analytical expressions for benchmark and standard good-deal bounds. For both models, our findings show that our benchmark results generate much tighter bounds. A deep analysis of the properties of bounds involving a sensitivity analysis and derivation of Greeks is also presented. These results provide strong practical applications taking into account the relevance of pricing and hedging strategies for traders, financial institutions, and risk managers.

Key words: Incomplete Markets, Stochastic Volatility Model, CIR Process, Ornstein-Uhlenbeck Process, Good-deal Bounds.

JEL classification: C61, G12, G13

\(^a\)Department of Applied Economics III (Econometrics and Statistics), University of the Basque Country, 48015 Bilbao, Spain. E-mail: naroa.marroquin@hotmail.com.

\(^b\) Corresponding author. Department of Economic Analysis and Finance, University of Castilla La-Mancha, Cobertizo de San Pedro Martir, s/n, 45071 Toledo, Spain. Phone: (34)92526 88 00. Fax: (34)925 26 88 01. E-mail: manuel.moreno@uclm.es
1 Introduction

Under the assumption of complete markets, every contingent claim can be replicated by a portfolio formed of the underlying basic assets of the market. In this case, the equivalent martingale measure and the market price of risk are unique and, then, the price of any security is uniquely determined by this martingale measure. Nevertheless, in the real world this situation barely happens and there can be (infinitely) many equivalent martingale measures.

When there are sources of risk that are not directly traded (such as stochastic volatility, jumps or weather) the assumption of complete market fails. As there will not exist a unique martingale measure, there will exist infinitely many arbitrage-free price processes for a certain financial security. Then, it can be interesting to derive no-arbitrage bounds on asset prices and define an interval where the price of the asset should lie.

In this type of situations, Cochrane and Saá-Requejo (2000) and Bernardo and Ledoit (2000) tried to find no-arbitrage bounds on prices as tight as possible using the stochastic discount factor (SDF) as starting point. The SDF was introduced in Hansen and Jaganathan (1991) who demonstrated that a bound on the maximum available Sharpe ratio is equivalent to a bound on the volatility of the admissible SDF. In more detail, Cochrane and Saá-Requejo (2000) measured the attractiveness of an investment using the Sharpe ratio and suggested to rule out usual arbitrage opportunities with too high Sharpe ratios. Thus, they obtained tighter price bounds that are named benchmark good-deal (BGD) bounds.

Bernardo and Ledoit (2000) also analyzed different investment opportunities but using the gain-loss ratio as a performance measure using a benchmark or reference asset pricing model. They demonstrated that a high gain-loss ratio is related to SDF’s that are “specially” far from the benchmark SDF. In this way, an appropriate benchmark SDF can tighten the no-arbitrage bounds and the corresponding no-arbitrage interval.

Several authors have proposed different methodologies to compute this type of bounds. For instance, Björk and Slinko (2006) extended the setting in Cochrane and Saá-Requejo (2000) by studying arbitrage-free good-deal pricing bounds for derivative assets and presented results for the Merton-jump diffusion model. Additionally, they derived extended Hansen-Jagannathan bounds for the Sharpe Ratio process in the point process setting.

Bondarenko and Longarela (2009) presented asset price bounds as the result of an optimization problem over a set of admissible SDF’s. They considered the option pricing model presented in Heston (1993) and assumed certain limits for the volatility risk premium. They derived closed-form solutions for the BGD bounds and for a particular case, standard good-deal (GD) bounds, showing that the former are much tighter than the latter.

Our paper focuses on computing and analyzing BGD and GD bounds for different asset prices
under two stochastic volatility option pricing models, that introduced in Heston (1993) and an extension of that posited in Schöbel and Zhu (1999). In this way, we can get an insight into the effects of different specifications for the stock volatility process on the aforementioned bounds.

Heston (1993) generalizes the classical model for stock prices presented in Black and Scholes (1973) allowing the stock volatility to follow a “square-root” (CIR-type) stochastic process as presented in Cox et al. (1985). Additionally, Schöbel and Zhu (1999) extended the stochastic volatility model of Stein and Stein (1991) where the stock volatility follows an Ornstein-Uhlenbeck process. They allow correlation to exist between the underlying stock returns and the instantaneous volatility and found a closed-form expression for option prices.

As these two models deal with stochastic volatility, markets are incomplete. However, both provide unique closed-form expressions for the prices of certain securities assuming a certain functional form of the market price(s) of risk of the corresponding factor(s). In fact, each functional form is associated to a martingale measure and, thus, to a price for the security.

This paper contributes to the existing literature in three ways: firstly, we analyze deeply Bondarenko and Longarela (2009) and, happily, fix different errors in their numerical analysis. One of our main results is that, now, the difference between GD and BGD bounds is stronger than that previously reported by these authors. Secondly, we extend the Schöbel and Zhu (1999) model allowing the market price of risk of the volatility to be different from zero. In this extended model, with no new mathematical ideas, we also obtain analytical expressions for option prices and their bounds. Numerical illustrations are shown for all the bounds obtained.

Our final contribution is that, for both models, extensive sensitivity analysis are carried out studying how changes in the models’ parameters affect bounds on option prices. Additionally, we also implement a hedging analysis by computing several Greeks for these bounds. Computation of these Greeks is relevant because, as shown in Carr (2001), these amounts can be interpreted as the values of certain quantoed contingent claims. Besides, as this author states, “this interpretation allows one to transfer intuitions regarding values to these Greeks and to apply any valuation methodology to determine them”.

The structure of the paper is as follows. Section 2 describes the theoretical framework that is needed to find the bounds on option prices. Stochastic volatility models are presented in Section 3. Section 4 derives analytical expressions for option prices and their bounds under these models. A deep analysis of the properties of bounds involving a sensitivity analysis and derivation of Greeks is included in Section 5. Finally, Section 6 summarizes the main findings and conclusions.
2 Theoretical Framework

We present now our theoretical framework. Consider a probability space \((\Omega, F, P)\) with the corresponding filtration \(\{F_t\}_{t \geq 0}\). Assume that we have a bond that pays the risk-free rate \(r_t\), a risky asset \(S_t\) (stock), and one (non-tradable) state variable \(V_t\). Let \((W^s_t, W^v_t)\) be two standard and independent Brownian motions and let \(h_t = (h^s_t, h^v_t)\) be an adapted two dimensional process, which satisfies the Novikov condition. Departing from the probability measure \(P\), we define the measure \(Q\) via the Radon-Nikodim derivative, that is,

\[
\frac{dQ}{dP} = \xi_t
\]

where, for all \(t\),

\[
\xi_t = \exp \left[ - \int_0^t h_u dW_u - \frac{1}{2} \int_0^t \|h_u\|^2 du \right]
\]

and \(\xi\) is a \(P\)-martingale with expected value equal to one. The SDF process is defined as

\[
\Lambda_t = B_t \xi_t
\]

where \(B_t = \exp \left( - \int_0^t r_u du \right)\). Applying Itô’s lemma, we get that

\[
\frac{d\Lambda_t}{\Lambda_t} = -r_t dt - h'_t dW_t
\]

We can define the benchmark model in terms of the vector process \(h^*_t = (h^s_t, h^v_t)\) with the corresponding martingale measure, \(Q^*\), and the SDF process, \(\Lambda^*\). Now, we will mention some important statements and restrictions that are necessary for our purposes.

By definition, \(\Lambda\) prices the risk-free bond and also the risky asset if it satisfies

\[
E^P_t \left[ \frac{d(\Lambda_t S_t)}{\Lambda_t S_t} \right] = -\delta_t dt
\]

where \(\delta_t\) is the continuous dividend yield. The market price of risk of the stock is given by

\[
\lambda_t^s = \frac{\mu_t - r_t}{\sigma_t}
\]

where \(\mu_t\) and \(\sigma_t\) denote the expected instantaneous rate of return and the instantaneous volatility of the stock, respectively. We now establish the set of admissible SDF’s as stated in this Lemma.

**Lemma 1** The SDF process \(\Lambda\) prices the stock if and only if it has an associated process satisfying \(h^s = \lambda^s\).

\[\Box\]

\[^1\text{This result corresponds to Lemma 1 in Bondarenko and Longarela (2009).}\]
For a fixed benchmark SDF process, $\Lambda^*$, the volatility constraint in continuous time is given by

$$E_t^Q \left[ \frac{d \left( \Lambda_t / \Lambda_t^* \right)}{\Lambda_t / \Lambda_t^*} \right]^2 \leq A_t^2 dt \quad (1)$$

where $A_t$ is some adapted process. Using that the change of measure does not affect the volatility process, the left-hand side of (1) can be simplified as

$$E_t^Q \left[ \frac{d \left( \Lambda_t / \Lambda_t^* \right)}{\Lambda_t / \Lambda_t^*} \right]^2 = E_t^P \left[ \frac{d \Lambda_t}{\Lambda_t} - \frac{d \Lambda_t^*}{\Lambda_t^*} \right]^2 = \| h_t - h_t^* \|^2 dt \quad (2)$$

Lemma 1 implies that, for an admissible SDF process, it holds that

$$\| h_t - h_t^* \|^2 = (\lambda_t^s - h_t^s)^2 + (h_t^v - h_t^v^*)^2 \quad (3)$$

Using (1)-(3), the volatility constraint becomes

$$(h_t^v - h_t^v^*)^2 \leq A_t^2 - (\lambda_t^s - h_t^s)^2 \quad (4)$$

In most applications, the benchmark is usually an admissible SDF. In this case, the volatility constraint discards those SDF processes for which $|h_t^v - h_t^v^*| \geq A_t$. $A_t \equiv A$ is sometimes called a ceiling process.

We can now define the BGD bounds for asset prices. Consider an asset that pays a stream of dividends given by an adapted process $X$ and a terminal payoff $X_T$. Its price $C_t$ under a candidate SDF process $\Lambda$ is given by

$$C_t(h) = E_t^P \left( \int_t^T \frac{\Lambda_u}{\Lambda_t} X_u du \right) + E_t^P \left( \frac{\Lambda_T}{\Lambda_t} X_T \right)$$

Let $H^v$ be the set of processes $h^v$ which satisfy (4) on the interval $[0,T]$. The lower bound is defined by the solution of a certain minimization problem, namely,

$$\underline{C}_0(h^*) = \inf_{h^v} C_0(h^v, h^*, A)$$

The equivalent maximization problem provides the corresponding upper bound. Formally, we can state the following Proposition.

**Proposition 1** The BGD bounds are given by

$$\underline{C}_0(h^*) = C_0(h^v), \quad \overline{C}_0(h^*) = C_0(h^v)$$

where

$$h_t^v = h_t^v^* + \sqrt{A_t^2 - (\lambda_t^s - h_t^s)^2}, \quad h_t^v^* = h_t^v^* - \sqrt{A_t^2 - (\lambda_t^s - h_t^s)^2}$$

**Remark 1** The GD bounds are derived as a particular case of the BGD bounds with $h_t^* = (0,0)$.
3 Stochastic Volatility Models

3.1 The Heston (1993) Model

Heston (1993) proposes to model the stock price evolution in time in a more general form than the standard classical model proposed in Black and Scholes (1973). In more detail, this author assumes a Geometric Brownian motion for the stock price evolution with the addition of a second state variable, namely, the variance of the stock return. Then, under the true probability measure $P$, the processes for the stock price $S_t$ and for the variance of the stock return $V_t$ are given by

$$\frac{dS_t}{S_t} = (r + sV_t)dt + \sqrt{V_t}dW_t^S$$

$$dV_t = (\alpha - \beta V_t)dt + \sigma \sqrt{V_t} (\rho dW_t^S + \sqrt{1-\rho^2} dW_t^V)$$

where $r$, $s$, $\alpha$, $\beta$, $\sigma$, and $\rho$ are constants and $W_t^S$ and $W_t^V$ are two uncorrelated standard Brownian motions. Then, the stock variance $V_t$ follows the square-root mean-reverting process posited in Cox et al. (1985). In more detail, $V_t$ converges to a long-run mean $\alpha/\beta$ with a certain speed of adjustment $\beta$. Additionally, the diffusion of the process is proportional to the variance level. The restriction $\sigma^2 \leq 2\alpha$ guarantees the positiveness of $V_t$. By construction, both processes are correlated with $\text{corr}(dS_t/S_t, dV_t) = \rho$.

Standard arbitrage arguments show that the price at time $t$ of any derivative asset on the stock, $U(S, V, t)$, must satisfy the following partial differential equation (PDE)

$$\frac{1}{2} V S^2 U_{SS} + \rho \sigma V S U_{SV} + \frac{1}{2} \sigma^2 V V_{VV} + r S U_S + [\alpha - \beta V_t - \lambda(S, V, t)] U_V - r U + U_t = 0$$

where subscripts indicate the corresponding partial derivative and with $\lambda(S, V, t)$ denoting the market price of risk related to the stock volatility.\(^3\) Moreover, similarly to Cox et al. (1985), we will assume that the risk premium of the variance is proportional to the variance level, that is,

$$\lambda(S, V, t) = \lambda V_t$$

This implies that, under the risk-neutral probability measure $Q$, the variance follows a mean-reverting process with long-run mean $\alpha/(\beta + \lambda)$ and speed of adjustment $(\beta + \lambda)$.

Under these assumptions, Heston (1993) derived a closed-form expression for the price of a European call stock option via Fourier inversion of the conditional characteristic functions. The Heston theoretical price at time $t = 0$ of this option will be denoted as

$$C^H_0(\lambda) = C^H(K, T, S_0, V_0, t = 0, \theta, \lambda)$$

\(^3\)Lamoureux and Lastrapes (1993) presented empirical evidence on the significativeness of this term when dealing with equity options.
where \( \theta = (\alpha, \beta, \sigma, \rho) \) is the set of parameters included in the process for the variance of the stock return, \( V_t \). As shown in Heston (1993), the expression for this price is stated as follows.

**Proposition 2** Under the Heston (1993) model, the closed-form expression for the price at time \( t \) of a European call stock option is given as

\[
C^H(K, T, S_t, V_t, t, \theta, \lambda) = S_t P_1 - e^{-r(T-t)} K P_2
\]

where, for \( j = 1, 2 \), we get

\[
P_j(S_t, V_t, \tau, K) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i \phi \ln(K) F_j(S_t, V_t, \tau, \phi)}}{i \phi} \right) d\phi
\]

\[
F_j(S_t, V_t, \tau, K) = \exp \{ C_j(\tau, \phi) + D_j(\tau, \phi) V_t + i \phi \ln(S_t) \}
\]

\[
C_j(\tau, \phi) = r \tau \phi + \frac{\alpha}{\sigma^2} \left( (\beta_j - \rho \sigma \phi i + h_j) \tau - 2 \ln \left( \frac{1 - g_j e^{h_j \tau}}{1 - g_j} \right) \right)
\]

\[
D_j(\tau, \phi) = \frac{\beta_j - \rho \sigma \phi i + h_j}{\beta_j - \rho \sigma \phi i - h_j}
\]

\[
g_j = \beta_j - \rho \sigma \phi i + h_j
\]

\[
h_j = \sqrt{(\rho \sigma \phi i) + \sigma^2 (2u_j \phi i - \phi^2)}
\]

where \( \tau = T - t \), \( u_1 = 0.5 \), \( u_2 = -0.5 \), \( \beta_1 = \beta + \lambda - \rho \sigma \), \( \beta_2 = \beta + \lambda \).

### 3.2 Extended Schöbel and Zhu (1999) Model

Now we will study the stochastic volatility model presented in Schöbel and Zhu (1999) in which the volatility follows an Ornstein-Uhlenbeck (O-U) process. Under the risk-neutral measure \( Q \), the processes for the logarithm of the stock price and the volatility for the stock return are given as

\[
dx_t = \left( r - \frac{1}{2} \nu_t^2 \right) dt + \nu_t d\tilde{W}_t^s
\]

\[
d\nu_t = \kappa (\theta - \nu_t) dt + \sigma \left( \rho d\tilde{W}_t^s + \sqrt{1 - \rho^2} d\tilde{W}_t^v \right)
\]

where \( \tilde{W}_t^s \) and \( \tilde{W}_t^v \) denote independent standard Brownian motions under the measure \( Q \). Hence, both processes are correlated with \( \text{corr}(dx_t, d\nu_t) = \rho \). The stock volatility tends to a long-term value \( \theta \) with speed \( \kappa \). As the volatility is a non-traded asset, the measure \( Q \) is not unique and depends on the market price of volatility, \( \lambda_t \), which is implicitly determined by the market participants. Schöbel and Zhu (1999) assumed \( \lambda_t = 0 \). Considering now that \( \lambda_t \neq 0 \) (further on we will assume that (7) holds), we can rewrite the processes (8)-(9) in terms of the stock price and variance to get

\[
\frac{dS_t}{S_t} = r dt + \sqrt{V_t} d\tilde{W}_t^s
\]

\[
dV_t = (2 \kappa \sqrt{V_t} (\theta - \sqrt{V_t}) + \sigma^2 - \lambda_t) dt + 2 \sigma \sqrt{V_t} (\rho d\tilde{W}_t^s + \sqrt{1 - \rho^2} d\tilde{W}_t^v)
\]
Thus, we have generalized the model of Schöbel and Zhu (1999). In this general model, we can obtain analytically the option price as stated in the following Proposition.

**Proposition 3** Under the extended Schöbel and Zhu (1999) model, the closed-form expression for the price at time \( t \) of a European call stock option is given as

\[
C^{OU}(K,T,S_0,V_0,t,\theta,\lambda) = S_tP_1 - e^{-r(T-t)}KP_2
\]

where the probabilities \( P_j, j = 1, 2 \) are given by

\[
P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i\phi \ln(K)f_j(\phi)}}{i\phi} \right) d\phi, \quad j = 1, 2
\]

where

\[
f_1(\phi) = \mathbb{E}^Q[\exp(-r(T-t) - x(t) + (1+i\phi)x(T))]
\]

\[
= \exp \left\{ i\phi(r(T-t) + x(t)) - \frac{1}{2}(1+i\phi)\rho [\sigma^{-1}v^2(t) + \sigma(T-t)] \right\}
\]

\[
\times \exp \left\{ \frac{1}{2}D(t,T,s_1,s_3)v^2(t) + B(t,T,s_1,s_2,s_3)v(t) + C(t,T,s_1,s_2,s_3) \right\}
\]

with

\[
s_1 = -\frac{1}{2}(1+i\phi)^2(1-\rho^2) + \frac{1}{2}(1+i\phi)(1-2(\kappa + \lambda')\rho\sigma^{-1})
\]

\[
s_2 = (1+i\phi)\kappa\theta\rho\sigma^{-1}
\]

\[
s_3 = \frac{1}{2}(1+i\phi)\rho\sigma^{-1}
\]

\[
\lambda' = \frac{\lambda}{2}
\]

and

\[
f_2(\phi) = \mathbb{E}^Q[\exp \{i\phi x(T)\}]
\]

\[
= \exp \left\{ i\phi(r(T-t) + x(t)) - \frac{1}{2}i\phi\rho [\sigma^{-1}v^2(t) + \sigma(T-t)] \right\}
\]

\[
\times \exp \left\{ \frac{1}{2}D(t,T,\hat{s}_1,\hat{s}_3)v^2(t) + B(t,T,\hat{s}_1,\hat{s}_2,\hat{s}_3)v(t) + C(t,T,\hat{s}_1,\hat{s}_2,\hat{s}_3) \right\}
\]

with

\[
\hat{s}_1 = \frac{1}{2}\phi^2(1-\rho^2) + \frac{1}{2}i\phi(1-2(\kappa + \lambda')\rho\sigma^{-1})
\]

\[
\hat{s}_2 = i\phi\kappa\theta\rho\sigma^{-1}
\]

\[
\hat{s}_3 = \frac{1}{2}i\phi\rho\sigma^{-1}
\]
Finally,

\[
D(t,T) = \frac{1}{\sigma^2} \left( \kappa + \lambda' - \gamma_1 \sinh \{\gamma_1(T-t)\} + \gamma_2 \cosh \{\gamma_1(T-t)\} \right)
\]
\[
B(t,T) = \frac{1}{\sigma^2\gamma_1} \left( \kappa\theta\gamma_1 - \gamma_2\gamma_3 + \gamma_3 \left( \sinh \{\gamma_1(T-t)\} + \gamma_2 \cosh \{\gamma_1(T-t)\} \right) - \kappa\theta\gamma_1 \right)
\]
\[
C(t,T) = \frac{1}{2} \ln \left( \cosh \{\gamma_1(T-t)\} + \gamma_2 \sinh \{\gamma_1(T-t)\} \right) + \frac{1}{2}(\kappa + \lambda')(T-t)
\]
\[
+ \frac{\kappa^2\theta^2\gamma_1 - \gamma_3^2}{2\sigma^2\gamma_1^3} \left( \frac{\sinh \{\gamma_1(T-t)\}}{\cosh \{\gamma_1(T-t)\} + \gamma_2 \sinh \{\gamma_1(T-t)\}} \right) - \gamma_1(T-t)
\]
\[
+ \frac{(\kappa\theta\gamma_1 - \gamma_2\gamma_3)\gamma_3}{\sigma^2\gamma_1^3} \left( \frac{\cosh \{\gamma_1(T-t)\} - 1}{\cosh \{\gamma_1(T-t)\} + \gamma_2 \sinh \{\gamma_1(T-t)\}} \right)
\]

with\(^5\)

\[
\gamma_1 = \sqrt{2\sigma^2 s_1 + (\kappa + \lambda')^2}, \quad \gamma_2 = \frac{1}{\gamma_1} \left[ \kappa + \lambda' - 2\sigma^2 s_3 \right], \quad \gamma_3 = (\kappa + \lambda')\kappa\theta - 2\sigma^2
\]

In this Proposition, for getting the probabilities \(P_1\) and \(P_2\), we derive their corresponding characteristic functions and follow the same schedule as Schöbel and Zhu (1999) until reaching the previous expressions in which the market price of risk of the volatility appears now explicitly.

Firstly, for deriving the characteristic functions, we need the volatility process. Applying the Itô’s lemma to the variance process (11), we get

\[dv_t = \kappa \left[ \theta - \left(1 + \frac{\lambda}{2\kappa} \right) v_t \right] dt + \sigma \left( pd\tilde{W}_t^s + \sqrt{1 - \rho^2} d\tilde{W}_t^r \right)\]

Secondly, the obtaining of the characteristic functions is similar to that shown in Schöbel and Zhu (1999) and involves solving the following system of ordinary differential equations

\[
D_t = -\sigma^2 D^2 + 2(\kappa + \lambda')D + 2s_1
\]
\[
B_t = [\kappa + \lambda' - \sigma^2 D] B - \kappa\theta B + s_2
\]
\[
C_t = -\frac{1}{2}\sigma^2 B^2 - \kappa\theta B - \frac{1}{2}\sigma^2 D
\]

subject to the terminal conditions \(D(T,T) = 2s_3, B(T,T) = C(T,T) = 0\).

Comparing the results obtained in Schöbel and Zhu (1999) with our Proposition, we can see that \(s_1\) and \(s_1\) have changed, thus the functions \(D(t,T), B(t,T), \) and \(C(t,T)\) have changed. Additionally, note that \(\gamma_1, \gamma_2,\) and \(\gamma_3\) have also changed.

\(^4\)Note that we are using interchangeably the notation for the functions \(B(\cdot), C(\cdot)\) and \(D(\cdot)\) for \(f_1\) and \(f_2\) with more arguments than right now. This is just to emphasize the variables that affect these functions.

\(^5\)In a similar way, we can define \(\hat{\gamma}_i, i = 1, 2, 3\) for the function \(f_2\).
4 Computation of Bounds for Option Prices

In this section we will apply our general theoretical framework to obtain analytically bounds on call option prices in the previous stochastic volatility models. We start with the Heston (1993) model.

4.1 Bounds for the Heston (1993) Model

Consider an adapted process $h_t = (h^s_t, h^v_t)$, the associated martingale measure $Q = Q(h)$, and the SDF process $\Lambda_t$ where

$$\frac{d\Lambda_t}{\Lambda_t} = -r dt - h^s_t dW^s_t - h^v_t dW^v_t$$

Under the risk-neutral measure $Q$, the stock price and variance processes (see (5)-(6)) are given as

$$dS_t = \left( r + sV_t - \lambda_t \sqrt{V_t} \right) dt + \sqrt{V_t} d\tilde{W}^S_t$$

$$dV_t = \left( \alpha - \beta V_t - \lambda_t \right) dt + \sigma \sqrt{V_t} \left( \rho d\tilde{W}^S_t + \sqrt{1 - \rho^2} d\tilde{W}^V_t \right)$$

The market price of risk of the variance is assumed to be

$$\lambda_t = \sigma \sqrt{V_t} (\rho^2 + \sqrt{1 - \rho^2} h^v_t)$$

for certain processes $h^s_t$ and $h^v_t$ to be obtained. By Lemma 1, it is known that $h^s_t = \lambda^s_t$ and, for making the process (12) risk-neutral, $\lambda^s_t$ has to satisfy $\lambda^s_t = s\sqrt{V_t}$. Hence,

$$h^s_t = s \sqrt{V_t}$$

Following equation (15), we get an expression of the form in (7) if

$$h^v_t = v \sqrt{V_t}$$

for some constant $v$. Replacing (15)-(16) in (14), we get that $\lambda = \sigma (s \rho + v \sqrt{1 - \rho^2})$.

The problem for the lower bound for a European call option with strike $K$ and maturity $T$ is

$$C_0 = \min C_0(h) = \min E^P_0 \left[ \frac{\Lambda_T}{\Lambda_0} \max \{0, S_T - K\} \right]$$

for some $h \in H^v$ where $H^v$ is the set of processes $h_t = (s \sqrt{V_t}, h^v_t)$ for which the volatility constraint in (4) is satisfied. Then, our benchmark is assumed to be of the form $h^*_1 = (s \sqrt{V_t}, v^* \sqrt{V_t})$ for some constant $v^*$, which will be chosen afterwards. This benchmark is admissible and it satisfies (7), so Heston’s formula applies. For the special case of standard GD bounds, we set $h^*_2 = (0, 0)$ and apply this to all the previous expressions wherever it is necessary.

Cochrane and Sáá-Requejo (2000) consider a case where, in the volatility constraint, $A$ is a positive constant. We assume the ceiling process is proportional to $V_t$, that is,

$$A_t = A \sqrt{V_t}$$

(17)
where $\overline{A}$ is a positive constant. This specification allows us to derive the analytical expression for the BGD bounds via the Heston (1993) formula.

According to the benchmark, as stated in Proposition 1, to compute the upper and lower bounds, we need two processes in which $h_{t}^{v} = \sqrt{V_{t}}, h_{t}^{v} = \sqrt{V_{t}}$, with $v$ and $\overline{v}$ certain constants.

For the BGD bounds, where $h_{t}^{*} = (s\sqrt{V_{t}}, v^{*}\sqrt{V_{t}})$, from Proposition 1 we can derive $v = v^{*} + A$ and $v = v^{*} - A$. Similarly, for the GD bounds, where $h_{t}^{2} = (0, 0)$, we get $v = \sqrt{A^{2} - s^{2}}, v = -\sqrt{A^{2} - s^{2}}$.

In both cases, the proper values for $A$ will be chosen later. With this result and Proposition 1, we are able to enunciate the following Proposition.\footnote{This result corresponds to Proposition 7 in Bondarenko and Longarela (2009).}

\textbf{Proposition 4} The lower and upper bounds for the price of a European call option are written as

\[ C_{0}(h^{*}) = C_{0}^{H}(\lambda), \quad C_{0}^{-}(h^{*}) = C_{0}^{H}(\overline{\lambda}) \]

where $\lambda = \sigma(s + v\sqrt{1 - \rho^{2}})$ and $\overline{\lambda} = \sigma(s + \overline{v}\sqrt{1 - \rho^{2}})$.

Consider a trader who wants to compute the call option price and the bounds for this price. However, she is concerned about the potential misspecification of the unobservable variance risk premium $\lambda_{t}$ in (14). In this case, she can allow some uncertainty assuming that the true variance risk premium is bounded as follows:

\[ \lambda_{l}V_{t} \leq \lambda V_{t} \leq \lambda_{h}V_{t} \]  \hspace{1cm} (18)

where $\lambda_{l}$ and $\lambda_{h}$ are constants given by $\lambda_{j} = \sigma(s + v\sqrt{1 - \rho^{2}}v_{j}, j = l, h$ and

\[ v_{l} = v - 0.5\Delta, \quad v_{h} = v + 0.5\Delta \]  \hspace{1cm} (19)

with $\Delta > 0$ indicating the distance between $v_{h}$ and $v_{l}$. Obviously, setting $\lambda_{l} = \lambda_{h}$, we get the Heston (1993) theoretical call price. The inequalities in (18) define the whole set of “plausible” candidate processes $h$, which produce a range of candidate call prices. As it is natural, we would like to have the price bounds as tight as possible.

It is easy to verify analytically that the optimal choices for $\overline{A}$ for BGD bounds and GD bounds are, respectively, $\overline{A} = \max \{|v_{l} - v^{*}|, |v_{h} - v^{*}|\}$ and $\overline{A} = \sqrt{s^{2} + \max \{v_{l}^{2}, v_{h}^{2}\}}$.

We will use Proposition 4 to compute analytically the bounds on call option prices, when $\Delta = 0, 2, 4$. Similarly to Bondarenko and Longarela (2009), we analyze the effect of correlation on the price bounds considering $\rho = -0.1, -0.53, -0.9$. We use their parameters $r = 0.05, S_{0} = 100, V_{0} = \alpha/\beta = 0.0137, T = 0.25, v^{*} = -22.6$ to compute the BGD bounds for the process $h_{t}^{*}$. The remaining parameters ($\alpha = 0.097, \beta = 7.1, \sigma = 0.32, s = 8.6, v = -22.6$) are based on Pan (2002), who fits the Heston (1993) model to a sample of S&P 500 index options over the period 1989-1996.\footnote{The parameter $v$ for the variance risk premium is the hardest one to estimate as the variance is a non-traded asset. However, the remaining parameters can be easily estimated as the stock price process $S_{t}$ can be observed.}
Table 1 shows the price bounds for call options with strikes from 85 to 115 and for the three correlations aforementioned. We notice analytically that if $\Delta = 0$, the BGD bounds are equal to the theoretical call price. Moreover, it is also easy to prove that the upper GD bounds coincide with the upper BGD bounds. Looking at the Heston prices, $C_0$, we see that correlation affects ITM and OTM options in a different way. As $\rho$ increases in absolute value, OTM option prices decrease, while ITM option prices first increase and then decrease. The same happens for BGD bounds and thus for upper GD bounds. The behavior of lower GD bounds is different: as $|\rho|$ gets higher, bounds on ITM option prices increase while, for OTM options, these bounds first decrease and then increase.

For the three cases of correlation and three values of $\Delta$, Figure 1 presents the difference between the bounds, $C_0^U$ and $C_0^L$, and the Black-Scholes price $C_0^{BS}$, in which the volatility is set to $\sqrt{V_0}$.

Figure 2 shows the size of bounds, that is, the difference between the bounds $C_0^U$ and $C_0^L$. As the graph shows, the largest size of the bounds corresponds to near at-the-money options.

It can be clearly seen that the bounds for $h_1^*$ are considerably tighter than those for $h_2^*$. This is particularly pronounced when $\Delta$ approaches zero, as the size of bounds is zero for $h_1^*$ and strictly positive for $h_2^*$. As expected, as $\Delta$ increases, the size of bounds increases as we are separating bounds further from the theoretical price. Note that the GD bounds are much wider than those reported in Bondarenko and Longarela (2009), thus our case of BGD versus GD bounds is now stronger.\footnote{The reason for these differences is due to some mistakes that, after code comparison, were identified in the programming codes of Bondarenko and Longarela (2009). We thank both authors for helpful interaction with us.}

As the correlation increases in absolute value, the bounds shrink in both GD and BGD bounds. It could be said that the larger $|\rho|$ is, the tighter the sizes of bounds are. For complete markets ($|\rho| = 1$), it can be proved analytically that the size of bounds turns zero. Intuitively, although the volatility is a “virtual” product, we know its behavior as changes in stock prices would transmit to changes in the volatility. To sum up, stochastic volatility involves incomplete market when $|\rho| < 1$.

Economically, we could deduce some results. The BGD bounds determine the range of possible prices for which the admissible SDFs lie in the “neighborhood” of a given benchmark. The “radius” of this neighborhood is determined by the ceiling $\bar{A}$. The ceiling must contain all the SDFs for which the variance risk premium is bounded as in (18). For the GD bounds, the neighborhood is symmetric with respect to the shock $dW_t^V$. Hence, even for $\Delta = 0$, their neighborhood must include
some SDFs that are economically implausible. Thus, the GD bounds are wider even when there is no uncertainty about the variance risk premium.

4.2 Bounds for the Extended Schöbel and Zhu (1999) Model

Now, we follow the same procedure as in the Heston (1993) model to compute the option prices and their bounds in the extended Schöbel and Zhu (1999) model. Under this extended model, the theoretical price of a European call option at time $t = 0$ is denoted as $C_{OU}^{0}(\lambda)$. We deduce from (11) that the market price of volatility will have the form $\lambda_t = 2\sigma\sqrt{V_t}(\rho h_t^* + \sqrt{1 - \rho^2} h_t^v)$ for some processes $h_t^*$ and $h_t^v$ to be obtained. As the spot price process is already risk-neutral we get $\lambda_t^* = 0 = h_t^*$ where the second equality comes from Lemma 1. As we want to obtain the form of the market price of volatility as in (7), we assume again (16). Thus, it is easy to verify that $\lambda = 2\sigma\sqrt{1 - \rho^2} v$.

Remember the form of the ceiling process in (17) and that $\lambda_t$ lies as in (18) so as to contain all the plausible candidate processes $h$ that generate the candidate call prices, where (19) holds.

Following Proposition 1, we will substitute $h_t^v = v\sqrt{V_t}$ and $\bar{h}_t^v = \bar{v}\sqrt{V_t}$ for computing the bounds. Moreover, for the case of BGD bounds, $h_1^* = (0, v^*\sqrt{V_t})$, thus we get $\bar{v} = v^* - \overline{A}$ and $\underline{v} = v^* + \overline{A}$. In a similar way, for GD bounds, $h_2^* = (0, 0)$ and we obtain $\bar{v} = -\overline{A}$ and $\underline{v} = \overline{A}$.

Then, we can state the next Proposition that arises directly from Proposition 1.

Proposition 5 The lower and upper bounds for the price of a European call option are written as

$$
\underline{C}_0(h^*) = C_{OU}^{0}(\lambda), \quad \overline{C}_0(h^*) = C_{OU}^{0}(\bar{\lambda})
$$

where $\lambda = 2\sigma\sqrt{1 - \rho^2} V$ and $\overline{\lambda} = 2\sigma\sqrt{1 - \rho^2} \bar{V}$.

As $\lambda_t$ lies as in (18), it is easy to prove that the optimal choice for the ceiling process for the BGD and GD bounds are, respectively, $\overline{A} = \max \{|v_h - v^*|, |v_l - v^*|\}$ and $\overline{A} = \max \{|v_h|, |v_l|\}$.

We will compute the prices and bounds for European call options. Following Schöbel and Zhu (1999), we consider the parameters $t = 0$, $T = 0.5$, $r = 0.093$, $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$ while, for computing the bounds, we choose $v = v^* = -22.6$ in line with Bondarenko and Longarela (2009). Additionally, we analyze a range of strikes $K$ from 90 to 120 and correlations $\rho = -0.25, -0.5, -0.75$.

Table 2 show prices and bounds for these correlations. As $|\rho|$ increases, prices and bounds decrease, except for the lower GD bounds that increase except for the very deep OTM options.

[INSERT TABLE 2 AROUND HERE]

Figure 3 graphs the difference between bounds and Black-Scholes prices for these correlations. The Black-Scholes prices will be denoted AVBS as they are based on the expected average variance AV that, as stated in Schöbel and Zhu (1999), is given by
This Figure shows that, as $|\rho|$ increases, the size of bounds minus the AVBS prices shrinks. Moreover, the bounds for the BGD case are much tighter than for the GD one.

Figure 4 shows the size of bounds. We see that as $|\rho|$ increases, the size of the bounds shrinks as we are closer to a complete market. If $|\rho| = 1$, we would be in the case of perfect correlation, and the market is complete as we explained for the Heston (1993) model.

If we compare Figures 1 and 3 and Figures 2 and 4, the size of bounds decreases as $|\rho|$ gets closer to one. Moreover, BGD bounds are always tighter than GD ones.

After having shown numerically and graphically these results, we will perform a more detailed sensitivity analysis of the bounds on option prices.

### 5 Properties of Bounds for Option Prices

This section analyzes the properties of the bounds for option prices in both stochastic volatility models. In more detail, we perform a detailed sensitivity analysis with respect to some parameters and a hedging analysis computing analytically and discussing several Greeks for these bounds.

For the sake of brevity, details of the sensitivity analysis are not included in the paper. For the Heston (1993) model, we find that, as $\alpha$, $\sigma$, and $V_0$ increase (and $\beta$ decreases), all the bounds increase and the distance between upper and lower GD bounds increases. Intuitively, we have a wider no-arbitrage interval for option prices and, hence, we have less information about the unique price we would find in complete markets. We also show that BGD bounds are tighter than GD ones. Similar qualitative conclusions are obtained for the extended Schöbel and Zhu (1999) model.

We report now analytical expressions of several Greeks for the bounds on option prices and provide numerical and graphical illustration. We focus on delta, gamma, and vega as traders and institutions use to design hedging schemes that involve these Greeks. The delta (vega) of an option indicates the rate of change in the option price with respect to the underlying price (volatility). The option’s gamma measures the rate of change in the option delta with respect to the underlying price.

For computing these Greeks for bounds we apply the same expressions as those for the theoretical ones, but replacing the input variable $\lambda$ by $\bar{\lambda}$ and $\lambda$ as stated in the following Proposition. We report just the expressions for the upper bound as those for the lower bound are completely similar.

---

$AV = \frac{\sigma^2}{2\kappa} + \theta^2 + \frac{1 - e^{-\kappa(T-t)}}{T-t} \left( \frac{2\theta}{\kappa} (v_t - \theta) - \frac{\sigma^2 - 2\kappa(v_t - \theta)^2}{4\kappa^2} \left(1 + e^{-\kappa(T-t)}\right) \right)$
Proposition 6 Under the Heston (1993) and extended Schöbel and Zhu (1999) models, the Greeks for the upper bound of the call option are given by

\[ \delta_s(S, V, t, T) = \frac{\partial C}{\partial S} = P_1(\lambda) \]

\[ \Gamma_s(S, V, t, T) = \frac{\partial^2 C}{\partial S^2} = \frac{\partial P_1(\lambda)}{\partial S} \]

\[ \delta_V(S, V, t, T) = \frac{\partial C}{\partial V} = S_t \left( \frac{\partial P_1(\lambda)}{\partial V} - e^{-r(T-t)} K \frac{\partial P_2(\lambda)}{\partial V} \right) \]

and, for computing the vega, the chain rule is needed:

\[ \nu = \frac{\partial C}{\partial \nu} = 2v_t \frac{\partial C}{\partial V} = 2v_t \delta_V \]

where \( C \) denotes the upper bound obtained for each model. Obviously, this bound presents a different closed-form expression in each model. 

The results will be reported in detail for the Heston (1993) model as we have obtained similar conclusions for the extended Schöbel and Zhu (1999) model.\(^{10}\)

Figure 5 presents, for different correlations, deltas for BGD and GD bounds as a function of the underlying price. Although it is hard to see the lines, going deeply into the graphs, we can see that \( \delta_s \) is above \( \delta_s \) (delta for the lower bound) just in the OTM area. Besides, it is clear that the theoretical \( \delta_s \) is always between \( \delta_s \) and \( \delta_s \). We have also checked that increasing the strike moves the graph to the right, keeping the same form and characteristics.

[INSERT FIGURE 5 AROUND HERE]

Figure 6 shows gammas for bounds as a function of the underlying price. When the option is deep OTM or deep ITM, \( \Gamma_s > \Gamma_s > \Gamma_s \) (gamma for the lower bound). However, when the call option is near ATM, the opposite happens. In the case of GD bounds, the lower ones for \( \Delta = 2, 4 \) take similar values and then, it is not possible to distinguish them visually. This Figure also shows that the maximum \( \Gamma_s \) is achieved in the OTM area. If we increased the strike, the graph moves to the right, maintaining the same shape and features except for the value that maximizes \( \Gamma_s \). As \( K \) increases, we have checked that the maximum \( \Gamma_s \) decreases, similarly to the Black-Scholes model.

[INSERT FIGURE 6 AROUND HERE]

Finally, Figure 7 presents vegas for BGD and GD bounds for different correlations. These Greeks respect the expected order, this is, \( \nu \) is always above \( \nu \) (vega for the lower bound) and the theoretical vega \( \nu \) is always located between the vegas for bounds.

\(^{10}\)Results are available upon request.
Graphing these vegas for bounds on options with different strikes, we have also checked that higher values of $K$ provide results that are qualitatively similar to those obtained for gammas for bounds except that the maximum $\nu$ increases with $K$.

Summarizing, we could say that Greeks for bounds do not behave exactly as bounds for prices. Sometimes what we have called “Greek for the upper bound” takes the place of the “Greek for the lower bound” and vice versa. Anyway, as expected, the theoretical Greek is always located between both. However, the two main conclusions observed for the bounds for prices keep for the Greeks for bounds. First of all, BGD bounds are much tighter than GD ones. Secondly, as $|\rho|$ increases, the size of the Greeks for bounds decreases and bounds get tighter. Thus, as $|\rho|$ gets closer to one, we are closer to a complete market, hence, both size of bounds and Greeks for bounds tend to zero.

6 Conclusions

As stated in the previous literature, it is possible to find an arbitrage-free interval for bounds of option prices under incomplete markets. We have analyzed the stochastic volatility models proposed in Heston (1993) and an extension of that introduced in Schöbel and Zhu (1999). For both models, we have obtained analytical expressions for BGD and GD bounds and have performed a detailed sensitivity and hedging analysis for these bounds. In both cases, the main qualitative conclusion is that BGD bounds are much tighter than GD ones. This also happens for Greeks for bounds.

Therefore, the answer for the question in the paper’s title is that the specification of the stochastic volatility does not seem to be relevant in qualitative terms when computing bounds on option prices and their Greeks. Moreover, as expected, approaching to the perfect correlation case (complete markets), the size of bounds becomes smaller.

Comparing the results obtained in both models for our chosen parameters, we notice that the correlation affects option prices in a different way depending on the degree of moneyness for the Heston (1993) model. Nevertheless, for the extended Schöbel and Zhu (1999) model, the correlation treats option prices in a similar way regardless the moneyness degree.

The main contributions of the paper are as follows: compared to Bondarenko and Longarela (2009), we have enlarged the analysis of bounds for option prices in the Heston (1993) model and fixed some errors in their numerical analysis finding now higher differences between GD and BGD bounds. Afterwards, we have extended the Schöbel and Zhu (1999) model and obtained analytical expressions for option prices and bounds. Our final contribution is that, for both models, we have performed extensive sensitivity and hedging analysis showing, respectively, the effect of changes in the model parameters on these bounds and the risk management properties of these bounds.
References


Appendix of Tables

Table 1: BGD and GD bounds for the Heston (1993) model.

<table>
<thead>
<tr>
<th>ρ = -0.10</th>
<th>K</th>
<th>C₀</th>
<th>C₀^BGD</th>
<th>C₀^GD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>105</td>
<td>1.6997</td>
<td>1.6520</td>
<td>1.6059</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>0.6601</td>
<td>0.6286</td>
<td>0.5986</td>
</tr>
<tr>
<td></td>
<td>115</td>
<td>0.2372</td>
<td>0.2207</td>
<td>0.2052</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ρ = -0.53</th>
<th>K</th>
<th>C₀</th>
<th>C₀^BGD</th>
<th>C₀^GD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>105</td>
<td>1.5757</td>
<td>1.5347</td>
<td>1.4950</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>0.4749</td>
<td>0.4509</td>
<td>0.4280</td>
</tr>
<tr>
<td></td>
<td>115</td>
<td>0.1133</td>
<td>0.1040</td>
<td>0.0954</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>ρ = -0.9</th>
<th>K</th>
<th>C₀</th>
<th>C₀^BGD</th>
<th>C₀^GD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>105</td>
<td>1.1679</td>
<td>1.1509</td>
<td>1.1343</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>0.1439</td>
<td>0.1379</td>
<td>0.1321</td>
</tr>
<tr>
<td></td>
<td>115</td>
<td>0.0029</td>
<td>0.0026</td>
<td>0.0023</td>
</tr>
</tbody>
</table>

This Table reports BGD and GD bounds for the Heston (1993) model. Strikes $K$ range from 85 to 115. $\rho = -0.1, -0.53, -0.9$. When $\Delta = 0$, lower and upper BGD bounds are equal to the theoretical call price $C_0$, that is why are not reported. $\alpha = 0.097$, $\beta = 7.1$, $\sigma = 0.32$, $s = 8.6$, $v = v^* = -22.6$. The remaining parameters are set as $r = 0.05$, $S_0 = 100$, $V_0 = \alpha/\beta = 0.0137$, $T = 0.25$. 

17
Table 2: BGD and GD bounds for the extended Schöbel and Zhu (1999) model.

<table>
<thead>
<tr>
<th>K</th>
<th>$C_0$</th>
<th>$C_0^{BGD}$</th>
<th>$C_0^{BGD}$</th>
<th>$C_0^{GD}$</th>
<th>$C_0^{GD}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\Delta = 2$</td>
<td>$\Delta = 4$</td>
<td>$\Delta = 2$</td>
<td>$\Delta = 4$</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>5.9943</td>
<td>5.8534</td>
<td>5.7168</td>
<td>2.3688</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>3.1736</td>
<td>3.0534</td>
<td>2.9377</td>
<td>0.5478</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>5.6049</td>
<td>5.4877</td>
<td>5.3737</td>
<td>2.4193</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>2.7700</td>
<td>2.6724</td>
<td>2.5780</td>
<td>0.5259</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>4.9556</td>
<td>4.8779</td>
<td>4.8018</td>
<td>3.0349</td>
</tr>
<tr>
<td></td>
<td>115</td>
<td>3.3452</td>
<td>3.2735</td>
<td>3.2034</td>
<td>1.2874</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>2.1609</td>
<td>2.0992</td>
<td>2.0393</td>
<td>0.5559</td>
</tr>
</tbody>
</table>

This Table reports BGD and GD bounds for the extended Schöbel and Zhu (1999) model. Strikes $K$ range from 90 to 120. $\rho = -0.25$, $-0.5$, $-0.75$. When $\Delta = 0$, lower and upper BGD bounds are equal to the theoretical call price $C_0$, that is why are not reported. The parameters are $t = 0$, $T = 0.5$, $r = 0.093$, $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$, $v = v^* = -22.6$. 
Appendix of Figures

Figure 1: Difference between option price bounds and Black-Scholes prices versus strike for the Heston (1993) model. Strikes $K$ range from 80 to 120. We consider correlations $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for $h_1^*$ and $h_2^*$, respectively. $\alpha = 0.097$, $\beta = 7.1$, $\sigma = 0.32$, $s = 8.6$, $v = v^* = -22.6$. The remaining parameters are set as $r = 0.05$, $S_0 = 100$, $V_0 = \alpha/\beta = 0.0137$, $T = 0.25$. 
Figure 2: Size of bounds $C_0 - C_0$ versus strike $K$. Heston (1993) model. Strikes $K$ range from 80 to 120. \(\rho = -0.1, -0.53, -0.9\). \(\Delta = 0\) (dashed lines), \(\Delta = 2\) (dotted lines), and \(\Delta = 4\) (solid lines). The left and right panels are for $h_1^*$ and $h_2^*$, respectively. \(\alpha = 0.097\), \(\beta = 7.1\), \(\sigma = 0.32\), \(s = 8.6\), \(v = v^* = -22.6\). The remaining parameters are set as \(r = 0.05\), \(S_0 = 100\), \(V_0 = \alpha/\beta = 0.0137\), \(T = 0.25\).
Figure 3: The difference between the option price bounds and the AVBS price versus strike $K$. Extended Schöbel and Zhu (1999) model. Strikes $K$ range from 90 to 120. $\rho = -0.25, -0.5, -0.75$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for $h^*_1$ and $h^*_2$, respectively. $t = 0$, $T = 0.5$, $r = 0.093$, $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$, $v = v^* = -22.6$. 
Figure 4: Size of bounds $\overline{C_0} - C_0$ versus strike $K$. Extended Schöbel and Zhu (1999) model. Strikes $K$ range from 90 to 120. $\rho = -0.25, -0.5, -0.75$. $\Delta = 0$ (dashed lines), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for $h_1^*$ and $h_2^*$, respectively. $t = 0$, $T = 0.5$, $r = 0.093$, $\kappa = 4$, $\theta = 0.2$, $\sigma = 0.1$, $v = v^* = -22.6$. 
Figure 5: Deltas for bounds versus $S_0$. Heston (1993) model. $K = 100$. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed line), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for $h^*_1$ and $h^*_2$, respectively. $\alpha = 0.097$, $\beta = 7.1$, $\sigma = 0.32$, $s = 8.6$, $v = -22.6$, $v^* = -22.6$. The remaining parameters are set as $r = 0.05$, $V_0 = \alpha/\beta = 0.0137$, $T = 0.25$. 
Figure 6: Gammas for bounds versus $S_0$. Heston (1993) model. $K = 100$. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed line), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for $h_1^*$ and $h_2^*$, respectively. $\alpha = 0.097$, $\beta = 7.1$, $\sigma = 0.32$, $s = 8.6$, $v = v^* = -22.6$. The remaining parameters are set as $r = 0.05$, $V_0 = \alpha/\beta = 0.0137$, $T = 0.25$. 
Figure 7: Vegas for bounds versus $S_0$. Heston (1993) model. $K=100$. $\rho = -0.1, -0.53, -0.9$. $\Delta = 0$ (dashed line), $\Delta = 2$ (dotted lines), and $\Delta = 4$ (solid lines). The left and right panels are for $h_1^*$ and $h_2^*$, respectively. $\alpha = 0.097$, $\beta = 7.1$, $\sigma = 0.32$, $s = 8.6$, $v = -22.6$, $v^* = -22.6$. The remaining parameters are set as $r = 0.05$, $V_0 = \alpha / \beta = 0.0137$, $T = 0.25$. 
