Term Structure Models with Differences in Beliefs

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ABSTRACT

In this paper we study both theoretically and empirically the implications of macroeconomic disagreement for bond market dynamics. If there is a source of heterogeneity in the belief structure of the economy then differences in beliefs can affect equilibrium asset prices. Using survey data on a unique data set we propose a new empirically observable proxy to measure macroeconomic disagreement and find a number of novel results. First, consistent with a general equilibrium model, heterogeneity in beliefs affect the price of risk so that belief dispersion regarding the real economy, inflation, short and long term interest rates predict excess bond returns with $R^2$ between 21%-43%. Second, macroeconomic disagreement explains the volatility of stock and bonds with high statistical significance with an $R^2$ ∼ 26% in monthly projections. Third, disagreement also contains significant information trading activity: dispersion in beliefs explains the growth rate of open interest on 10 year treasury notes with $R^2$ equal to 21%. Fourth, while around half the information contained in the cross-section of expectations is spanned by the yield curve, there remains large unspanned component important for bond pricing. Finally, we control for an array of alternative predictor variables and show that the information contained in the belief structure of the economy is different from either consensus views or fundamentals.

JEL classification: D9, E3, E4, G12

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Introduction

This paper investigates the empirical implications of macroeconomic disagreement for the time variation in bond market risk premia. When moving from single agent to heterogeneous agent models several important properties of asset prices change. Differences in beliefs can affect the stochastic discount factor, thus equilibrium asset prices. This is important since the dynamics of macroeconomic disagreement may become a source of predictable variation in bond excess returns. A growing body of evidence indicates that heterogeneity plays an important role in a variety of settings, including equity, foreign exchange, and derivative markets. However little is known about its affect on bond markets. In this paper we test the link between macroeconomic disagreement and expected bonds returns using a data set we constructed by merging the original paper archives of BlueChip. The dataset contains monthly observations on macroeconomic forecasts and allows us to directly look at market participants expectations regarding real, nominal, and monetary components of the economy.

The term structure literature is vast. Traditional reduced-form and structural models have provided significant insights that have improved our understanding of the dynamics of interest rates and are used in a number of applications, including risk management, trading, and monetary policy. At the same time, however, the literature highlights several empirical regularities that are difficult to reconcile with traditional homogeneous economies with no frictions. First, long term bond yields appear too volatile to accord with standard representative agent models (Shiller (1979)). At the same time, model implied Sharpe ratios from medium and long term bonds are small with respect those observed in the data (Duffee (2011)). Second, there appears to be a large degree of predictability in returns by several yield curve factors. An extensive literature has evolved from the univariate regression approach of Fama and Bliss (1987) to multidimensional predictive regressions as in Cochrane and Piazzesi (2005). The results suggest a substantial in-sample variability of conditional expected excess returns. However, the term structure appears to contain information on future term structures in a direction and magnitude that is difficult to explain within some classes of models (Campbell and Shiller (1991)). The dynamics of bond market risk compensation are complex and demand a rich specification for the price of risk. For example, Duffee (2002) proposes the ‘essentially affine’ class that allows for a flexible specification for the price of risk. However, while the essentially affine class can better match some salient features of the data, they are unable to match at the same time first and second moments of yields. To address this issue a stream of the literature has investigated non-linear models which are ca-

1 The commercially available digital monthly economic BlueChip files include only the post-2007 period.
pable of generating counter-cyclical risk-premia (as in habit models, see Buraschi and Jiltsov (2006)). However, while consumption habits can generate more realistic time variation in risk premia, they also imply a tight link between past consumption and bond expected excess returns which is not fully reflected in the data. Third, in their canonical form, essentially affine models imply that primitive shocks underlying the economy are perfectly spanned by yield curve inversion so that macroeconomic aggregates contain no incremental information useful for bond pricing. However, the literature shows that the interest rate dynamics display unspanned stochastic volatility: bond portfolios appear unable to hedge interest rate derivatives, thus suggesting some form of market incompleteness. It appears that the set of state variables driving volatility is not the same set driving yields. Last, little is known about the link between the previous questions and the trading activity of Treasury bond. While most of the empirical evidence focus on fitting bond yields and returns, one may argue that the dynamics of bond trading volumes is equally important as a source of information to help distinguish alternative models.\(^2\)

This paper takes a different route by focusing on the cross-sectional and time series relationship between heterogeneity in beliefs and bond markets. A growing literature highlights that heterogeneity in beliefs can induce significant changes to the dynamics of the stochastic discount factor. We provide a comprehensive empirical study in the context of bond markets. To provide a framework for our empirical analysis and highlight better the marginal effect of different assumptions, we cast our questions within a general term structure model that nests previous theoretical results as special cases. We start from a simple benchmark Vasicek general equilibrium economy and introduce multiple agents, dynamic disagreement and learning. We derive the implications in terms of (a) bond risk premia; (b) bond volatility; and (c) trading activity when agents have incomplete information. It is known that when agents have log-utility, bond prices can deviate from those implied by the average consensus beliefs (Xiong and Yan (2010)). The beliefs of the representative agent include an aggregation bias (Jouini, Marin, and Napp (2010)) which could make the representative agent to act as if pessimistic with respect to the consensus belief. When agents are not log-utility investors and differences in beliefs follow a dynamic process, however, trading

\(^2\)Solutions posed in the literature can be roughly sorted into three strands: i) statistical models include either extensions to the price of risk (essentially affine, extended affine, quadratic models) or extensions to the state space (time-varying covariances, Wishart or multi-frequency dynamics); ii) reduced form economic models which either use new econometric methods for measuring state variables (dynamic factor analysis, least absolute shrinkage and selection operator (lasso) approaches), or introduce observable information (monetary policy shocks extracted from high frequency data, spanned and/or unspanned risk factors); or iii) structural macro-finance models that include a richer preference structure (habit formation, ambiguity aversion, or recursive preferences).
includes also an additional intertemporal risk sharing term that makes differences in beliefs priced in equilibrium. We derive in closed-form the term structure of bond prices for this more general case. The solution is exponential quadratic in differences in beliefs. In this economy bond expected excess returns are predictable even if the benchmark homogeneous Vasicek economy has homoskedastic discount factors. Predictability is generated by time-variation in differences in beliefs. Moreover, in this economy the formation of expectations directly affects bond volatility, even if in the (fictitious) homogeneous economy volatility is constant. Finally, differences in beliefs have been used in the empirical finance literature to proxy for both disagreement and ambiguity. While it is not easy to distinguish these two approaches based on risk premia, an important element of distinction is their implications in terms of trading activity. In absence of frictions, a larger heterogeneity in beliefs induce more trading aimed to better information-risk sharing. The larger the disagreement, the greater the trading activity. Models with knightian uncertainty and ambiguity, however, have the opposite implications: greater ambiguity induce portfolio inertia as discussed in Illeditsch (2011), de Castro and Chateauneuf (2010), and Chen, Ju, and Miao (2011).

We use a unique dataset on individual professional macroeconomic forecasts to address four main empirical questions. First, we revisit the predicability literature in bond returns and show that the cross-section of agents expectations contains economically important and statistically significant information on expected excess bond returns on a 1-year horizon. The combination of real, inflation and monetary disagreement measures forecast excess bond returns with $R^2$ equal to 43% and 21% on 2-year and 10-year bonds, respectively. We find that disagreement about the real economy is highly statistically significant in a number of specifications and loads positively on expected excess returns, while disagreement about inflation appears less important and is subsumed by monetary components. Disagreement about short term, after controlling for long term disagreement loads positively and is always highly statistically significant. Controlling for consensus views and realisations of fundamentals we test whether the information content in belief dispersion is subsumed by more traditional predictor variables and find the results are robust to the inclusion of a number of alternatives. Additionally, we recast our return predictability tests in terms of reverse regressions a’ la Hodrick (1992) and confirm its statistical significance. These findings are important since they show that information contained in the belief structure of the economy, which is not contained in consensus expectations or in macro aggregates, is relevant for risk bond risk premia; this helps to explain why single agent homogeneous economies (and their reduced form counter parts) find it difficult to fully explain the term structure puzzles. The evidence is consistent with heterogeneous models with dynamic disagreement and non-myopic preferences.
Second, we examine the role of heterogeneity for second moments by running regressions of stock and bond volatility measured from squared daily returns between $t \rightarrow t + 1$ on disagreement recorded at $t$ and find a strong result. Consistent with our theoretical framework, relative wealth fluctuations between agents who ‘agree to disagree’ generate a source of endogenous return volatility. In monthly projections disagreement about the real economy and inflation load positively on realised future volatility of stocks and bonds, with t-stats significant at the 1% level, and $R^2$ of 26% and 23%, respectively. Symmetrically to the results on return predictability, in specifications including monetary components we find no marginal increase on the explanatory power above pure macro disagreement. Controlling for macro expectations and fundamentals has no effect on real disagreement while disagreement about inflation loses some significance for stock return volatility. Overall, the results on volatility are striking, statistically significant, and consistent with economic theory.

Third, we focus on the relationship between investor heterogeneity and trading activity by running regressions of the growth rate of open interest on belief dispersion. We find robust evidence of a positive correlation between investor heterogeneity and trading activity. Considering open interest from options and futures on 10-year Treasury notes, including only disagreement on the right hand side, we find that heterogeneity explains the time variation in open interest growth with $R^2$ equal to 21% while adding macro fundamentals raises this $R^2$ to 35%. Importantly, the t-stats on real disagreement and inflation are significant at the 5% level or higher. These results are interesting since they help to distinguish models with differences in beliefs from models with ambiguity. These two streams of the literature study different aspects of uncertainty but they both predict a positive correlation between uncertainty and risk premia. However, they generate opposite predictions in terms of trading activity.

Fourth, we study the spanning properties of macroeconomic disagreement. In traditional general equilibrium models of the term structure, the yield curve can be inverted to reveal the state variables that drive expected returns. Cochrane and Piazzesi (2005) uncover a tent-shape factor from forward interest rates. We find that time variation in the shape of the forward curve in part represents heterogeneity in the belief structure of the economy. These beliefs-driven components reveals properties of the stochastic discount factor which are significant for the time variation in the price of risk, thus proving to be a forecasting factor for excess bond returns. However, disagreement is only partially spanned by the yield curve in the sense that important components of disagreement, which are orthogonal to the first 5 principle components of yields, contain economically and statistically important information on expected returns. In a return predictability regression including the unspanned components of disagreement as right hand variables we find that disagreement about infla-
tion and real GDP are unimportant but that disagreement about short and long ends of the yield curve are statistically significant at the 5% level, forecasting bond returns with an $R^2$ of between 27% and 29%, on 2 – 5 year bonds. Furthermore, exploring the relationship between the time-series dynamic of yields and disagreement we project the hidden risk premium factor from Duffee (2011) on unspanned components and find that disagreement about short term interest rates is statistically significant at the 1% level with an $R^2$ statistic of 6%. Finally, using information orthogonal to a space spanned by both the cross-section and time-series dynamics of yields we document an ‘above’ component linked to disagreement about long term interest rates which retains economically important forecasting power for expected returns. This is consistent to the term structure model presented in the theory section which is exponentially quadratic, thus non-invertible, in disagreement.

I Economies with Differences in Beliefs

An increasingly important part of the asset pricing literature focus on the role of heterogeneity in beliefs. In two seminal papers, Harrison and Kreps (1978) and Harris and Raviv (1993) develop a model of speculative trading based on difference of opinion in which investors receive common information but differ in the way in which they interpret information. All investors in their economy agree on the nature of the information, be it positive or negative, but disagree on its importance. They show that the heterogeneity in beliefs has important implications for asset prices. Similar settings have been studied by Detemple and Murthy (1994) and Zapatero (1998) in the context of a continuous time economy. Buraschi and Jiltsov (2006) allow for Bayesian learning and dynamic disagreement and show that realistic levels of heterogeneous beliefs can generate an option-implied volatility smile and help to explain the dynamics of option prices (see for a survey Basak, 2005).

A second stream in this literature builds on the interaction between behavioral biases and trading frictions. Scheinkman and Xiong (2003) study a model with overconfident and risk-neutral agents. They show that, in this context, short-selling constraints can support rational asset price bubbles in equilibrium. Additional contribution include Hong and Stein (2003). Empirically, Anderson, Ghysels, and Juergens (2005) use data on equity returns and find evidence supporting a neoclassical (i.e., risk-based) interpretation of the impact of differences in beliefs.

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3Kurz (1994) motivates belief disagreement from the difficulties to distinguish different models using existing data.

4Equilibrium treatments of heterogeneity in beliefs include David (2008), who develops a model with counter-cyclical consumption volatility and cross-sectional consumption dispersion where agents assume different models for the underlying data generating process; (Buraschi, Trojani, and Vedolin, 2011, 2010), Bhamra and Uppal (2011); Gallmeyer and Hollifield (2008), Dumas, Kurshev, and Uppal (2010), and
Surprisingly little is known in the context of bond markets. The first article that studies bond markets is Xiong and Yan (2010), who provide a theoretical treatment of bond risk premia in a heterogeneous agent economy with log-utility investors. The authors develop a model of speculative trading in which two types of investors hold different beliefs regarding the central bank’s inflation target. In the model, the inflation target is unobservable so investors form inferences based on a common signal. Although the signal is actually uninformative with respect to the inflation target, heterogeneous prior knowledge causes investors to react differently to the signal flow. Investor trading drives endogenous wealth fluctuations that amplify bond yield volatilities and generates a time varying risk premium. They provide a calibration exercise and show that a simulation of their economy can reproduce the Campbell and Shiller (1991) regression coefficients and the tent shaped linear combination of forward rates from Cochrane and Piazzesi (2005). No empirical study, however, provides empirical evidence on these questions. In what follows, we extend Xiong and Yan (2010) setting to non-myopic agents, derive closed-form solutions for the terms structure of interest rates, and investigate empirically these questions.

A The Homogeneous Benchmark Economy

Let us consider a simple endowment economy in which agents have constant RRA preferences $u'(c_t) = e^{-\rho t}c_t^{-\gamma}$. The growth rate of endowment is a function of a vector of factors $g_t$, with

$$
\frac{dD_t}{D_t} = \beta' g_t dt + \sigma_D dW_t^D
$$

(1)

$$
\frac{dg_t}{(k \times 1)} = -\kappa_g \left( g_t \frac{(k \times k)}{k \times 1} - \theta \frac{(k \times 1)}{k \times 1} \right) dt + \sigma_g \frac{(k \times q)}{(q \times 1)} dW_t^g.
$$

(2)

When agents have common beliefs about the data generating process, it is well known that bond prices satisfy a simple representation. This solution has been studied extensively and is known as the Vasicek (1977) model of the term structure of interest rates. Given the pricing kernel $M_t^*$, with $dM_t^*/M_t^* = -r_t dt - \kappa' dW_t^*$, since $M_t^* = u'(D_t)$ from Ito’s Lemma one finds that $r_t$ must satisfy

$$
r_t = \delta + \gamma \beta' g_t - \frac{1}{2} \gamma(1 + \gamma) \sigma_D^2.
$$

If growth rates are constant, i.e. $\beta' g_t = g_0$, so are interest rates and the term structure is flat. When $g_t$ is stochastic, however, bond prices can be computed from the Euler equation $B_t^{(T-t)} = E_t^* \left[ \frac{M_T}{M_t} \right]$, which gives rise to the simple well know affine representation $B_t^{(T-t)} =
exp \[ A(t, T) + G(t, T)g_t \], which implies that bond excess returns are equal to

\[ r_{x,t}^{(T)} = -\gamma G(t, T)\sigma_D\sigma_G E \left( dW_t^D dW_t^g \right). \] (3)

The dynamics of bond prices \( dB_t^{(T-t)} \) and \( dM^* \) depend, respectively, on \( dW_t^g \) and \( dW_t^D \). If \( E \left( dW_t^D dW_t^g \right) \neq 0 \) long-term bonds command a risk premium\(^5\), which is, however, constant in this economy. A vast empirical literature have shown that the presence of factors with different spectral density can generate realistic cross-sectional shapes of the term structure. An equally vast literature, however, show that its dynamic properties are difficult to be reconciled with the data. Even when \( dW_t^D \) and \( dW_t^g \) are perfectly correlated, the simple benchmark model restricts expected excess returns to be proportional to the volatility of macroeconomic fundamentals. This tight connection makes the model able to reproduce only a small fraction of the predictable variations in expected excess returns found in the data (Duffee (2002)) as the dynamic properties of conditional volatilities depart quite substantially from those of conditional first moments. To break this link, the affine literature has investigated flexible specifications of the price of risk (as in Cheridito, Filipovic, and Kimmel (2007)). We explore a different channel of predictability which is generated by the aggregation properties of the belief dynamics of agents with different priors.\(^6\)

**B Disagreement and the Term Structure**

Now, growth rates are unobservable and suppose further that agents agree on \( \sigma_g \) and \( \theta_g \) but disagree on growth persistence \( \kappa_g \). Notice that when perceived growth rates are below their long run mean (\( g_t < \theta_g \)) the most optimistic investor in the economy is the one with the largest \( \kappa_g \) and vice-versa when when growth rates are above their long run mean (\( g_t > \theta_g \)), i.e., the most optimistic investor holds a smaller value of \( \kappa_g \). For our purposes it suffices to think of a two factor economy driven by a 2-dimensional standard Brownian motion: the first, \( g_1^t \), being a short-term factor that quickly mean-reverts with \( \kappa_g^1 > 0 \). This unobserved hidden state could be related to short-term uncertainty about monetary policy decisions (and their effects on the productivity of capital \( g_1^t \)). The second factor, on the other hand, can be thought as a very persistent factor (as often assumed in the long-run literature) with \( \kappa_g^2 \) small but positive. It may be natural to think this unobserved hidden state as a factor

\(^5\)A common assumption is to restrict \( D_t \) to be an affine transformation of \( g_t \), i.e. \( D_t = \exp(\beta' g_t) \).

\(^6\)Previous literature has investigated models with preference shocks, see Bekaert and Grenadier (2000), and models with more flexible preferences with habit formation, Buraschi and Jiltsov (2006) and Wachter (2006).
related to technological innovations and their uncertain effect on the long-run component $g_t^2$:

$$dg_t = -\kappa_g (g_t - \theta_g) dt + \sigma_g d\hat{W}_t^{g,i}.$$ 

Let the two subjective probability measures associated to the two posteriors be $dP^a_t$ and $dP^b_t$. In this context the two probability measures are absolutely continuous; the difference in beliefs between the two agents can be conveniently summarized by the Radon-Nikodym derivative $\eta_t = \frac{dP^b_t}{dP^a_t}$, so that for any random variable $X_t$ that is $\mathcal{F}_t$-measurable,

$$E^b(X_T|\mathcal{F}_t) = E^a\left(\frac{\eta_T}{\eta_t}X_T|\mathcal{F}_t\right).$$

All agents observe the same variable $D_t$, so that $\mathcal{F}_t$ is common knowledge among all agents and there is no private information: agents simply agree to disagree, as in Detemple and Murthy (1994).\footnote{A large literature study economies where agents agree to disagree (cite relevant literature)}

In this setting, the first agent may think that the economy is dominated by long-run risk components while the second agent may think that it is more exposed to short-term shocks. Since $g_t$ is not observable, it may be difficult for the agents to agree on its true value (see Hansen, Heaton and Li (2008) and Pastor and Stambaugh (2009)).\footnote{Hansen, Heaton and Li (2008) argue about the existence of significant measurement challenges in quantifying the long-run risk-return trade-off and that “the same statistical challenges that plague econometricians presumably also plague market participants.” Pastor and Stambaugh (2009) discuss the statistical properties of predictive systems when the predictors are autocorrelated but $\kappa$ is not known.}

Xiong and Yan (2010) notice that disagreement can have non-trivial effects on bond prices induced by the fact that agents have ex-ante incentives to trade with each others. In turn, agents relative wealth will ex-post be affected by their trading ex-ante. The first order effect of disagreement can be immediately appreciated by noticing that, with no further assumptions:

**Proposition 1** (Xiong and Yan). If agents have logarithmic preferences, $u(c_t) = e^{-\rho(T-t)} \ln c_t$, they will trade until their wealth ratio is equal to $\eta_T$, i.e. $\eta_T = W^b_T/W^a_T$. Furthermore, in equilibrium the price of a zero-coupon bond $B_t^{(T-t)}$ with time to maturity $T-t$ is equal to the $\eta_t$-weighted average of the zero-coupon bonds prices prevailing in the (fictitious) homogeneous economies populated only by each of the two agents, $B_t^{(T-t),a}$ and $B_t^{(T-t),b}$, with

$$B_t^{(T-t)} = \frac{1}{1 + \eta_t} B_t^{(T-t),a} + \frac{\eta_t}{1 + \eta_t} B_t^{(T-t),b}.$$  

(4)

One may notice that even if $\eta_t$ were constant, bond prices in the heterogeneous econ-
omy would not be affine. The affine class is not robust to aggregation when agents have different probability measures. Moreover, if $\eta_t$ were to be stochastic, equilibrium bond prices may greatly differ from those prevailing in a (fictitious) economy populated by only one agent. Xiong and Yan (2010) calibrate a model with log-utility investors assuming an affine specification for $B^\eta_t$ and show that a realistic parametrization can generate a rich set of cross-sectional shapes of the term structure.

While adequate for their purposes, log-preferences make agents myopic and the absence of intertemporal hedging demands can restrict the link between risk premia and the dynamics of differences in beliefs. For instance, David (2008) show that in a heterogeneous agent economy a necessary condition for equity risk premia to be increasing in differences in beliefs is that relative risk aversion is greater than 1. Given our focus, we explore the empirical implications when one relaxes this assumption and allows for general power utility preferences and dynamic rational learning.\footnote{Scheinkman and Xiong (2003) and Buraschi and Jiltsov (2006), Dumas, Kurshev, and Uppal (2010), Buraschi, Trojani, and Vedolin (2011), and Xiong and Yan (2010) study economies in which a process $\eta_t$ arise from investors’ different prior knowledge about the informativeness of signals and the dynamics of unobservable economic variables. Kurz (1994) argues that non-stationarity of economic systems and limited data make it difficult for rational investors to identify the correct model of the economy from alternative ones.} Two main results will emerge. First, the presence of signals can increase the state-space and disagreement on these signals becomes an additional priced risk-factor above and beyond the volatility of fundamentals. This is potentially important since it can directly affect the dynamics of risk premia. Second, we derive a closed-form solution for the term structure of bond prices that encompass some of the results in the literature as special cases. Expected excess returns are time-varying and driven by the dynamics of the difference in beliefs $\eta_t$. We will then use these properties to guide and interpret our empirical study.

C A General Model

We assume that investors can improve their forecast on $\hat{g}_t^i$ by using signals $dS^i_t$, whose drifts are correlated with with the growth components of the economy

\begin{align}
    dS^i_t &= (\phi^i g^i_t + (1 - \phi^i) \varepsilon_t) \, dt + \sigma^i dW^S_t, \\
    d\varepsilon_t &= dW^\varepsilon_t. \tag{5}
\end{align}

Agents are uncertain about $g^i_t$ and compute posterior estimates (in Bayesian fashion) given initial priors and all available information $\mathcal{G}_t = \sigma(D_u, S^1_u, S^2_u; 0 \geq u \geq t)$. The larger the
value of $\phi^i$ the more weight agents place on the signals $S^i_t$ when estimating the growth of the economy. The optimal drift forecasts can be conveniently computed by writing the economy in a state-space representation: $X_t = [\log D_t, S^1_t, S^2_t]'$ and $\mu_t = [g^1_t, g^2_t, \epsilon_t]'$, with Gaussian diffusions following

$$dX_t = (A_0 + A_1\mu_t)\,dt + BdW^X_t$$

(7)

$$d\mu_t = (a_0 + a_1\mu_t)\,dt + bdW^\mu_t,$$

(8)

where the matrices $A_0, A_1, a_0, a_1$ are given in the appendix, and $W^X_t$ and $W^\mu_t$ are 3 dimensional standard Brownian motions. Denote subjective posterior beliefs of the unobservables states $m^n_t := E^n(\mu_t | \mathcal{F}_t)$ and posterior covariance matrix $\nu^n_t = E^n[(\mu_t - m^n_t)(\mu_t - m^n_t)'] | \mathcal{F}_t.$

**Lemma 1. (Beliefs)** Under technical conditions discussed in the appendix, $m^n_t$ and $\nu^n_t$ are $\mathcal{F}_t$ measurable, unique, and continuous processes solving

$$dm^n_t = (a_0 + a_1m_t)dt + v_tA_1'B^{-1}dW^{X,n}_t$$

(9)

$$\dot{v}_t = a_1v_t + v_tA_1' + bb' - v_tA_1'(BB')^{-1}A_1v_t$$

(10)

where $dW^{X}_t = B^{-1}[dX_t - (A_0 + A_1\mu_t)\,dt].$

When $v_t \neq 0$, a rational agent will make use of observations on $dS^i_t$ to update their prior beliefs so that $m^n_t$ depends on the characteristics of matrix $A_1$, which in turn depend on the prior beliefs on both $v^i, \sigma^i_s$, and subjective parameters. However, since agents must agree on the observables $X_t$, if $A_0$ is common it must be true that $d\hat{W}^{X,a}_t = d\hat{W}^{X,b}_t + B^{-1}A_1(m^n_a - m^n_b)dt$. Spreads in the expected unobserved states $m_t$ drive a wedge in the perceived shocks $d\hat{W}^{X,n}_t$. This drift plays a key role in describing the difference in the probability measures of the two agents. Thus, let us define $\Psi_t \equiv B^{-1}A_1(m^n_a - m^n_b)$ as the standardized difference in beliefs. Those agents with relatively lower posterior estimates for signal drifts $m_t^{S_i,n}$ interpret any signal shock as relatively better news for productivity and will update more their posterior to higher values.

From equation (9) one can derive the diffusion process for $d\Psi_t$. One can immediately notice that when $v^n_a \neq v^n_b$ the process $\Psi_t$ is stochastic. Moreover, when $A^n_1 \neq A^n_1$ disagreement does not converge to zero asymptotically (see Appendix).\textsuperscript{10}

\textsuperscript{10}The intuition is nicely developed in Acemoglu, Chernozhukov, and Yildiz (2008). When agents are uncertain about the signals they use to improve their forecasts, they show that observing an infinite sequence of signals does not guarantee degenerate asymptotic disagreement. This is because investors have to update beliefs about two sources of uncertainty using one sequence of signals.
D Individual Investor Problem

To solve for the equilibrium SDF, consider an economy in which agents have \( u_t' = c_t^{-\gamma} \), a time preference discount \( \theta_t = \text{exp}[-\int_0^t \rho(s)ds] \), and a sequence of endowments \( c_t \). When markets are complete, an equilibrium is defined by a unique stochastic discount factor \( \mathcal{M}_t \) for each agent and a consumption plan \( c_t \) that solves the following intertemporal problem

\[
\max_{c_t,\mathcal{M}_t} E_0 \int_0^\infty \theta_t u(c_t) \, dt \text{ subject to } E_0 \int_0^\infty \mathcal{M}_t [c_t - c_t^*] \, dt \leq 0 \text{ and such that markets clear, i.e. } \sum_i c_t^i = D_t \text{ for } \forall t. \]

The first order conditions imply that the optimal consumption policies are of the form \( c_t^i = (\theta_t / (\alpha_i \mathcal{M}_t^i))^{1/\gamma} \), where \( \alpha_i \) is the Lagrange multiplier associated with the static budget constraint of agent \( i \). It is easy to show that in equilibrium the Radon-Nikodym derivative \( \eta_t \) must be equal to the ratio of the stochastic discount factors of the two agents:

\[
\eta_t = \frac{\alpha_i u_t'(c_t)}{\alpha_a u_t'(c_a)} = \frac{\mathcal{M}_t^a}{\mathcal{M}_t} \quad 11
\]

Moreover, its diffusion satisfies (see Appendix):

\[
d\eta_t/\eta_t = -\Psi_t d\hat{W}_t^{X,a}
\]

An important implication follows: since the level of \( \Psi_t \) affects the \( \eta_t \) process, it directly affects the evolution of bond prices. The special case of log-investors in equation (4) illustrates this point.

E Bond Market Implications

The dynamic properties of bond prices \( B_t^{(T-t)} \) depend on the characteristics of the stochastic discount factor of the representative agent under the econometrician measure \( B_t^{(T-t)} = E_t^s (\mathcal{M}_t^s / \mathcal{M}_t^r) \), with \( d\mathcal{M}_t^s / \mathcal{M}_t^r = -r_f(t)dt - \kappa_t d\hat{W}_t^{X,s} \). The representative investor utility function is a weighted average of each individual utilities with weight \( \lambda_t \): \( U^s(D(t), \lambda) := \max_{c_a(t) + c_b(t)} \{ \theta_t u_a(c_a(t)) + \lambda_t \theta_t u_b(c_b(t)) \} \). Since a necessary condition for a social optimum is that \( u_a'(c_a(t)) = \lambda_t u_b'(c_b(t)) \), from the first order condition of each individual agent, one can immediately see that this can be achieved if the representative agent sets a stochastic weight equal to \( \lambda_t = \frac{u_a'(c_a(t))}{u_b'(c_b(t))} = \frac{\alpha_a \mathcal{M}_t^a(t)}{\alpha_b \mathcal{M}_t^b(t)} \). This implies that the relative weight of the second agent must be proportional to the Radon-Nikodym process \( \eta_t \): i.e. \( \lambda_t = \frac{\alpha_a}{\alpha_b} \eta_t \). Moreover, since the Lagrange multipliers are constant, the diffusion of the Radon-Nykodym process coincides with the dynamics of the relative weight: \( d\eta_t/\eta_t = d\lambda_t / \lambda_t \). \quad 12

---

11 Consider a tradable asset with terminal payoff \( B_T \). In equilibrium, both agents must agree on its price. Under general preferences, \( u_a(c_t) \), from the Euler equation it must be true that \( E_t^b \left( \frac{u_a'(c_T)}{u_b'(c_T)} B_T \right) = E_t^a \left( \frac{u_a'(c_T)}{u_a'(c_T)} B_T \right) \). Thus \( E_t^b \left( \frac{u_a'(c_T)}{u_b'(c_T)} B_T \right) = E_t^a \left( \frac{u_a'(c_T)/u_a'(c_T)}{u_a'(c_T)/u_a'(c_T)} \right) B_T \), which implies that \( \eta_T = \frac{\alpha_a}{\alpha_b} \eta_T \).

12 For applications of the martingale approach to heterogeneous beliefs models, see Cuoco and He (1994), Karatzas and Shreve (1998), and Basak and Cuoco (1998).
It can be shown that the stochastic discount factor of the representative agent is therefore 
\( M_t^* = \alpha_a M_a(t) = \lambda(t) \alpha_b M_b(t) \), which is proportional to the first agent’s state price density.\(^{13}\) Combining the first order conditions from the individual agent’s problems with the Radon-Nikodym \( \eta_t \) and imposing market clearing one obtains the stochastic discount factors for the representative agent:

\[
M_t^* = g_t D_t^{-\gamma} \left( 1 + \left( \frac{\alpha_a}{\alpha_b} \eta_t \right)^{1/\gamma} \right)^\gamma
\]  
(11)

The drift of \( dM_t^* \) provides the risk free rate, which is equal to:

\[
r_f = \rho + \gamma \beta' \left( \omega_a(t) \hat{g}_t^a + \omega_b(t) \hat{g}_t^b \right) - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2 + \frac{\gamma - 1}{2\gamma} \omega_a(t) \omega_b(t) \Psi_t \Psi_t',
\]  
(12)

where \( \omega_i(t) = c_i^t / D_t \) is investor’s \( i \) total consumption share. This result highlights an immediate implication which is relevant for bonds markets. With differences in beliefs, the formation of expectations impact short term interest rates in two different ways: (a) via a consumption-weighted consensus belief, and (b) a quadratic term in differences in beliefs \( \Psi_t \).

When \( \Psi_t = 0 \) the model reduces to the special case of a standard Vasicek economy in which \( r_f = \rho + \gamma \beta' \hat{g}_t - \frac{1}{2} \gamma (\gamma + 1) \sigma_D^2 \). The cross-sectional distribution of consumption is degenerate and state prices, \( M_t^a = M_t^b \), depend exclusively on the diffusion \( dD_t \). When \( \Psi_t \neq 0 \) and \( \gamma > 1 \), there is a fourth term quadratic in \( \Psi_t \). When preferences are logarithmic (i.e. \( \gamma = 1 \)), as in Xiong and Yan (2010), the last term in equation \( 12 \) disappears and disagreement impacts the risk free rate only because of an aggregation bias due to the consumption weights \( \omega_n(t) \). When agents are not myopic, the dynamics in \( \Psi_t \) has a direct effect on \( r_f \). In particular, when \( \gamma > 1 \) interest rates are increasing in \( \Psi_t \).

The second implication is about risk premia and bond excess returns. In a Vasicek economy, the only priced shocks are \( dW_a(t) \) and since \( dD_t / D_t \) is homoskedastic, the price of risk is constant. In the partial information heterogeneous economy with learning, however, the dynamics of \( dM_t^* \) also depend on \( d\eta_t \). This creates a potential channel for \( \Psi_t \) to play a role in the predictability of bond excess returns. The intuition is simple. When agents have different subjective beliefs, relative consumption is stochastic.\(^{14}\) Optimistic (pessimistic) investors consume more (less) in states of high aggregate cash flows, at a lower (higher) marginal

\(^{13}\)This follows from: \( U^\ast(D(t), \lambda(t)) = u'_a(c_a(t)) \frac{\partial \lambda}{\partial D} + \lambda(t) u'_b(c_b(t)) \frac{\partial \lambda}{\partial D} = u'_a \left( \frac{\partial \lambda}{\partial D} + \frac{\partial \lambda}{\partial D} \right) = \alpha_a M_a(t) = \lambda(t) \alpha_b M_b(t) \)

\(^{14}\)The implied optimal consumption policies are \( c_a(t) = D_t \left( 1 + \left( \frac{\alpha_a}{\alpha_b} \eta_t \right)^{1/\gamma} \right)^{-1} \) and \( c_b(t) = D_t \left( \frac{\alpha_a}{\alpha_b} \eta_t \right)^{1/\gamma} \).
utility, because they perceive those states as more (less) likely. This also implies that the consumption volatility of the optimist is higher than the pessimist.\footnote{Notice that differences in beliefs make consumption volatility of each agent higher even if markets are complete due to imperfect risk sharing. Individual consumption volatilities determined endogenously as $\sigma_{c_n} = \frac{\sigma_n}{\gamma}$. This compares to an average aggregate consumption volatility of $\bar{\sigma}_C = \frac{1}{2} (\sigma_{c_a} + \sigma_{c_b}) = \sigma_D + DiB^X_t$. Incomplete consumption risk sharing makes individual consumption volatility higher than aggregate endowment volatility.}

This can be summarized by the prices of risks, which are $\kappa'_a(t) = \gamma \sigma_D + \omega_b(t) \Psi_t$ and $\kappa'_b(t) = \gamma \sigma_D - \omega_a(t) \Psi_t$. This intuition can be formalized by noticing that the stochastic discount factor of the individual agent can be factorized as the product of two components, the stochastic discount factor that would survive in Vasicek economy with no differences in beliefs $\tilde{M}_i^t \equiv \frac{1}{\alpha_i} q_t D_t^{-\gamma}$, which only depends on aggregate consumption shocks, and the stochastic consumption share $\omega_i(t)$: $M_i^t = \tilde{M}_i^t \times \omega_i(t)^{-\gamma}$. In order to finance ex-ante the different individual consumption plans, pessimistic investors have to buy financial protection against low aggregate cash flow states from optimistic investors. If a negative state occurs ex-post, optimistic investors are hit twice: First, because the aggregate endowment is lower; second, because their consumption share is lower due to the protection agreement. The size of this risk transfer is proportional to the degree of disagreement among agents. In the economy with homogeneous beliefs, $\omega_i(t)$ is constant and the discount factor is proportional to the marginal utility $D_t^{-\gamma}$ of aggregate consumption.\footnote{The relative consumption of the two agents $c_2/c_1$ is constant since now complete markets allow for perfect risk sharing.} In the economy with disagreement, $\omega_i(t)$ depends on the trading occurred ex-ante between the agents.

The most interesting aspect of this risk transfer is that it does not cancel out at the representative agent level. The prices of risk under the econometrician measure are equal to

$$
\kappa_g^*(t) = \gamma \sigma_D + \omega_b(t) \psi_b(t) \quad \kappa_s^*(t) = \omega_b(t) \psi_{s_i}(t),
$$

which implies that bond excess returns explicitly depend on the dynamics of $\Psi_t$ (see Appendix). Different than in the Vasicek benchmark economy the price of risk is time varying, which gives rise to our first set of empirical questions. Moreover, in this economy disagreement on market-wide signal (not just on fundamentals) are priced. This is because agents use signals to determine their belief-dependent optimal consumption plans. This channel is absent in Basak (2005) and Jouini and Napp (2010) who specialize their analysis to the case of constant beliefs and no learning.
The Term Structure of Bond Prices

The price of a default-free zero coupon bond is given by

\[ B(t, T) = E_t^* \frac{\mathcal{M}^*(T)}{\mathcal{M}^*(t)}. \]

One can notice that solving for the term structure of interest rates is complicated by the fact that it requires knowledge of the joint density of \( D(t) \) and \( \lambda(t) \), which is not available in closed-form. The solution method suggested in Xiong and Yan only applies to the case of log-investors. Fortunately, it is possible to calculate the joint Laplace transform of \( D(t) \) and \( \lambda(t) \).

This can be used, in a second step, to obtain bond prices in closed-form by Fourier inversion.

The result is remarkably simple. The price of a zero coupon bond is the product of two components: the first component depends on the posterior \( m_d \), the second component depends on the vector of differences in beliefs \( \Psi \). The following Theorem summarizes the results.

**Theorem 1 (The Term Structure of Bond Prices).** The term structure of bond prices is equal to the product of two deterministic functions. The first is exponentially affine in the posterior growth rate of the endowment; the second is exponentially quadratic in the level of differences in beliefs:

\[
B(t, T) = \varrho_{T-t} F_m (m^1_D, t, T; -\gamma) G(t, T, -\gamma; \Psi_d; \Psi_s),
\]

\[
G(t, T, -\gamma; DiB_g; DiB_S) = \int_0^\infty \left( \frac{1 + \lambda(T)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} \right)^\gamma \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\lambda(T)}{\lambda(t)} \right)^{-i\chi} F_{\lambda, \Psi_d; \Psi_s} d\chi \right] d\lambda(T)/\lambda(T),
\]

\[
F_{m^D_g}(m^1_D, \tau, \epsilon) = \exp(A(\epsilon, \tau)m^1_D + B(\epsilon, \tau)),
\]

\[
A(\epsilon, \tau) = -\frac{\epsilon(e^{-\kappa^g_D\tau} - 1)}{\kappa^g_D}
\]

\[
B(\epsilon, \tau) = \frac{1}{2}\epsilon(\epsilon - 1)\sigma^2_d\tau - \left( \theta_g + \frac{\epsilon \gamma^g_D}{\kappa^g_D} \right) \left( e^{\kappa^g_D\tau} + \tau \right)
\]

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17 Obviously, in equilibrium, the solution can be equivalently computed with respect to any probability measure since \( B(t, T) = E^v(q_\tau^T) = E^h(q_\tau^T) = E^*(q_\tau^T) \).

18 See also Dumas, Kurshev, and Uppal (2009) and Buraschi, Trojani and Vedolin (2009) for the use of this technique in different contexts.

19 The spirit of the Fourier inversion approach is similar to the one used to price derivatives in stochastic volatility models, such as Heston (1993), Duffie, Pan, and Singleton (2000), and Carr, Geman, Madan, and Yor (2001), or in interest-rate models, such as Chacko and Das (2002).

20 We reduce the system of ordinary differential equations for functions \( A_0, B_1, B_2, C_1, C_2 \) and \( D_0 \) in Lemma 5 to a system of matrix Riccati equations, which can be linearized using Radon’s Lemma. In this way, we obtain explicit expressions for the coefficients in the exponentially quadratic solution of the Laplace transform.
\[-\frac{1}{\kappa_g} \left( \left( \frac{\gamma_g}{\sigma_D} \right)^2 + \left( \frac{\phi \gamma_g + (1 - \phi) \gamma_g \epsilon}{\sigma_S} \right)^2 \right) \left( \frac{3}{2} e^{-\kappa_g \tau} + \kappa_g \tau \right) \]

with \( \tau = T - t \) and

\[ F_{\psi_g, \psi_S} = e^{C(\tau)+D(\tau)\Psi_t+E(\tau)\Psi_t+F(\tau)\Psi_t(i)\Psi_t(j)}, \]  

for a vector of functions \( C, D, E, \) and \( F, \) where \( \Psi_t, (\Psi_t'\Psi_t), \) and \( \Psi_t(i)\Psi_t(j) \) are linear, quadratic, and covariance terms in disagreement, respectively. (see Appendix).

The dependence of bond prices on \( m^2_H(t) \) is exponentially affine: this is due to the fact that the dividend process, conditional on an estimate for \( \hat{\theta}_t, \) is lognormal (Vasicek economy). Under incomplete information and learning, the term structure also explicitly depends on the difference in beliefs \( \Psi_d(t) \) and \( \Psi_s(t). \) The dependence on these factors is exponentially quadratic. A few important observations emerge which lead to our empirical questions.

**Hypothesis 1.** Disagreement can affect bond prices in two ways. (a) They can create an aggregation bias (the first term in equation (12)) so that the representative agent behave as more pessimist (or optimist) agent with respect to an agent with consensus beliefs. (b) If \( \gamma > 1 \) and disagreement is stochastic, agents also engage in risk sharing that make expected excess returns a function of disagreement. Moreover, if agents use signals but disagree on their informativeness, disagreement on both fundamentals and signals are priced (see equation (13)). Thus, the dimensionality of the vector of priced factors is increased by the use of signals.

**Hypothesis 2.** Second, an important literature investigates the extent to which the shape of the term structure spans the state variables that are responsible for the predictability of bond excess returns (Cochrane and Piazzesi (2005), Duffee (2011), Joslin, Priebsch, and Singleton (2009)). When spanning can be achieved, bond prices (or forward rates) provide sufficient information to explain the dynamics of bond excess returns. In our context, the mapping \( G \) that links differences in beliefs to bond prices is not invertible (see equation (??)). Thus, this economy is consistent with evidence of bond return predictability induced by disagreement and, at the same time, only partial spanning by the cross-section of bond prices.

**Hypothesis 3.** The third implication we study is related to bond return volatility. Even if in the benchmark homogeneous Vasicek economy bond yields have constant volatility, the heterogeneous economy is characterized by stochastic bond yield volatility. Moreover, this
volatility is linked to $\Psi_t$. This effect is discussed in Buraschi and Jiltsov (2006) and Buraschi, Trojani and Vedolin (2011) who investigate equilibrium volatility and correlation risk premia in the equity market and Dumas, Krushev, Uppal (2010) who suggest an explanation for excess equity market volatility. However, it has never been studied in the bond market.

**Hypothesis 4.** The previous effects arise because disagreement induce agents to engage in speculative trading. This gives rise to a fourth testable implication which is an essential part of any model with heterogeneous beliefs: differences in beliefs are correlated with trading activity (as in Buraschi and Jiltsov (2006), Banjeree (2008)). This implication is of particular interest since it distinguishes this class of models from those with knightian uncertainty and ambiguity.\(^{21}\) It has been shown that greater ambiguity induce portfolio inertia and limited participation, as discussed in Illeditsch (2011), de Castro and Chateauneuf (2008), and Chen (2010), as opposed to greater trading volume.

**II Data**

**A Disagreement Data**

We obtain measures of heterogeneity directly from market participants’ expectations of future fundamentals. Survey data provides a rich source to learn how agents form beliefs about economic variables but few sources exist with large sample periods or appropriate frequencies; BlueChip Economic Indicators (BCEI) does provide an extensive panel of data on expectations by agents who are woking at institutions who are active in financial markets and importantly it allows a simple aggregation procedure (discussed below) that mitigates problems associated with rolling forecast horizons. Unfortunately, digital copies of BCEI are only available since 2007. The complete original BCEI paper archive was obtained from Wolters Kluwer and entered into a digital format. The digitisation process required inputting around 350,000 entries of named forecasts plus quality control checking. The resulting dataset represents an extensive and unique dataset to investigate the role of formation of expectations in asset pricing.

Each month BlueChip carry out surveys of professional economists from leading financial institutions and service companies regarding a large set of economic fundamentals covering real, nominal, and monetary variables. While exact timings of the surveys are not published, the survey is conducted over the first two days of the beginning of each month and mailed to

\(^{21}\)Dispersion in beliefs have been used to proxy for entropy at the individual level.
subscribers on the third day our empirical analysis is therefore not affected by biases induced by overlapping observations of returns and responses. The sample period for which we have a fully digitised dataset is 1.1.1990 - 1.12.2011. Forecasts are available for:

1. **Real**: Real GDP, Disposable Personal Income, Non-residential Fixed Investment, Unemployment, Industrial Production, Corporate Profits, Housing Starts, Auto/Truck Sales.

2. **Nominal**: Consumer Price Inflation, Nominal GDP.

3. **Monetary**: 3 Month Treasury Rate, 10 year Treasury, AAA corporate Bond.

Furthermore, for each variable two types of forecast are made:

1. **Short-Term**: an average for the remaining period of the current calendar year;

2. **Long-Term**: an average for the following year.

For example, in July 2003 each contributor to the survey made a forecast for the percentage change in total industrial production for the remaining two quarters of 2003 (6 months ahead), and an average percentage change for 2004 (18 months ahead). The December 2003 issue contains forecasts for the remaining period of 2003 (1 month ahead) and an average for 2004 (13 months ahead). The moving forecast horizon induces a seasonal pattern in the survey which can be adjusted in two simple ways: i) one can adjust cross-sectional statistics for both long and short term forecasts using an X-12 ARIMA filter and subsequently take some linear combination of the resulting seasonally adjusted measures; or ii) one can compute an implied constant maturity forecast for each individual forecaster, compute summary statistics, then adjust any residual seasonality with an X-12 ARIMA filter. Testing both methods we find that combining long and short term forecasts at the individual level removes the vast majority of the observable seasonality and we proceed with this method which is outlined in detail in the appendix. On average 51 respondents are surveyed for short term forecasts and 49 for long term forecasts with standard deviations of 1.6 and 3.3 respectively. Figures 2 and 3 plot the distributions and time series properties of respondent numbers which show that only on rare occasions are survey numbers less than 40 and no business cycle patterns are visible. For comparison, figure 4 plots the time series and distribution of respondents.

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22An exception to the general rule was the survey for the January 1996 issue when non-essential offices of the U.S. government were shut down due to a budgetary impasse and at the same time a massive snow storm covered Washington, DC: [www.nytimes.com/1996/01/04/us/battle-over-budget-effects-paralysis-brought-shutdown-begins-seep-private-sector.html](http://www.nytimes.com/1996/01/04/us/battle-over-budget-effects-paralysis-brought-shutdown-begins-seep-private-sector.html). As a result, the survey was delayed a week.
to the more traditional ‘Survey of Professional Forecasters’.\textsuperscript{23} Compared with the BlueChip dataset the distribution of respondents displays significant variability; for example, while the mean number of respondents is around 40, the standard deviation is 13 and in some years the number of contributors is as low as 9. Furthermore, while forecasts are available since the 4th quarter of 1968 the survey is conducted quarterly meaning that including the 4th quarter of 2011, 169 observations are available compared with 252 from BlueChip over the sample period considered here. The survey has been administered by different agencies over the years. While at the beginning of the sample the number of forecasters was around 60, it decreased in two major steps in the mid 1970s and mid 1980s to as low as 14 forecasters in 1990 and if one restricts the attention to forecasters who participated to at least 8 surveys, this limits the number of data point considerably. There are additional problems, for example, it is suggested that quarters may not be comparable across years since the forecasting horizon shifted in a non-systematic way. For a detailed discussion on the issues related to the Survey of Professional Forecasters see D’Amico and Orphanides (2008) and Giordani and Soderlind (2003). Other well known surveys, such as the ‘University of Michigan Survey of Consumers’ do not provide point estimates from individual survey respondents.

Macroeconomic disagreement is then measured as the cross-sectional mean-absolute-deviation (MAD) in forecasts. Finally, we proxy for disagreement about the real economy from the first principle component of the filtered MAD regarding Industrial Production Growth and Real GDP, and disagreement about inflation from the first principle component of filtered MAD about the GPD deflator and the Consumer Price Index.

Figure 5 plots the first principle component from the individual disagreement measures shown in figures 6 - 8.\textsuperscript{24} One can observe a general decline in the level of disagreement since the early 1990’s accompanied by economically interesting periods characterised by large spikes. Swanson (2006) makes a similar observation using a different measure of cross-sectional uncertainty on the same data set. The purpose of Swanson (2006) was to study the effect of central bank transparency with respect to private sector interest rate forecasts. Using various measures of forecast accuracy the author shows that since the 80’s private sector

\footnotesize{\textsuperscript{23}available here: www.philadelphiafed.org/research-and-data/real-time-center/survey-of-professional-forecasters/.

\textsuperscript{24}As usual the 1st PC is essentially a level factor which in this instance explains $\sim 45\%$ of the variance from the underlying measures}
agents have a) improved projections of the federal funds rate; and b) are more unanimous (cross-sectionally) in forming expectations. In unreported results we find that not only are agents more unanimous regarding interest rate forecasts but are more unanimous regarding real, nominal, and monetary elements of the economy. Figures 6 - 8 plot the time series dynamics for inflation, real, and monetary disagreement measures. Comparing the plots with the summary statistics from table I we find that while measures of real and inflation disagreement are highly correlated, disagreement across real and inflationary components is not especially high. Furthermore, the dynamics of disagreement regarding long and short ends of the yield curve appear quite distinct.

[Insert table I here.]

B Stock and Bond Data

For Treasury bonds data, we use both the (unsmoothed) Fama-Bliss discount bonds dataset, for maturities up to five years, and the (smoothed) Treasury zero-coupon bond yields dataset of Gürkaynak, Sack, and Wright (2006) (GSW). The GSW data set includes daily yields for longer maturities: 1-15 years pre-1971 and 1-30 years post-1971. We introduce notation along the lines of Cochrane and Piazzesi (2005) by defining the date \( t \) log price of a \( n \)-year discount bond as:

\[
p_t^{(n)} = \log \text{price of } n\text{-year zero coupon bond.} \tag{15}
\]

The yield of a bond, the known annual interest rate that justifies the bonds price is given by

\[
y_t^{(n)} = -\frac{1}{n} p_t^{(n)}. \tag{16}
\]

The date \( t \) 1-year forward rate for the year from \( t + n - 1 \) and \( t + n \) is

\[
f_t^{(n)} = p_t^{(n)} - p_t^{(n+1)}. \tag{17}
\]

The log holding period return is the realised return on an \( n \)-year maturity bond bought at date \( t \) and sold as an \((n - 1)\)-year maturity bond at date \( t + 12 \):

\[
r_{t,t+12}^{(n)} = p_{t+12}^{(n-1)} - p_t^{(n)}. \tag{18}
\]

Excess holding period returns are denoted by:

\[
r x_{t,t+12}^{(n)} = r_{t,t+12}^{(n)} - y_t^{(1)}. \tag{19}
\]

The realised second moments of stock and bond returns are measured at daily frequency following Schwert (1990) and Viceira (2007) among many others. Integrated instantaneous

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\[25\] In constructing disagreement about long term rates we use forecasts of AAA rated corporate bonds before 1996 since 10 year Treasury rate forecasts are unavailable before then.

\[26\] The dataset is available at: [www.federalreserve.gov/econresdata/researchdata.htm](http://www.federalreserve.gov/econresdata/researchdata.htm)
volatility is proxied by realised volatility between month $t$ and $t+1$ as

$$\hat{\sigma}^2_{S,B}(t) = \frac{1}{n - 1} \sum_{i=1}^{n} r^2_{S,B}(t,i). \quad (18)$$

Integrated instantaneous covariance is then proxied by realised covariance on stocks and bonds between month $t$ and $t+1$:

$$\hat{\sigma}_{S,B}(t) = \frac{1}{n - 1} \sum_{i=1}^{n} r_S(t,i) \times r_B(t,i), \quad (19)$$

and stock-bond correlation then estimated as:

$$\hat{\rho}_{S,B}(t) = \frac{\hat{\sigma}_{S,B}(t)}{\hat{\sigma}_S(t)\hat{\sigma}_B(t)}. \quad (20)$$

All estimates are then annualised appropriately. For volatility and correlation estimates, we use squared daily returns from the GSW dataset. As proxy for the equity market portfolio we take the value-weighted index from the daily CRSP files which consists of stocks traded in NYSE, AMEX, and NASDAQ. Stock and Bond data sample periods run from 31.12.1989 - 30.11.2011.

C Open Interest

In order to gauge a measure of trade we collect total combined (options and futures) open interest on CBOT 10-year Treasury Notes and the CME S&P 500 index from the ‘Commitments of Traders in Commodity Futures’ which is available here: [http://www.cftc.gov/OCE/WEB/index.htm](http://www.cftc.gov/OCE/WEB/index.htm). Following Hong and Yogo (2011) we compute a 12-month geometrically averaged growth rate of bond and stock market open interest. Sample Period includes 28.03.96 - 30.11.11

D Macro Data

Dynamic macroeconomic theory suggests a small set of common factors are responsible for the co-movement of a large set of economic and financial time series. However, until recently the search for these factors has been carried out with limited success. The limited success of linking the macro economy to term premia led researchers to explore alternative empirical routes to pin down the state variables priced in bond markets. For example, Ang and Piazzesi (2003) estimate a VAR with identifying restrictions derived from the absence of arbitrage and find the combination of macro and yield curve factors improves performance over a
model including yield factors only. More recently Ludvigson and Ng (2009b) find strong
evidence linking variations in the level of macro fundamentals to time variation in the price
of risk. We adopt the procedure of Ludvigson and Ng (2009b) by estimating macro-activity
factors using static factor analysis on a large panel of macroeconomic data. The panel used
in our estimation is an updated version of the one in Ludvigson and Ng (2009a), except that
we exclude price based information in order to interpret factors as pure ‘macro’ and allow
clearer distinction between information contained in agents’ beliefs from that contained in
macroeconomic aggregates. After removing price based information from the panel we end
up with a 99 macro series. Classical understanding of risk compensation for nominal bonds
also says that investors should be rewarded for the volatility of inflation and consumption
growth. We proxy for these by estimating a GARCH process for monthly log differences of
CPI All Urban Consumers: Non-Durables (NSA) and Industrial Production and Capacity
Utilisation: All Major Industry Groups (NSA). Finally, from Campbell, Sunderam, and
Viceira (2009) we know that an important driver of bond risk premia is the real-nominal
covariance which we proxy for by estimating a dynamic correlation MV GARCH process
for inflation and consumption growth. All macro data is either from Global Insight or the

III Empirical Results

In the following section we study the role of heterogeneity across a number of dimensions
of asset pricing: i) risk premia; ii) volatility; iii) trade; iv) the spanning properties of bond
prices. Specifically, we run multivariate regressions that focus on differences in belief about
the real economy and inflation, and augment these measures with monetary measures that
potentially reveal important information for bond pricing. The estimated coefficients in
the tests that follow are both economically and statistically significant and survive a host of
robustness tests. The overall results of the joint tests on both bond returns and volume can
be rationalised within the existing theoretical literature on investor heterogeneity.

A Disagreement and Bond Risk Premia

Models with homoskedastic stochastic discount factors imply that excess returns are not pre-
dictable. This restriction has been widely rejected in the literature. Moreover, even models

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27Examples of price variables removed include: S&P dividend yield, the Federal Funds (FF) rate; 10 year
T-bond; 10 year - FF term spread; Baa - FF default spread; and the dollar-Yen exchange rate. A small
number of discontinued macro series were replaced with appropriate alternatives or dropped.

28one may worry about the inclusion of persistent interest rates as right hand variables, disagreement
about interest rates however is not that persistent (see table 1) compared to, for example, dividend yields.
with heteroskedastic discount factors find it difficult to reproduce how this rejection occurs. We investigate if some of the components of the time-varying stochastic discount factor that generate time-variation in risk premia are linked to the dynamics of the differences in beliefs. As summarized by Hypothesis 1, when disagreement is time-varying and the risk aversion coefficient is $\gamma > 1$ differences in beliefs can become an explicit source of predictability. We consider four sources of disagreement. The first two capture disagreement on real and nominal macro variables (real gdp growth and inflation).

To proxy for disagreement on signals relevant for bonds, we consider directly disagreement on future bond yields. In all cases we control for consensus beliefs on future yields to separate the relative effects. Since disagreement on short-term bond yields may be linked to transitory shocks to $g_t$, while disagreement on long-term bond yields may be linked to long-run persistent components of $g_t$, we construct a proxy of disagreement in both short-term and long-term yields. Disagreement on yields at time $\tau$ capture time-$t$ disagreement about the time-$\tau$ (equilibrium) expected marginal utility of a later cash flow (i.e. time-$t$ disagreement on $E^{*}_{t+\tau}M^{*}_{t+\tau,T}$). Thus, it synthetizes different information than what already contained in the time-$t$ disagreement on the realization of economic variables such as gdp and inflation at time $\tau$. We use this information to capture the extra dimensionality played by signals if these are used by learning agents.

We then proceed by running multivariate forecasting regressions of 1-year excess returns from 2, 5 and 10 year maturity bonds and control for a number of factors proposed by the recent literature on bond return predictability. We run regressions of the following form:

$$r_{x,n}^{(t)} = \text{const}^{(n)} + \sum_{i=1}^{4} \beta_{i}^{(n)} DiB_{t}^{i}(\star) + \sum_{i=1}^{2} \gamma_{i}^{(n)} E_{t}^{i}(\star) + \sum_{i=1}^{3} \phi_{i}^{(n)} Macro_{t}^{i}(\star) + \varepsilon_{t+12}^{(n)},$$

where $DiB_{t}(\star)$ includes the set of disagreement measures as discussed above, $E_{t}(\star)$ is the consensus estimate of either expected inflation or expected RGDP, and $Macro_{t}(\star)$ includes a set of controls as outlined in section D.

[Insert table II here.]

Table II columns (i) – (iii) report that disagreement about the real economy, long and short term interest rates are statistically significant with slope coefficients increasing in

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29 For a recent theoretical motivation of our use of disagreement on future expected prices as a proxy for signal disagreement see Banerjee and Kremer (2010).
magnitude with bond maturity, indicating a larger change in the term premium for longer maturity bonds given a shock to any one factor. In terms of predictable variation, the results are striking: dispersion in beliefs forecasts excess returns with $R^2$'s ranging from 21% on 10-year bonds to 43% on 2-year bonds. For 2-year bonds the t-statistics for the slope coefficient on the real economy (long rates, and short rates) is equal to 3.82 (-3.84, and -5.23) respectively. However, while disagreement about inflation appears significant in specification (i) for 2-year bonds, it does not appear important for expected bond returns elsewhere. The signs of the slope coefficients on disagreement about the real economy and short rates are positive, while the signs of the slope coefficient on disagreement about long rates is negative. This is interesting and may depend on whether long term U.S. Treasury bonds are viewed by the agent as bets or hedge against long-run permanent shocks. To investigate this further, we run regressions on the disagreement on short-term rates per unit of disagreement on long-term rates and find that the slope coefficient is positive. We find that when agents more disagree on short term interest rates, per unit of disagreement on long term rates, expected excess returns are positive.

To clarify the economic significance of the estimated loadings, Table ?? documents the effect on risk premia given a shock to any one factor compared to unconditional mean returns. According to the model expected excess returns are highly variable; for example, 10-year bond returns averaged 4.98% above the risk free 1-year bond return but with a standard deviation of 2.16%. A 1-standard deviation shock to disagreement about the real economy raises expected returns on these bonds by 1.42% while a 1-standard deviation shock to disagreement about short term rates raises expected returns by 2.39%

Columns (iv) and (v) control for information contained in macro expectations and in macro aggregates, respectively. The information contained in the cross section of agents expectations is largely orthogonal to either consensus views or realisations of fundamentals themselves. Specifically, the consensus does not enter significantly alongside disagreement and has virtually no effect on $R^2$'s. Furthermore, while many of the macro factors enter significantly the only loss of statistical significance is for real disagreement for 2-year bonds only, while there is very little increase in $R^2$'s. These results suggest that a sizeable proportion of time-variation in expected returns is due to changes in the level of macroeconomic disagreement and that this result is not subsumed by more traditional risk factors that have been studied recently in the fixed-income literature.
A.1 Robustness

The standard approach in the predictability literature relies on compounding returns and conducting significance tests of explanatory variables using overlapping observations. It is well known however that the use of overlapping returns is not innocuous from a statistical point of view. Compounding returns induces an MA(12) error structure under the null of no predictability which must be corrected for during estimation. In the above we conducted tests of return predictability using a robust GMM generalisation of Hansen and Hodrick (1983) with an 18-lag Newey-West correction. While most researchers agree that risk premia are time-varying the size of the observed predictability is a topical question. A good summary for the arguments against a ‘large’ predictable component in asset returns are given by Ang and Bekaert (2007) in the space of stock returns or by Wei and Wright (2010) in the space of bond returns. Ang and Bekaert find that the evidence of long horizon predictability using Hansen-Hodrick or Newey-West errors disappears once robust correction of heteroskedasticity and autocorrelation is conducted, while Wei and Wright argue that long-horizon predictive regressions using overlapping observations induce serious size distortions even after correction. Both sets of authors advocate use of an alternative inference procedure proposed by Hodrick (1992).

Hodrick (1992) proposes an alternative estimator for the point estimate, \( \hat{\beta} \), in return predictability regressions. The numerator of the estimator \( \hat{\beta}^{(n)} \) is a covariance. Hodrick suggests to project 1-period returns on a lagged summation of right hand variables as opposed to the traditional projection of the future overlapping returns on time \( t \) observations. Covariance stationarity should lead to the same result:

\[
\text{cov}(r_t, r_{t+k}, x_t) = \text{cov}(r_t + \ldots r_{t+k}, x_t) = \sum_{j=0}^{k-1} \text{cov}(r_{t+j}, x_t) = \text{cov}(r_t, \sum_{j=0}^{k-1} x_{t-j}).
\]

(21)

The last term is the numerator of the slope coefficient of a regression of 1-period returns on a lagged summation of right hand variables. Long (\( \hat{\beta}^{(n)} \)) and short (\( \hat{\gamma} \)) horizon regression coefficients are therefore linked by the relation:

\[
\hat{\beta}_k = V_o^{-1} \text{cov}(r_{t,t+k}, x_t) = V_n^{-1} V_n \hat{\gamma}.
\]

(23)

where \( V_o \) is the parameter covariance matrix from the overlapping regression and \( V_n \) is the parameter covariance matrix from the non-overlapping regression. Therefore, a necessary
and sufficient condition to reject the null of no-predictability using overlapping annual horizon returns is that the loading on a 12-period lagged sum of past disagreement measures be different from zero in a monthly forecasting regression. We call this the ‘reverse regression’.

In addition to testing the robustness of the above findings we also investigate the extent to which macroeconomic disagreement is exogenous to time $t$ price innovations. One may worry that heterogeneity in beliefs might be correlated with contemporaneous return volatility so that disagreement would map risk premia associated with some other unobserved fundamental factor. If this were the case, date $t \rightarrow t + h$ returns and date $t$ disagreement would be correlated by construction and no causal interpretation could be attached. Table IV addresses the issue of size and exogeneity simultaneously.

We consider projections of 1-period returns on $h = 3$, 6 and 12 month summations of past disagreement measures (dropping disagreement about inflation) corresponding to implied forecast horizons of 3, 6 and 12 months. In addition to casting predictability tests in terms of reverse regressions we consider lags of $k = 1$, 2 and 3 of disagreement for each forecast horizon. Mindful of the so-called ‘Richardson’s Critique’ who argues that interpretation of such results should take into account correlation in the test statistics, we estimate all the regressions simultaneously in a GMM framework and test the hypothesis that loadings on $DiB^{RGDP}$, $DiB^{LR}$, and $DiB^{SR}$ are jointly different from zero. Considering the loadings on real disagreement, the t-statistics are significant at the 5% level or above for 8 out of 9 of the loadings. For example, consider a 2-month lag of disagreement for horizons $h = 3$, 6 and 12 the t-stats are 2.31, 2.70, and 2.52, respectively. Furthermore, considering a joint restriction for real disagreement we strongly reject the null of no predictability with asymptotic $\chi^2(3)$ values of 8.54 ($p = 0.04$), 8.78 ($p = 0.03$), and 8.66 ($p = 0.03$), respectively. The conclusion here then is that disagreement about real consumption growth contains substantial information on expected bond returns for horizons up to 1-year. Moving to robustness tests of monetary components the results are convincing: the estimated loadings are mainly individually significant at the 1% level, and jointly have $\chi^2(3)$ values that are always above the 5% threshold.

B Disagreement and Volatility

To study the role played by disagreement for the second moments of stock and bond returns we estimate stock / bond volatility and stock bond correlation according to equations 18 -
and run regressions of the type:

\[
Vol/Corr_{t,t+1} = \text{const} + \sum_{i=1}^{4} \beta_i DiB_{i,t}(\star) + \sum_{i=1}^{2} \gamma_i E_{i,t}(\star) + \sum_{i=1}^{3} \phi_i \text{Macro}_{i,t}(\star) + \varepsilon_{t+1},
\]

where as in the previous section \(DiB_t(\star)\) is a set of disagreement measures, \(E_t(\star)\) is consensus estimates, and \(\text{Macro}_t(\star)\) is a set of controls estimated from fundamentals.

[Insert table V here.]

Table V reports estimates for second moment regressions. Considering first bond volatility, in contrast with the return predictability regressions where monetary disagreement was the strongest predicting factor, disagreement about long term rates is now insignificant, and while the loading on disagreement about short rates enters significantly in specifications (ii) - (iv) it does not survive the inclusion of \(\text{Macro}_t(\star)\) and contributes very little in \(R^2\). However, symmetrically, the coefficients on inflation and real disagreement are both positive and highly statistically significant. In terms of \(R^2\) real and inflationary dispersion measures explain 26\% of the time variation in 10-year treasury volatility. Table VI shows that the estimated loadings are also economically meaningful: compared to the sample mean a 1-standard deviation shock to disagreement about inflation or the real economy increases 10 year bond volatility by approximately 10\%. Consensus views enter significantly with negative signs but add just 4\% in \(R^2\) while real and inflationary disagreement remain significant. Finally, controlling for information in macro aggregates neither the level of macro activity or the volatility of inflation are significant. The volatility of consumption however, is positive and significant consistent with a standard single agent Lucas Tree economy where asset volatilities are equal to the volatility of the endowment process. Consistent with heterogeneous agent Lucas Tree economies such as Buraschi and Jiltsov (2006), Xiong and Yan (2010), or Buraschi, Trojani, and Vedolin (2011), belief dispersion results in relative wealth fluctuations which amplify asset volatilities or can even generate heteroscedastic second moments when the endowment process is homoscedastic.

Considering now stock volatility, real and inflationary dispersion measures are again consistently positive and significant in specifications (i) and (iii). However, after controlling for consensus estimates which are negative and significant, and the volatility of consumption growth, which is positive and highly significant, we find that disagreement about inflation is

\footnote{this evidence is in contrast with findings in Schwert (1990) who finds weak evidence to support the hypothesis that macroeconomic volatility can help predict stock and bond volatility}
driven out while disagreement on the real economy survives at the 10% level.

Finally, we find a number of results which contribute to the existing debate of the determinants of stock bond correlation. Firstly, consistent with the results David and Veronesi (2008) we do find a statistical relationship between dispersion in inflation expectations and the second moments of stocks and bonds \(^{31}\). However, like Viceira (2007) we also find that this result is not robust to the inclusion of other predicting factors, in our case the consensus view of inflation, disagreement about short term interest rates, and a macro activity factor \(^{32}\). In light of these findings, in unreported results, we also control for the level of the short term interest rate and find that the significance of expected inflation and its loading are cut in half (\(\phi = 0.16\), t-stat= 2.25) in specification (\(iii\)), that the short rate does indeed drive out dispersion on inflation in specification (\(i\)), but that the significance and economic impact of disagreement about short rates is unaffected by its inclusion.

C Disagreement and Trade

In this section we examine the relationship between investor heterogeneity and trade by running regressions of opening interest (our gauge of trading activity) on belief dispersion. Both ambiguity and differences in beliefs can generated larger expected risk premia. An important distinguishing feature, however, is their implication in terms of trading volumes. Heterogeneous beliefs models imply greater trading between disagreeing agents (see Buraschi and Jiltsov, 2006, and Banjeree, 2008), who try to finance different expected marginal utilities. On the other hand, Illeditsch (2011), de Castro and Chateauneuf (2010), and Chen, Ju, and Miao (2011) show that models with knightian uncertainty and ambiguity generate portfolio inertia.

Since the level of open interest is non-stationary we follow Hong and Yogo (2011) and compute a 12-month geometrically averaged growth rate of bond and stock market open interest. We then run regressions of the type:

\(^{31}\)David and Veronesi (2008) take a structural approach to forecasting second moments by specifying a joint macroeconomic relationship between nominal and real variables within a bayesian learning setting. Investor ‘uncertainty’ about fundamentals as proxied by dispersion in beliefs forecasts second moments with strong statistical significance after controlling for lags of second moments or macro aggregates.

\(^{32}\)Viceira (2007) notes that inflation uncertainty proxied cross-sectional dispersion is driven out as a significant predictor once the short rate is included. Viceira suggests this result is because the level of the short rate is a general proxy for aggregate economic uncertainty.
\[
\frac{OI(t+1) - OI(t)}{OI(t)} = \text{const} + \sum_{i=1}^{4} \beta_i DiB_t(\star) + \sum_{i=1}^{2} \gamma_i E_t(\star) + \sum_{i=1}^{3} \phi_i \text{Macro}_t(\star) + \varepsilon_{t+1}. \tag{24}
\]

Table VII reports the results. Column (i) reports the baseline specification including disagreement about inflation and the real economy on the right hand side. For the growth rate of open interest on treasury note futures and options disagreement on both inflation and the real economy loads positively with high statistical significance (4.38 and 3.62, respectively) with an $R^2$ of 19%, while the results for the S&P are insignificant. Column (ii) introduces disagreement about monetary components which enter with statistically insignificant coefficients for disagreement on long term rates for treasury open interest with an $R^2$ of 8% but again insignificant for the S&P. Moving to column (iii), which includes all disagreement measures as explanatory variables, we find a marginal contribution for disagreement about long term rates for treasury open interest ($DiB^{LR}$ loads positively with a t-stat of 1.86) while the point estimates and t-stats for real and inflation dispersion measures on treasury open interest are largely unaffected. Finally, moving to columns (iv) and (v) we control for consensus expectations and then macro fundamentals. Including consensus views on treasury open interest, has little effect of disagreement about inflation (the point estimate is unaffected and with a t-stat of 3.79) while real disagreement becomes insignificant. However, one notices that real expectations themselves are contribute nothing: both the point estimate and its significance are almost zero. The result is therefore entirely driven by inflation expectations and thus have little theoretically to do with real uncertainty. In terms of predictable variation the addition of consensus view to disagreement raises the $R^2$ just 4%, from 21% to 25%. Considering the inclusion of macro fundamentals in column (v) both disagreement about the real economy and inflation remain highly statistically significant, while the volatility of inflation and real consumption growth also enter significantly and raise the $R^2$ to 35%. In summary, the results on disagreement and trade strongly suggest an economically important and statistically robust positive correlation between investor heterogeneity and trade.

The theoretical origins of disagreement and uncertainty are distinct. The last refers to unknown unknowns and studies the role of the lack of knowledge regarding the reference model on the equilibrium demand at the individual level. The first focuses, instead, on the pricing implications of state-contingent trading among disagreeing agents. Empirically, while the last relies on proxies of dispersion of individual priors (or empirical measures of entropy) at the level of the individual agent, the first relies on the difference in the mean forecasts of different agents. While these concept are different, it is reasonable to argue, however, that
they are conditionally correlated. In a world of certainty, after all, agents would not disagree. For this reason differences in beliefs have been used to proxy for ambiguity. An important contribution to this literature is Ulrich (2010). He considers a single agent economy in which the investor has multiple priors about the inflation process and is ambiguity averse. The agent is assumed to observe the expected change in relative entropy between the worst-case and the approximate model for trend inflation. The observed set of multiple forecasts on trend inflation exposes the investor to inflation ambiguity. In the context of a min-max recursive multiple-prior solution, Ulrich (2010) shows that risk premia can be generated if changes in aggregate ambiguity are correlated with changes in the real value of a nominal bond. He uses the quarterly Survey of Professional Forecasters to obtain a measure of variance across individuals for inflation expectations which is then used to proxy for ambiguity at the individual level and fit the yield curve. He finds that the inflation ambiguity premium is upward sloping and peaked during the mid 1970s and early 1980s.

The empirical results support a positive link between differences in beliefs and bond trading volume. Either ambiguity is less relevant for modelling bond markets or differences in beliefs may not be a good proxy for ambiguity. We prefer the latter interpretation as D’Amico and Orphanides (2008), using individual level forecasts, show that the link between differences in beliefs and ambiguity is not strong.

IV The Information ‘In’, ‘Not In’, and ‘Above’ the Term Structure

We study Hypothesis 2, which states that when $\gamma > 1$ and differences in beliefs are stochastic the closed-form solution of bond prices is not invertible (exponentially quadratic) even when the equivalent homogeneous economy would support an affine solution. This implies that the stochastic discount factor may not be completely spanned by observations on bond prices. This is an interesting implication of this class of models at the light of an important recent stream of the literature that focus on the spanning properties of bond prices.

Consider a $N$-factor affine term structure model admitting as a solution for bond prices $P(X_t, \tau) = \exp(a(\tau) + b(\tau)'X_t)$, where $a(\tau)$ is a scalar function and $b(\tau)$ is an $N$-valued function, expected excess returns to holding a $T$ period bond are equal to $rx_{t,t+dt}^{(T)} = -b(T)'\Sigma\sqrt{S_t}\Lambda_t$, where $\Sigma\sqrt{S_t}$ is the factor loading of the affine process for the factors $X_t$ and $\Lambda_t$ is the price of risk. If the price of risk is ‘completely affine’, i.e. $\Lambda_t = \sqrt{S_t}\lambda_1$, then expected excess returns are proportional to factor variance. Dai and Singleton (2000) denote the admissible subfamily of completely affine models as $A_m(N)$ which are those with
m state variables driving N conditional variances $S_t$. Although convenient, the completely affine specification still imposes important restrictions on the link between conditional first and second moments of bond yields and expected bond returns. Specifically, elements of the state vector $X_t$ that do not affect factor volatility (and hence bond volatility) cannot affect expected returns, thus factor variance and expected returns still go somewhat hand-in-hand. Motivated by this observation, Duffee (2002) extends the completely affine class to a set of ‘essentially’ affine models in which the risk factors in the economy enter the market price of risk directly and not just through their factor volatilities. He suggests to consider $\Lambda_t = \sqrt{S_t} \lambda^0 + \sqrt{S_t^{-1}} X_t$, where $\lambda^0$ is an $n \times n$ matrix of constants and $S_t^{-1}$ is a diagonal matrix such that $[S_t^{-1}]_{ii} = (\alpha_i + \beta_i' X_t)^{-1}$ if $\inf(\alpha_i + \beta_i' X_t) > 0$ and zero otherwise. The additional flexibility of non-zero entries in $S_t^{-1}$ translates into additional state dependent flexibility for the price of risk such that the tight link between risk compensation and factor variance is broken.

A shared characteristic of the $A_m(N)$ subfamily of affine term structure models is that the cross-section of bond yields follows a Markov structure so that all current information regarding future interest rates (and thus expected returns) is summarised in the shape of the term structure today. Linear combinations of date $t$ bond yields thus suffice to characterise date $t$ risk factors through so-called yield curve inversion. Building on this notion Cochrane and Piazzesi (2005) show that the shape of the term structure embeds substantial information that explains the dynamics of bond excess returns.

The Cochrane-Piazzesi return forecasting factor, $CP_t$, is a tent-shaped linear combination of forward rates that embeds all spanned information on 1-year risk premia predicts excess returns on bonds with $R^2$ statistics as high as 43% (in their sample period). More recently evidence presented by Ludvigson and Ng (2009b) and Cooper and Priestley (2009) suggest that yield inversion is not enough to reveal all relevant dynamics for underlying state variables and thus crucial ingredients for term structure models are unspanned by the space of yields. Recent work along these lines is found in Duffee (2011) and Joslin, Priebisch, and Singleton (2009) who highlight the importance of studying hidden factor models, or models

33 Cheridito, Filipovic, and Kimmel (2007) extend even further this class to yield models that are affine under both objective and risk-neutral probability measures without permitting arbitrage opportunities.

34 Specifically, assume $N$ bond yields are measured without error. Then, stacking these yields into the vector $y^N = A^N + B^N X_t$, we can solve for the risk factors through inversion as $X_t = (B^N)^{-1} (y^N - A^N)$ so long as the matrix $B^N$ is non-singular.

35 For a detailed discussion of $CP_t$ we refer the reader to Cochrane and Piazzesi (2005). Briefly, the single factor construction begins with projecting average excess return (across maturity) on a constant plus available forward rates: $\sum_{n=2}^{5} r_x^{(n)}_{t+12} = \gamma_0 + \gamma_1 y_t^{(1)} + \gamma_2 f_t^{(1)} + \gamma_3 f_t^{(2)} + \gamma_4 f_t^{(3)} + \gamma_5 f_t^{(4)} + \bar{\varepsilon}_{t+12} = \gamma_f t + \varepsilon_{t+12}$. Next, the fitted regression coefficients are used as loadings in forming a linear combination of forward rates that serves as a state variable in restricted univariate and multivariate regressions: $r_x^{(n)}_{t+12} = \beta(CP_t + \phi X_t + \varepsilon_{t+12}$. 30
with unspanned macro risk, in which time variation in macro variables orthogonal to the cross-section of yields (and thus absent from date $t$ prices) contains substantial forecasting power for future excess returns on bonds.

**Hypothesis 2** is linked to this literature. It suggests that lack of spanning could be linked to disagreement. In a single agent Gaussian economy, term structure inversion reveals the dynamics of risk factors and thus expected returns. In our specification, however, bond yields are: (a) non invertible (exponentially quadratic); (b) function of a number of signals (which is potentially infinite) that extend the dimension of the state space. Disagreement may be unspanned by the cross-section of prices yet reveal information about expected returns. Thus, a natural question to ask is which component of disagreement relevant for expected returns is revealed by the cross-section of prices (Cochrane-Piazzesi) versus the time-series of prices (Duffee; Joslin, Priebsch, Singleton). Proceeding in two steps, we first define the information set $G_1 \subseteq \sigma(\text{PC}(1-5))$ and compute the unspanned component of $DiB$ which is not explained by the cross-section of bond prices (the first five principal component, as used in Cochrane and Piazzesi (2005)): $UN_{DiB_t} = DiB_t - P_j [DiB_t | G_1]$. After computing $UN_{DiB_t}$, we proceed to test the content of unspanned, i.e. ‘Not-In’, disagreement as follows:

$$rx_{t,t+12}^{(n)} = const + \beta_1^{(n)} UN_{DiB_t}^{\text{INF}} + \beta_2^{(n)} UN_{DiB_t}^{\text{RGDP}} + \beta_3^{(n)} UN_{DiB_t}^{\text{LR}} + \beta_4^{(n)} UN_{DiB_t}^{\text{SR}} + \varepsilon_{t+12}^{(n)}.$$  \tag{25}

Second, we define $G_2 \subseteq [G_1 \cup \sigma(y^{(n)})] \setminus G_1$ where $G_2 \sim \sigma(H_t)$ is the ‘Hidden’ factor filtered from the time-series of prices from a 5-factor Gaussian term structure model studied in Duffee (2011). Then, we estimate the component of disagreement unspanned neither by the cross-section of prices nor by information related to the hidden factor $H_t$. We define $AB_{DiB_t} = UN_{DiB_t} - P_j [UN_{DiB_t} | H_t]$ and test the predictive content of macroeconomic disagreement which is ‘Above’ the yield curve as

$$rx_{t,t+12}^{(n)} = const + \beta_1^{(n)} AB_{DiB_t}^{\text{INF}} + \beta_2^{(n)} AB_{DiB_t}^{\text{RGDP}} + \beta_3^{(n)} AB_{DiB_t}^{\text{LR}} + \beta_4^{(n)} AB_{DiB_t}^{\text{SR}} + \varepsilon_{t+12}^{(n)}.$$  \tag{26}

Table VIII reports a contemporaneous projection of disagreement measures on the first 5 principle components from an eigenvalue decomposition of the unconditional covariance matrix of yields (from the Fama-Bliss data set as in Cochrane-Piazzesi). The results show that a substantial proportion of the time-variation in disagreement about the real economy and short term interest rates is spanned by the yield curve, specifically, $DiB^{\text{REAL}}$ and $DiB^{\text{SR}}$

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36 More specifically, $G_1$ is the sigma algebra (information set) generated by the eigenvalue decomposition of the unconditional covariance matrix of yields, or, alternatively, since there exists a linear mapping between yields and forward rates, $G_1$ is the space spanned by the return forecasting factor $CP$.

37 We thank G. Duffee for providing the data on the hidden factor $H_{\text{Hidden}}$.  


both load significantly on PCs 1-4 with $R^2$'s of 36% and 38% respectively. The first learning point, then, is that time variation in the shape of the forward curve can in part represent heterogeneity in the belief structure of the economy, thus lending economic support to the empirical results of Cochrane and Piazzesi (2005) and the theoretical results of Xiong and Yan (2010). Panel A and B of table X documents the impact on return predictability when one removes the component of $DiB$ spanned by the yield curve. Note first that repeating the return predictability regressions of section A on a different dataset, on a different sample period (pre-2008 crisis), we obtain almost identical results both in terms of point estimates, t-statistics, and $R^2$'s. The second learning point with respect to this section is that more than half of the time-variation in expected returns attributable to disagreement is unspanned and that component is entirely due to monetary disagreement. For example, in moving from spanned to unspanned disagreement the $R^2$ for 2-year bonds goes from 42% to 27%.

Next, we examine the time-series characteristics of unspanned disagreement by running projections of unspanned disagreement on the $Hidden_t$ risk premium component from Duffee (2011). Table IX reports the following multivariate regression:

$$Hidden_t = const + \sum_{i=1}^{4} \beta_i UN_{DiB}^i + \varepsilon_t,$$

The results show that information contained in dispersion in beliefs that is orthogonal to the yield curve explains time variation in the hidden factor with high statistical significance (t-stat: 2.96) with an $R^2$ statistic of 8% 38. The third learning point is that after controlling for information extracted from the time-series of prices there still exists a substantial proportion of information contained in the cross-section of agents expectations that is relevant for bond pricing. Table X documents the predictive power of the above components, as defined in equation 26, for expected bond returns. This particular unspanned component is specific to disagreement regarding the long end of the yield curve and is orthogonal to i) the cross-section of yields; and ii) a risk premium component embedded in the time series of yields. Still, it contains substantial information for future expected bond returns, with t-statistics significant at the 1% level, and $R^2$ between 20% and 22%. Importantly, this component is also economically important for bond risk premia: a 1-standard deviation shock to $AB_{DiB}$ lowers expected excess returns on 5-year bonds by 2.38%.

38This compares with an $R^2$ of 10% in a projection of $H_t$ on the real activity factor (PC1) from Ludvigson and Ng (2009b).
V Concluding Remarks

In summary, what do we learn? Theoretically, expected returns are predictable if i) subjective updating of the underlying state-space generates non-degenerate disagreement; and ii) investors risk share intertemporally ($\gamma > 1$) based on filtered distributions. Empirically we confirm that differences in belief is a prices risk factor, a result absent in multiple agents models with log utility investors or constant beliefs/disagreement. Turning to second moments of return distributions we learn (both theoretically and empirically) that the dynamics belief dispersion are tightly linked to the dynamics of volatility. Furthermore, this result is in contrast to models of differences in belief that do not explicitly consider the learning process; for example, in the very general belief and preference heterogeneity framework of Bhamra and Uppal (2011), differences in belief only has a bite for second moments when optimists are more risk averse pessimists. The results on first and second moments of bond returns depend crucially on risk sharing: optimists insure pessimists against bad states of the world by trading state-contingents claims. The immediate implication is that differences should matter for trade. Testing this hypothesis for a long time series of futures market open interest we find strong support for this intuitive theory. Finally, we propose a linear-quadratic term structure model that incorporates subjective cross-sectional learning about short and long run components of the economy, is capable of a capturing all empirically observed term structure shapes, and importantly contains components related to signal risk that are unspanned by the cross-section of bond prices. Theoretically, this is distinct from heterogeneous investor models with log investors, such as Xiong and Yan (2010), where dynamics are non-affine but whose pricing implications are perfectly spanned.
A Appendix A: Proofs

State-space representation. The economy can be written as a conditionally Gaussian state-space representation with \( X_t = [\log D_t, S_t]' \) and \( \mu_t = [g_t, \varepsilon_t]' \), with Gaussian diffusions following

\[
dX_t = (A_0 + A_1 \mu_t) \, dt + BdW^X_t
\]
\[
d\mu_t = (a_0 + a_1 \mu_t) \, dt + bdW^\mu_t
\]

where

\[
A_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ \phi & (1 - \phi) \end{bmatrix},
\]
\[
a_0 = \begin{bmatrix} \kappa_g \theta_g \\ 0 \end{bmatrix}, \quad a_1 = \begin{bmatrix} -\kappa_g & 0 \\ 0 & 0 \end{bmatrix},
\]

\( B = \text{diag}(\sigma_D, \sigma_S) \) and \( b = \text{diag}(\sigma_g, 1) \). Under technical conditions the subjective solutions to the Kalman-Bucy filters are follow

\[
dm^n_t = (a_0 + a_1 m_t)dt + v_t A'_1 B^{-1} d\hat{W}^{X,n}_t
\]
\[
\dot{v}_t = a_1 v_t + v_t a'_1 + b' - v_t A'_1 (BB')^{-1} A_1 v_t
\]

where \( d\hat{W}^{X}_t = B^{-1}[dX_t - (A_0 + A_1 \mu_t) \, dt] \). Closed form solutions of the matrix Riccati equation for \( v(t) \) are obtained via Radon’s Lemma by linearizing the flow of the differential equation 31.

Lemma 2. Let

\[
\begin{pmatrix} g_{11}(t) & g_{12}(t) \\ g_{21}(t) & g_{22}(t) \end{pmatrix} = \exp\left(t \begin{pmatrix} a_1 & A'(BB'^{-1}A) \\ b'b' & -a'_1 \end{pmatrix}\right).
\]

Then the solution for the posterior variances can be written as:

\[
v(t) = (v(0)g_{12}(t) + g_{22}(t))^{-1}(v(0)g_{11}(t) + g_{21}(t)).
\]

Proof: Let \( v(t) = F(t)^{-1}G(t) \), for two differentiable functions \( F(t) \) and \( G(t) \) such that \( G(t) \) is invertible and \( F(0) = id_{2 \times 2} \). It then follows:

\[
\frac{d}{dt} [F(t)v(t)] - \frac{d}{dt} [F(t)] v(t) = F(t) \frac{d}{dt} v(t),
\]
and
\[ \frac{d}{dt} G(t) - \frac{d}{dt} [F(t)]v(t) = bb' F(t) + G(t) a_1 + (F(t)a_1' + G(t)A'(BB'^{-1}A))v(t). \] (33)

The last ordinary differential equation leads to the following system of linear equations:
\[ \frac{d}{dt} G(t) = F(t)bb' + G(t)a_1', \]
\[ \frac{d}{dt} F(t) = -F(t)a_1 + G(t)A'(BB'^{-1}A). \]

The solution of this system of differential equations is:
\[
\begin{pmatrix} G(t) \\ F(t) \end{pmatrix} = \begin{pmatrix} G(0) \\ F(0) \end{pmatrix} \exp \left(t \begin{pmatrix} a_1 & A'(BB'^{-1}A) \\ bb' & -a_1' \end{pmatrix} \right),
\]
\[
= \begin{pmatrix} v(0)g_{11}(t) + g_{21}(t) \\ v(0)g_{12}(t) + g_{22}(t) \end{pmatrix}.
\]

It then follows that the closed form solution of the matrix Riccati equation 31 is:
\[ v(t) = (v(0)g_{12}(t) + g_{22}(t))^{-1}(v(0)g_{11}(t) + g_{21}(t)). \]

If \( v(t) \) is at the steady-state, then the matrix Riccati becomes an algebraic Riccati equation (ARE) of the form:
\[ 0 = a_1 v + va_1' + bb' - vA'(BB'^{-1}A) v, \] (34)

**Lemma 3.** Let
\[ F = bb' A'(BB'^{-1})A + a_1a_1', \] (35)
then the solution to equation (34) is given by:
\[ v = \left(F^{1/2} - (a_1' + a_1)/2 \right)^{-1} bb'. \] (36)

**Proof:** Analytical solutions to ARE are in general not available, however, under the condition that \( bb' \) is non-singular and \( bb'a_1 \) is a symmetric matrix, Incertis (1981) derives analytical solutions for equation (34). Applying the steps in the paper, we find that the closed-form solution of \( v \) is given by (36), where \( F^{1/2} \) is the square root of the matrix \( F \), which is defined in 35). Computing the \( F \) matrix explicitly for our economy ones finds:
\[ F = \begin{pmatrix} \kappa_g^2 + \sigma_g^2 \left( \frac{1}{\sigma_B^2} + \frac{\phi^2}{\sigma_S^2} \right) \\ \frac{(1-\phi)\sigma_g^2}{\sigma_S^2} \end{pmatrix} \]  

The \( F \) matrix above is positive semi-definite and symmetric up to the transformation \( \text{diag}(1, \frac{1}{\sigma_g^2}) \); thus, a unique positive definite square root exists. Computing this matrix we find the following form for the steady state solution:

\[ v_t^n = \begin{pmatrix} v_g^n \\ v_{g\varepsilon}^n \\ v_{\varepsilon}^n \end{pmatrix} \quad (t) \quad (37) \]

Define the standardized disagreement process as \( \Psi_t = B^{-1}A_1(m_t^a - m_t^b) \). Since \( X_t \) is observable we can rewrite the system in terms of agent \( a \)'s beliefs:

\[
\begin{align*}
    dW_t^{X,b} &= dW_t^{X,a} + B^{-1}A_1(m_t^a - m_t^b)dt \\
    dW_t^{X,b} &= dW_t^{X,a} + \Psi_t dt 
\end{align*}
\]

So that we can re-write the posterior mean dynamics in terms on agent \( a \)'s measure:

\[
\begin{align*}
    dm_t^a &= (a_0 + a_1 m_t^a)dt + v_t^a A_1'B^{-1}dW_t^{X,a} \\
    dm_t^b &= (a_0 + a_1 m_t^b)dt + v_t^b A_1'B^{-1} \left[dW_t^{X,a} + \Psi_t dt \right] \\
    dm_t^b &= (a_0 + a_1 m_t^b + v_t^b A_1'B^{-1}\Psi_t)dt + v_t^b A_1'B^{-1}dW_t^{X,a} 
\end{align*}
\]

Spelling out the filtered dynamics for agent \( a \) explicitly we have

\[
\begin{pmatrix} d\hat{g}_t^a \\ d\hat{x}_t^a \end{pmatrix} = \begin{pmatrix} -g_a\kappa_g + \theta_g\kappa_g \\ 0 \end{pmatrix} dt + \begin{pmatrix} \gamma_g^2 \sigma_B d\hat{W}_t^{D,a} + \left( \frac{\phi\gamma_g^2 + (1-\phi)\gamma_a^2}{\sigma_S} \right) d\hat{W}_t^{S,a} \\ \gamma_a^2 \sigma_B d\hat{W}_t^{D,a} + \left( \frac{\phi\gamma_a^2 + (1-\phi)\gamma_a^2}{\sigma_S} \right) d\hat{W}_t^{S,a} \end{pmatrix}
\]

and for \( b \) we have a disagreement adjustment to the drift. The diffusion process for disagreement is \( d\Psi_t = B^{-1}A_1(dm_t^a - dm_t^b) \):

\[
\begin{align*}
    d\Psi_t &= B^{-1}A_1 \left( (a_1(m_t^a - m_t^b) - v_t^b A_1'B^{-1}\Psi_t)dt + (v_t^a - v_t^b) A_1'B^{-1}dW_t^{X,a} \right), \\
    d\Psi_t &= B^{-1}A_1 \left[ (a_1(A_1'B^{-1}A_1)^{-1}(A_1'B^{-1}A_1)(m_t^a - m_t^b) - v_t^b A_1'B^{-1}\Psi_t) \right] dt + B^{-1}A_1(v_t^a - v_t^b) A_1'B^{-1}dW_t^{X,a}, \\
    d\Psi_t &= B^{-1}A_1 \left[ (a_1(A_1'B^{-1}A_1)^{-1}(A_1'B^{-1}A_1)(m_t^a - m_t^b) - v_t^b A_1'B^{-1}\Psi_t) \right] dt + B^{-1}A_1(v_t^a - v_t^b) A_1'B^{-1}dW_t^{X,a}, \\
\end{align*}
\]
= B^{-1}A_1 \left[ (a_1(A^t_1A_1)^{-1}(A^t_1B))\Psi_t - v^b_t A^r_1 B^{-1} \Psi_t \right] dt + B^{-1}A_1(v^a_t - v^b_t)A^r_1 B^{-1} dW^r_t, \\
= B^{-1}A_1 \left[ (a_1 A^r_1 B - v^b_t A^r_1 B^{-1})\Psi_t dt + (v^a_t - v^b_t)A^r_1 B^{-1} dW^{X,a}_t \right].

Writing out the diffusions for each disagreement explicitly we have

\[ d\psi^g_t = \left[ \left( -\frac{\gamma^b_g}{\sigma^2_D} - \kappa_g \right) \psi^g_t + \left( -\frac{\phi \gamma^b_g(1 - \phi)\gamma^b_{ge}}{\sigma_D \sigma_S} - \phi \kappa_g \sigma_S \sigma_D \right) \psi^S_t \right] dt \\
+ \left[ \frac{\gamma^a_g - \gamma^b_g}{\sigma^2_D} \right] d\hat{W}^{D,a}_t + \left[ \frac{\phi(\gamma^a_g - \gamma^b_g) + (1 - \phi)(\gamma^a_{ge} - \gamma^b_{ge})}{\sigma_D \sigma_S} \right] d\hat{W}^{S,a}_t, \]

and

\[ d\psi^S_t = \left[ \left( -\frac{\gamma^b_g}{\sigma_D \sigma_S} + \phi \left( -\frac{\gamma^b_g}{\sigma_D \sigma_S} - \kappa_g \sigma_S \sigma_D \right) \right) \psi^g_t \\
+ \left( -\frac{(1 - \phi)(\phi \gamma^b_{ge} + (1 - \phi)\gamma^b_{e})}{\sigma^2_S} \right) \psi^S_t + \phi \left( -\frac{\phi \gamma^b_g(1 - \phi)\gamma^b_{ge}}{\sigma^2_S} - \phi \kappa_g \right) \psi^S_t \right] dt \\
+ \left[ \frac{\phi(\gamma^a_g - \gamma^b_g) + (1 - \phi)(\gamma^a_{ge} - \gamma^b_{ge})}{\sigma_D \sigma_S} \right] d\hat{W}^{D,a}_t \\
+ \left[ \frac{(\phi^2(\gamma^a_g - \gamma^b_g)^2 + 2(1 - \phi)(\gamma^a_{ge} - \gamma^b_{ge}) + (1 - \phi)^2(\gamma^a_{e} - \gamma^b_{e})}{\sigma^2_S} \right] d\hat{W}^{S,a}_t. \]

**Lemma 4** (Radon-Nikodym derivative). The diffusion process \( d\eta_t/\eta_t \) follows, under the probability measure of agent \( n = \{a,b\} \),

\[ d\eta_t/\eta_t = - \left[ \Psi^t d\hat{W}^{X,a}_t \right]_{(1 \times 2)} (2 \times 1) \]

where \( \Psi_t = B^{-1}A_1(m^a_t - m^b_t) \) and \( m^a_t := E^n(\mu_t|\Sigma_t) \).

Proof. We proceed first by deriving the dynamics of the Radon-Nikodym derivative \( \eta_t = \frac{M^a_t}{M^b_t} \). Since markets are complete, there exists a unique stochastic discount factor for each agent. Absence of arbitrage implies the existence of a stochastic discount factor \( M^a_t \), with \( dM^a_t = -\left( \frac{d^a_t}{d^b_t} dt + \kappa^a_t dW^{X,a}_t \right) \), where \( \kappa^a_t \) is a vector process in \( \mathbb{R}^2 \) containing the price of risk for dividend and signal shocks. From these dynamics it follows that

\[ \eta(t) \eta(0) = \frac{-\int_0^t (r(s) + \frac{a^a(s)^2}{2}) ds - \int_0^t \kappa^a(s) dW^{X,a}(s)}{e^{-\int_0^t (r(s) + \frac{a^b(s)^2}{2}) ds - \int_0^t \kappa^b(s) dW^{X,b}(s)}} \]

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The dynamics
\[ d\eta_t/\eta_t = - (\kappa_t^a - \kappa_t^b) \, d\hat{W}_t^{X,a} \tag{38} \]
follow immediately. Notice, the Radon-Nikodym derivative defined as \( \eta_t = \frac{M_t}{M_0} \) is only an exponential martingale under the representative agent’s (and agent \( a \)’s) measures. In order to find a valid change of measure from the perspective of agents \( b \) one simply re-defines the Radon-Nikodym derivative as \( \eta_t = \frac{M_t^b}{M_0^b} \). Next, for agents measures \( n = a, b \), the state vector follows \( dX_t/X_t = (A_0 + A_1 m_t^a) \, dt + Bd\hat{W}_t^{X,n} \) where subjective uncertainties are related by the disagreement process, \( \Psi_t := [\psi_t^g, \psi_t^S] \) with \( \Psi_t = B^{-1} A_1 (m_t^a - m_t^b) \).

\[ dW_t^{X,b} = \hat{\dot{W}}_t^{X,a} + \Psi_t \, dt. \]

where explicitly the elements of the disagreement process are

\[ d\hat{W}_t^{D,b} = d\hat{W}_t^{D,a} + \left( \frac{\hat{g}_t^a - \hat{g}_t^b}{\sigma_D} \right) \, dt = d\hat{W}_t^{D,a} + \psi_t^g \, dt \]
\[ d\hat{W}_t^{S,b} = d\hat{W}_t^{S,a} + \phi \left( \frac{\hat{g}_t^a - \hat{g}_t^b}{\sigma_S} \right) \, dt + (1 - \phi) \left( \frac{\hat{\epsilon}_t^a - \hat{\epsilon}_t^b}{\sigma_S} \right) \, dt \]
\[ = d\hat{W}_t^{S,a} + \phi \frac{\sigma_D}{\sigma_S} \psi_t^g \, dt + (1 - \phi) \psi_t^S \, dt \]

where \( \psi_t^g \) is disagreement on the growth rate of the economy, \( \psi_t^\xi \) is disagreement on the noise component of the signal, and \( \psi_t^S \) is disagreement on the growth rate of the signal (being a weighted sum of disagreement of economic growth and the noise in the economy). Consider an asset \( \zeta \) with price \( P_\zeta \). Let us write his generic diffusion under the beliefs of agent \( b \) as

\[ \frac{dP_\zeta(t)}{P_\zeta(t)} = \mu_\zeta^b(t) \, dt + \Sigma_{\zeta,D} d\hat{W}_t^{D,b} + \Sigma_{\zeta,S} d\hat{W}_t^{S,b}. \]

From the perspective of agent \( a \) the diffusion follows

\[ \frac{dP_\zeta(t)}{P_\zeta(t)} = \mu_\zeta^a(t) \, dt + \Sigma_{\zeta,D} (d\hat{W}_t^{D,a} + \psi_t^g \, dt) + \Sigma_{\zeta,S} \left( d\hat{W}_t^{S,a} + \psi_t^S \, dt \right). \]
which implies that the difference in expected return between agents is

\[ \hat{\mu}_a^a(t) - \hat{\mu}_b^b(t) = -\Sigma_\zeta \Psi_t. \]

where \( \Sigma_\zeta = [\Sigma_{\zeta, D}, \Sigma_{\zeta, S}] \). Since in equilibrium expected excess returns are compensation for risk, i.e. \( \hat{\mu}_n^n(t) - r_f(t) = \Sigma_\zeta \kappa^n_i \), then

\[ \Sigma_\zeta (\kappa^a_i - \kappa^b_i) = \Sigma_\zeta \psi_t \]

and combining this with equation 38 we obtain

\[ d\eta_t/\eta_t = -\Psi'_t d\hat{W}_t^X,a \]

**Theorem 2 (Equilibrium Prices of Risk).** For agents \( n = a, b \) the perceived prices of growth risk and signal risk are

\[ \begin{align*}
\kappa_{g,a}(t) &= \gamma \sigma_D + \omega_b(t) \psi^g_t, \\
\kappa_{g,b}(t) &= \gamma \sigma_D - \omega_a(t) \psi^g_t, \\
\kappa_{S,a}(t) &= \omega_b(t) \psi^S_t, \\
\kappa_{S,b}(t) &= -\omega_a(t) \psi^S_t,
\end{align*} \]

for \( i = 1, 2 \). Equilibrium consumption volatilities are

\[ \sigma_c^n(t) = \frac{\kappa^n(t)}{\gamma}, \]

where \( \sigma_c^n(t) = [\sigma_{g,c}^n(t), \sigma_{S,c}^n(t)] \) is a vector containing factor loadings for innovations to the consumption processes. And, the heterogeneous Vasicek short rate is

\[ r(t) = \rho + \gamma \beta' (\omega_a \hat{g}_a^a + \omega_b \hat{g}_b^b) - \frac{1}{2} \gamma (1 + \gamma) \sigma_D^2 - \frac{1}{2} \frac{(\gamma - 1)}{\gamma} \omega_a \omega_b \Psi'_t \Psi_t \]

where \( \omega_i(t) = c_i^t/D_t \) is investor \( i \)'s consumption share.

Proof. Applying Itô’s Lemma to individual first order conditions \( u'_n = \alpha_n M_n, \ n = a, b \):

\[ dM^n_t = - \left( r^n_t dt + \kappa^n(t) d\hat{W}_t^X,n \right), \]

\[ = \frac{u''_n}{u'_n} d\eta_n + \frac{1}{2} \frac{u'''_n}{u'_n} (d\eta_n, d\eta_n), \]

\[ = \frac{u''_n}{u'_n} t^n_c \left( \mu_c^n(t) dt + \sigma_c^n(t) d\hat{W}_t^X, a \right) + \frac{1}{2} \frac{u'''_n}{u'_n} (c^n_t)^2 \sigma_c^n(t) \cdot \sigma_c^n(t) dt. \]
Equating drift and diffusion coefficients, it follows, for $n = a, b$:

$$
\sigma_c^n(t) = \left(-\frac{c_n u_n''}{u_n'}\right)^{-1} \kappa^n(t) = \frac{\kappa^n(t)}{\gamma},
$$

$$
r_t = \frac{u_n''}{u_n'} \left( c_t^n \mu_t^{c_n} + \frac{1}{2} u_n'' \left( c_t^n \right)^2 \sigma_c^n(t) \cdot \sigma_c^n(t) \right),
$$

Since diffusion terms are identical across perceived subjective dynamics, this fixes the volatility of individual optimal consumptions. Next, differentiating the market clearing condition $c_t^a + c_t^b = D_t$ under agent $a$’s beliefs

$$
dD_t = D_t(\hat{g}_t^a dt + \sigma_D d\hat{W}_t^{D,a}),
$$

$$
= c_t^a \left( \mu_t^{c_a} dt + \sigma_{X,c}^a(t) d\hat{W}_t^{X,a} \right) + c_t^b \left( \mu_t^{c_b} dt + \sigma_{X,c}^b(t) d\hat{W}_t^{X,b} \right),
$$

$$
= \left( c_t^a \mu_t^{c_a} + c_t^b \mu_t^{c_b} + c_t^b \sigma_{X,c}^b(t) \Psi_t^X \right) dt + \left( c_t^a \sigma_{X,c}^a(t) + c_t^b \sigma_{X,c}^b(t) \right) d\hat{W}_t^{X,a},
$$

which is a set of restrictions tying drifts and diffusion components across the dividend and consumption processes. Writing explicitly, these are

$$
c_t^a \sigma_{g,c}^a(t) + c_t^b \sigma_{g,c}^b(t) = D_t \sigma_D, \tag{41}
$$

$$
c_t^a \sigma_{S,c}^a(t) + c_t^b \sigma_{S,c}^b(t) = 0, \tag{42}
$$

and

$$
c_t^a \mu_t^{c_a} + c_t^b \mu_t^{c_b} = D_t \hat{g}_t^a - c_t^b \left[ \sigma_{g,c}^b \sigma_{S,c}^b \right] \left[ \begin{array}{c} \psi_t^g \\ \psi_t^S \end{array} \right]
$$

$$
= \left[ c_t^a \sigma_{g,c}^a(t) + c_t^b \sigma_{g,c}^b(t) \right] \hat{g}_t^a - c_t^b \left[ \sigma_{g,c}^b \sigma_{S,c}^b \right] \left[ \begin{array}{c} \frac{\tilde{g}_t^g - \hat{g}_t^g}{\sigma_D} \\ \psi_t^S \end{array} \right]
$$

$$
= \left( \frac{c_t^a \sigma_{g,c}^a(t)}{\sigma_D} \hat{g}_t^a + \frac{c_t^b \sigma_{g,c}^b(t)}{\sigma_D} \hat{g}_t^b \right) - c_t^b \sigma_{S,c}^b \psi_t^S.
$$

One finds the subjective prices of risk for dividend shocks by summing up the first element
\[ D_t \sigma_D = c_t^a \kappa_{g,a}(t) \frac{\gamma}{\gamma} + c_t^b \kappa_{g,b}(t) \frac{\gamma}{\gamma}. \]

Then combining with the consistency relation \( \kappa_{D,a}(t) - \kappa_{D,b}(t) = \psi_g(t) \) and the preferences parameters for power utility we obtain:

\[
\kappa_{g,a}(t) = \gamma \sigma_D + \omega_b(t) \psi_g(t), \\
\kappa_{g,b}(t) = \gamma \sigma_D - \omega_a(t) \psi_g(t),
\]

where \( \omega_i(t) = c_t^i / D_t \) is investor \( i \)'s consumption share. To find the prices of signal risk sum up the second elements of over agents and impose diffusion consistency condition 41, and price of risk consistency \( \kappa_{S,a}(t) - \kappa_{S,b}(t) = \psi^S_t \)

\[
\kappa_{S,a}(t) = \omega_b(t) \psi^S_t, \\
\kappa_{S,b}(t) = -\omega_a(t) \psi^S_t,
\]

To obtain the heterogeneous Vasicek short rate re-write equation 39 as

\[
r_t = \rho + \left( -\frac{\gamma}{c_t^a} \right) \hat{\mu}_c^a c_t^a - \frac{1}{2} \gamma (1 + \gamma) \sigma_c^a \cdot \sigma_c^a.
\]

Then summing up over agents and imposing market clearing, \( c_t^a + c_t^b = D_t \) we have

\[
(r - \rho) \frac{D_t}{\gamma} = c_t^a \hat{\mu}_t^a c_t^a + c_t^b \hat{\mu}_t^b c_t^b - \frac{1}{2} (1 + \gamma) \left( c_t^a \sigma_c^a \cdot \sigma_c^a + c_t^b \sigma_c^b \cdot \sigma_c^b \right)
\]

\[
r = \frac{\gamma}{D_t} \left( c_t^a \hat{\mu}_t^a c_t^a + c_t^b \hat{\mu}_t^b c_t^b \right) - \frac{1}{2} \frac{(1 + \gamma)}{\gamma^2} \frac{\gamma}{D_t} \left( c_t^a \kappa_t^a(t) \cdot \kappa_t^a(t) + c_t^b \kappa_t^b(t) \cdot \kappa_t^b(t) \right)
\]

Moreover, imposing the drift restriction from above, and substituting in equation ??:

\[
r = \rho + \gamma \left( \omega_a \frac{\sigma_a^g(t)}{\sigma_D} \hat{g}_t^a + \omega_b \frac{\sigma_b^g(t)}{\sigma_D} \hat{g}_t^b \right) \right) - \gamma \omega_b \sigma_b^S \psi^S_t - \frac{1}{2} \frac{(1 + \gamma)}{\gamma} \left( \omega_a \kappa_t^a(t) \cdot \kappa_t^a(t) + \omega_b \cdot \kappa_t^b(t) \cdot \kappa_t^b(t) \right)
\]

\[
r = \rho + \gamma \left( \omega_a \hat{g}_t^a + \omega_b \hat{g}_t^b \right) + \omega_a \omega_b (\psi^S_t)^2 - \gamma \omega_b \sigma_b^S \psi^S_t - \frac{1}{2} \frac{(1 + \gamma)}{\gamma} \left( \omega_a \kappa_t^a(t) \cdot \kappa_t^a(t) + \omega_b \cdot \kappa_t^b(t) \cdot \kappa_t^b(t) \right)
\]

where \( \omega_n = c_t^n / D \) is the consumption share of agent \( n = a, b \). Noticing that the prices of risk
We see that the standard precautionary savings term is contained within the consumption share weighted subjective prices of risk. However, there are now additional terms to the standard result which are quadratic in total disagreement risk. To see this explicitly substitute in \( \psi \) and after some algebra one obtains:

\[
\begin{aligned}
\kappa_{g,a}(t) &= \gamma \sigma_D + \omega_a(t) \psi_t^g, \\
\kappa_{g,b}(t) &= \gamma \sigma_D - \omega_a(t) \psi_t^g, \\
\kappa_{S,a}(t) &= \omega_b(t) \psi_t^S, \\
\kappa_{S,b}(t) &= -\omega_a(t) \psi_t^S.
\end{aligned}
\]

This concludes the proof.

**Lemma 5.** The joint Laplace transform of \( D(t) \) and \( \lambda(t) \) under the belief of agent \( a \) is given by:

\[
K_{m,a}(m_D^a, t, T; \epsilon) = \int_0^T \int_0^T \epsilon(t)(\lambda(t))^{\epsilon} d\lambda(t) d\epsilon(t)
\]

where

\[
K_{m,a}(m_D^a, \tau, \epsilon) = \exp(A(\epsilon, \tau) m_D^a + B(\epsilon, \tau))
\]

with explicit solutions:

\[
A(\epsilon, \tau) = -\frac{\epsilon(\epsilon - \kappa^a_D \tau - 1)}{\kappa^a_g}
\]

\[
B(\epsilon, \tau) = \frac{1}{2} \epsilon(\epsilon - 1) \sigma_D \tau - \left( \theta_g + \frac{\epsilon \gamma^a_D}{\kappa^a_g} \right) (e^{\kappa^a_D \tau} + \tau)
\]

\[
- \frac{1}{\kappa^a_g} \left( \frac{\gamma^a_g}{\sigma_D} \right)^2 + \left( \frac{\phi \gamma^a_g + (1 - \phi) \gamma^a_S}{\sigma_S} \right)^2 \left( \frac{3}{2} e^{-\kappa^a_D \tau} + \kappa^a_g \tau \right)
\]

with \( \tau = T - t \) and

\[
K_{\psi,g,\psi,S_1} = e^{C(\tau) + D(\tau) \Psi_t + E(\tau) \Psi_t^i + F(\tau) \Psi_t^j}(i) \Psi_t^j(j)
\]

for a vector of functions \( C, D, E, \) and \( F \), where \( \Psi_t, (\Psi_t^i \Psi_t^j), \) and \( (\Psi_t^i \Psi_t^j) \) are linear, quadratic, and covariance terms in disagreement, respectively.
PROOF: We need to compute the following moment generating function

\[ F_{(D, \eta, m_D^a, \psi_S)} = E_{D, \eta, m_D^a, \psi_S, (D(t)^\gamma \eta(u))^\chi}. \]

From Feynman-Kac this function satisfies the following partial differential equation

\[ 0 = \mathcal{D} F_{(D, \eta, m_D^a, \psi_S)} + \frac{\partial F}{\partial t} (D, \eta, m_D^a, \psi_S, t, u; \epsilon, \chi) \]

with initial condition \( F_{(m_D^a, \eta, \psi_S)} = D(t)^\gamma \eta(t)^\chi \), and where \( \mathcal{D} \) is the differential operator of the multivariate process \( (D, \eta, m_D^a, \psi_S, t, u; \epsilon, \chi) \) under agent \( a \)'s beliefs. Applying the operator we have

\[
\begin{align*}
0 &= \frac{\partial F}{\partial D} D g^a \delta t - \frac{\partial F}{\partial m_D^a} \delta g^a + \frac{\partial F}{\partial \eta} \left( (1 - \phi) g^a - \frac{\partial D}{\partial \eta} \right) + \frac{\partial F}{\partial \psi_S} \left( 2 \frac{\partial F}{\partial \psi_S} \right) + \frac{\partial^2 F}{\partial \psi_S^2} \left( \frac{\partial \psi_S}{\partial \sigma_D^a} \right) + \frac{\partial^2 F}{\partial \psi_S \partial \sigma_D^a} \left( \frac{\partial \psi_S}{\partial \sigma_D^a} \right) + \frac{\partial^2 F}{\partial \psi_S \partial \sigma_S} \left( \frac{\partial \psi_S}{\partial \sigma_S} \right) \right.
\end{align*}
\]
The solution to this partial differential equation takes the following form:

\[ F(D, m_D^a, \psi_g, \psi_S, t; \epsilon, \chi) = D^r \eta F_{m_D^a} (m_D^a, t, T; \epsilon) F_{\psi_g, \psi_S} (\psi_g, \psi_S, t, T; \epsilon, \chi) = D^r \eta F (m_D^a, \psi_g, \psi_S, t, T; \epsilon, \chi) \]

Substituting in above we obtain

\[
\begin{align*}
0 &= \partial F / \partial m_D^a \left( \dot{\phi}_1^a - \dot{\theta}_g \right) + \partial F / \partial \psi_g \left[ \left( - \frac{\gamma_g^b}{\sigma_D} - \kappa_g \right) \psi_t^g + \left( \frac{\phi \gamma_g^b + (1 - \phi) \gamma_{ge}^b}{\sigma_D \sigma_S} - \phi \kappa_g \right) \right] \\
&\quad + \partial F / \partial \psi_S \left[ \left( \frac{\gamma_{ge}^b}{\sigma_D \sigma_S} + \phi \left( - \frac{\gamma_g^b}{\sigma_D \sigma_S} - \kappa_g \right) \right) \psi_t^g + \left( \frac{\phi \gamma_g^b + (1 - \phi) \gamma_{ge}^b}{\sigma_D \sigma_S} - \kappa_g \right) \psi_t^S + \phi \left( \frac{\phi \gamma_g^b + (1 - \phi) \gamma_{ge}^b}{\sigma_D \sigma_S} - \kappa_g \right) \right] \\
&\quad + \frac{1}{2} \epsilon (\epsilon - 1) F^2 \sigma_D^2 + \frac{1}{2} \chi (\chi - 1) F \left[ \psi_t^g + \psi_t^S \right] + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left[ \left( \frac{\gamma_g^a - \gamma_{ge}^b}{\sigma_D^2} \right)^2 + \left( \frac{\phi \gamma_g^a - \gamma_{ge}^b}{\sigma_D \sigma_S} \right)^2 \right] \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left[ \left( \frac{\gamma_g^a - \gamma_{ge}^b + (1 - \phi)(\gamma_{ge}^a - \gamma_{ge}^b)}{\sigma_D} \right)^2 + \left( \frac{\phi \gamma_g^a - \gamma_{ge}^b + (1 - \phi)(\gamma_{ge}^a - \gamma_{ge}^b)}{\sigma_D \sigma_S} \right)^2 \right] \\
&\quad + \hat{F} \chi \psi_g \left[ \gamma_g^a - \phi \gamma_{ge}^b \right] + \partial F / \partial m_D^a \left( \gamma_g^a + \phi \gamma_{ge}^b \right) \psi_t^g + \partial F / \partial \psi_g \left[ \gamma_g^a + (1 - \phi)(\gamma_{ge}^a - \gamma_{ge}^b) \right] \\
&\quad - \partial F / \partial m_D^a \chi \left[ \gamma_g^a \psi_t^g + (\phi \gamma_g^a + (1 - \phi)(\gamma_{ge}^a)) \psi_t^g \right] - \partial F / \partial \psi_g \chi \left[ \gamma_g^a - \phi \gamma_{ge}^b \psi_t^g + (\phi \gamma_g^a + (1 - \phi)(\gamma_{ge}^a) \psi_t^g \right] \\
&\quad - \partial F / \partial \psi_S \left[ \phi \gamma_{ge}^b + (1 - \phi)(\gamma_{ge}^a - \gamma_{ge}^b) \psi_t^g \right] + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{\phi \gamma_g^a + (1 - \phi)(\gamma_{ge}^a - \gamma_{ge}^b)}{\sigma_D \sigma_S} \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \gamma_g^a - \phi \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \\
&\quad + \frac{1}{2} \partial \hat{F} / \partial \psi_g \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g + \frac{1}{2} \partial \hat{F} / \partial \psi_S \left( \phi \gamma_g^a - \gamma_{ge}^b \right) \psi_t^g \right] + \partial \hat{F} / \partial t \\
\end{align*}
\]

First, we factor out the terms that do not involve \( \eta \) or \( \Psi \) (in red) and solve for \( m_D^a \) by direct integration with the following ansatz:

\[
F_{m_D^a} (m_D^a, \tau, \epsilon) = \exp(A(\epsilon, \tau)m_D^a + B(\epsilon, \tau))
\]
with explicit solutions

\[ A(\epsilon, \tau) = \frac{-\epsilon (e^{\kappa^a} - 1)}{\kappa^g}, \]

\[ B(\epsilon, \tau) = \frac{1}{2} \frac{1}{\epsilon} \left( \theta_D + \frac{\epsilon \gamma_D}{\kappa^a} \right) \left( e^{\kappa^a} + \tau \right) \]

\[ - \frac{1}{\kappa^g} \left( \left( \frac{\gamma_D^a}{\sigma_D} \right)^2 + \left( \frac{\phi \gamma_D^a + (1 - \phi) \gamma_S^a}{\sigma_S} \right)^2 \right) \left( \frac{3}{2} e^{\kappa^a} + \kappa^a \right), \]

where \( \tau = T - t \). The remaining solution method for this partial differential equation draws upon the insights of Dumas, Kurshev, and Uppal (2010). In the interests of space this appendix describes the steps that are taking in obtaining the reported closed form result, while a detailed technical appendix that fills in the details is available on the authors websites.

The remaining steps are as follows, guessing the form:

\[ F_{\psi_g, \psi_S} = e^{C(\tau) + D(\tau) \Psi_t + E(\tau) \Psi_S + F(\tau) \Psi_t(i) \Psi_t(j)}, \]  

(48)

for a vector of functions \( C, D, E, \) and \( F \), where \( \Psi_t, (\Psi_t \Psi_t), \) and \( \Psi_t(i) \Psi_t(j), \) one can reduce the residual PDE to a system of ODE’s. An application of Radons lemma and direct integration yield explicit results for the remaining functions. Moving to the bond pricing function, by definition, risk-less zero coupon bond prices are given by:

\[ B(t, T) = \frac{1}{\xi^a(t)} E_t^a \left( e^{-\rho(T-t) \xi^a(T)} \right). \]

Using the expression for \( \xi^a(t) \), we get:

\[ B(t, T) = E_t^1 \left( e^{-\rho(T-t)} \left( \frac{D(T)}{D(t)} \right)^{-\gamma} \left( \frac{1 + \lambda(T)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} \right)^\gamma \right). \]  

(49)

Let

\[ G(t, T, x; \psi_g, \psi_S) \equiv \int_0^\infty \left( \frac{1 + \lambda(t)^{1/\gamma}}{1 + \lambda(t)^{1/\gamma}} \right) \gamma \left[ \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\lambda(T)}{\lambda(t)} \right)^{-i\chi} F_{\psi_g, \psi_S}(\psi_g, \psi_S, t, T; -\gamma, i\chi) d\chi \right] d\lambda(T). \]

By Fourier inversion, it then follows:

\[ B(t, T) = e^{-\rho(T-t)} F_{m_D^a}(m_D^a, t, T; -\gamma) G(t, T, -\gamma; \psi_g, \psi_S). \]

This concludes the proof.
In order to construct a constant 1-year maturity disagreement measure for each forecaster, we take a weighted average of the short and long term forecasts from the BlueChip survey. Figure 1 gives a visual explanation to the construction of the constant maturity proxy. Let \( j \) be the month of the year, so that \( j = 1 \) for January and \( j = 1, 2, \ldots, 12 \). A constant maturity disagreement is formed taking as weight \( (1 - \frac{j}{12}) \), for the short term disagreement (the remaining forecast for the same year), and \( \frac{j}{12} \), for the long-term disagreement (the forecast for the following year). As an example, in April each year the approximate 1-year difference in belief is constructed from \( 9/12 \)th’s of the short term forecast and \( 3/12 \)th’s of the long term forecast.

![Figure 1 - Constant Maturity Disagreement](image-url)
Figure 2 – Short Term Forecast Respondent Numbers

Figure 3 – Long Term Forecast Respondent Numbers
Figure 4 – Forecast Respondent Numbers: Survey of Professional Forecasters
Figure 5 – Disagreement

Figure 6 – Disagreement on Inflation
Figure 7 – Disagreement on Real Growth

Figure 8 – Disagreement on Interest Rates
## Appendix C: Tables

### Table I – Summary Statistics: Disagreement

<table>
<thead>
<tr>
<th></th>
<th>Di(B^{RGDP})</th>
<th>Di(B^{IP})</th>
<th>Di(B^{CPI})</th>
<th>Di(B^{GDPI})</th>
<th>Di(B^{SR})</th>
<th>Di(B^{LR})</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A:</strong></td>
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<td></td>
<td></td>
<td></td>
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<td></td>
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<tr>
<td>Mean</td>
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<td>0.24</td>
<td>0.23</td>
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<td>0.05</td>
<td>0.06</td>
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<td>0.28</td>
</tr>
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<td>Kurt</td>
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<td>4.47</td>
<td>15.18</td>
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<td>3.42</td>
<td>2.88</td>
</tr>
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<td>AC(1)</td>
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<td>0.86</td>
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<tr>
<td><strong>Panel B:</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Di(B^{RGDP})</td>
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<td>0.79</td>
<td>0.45</td>
<td>0.42</td>
<td>0.19</td>
<td>0.42</td>
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<td>Di(B^{IP})</td>
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<td>1.00</td>
<td>0.53</td>
<td>0.58</td>
<td>0.16</td>
<td>0.29</td>
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<tr>
<td>Di(B^{CPI})</td>
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<td>0.53</td>
<td>1.00</td>
<td>0.69</td>
<td>0.01</td>
<td>−0.07</td>
</tr>
<tr>
<td>Di(B^{GDPI})</td>
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<td>0.58</td>
<td>0.69</td>
<td>1.00</td>
<td>0.20</td>
<td>0.01</td>
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<tr>
<td>Di(B^{SR})</td>
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<tr>
<td>Di(B^{LR})</td>
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<td>0.01</td>
<td>0.38</td>
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### Table II – Return Predictability Regressions

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<td>(-0.25)</td>
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<td>$D_iB_t^{REAL}$</td>
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<td>(3.82)</td>
<td>(1.57)</td>
</tr>
<tr>
<td>$D_iB_t^{LR}$</td>
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<td>-0.42</td>
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<td></td>
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<td>(5.23)</td>
<td>(3.47)</td>
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<td>(0.78)</td>
<td>(0.73)</td>
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<tr>
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<td>$\rho_t^{INF,REAL}$</td>
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<td>(2.57)</td>
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<td>$R^2$</td>
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### Table III – Economic Significance: Expected Returns

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<tr>
<th>Maturity(n)</th>
<th>$E(r_x^{(n)})$</th>
<th>$\sigma(E(r_x^{(n)}))$</th>
<th>$\sigma(D_iB_t^{INF})$</th>
<th>$\sigma(D_iB_t^{REAL})$</th>
<th>$\sigma(D_iB_t^{LR})$</th>
<th>$\sigma(D_iB_t^{SR})$</th>
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<td>2yr</td>
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<td>5yr</td>
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## Table IV – Reverse Regressions

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<th>horizon(h)</th>
<th>$DiB_1^{REAL}$</th>
<th>$DiB_1^{LR}$</th>
<th>$DiB_1^{SR}$</th>
<th>$DiB_1^{REAL}$</th>
<th>$DiB_1^{LR}$</th>
<th>$DiB_1^{SR}$</th>
<th>$DiB_1^{REAL}$</th>
<th>$DiB_1^{LR}$</th>
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<td>-0.22</td>
<td>0.19</td>
<td>0.12</td>
<td>-0.21</td>
<td>0.16</td>
<td>0.13</td>
<td>-0.17</td>
<td>0.15</td>
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<td></td>
<td>(1.63)</td>
<td>(-3.16)</td>
<td>(3.08)</td>
<td>(2.31)</td>
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<td>0.13</td>
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### Table V – Volatility and Correlation Regressions

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### Table VI – Economic Significance: Volatility

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<th>$\sigma^{DiB_t^{LR}}$</th>
<th>$\sigma^{DiB_t^{SR}}$</th>
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Table VII – Open Interest Regressions

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### Table VIII – Spanned Disagreement

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\[ R^2 \]

0.03 0.36 0.14 0.38

### Table IX – Unspanned Disagreement and the Hidden Factor

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Table X – Return Predictability, Spanned, Unspanned, and Above Disagreement

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References

Acemoglu, D., V. Chernozhukov, and M. Yildiz, 2008, *Fragility of asymptotic agreement under Bayesian learning*. (Massachusetts Institute of Technology, Dept. of Economics).


Schwert, G.W., 1990, Why does stock market volatility change over time?, .


