A Multidimensional Dependent Jump-Diffusion Approach for Pricing Barrier Reverse Convertibles

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Abstract

We propose a simple multidimensional jump-diffusion model for pricing single- and multi-asset barrier reverse convertibles. Our model ensures the possibility of sudden drops in prices for underlying assets, fits empirical properties of asset returns, and incorporates dependencies between the diffusions, the number of jumps, and the jump-size. Our results indicate that models without jumps tend to underestimate the risk associated with these investments when the maturity is short. This feature is likely to have consecutive effects for studies on the overpricing of exotic multi-asset financial derivatives.

JEL-Classifications: C13, G13
Keywords: Jump-Diffusion Model, Multi Barrier Reverse Convertibles, Structured Products, Exotic Derivatives, Knock-in Derivatives

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1 Introduction

Continuous time stochastic processes have been extensively used for pricing contingent claims for decades. Examples include linear diffusions of Black and Scholes (1973) and Vasicek (1977); bivariate diffusions proposed by Hull and White (1987), Melino and Turnbull (1990), and Heston (1993); nonlinear diffusions with deterministic volatility functions proposed by Derman and Kani (1994) and Dupire (1994); and many others. An important disadvantage of these specifications is that they neglect abrupt drops in prices of underlying assets, such that there is no possibility that a given barrier level is crossed by surprise. This feature has important effects on pricing barrier-dependent short-term securities with conditional capital protection, such as barrier reverse convertibles (BRCs hereafter) that belong to very popular structured products in Switzerland. Since investors have experienced substantial losses in BRCs during the financial crisis of 2007, there is evidence that the associated risk of these investments has been underestimated – see Lindauer and Seiz (2008) and Wallmeier and Diethelm (2009). Regarding significant pricing errors between theoretical and observed prices for structured products in favour of the issuing institution within the most cases – reported by Chen and Sears (1990), Baubonis et al. (1993), Burth et al. (2001), Wilkens et al. (2003), Stoimenov and Wilkens (2005), and particularly for BRCs reported by Lindauer and Seiz (2008), Wallmeier and Dietheim (2009), and others –, the choice of a model that is superior in fitting empirical properties of asset returns can be of greater relevance. On the other hand, an appropriate model has to meet the requirements of simplicity and analytical tractability, especially when pricing multi-asset derivatives which still remains challenging; see e.g. Leoni and Schoutens (2007).

This paper proposes a simple pricing framework for single- and multi-asset BRCs using a multidimensional jump-diffusion related to Merton (1976), and the structural approach of Zhou (1997, 2001), while incorporating dependencies between the underlying stocks. In particular, we construct dependent Poisson processes by using dependent binary random variables; see e.g. Teugels (1990). In accordance with Huang (1985) and the empirical evidence for jumps in asset returns documented by Bates (1996), Andersen et al. (2002), Eraker et al. (2003), Huang and Tauchen (2005), Broadie et al. (2007), and many others, our specification allows large, sudden changes for stock prices at the time scale of interest. Furthermore, our model ensures heavy tails in the distribution of the underlying stock returns. While diffusion-based stochastic volatility models are also able to reproduce well-known skew/smile patterns of implied volatilities, see e.g. Heston (1993), our specification also captures the non-flat profile of its’ term structure – see e.g. Cont and Fonseca (2002) for a survey – and proposes explanations in terms of market anticipations regarding stock market crashes; see e.g. Bates (1991, 2000). As our results show, neglecting the possibility sudden
drops can result in underestimation of yield spreads for BRCs when the maturity is short, even if jumps are rare. On the other hand, once the barrier has been hit, the presence of jumps enables the stock to reach a higher level again, resulting in lower spreads than for pure diffusion-based models. As a consequence, pricing error will tend to increase within a short-term time horizon and it will tend to decrease for larger maturities when jumps are incorporated. This effect will likely to be larger in case of multi-asset BRCs, since jumps tend to occur more often, whereas jumps become less important in the long run. Our model does not rely on implied volatilities, and thus, positive definiteness of covariance-matrices can be ensured. This is essential when the dependence structure – to which structured products are sensitive, see e.g. Engle (2002) – of underlying assets is determined. By using dependent diffusions and dependent jumps with dependent jump-sizes, we are able to model both, long- and short-term dependence structures of the underlying stocks. We provide a detailed description for the implementation of the model that incorporates parameter estimation and Monte Carlo simulations.

The structure of this paper is as follows. Section 2 builds the parameter estimation followed by the pricing framework for single BRCs which is extended in Section 3 for multi-asset BRCs that have multiple underlying stocks. Empirical implications of the model are illustrated in Section 4. Section 5 concludes.

2 Single Barrier Reverse Convertibles

Let \( S_t \) denote the stock price of the associated firm at time \( t \). For simplification, the notation neglects theoretical differences between random variables, stochastic processes, and observed samples. In particular, the representation of compound Poisson processes is simplified.

\textbf{Assumption 1:} The stock price \( S_t \) at time \( t \) satisfies

\[
\frac{dS_t}{S_t} = (\mu - \lambda \cdot E(Z - 1)) dt + \sigma dW_t + (Z - 1)dY_t,
\]

where \( \mu \) and \( \sigma \) are real constants, \( E(\cdot) \) is the expectation operator, \( W_t \) is a Wiener process, \( Y_t \) is a Poisson process with intensity \( \lambda \), and \( Z \) is the jump-amplitude with \( \ln Z \sim N(\alpha, \gamma^2) \). The initial value \( S_0 \) is fixed and positive and we postulate that \( W_t, Y_t, \) and \( Z \) are mutually independent.

The diffusion component of (1) describes the ordinary changes in the stock price, whereas the jump component reflects sudden, extraordinary changes due to surprising information that has relatively large effects. The distribution of the jump amplitude from (1) implies
The parameter $\mu$ represents the instantaneous return on the stock, $\sigma^2$ denotes the instantaneous variance of the return in absence of jumps and $\lambda$ represents the average number of jumps per unit time. Furthermore $\alpha$ and $\gamma^2$ are the mean and the variance of the jump-size of the return. The expression $(Z - 1) dY_i$ symbolizes a compound Poisson process denoted as

$$(Z - 1) dY_i = d\left(\sum_{i=1}^{Y_i} Z_i - 1\right),$$

where $Z_i$ are i.i.d. jump-sizes. The following proposition provides a tractable way to estimating the parameters from a finite sample of stock returns.

**Proposition 1:** Under Assumption 1, the probability density function $\varphi(\theta, \ln S_t)$ of $\{\ln S_t\}_{t \geq 0}$ can be expressed as

$$\ln \varphi(\theta, \ln S_t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \ln g(\theta, \Delta \ln S_i),$$

where $\theta = (\mu, \sigma^2, \lambda, \alpha, \gamma^2)$, $\Delta \ln S_i = \ln S_t - \ln S_{t-1}$, $t_0 = 0$, $t_n = \tau$, and the time change is $\Delta t = t_i - t_{i-1}$ for $i = 1, \ldots, n$. Here, $g(\theta, \Delta \ln S_i)$ is defined as

$$g(\theta, \Delta \ln S_i) = \sum_{k=0}^{1} p(k) \cdot f(\theta, \Delta \ln S_i, k),$$

where

$$p(k) = (1 - \lambda \cdot \Delta t)^{1-k} \cdot (\lambda \cdot \Delta t)^k,$$

$$f(\theta, \Delta \ln S_i, k) = \frac{1}{\sqrt{2\pi v(k)}} \cdot \exp\left(-\frac{(\Delta \ln S_i - m(k))^2}{2 \cdot v(k)}\right),$$

$$m(k) = (\mu - \sigma^2/2 - \lambda \cdot E(Z - 1)) \cdot \Delta t + k \cdot \alpha,$$

$$v(k) = \sigma^2 \cdot \Delta t + k \cdot \gamma^2,$$

with $k \in \{0, 1\}$ and $E(Z - 1)$ is given in (2).

**Proof:** Appendix.

The functions $f(\theta, \Delta \ln S_i, k)$, $m(k)$, and $s(k)$ represent the probability density, the mean, and the variance of each $\ln S_i$ with $i = 1, \ldots, n$ conditioning on $\ln S_{t_{i-1}}$ and $k \in \{0, 1\}$ number of jumps between $t_i$ and $t_{i-1}$. Since the
probability of additional jumps can be neglected, \( \Delta Y_t \) degenerates to a Bernoulli distributed random variable with mean \( \lambda \cdot \Delta t \), i.e. \( \Delta Y_t \sim \text{Be}(\lambda \cdot \Delta t) \), and probability mass function \( p(k) \).

Based on a finite observed sample of size \( n \) and equidistant points of time, the resulting likelihood estimate \( \hat{\theta} \) of \( \theta \) can be computed numerically from

\[
\hat{\theta} = \arg \max_{\theta} \left\{ \sum_{i=1}^{n} \ln g(\theta; \Delta \ln S_i) \right\}.
\]

The following assumption concerns the pricing framework within this context.

**Assumption 2**: The jump component of the stock price is non-systematic. In addition to stocks, a risk-free asset with rate \( r \), which is constant over time, is also traded. Arbitrage opportunities are excluded.

According to Merton (1976), there is no possibility to eliminate the jump risk within the risk-neutral setting of Black and Scholes (1973). If the jump component represents a non-systematic risk, a portfolio, which eliminates the uncertainty of the diffusion, must earn the risk-free rate \( r \). However, this can be extended by introducing utility functions – an aspect which is ignored here in order to simplify matters. The absence of arbitrage opportunities ensures the existence of equivalent martingale measures for pricing derivatives such as single BRCs given in

**Definition 1**: A single barrier reverse convertible linked to \( S_t \) is a derivative security on \( S_t \) issued at time \( 0 \) with payoff \( V_T \) at maturity \( T \geq 0 \) given by

\[
V_T = c \cdot D + L \cdot S_T \cdot 1_A + D \cdot (1 - 1_A),
\]

where \( A \) is an event given by

\[
A = \{ S_T < D/L \} \cap \{ \inf \{ S_t \mid 0 \leq t \leq T \} \leq B \},
\]

and \( c \), \( D \), \( L \), and \( B \) are positive constants.

A buyer of a single BRC receives \( L \) identical stocks at maturity \( T \) if \( S_T \) is below \( D/L \) and if a barrier event has occurred; i.e. the stock has been traded at or below the barrier level \( B \) during the lifetime of the contract. Otherwise, the nominal value \( D \) is paid. In both cases, the buyer also receives a coupon payment \( c \cdot D \) on the nominal value which is guaranteed by the issuer. Generally, the contract is specified so that \( B < D/L \) and the coupon ranges commonly far above the risk free rate.
Using the standard risk-neutral pricing approach under Assumptions 1 and 2, the price \( V_0 \) of the a single BRC at issuing time is

\[
V_0 = \exp(-r \cdot T) \cdot c \cdot D + \exp(-r \cdot T) \cdot E_0^Q(L \cdot S_T \cdot 1_A + D \cdot (1 - 1_A)) \tag{3}
\]

where \( E_0^Q(\ldots) \) denotes the expectation at time 0 under the risk-adjusted martingale measure \( Q \) conditioning on information currently available. Under this measure, the stock price satisfies

\[
dS_t / S_t = (r - \lambda \cdot E(Z - 1)) \, dt + \sigma \, dW_t + (Z - 1) \, dY_t.
\]

The expression \( E_0^Q(1_A) \) derived from (3) represents the risk-neutral probability at time 0 that \( L \) stocks are paid at maturity to a buyer of the contract. Due to path-dependency, a closed form solution of \( V_0 \) is not attainable under Assumption 1. Thus, for pricing single BRCs, a Monte Carlo approach is necessary which is provided by

**Lemma 1:** Under Assumptions 1 and 2, the price \( V_0 \) of a single barrier reverse convertible at the issuing time 0 can be expressed as

\[
V_0 = \exp(-r \cdot T) \cdot c \cdot D + \exp(-r \cdot T) \times \lim_{n \to \infty} E_0^Q(L \cdot S_T^* \cdot 1_{A^*} + D \cdot (1 - 1_{A^*})),
\]

where \( t_0 = 0, \ t_n = T, \ \Delta t = t_i - t_{i-1} = T/n, \) and

\[
\begin{align*}
S_{t_0}^* &= S_0, \\
\Delta \ln S_{t_i}^* &= \ln S_{t_i}^* - \ln S_{t_{i-1}}^* = \Delta s_i + \Delta z_i + \Delta y_i, \\
A^* &= \{S_{t_i}^* < D / L \} \cap \{\min(S_{t_i}^* \mid i = 0, \ldots, n) \leq B\}.
\end{align*}
\]

Here \( s_i, z_i, \) and \( y_i \) are mutually and serially independent random variables with distributions

\[
\begin{align*}
s_i &\sim N((r - \sigma^2 / 2 - \lambda \cdot E(Z - 1)) \cdot \Delta t, \sigma^2 \cdot \Delta t), \\
z_i &\sim N(\alpha, \gamma^2), \\
y_i &\sim \text{Be}(\lambda \cdot \Delta t),
\end{align*}
\]

for all \( i = 1, \ldots, n, \) and \( E(Z - 1) \) is given in (2).

**Proof:** Appendix.
Lemma 1 holds because the probability of additional jumps within the sampling points $t_i$ and $t_{i-1}$ can be neglected. Based on Lemma 1, a single BRC can be valued at the issuing time by generating simulated samples of $\{S_i^t\}_{i=0,1,\ldots,n}$ using the current observation $S_0$ of the stock as the starting value, whereby the payoff of the contract is determined for each sample according to Definition 1. The price of the contract is then simply the discounted mean of the resulting payoffs.

It should be noted that Lemma 1 is not directly applicable if the price is not calculated at the issuing time. If the stock has hit the barrier previously, then a single BRC degenerates to a portfolio consisting of the coupon payment, $L$ stocks, and $-L$ call options, each with strike price $D/L$. The prices of the options are then calculated using Merton (1976).

3 Multi Barrier Reverse Convertibles

Multi BRCs are based on two or more underlyings and consequently on several barriers and numbers of shares delivered in case of a barrier event. This involves multidimensional normal and Poisson distributions. For the former, we only have to specify covariances in addition to marginal distributions. However, Poisson processes are not necessary independent if the pairwise correlations are zero.

We generalize the previous pricing framework for multiple underlyings as follows. Let $S_t$ denote a vector containing the stock prices of $J$ associated firms at time $t$, i.e. $S_t = (S_{1,t},\ldots,S_{J,t})'$. We use binary expansions as follows. We write each integer $k$ with $1 \leq k \leq 2^J$ as

$$k = 1 + \sum_{j=1}^{J} k_j \cdot 2^{j-1},$$

where $k_j \in \{0,1\}$ for all $j = 1,\ldots,J$. This expansion induces a one-to-one correspondence between $k$ and the vector $(k_1,\ldots,k_J)$. Note that for $k=1$ we have $k_1 = \ldots = k_J = 0$ and for $k=2^J$ we obtain $k_1 = \ldots = k_J = 1$. We then express the intensity $h$ of $Y_t$ in terms of ordinary moments per unit time, representing intensities of marginal and common jumps, i.e.

$$h = (h_1, h_2, \ldots, h_{2^J}),$$

where

$$h_k = E(Y_{1,t}^{k_1} \ldots Y_{J,t}^{k_J}) \text{ with } k = 1,\ldots,2^J,$$
and the vector \((k_3,...,k_d)\) linked to \(k\) is obtained by means of (4). Note that \(h_1=1\) because in this case \(k_1=...=k_d=0\). Each of the corresponding marginal intensities \(\lambda_1,...,\lambda_d\) of \(Y_{1,t},...,Y_{d,t}\) can be obtained from \(h\) by setting exactly one component of \((k_1,...,k_d)\) equal to 1. The total number of the parameters linked to a multidimensional Poisson process is \(2^d-1\).

The considerations above enable us to generalize Assumption 1 for the multidimensional framework as follows.

**Assumption 3:** The vector of stock prices \(S_t\) at time \(t\) satisfies

\[
d\ln S_t = (\mu - \Sigma/2 - \lambda \cdot E(Z - 1)) dt + \Sigma^{0.5} dW_t + \ln Z dY_t,
\]

where \(t\) is a vector of length \(J\) representing the time, \(\Sigma\) is a positive definite matrix, and \(\mu\) and \(\lambda\) are given by

\[
\mu = \text{diag}(\mu_1,...,\mu_J) \quad \text{and} \quad \lambda = \text{diag}(\lambda_1,...,\lambda_J).
\]

Moreover, \(W_t\) is vector of \(J\) independent Wiener processes, \(Y_t\) contains \(J\) dependent Poisson processes with intensity vector \(h\) given in (5), and \(Z\) is a random matrix satisfying

\[
Z = \text{diag}(Z_1,...,Z_J),
\]

where

\[
(\ln Z_1,...,\ln Z_J)' \sim N(\alpha, \Gamma),
\]

\[
\alpha = (\alpha_1,...,\alpha_J)',
\]

and \(\Gamma\) is a positive definite matrix. The matrices \(\Sigma\) and \(\Gamma\) are of the form

\[
\Sigma = \begin{pmatrix}
\sigma_1^2 & \sigma_{1,2} & \cdots & \sigma_{1,J} \\
\sigma_{1,2} & \sigma_2^2 & \cdots & \sigma_{2,J} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{1,J} & \sigma_{2,J} & \cdots & \sigma_J^2
\end{pmatrix}
\quad \text{and} \quad
\Gamma = \begin{pmatrix}
\gamma_1^2 & \gamma_{1,2} & \cdots & \gamma_{1,J} \\
\gamma_{1,2} & \gamma_2^2 & \cdots & \gamma_{2,J} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{1,J} & \gamma_{2,J} & \cdots & \gamma_J^2
\end{pmatrix}.
\]

The initial value \(S_0\) is fixed and positive and \(W_t, Y_t,\) and \(Z\) are mutually independent.

The diffusion components of the stocks \(i\) and \(j\) are independent if and only if \(\sigma_{i,j} = 0\), where \(\sigma_{i,j}\) represents the instantaneous covariance of the corresponding returns in absence of jumps. The matrix \(\Gamma\) contains the covariance \(\gamma_{i,j}\) of the jump-sizes linked to the stocks \(i\) and \(j\) which are independent if and only if \(\gamma_{i,j} = 0\). The distribution of \(Z\) implies
The resulting parameter vector \( \theta \) contains \( J^2 + 3 \cdot J + 2^J - 1 \) parameters that can be estimated using

**Proposition 2:** Under Assumption 3, the probability density function \( \varphi(\theta, \ln S_t) \) of \( \{\ln S_t\}_{0 < t < \tau} \) can be expressed as

\[
\ln \varphi(\theta, \ln S_t) = \lim_{n \to \infty} \sum_{i=1}^{n} \ln g(\theta, \Delta \ln S_i),
\]

where \( \Delta \ln S_i = \ln S_{n_i} - \ln S_{n_i-1} \), \( t_0 = 0 \), \( t_n = \tau \), and the time change is \( \Delta t = t_i - t_{i-1} \) for \( i = 1, \ldots, n \). Here, \( g(\theta, \Delta \ln S_i) \) is defined as

\[
g(\theta, \Delta \ln S_i) = \sum_{k=1}^{2^J} p_k \cdot f(\theta, \Delta \ln S_i, k).
\]

Furthermore, \( p_k \) with \( k = 1, \ldots, 2^J \) is the \( k \)-th component of \( p = (p_1, \ldots, p_{2^J})' \) given by

\[
p = \begin{pmatrix} 1 & -1 \end{pmatrix}^{\otimes J} \cdot \mathbf{h}_M
\]

and

\[
\mathbf{h}_M = \text{diag}(1, \Delta t, \ldots, \Delta t) \cdot \mathbf{h},
\]

where \( \otimes \) is the Kronecker-product and \( \mathbf{h} \) is given in (5). Moreover

\[
f(\theta, \Delta \ln S_i, k) = \frac{1}{(2 \cdot \pi)^{J/2} \cdot \sqrt{\det V}} \cdot \exp \left( -\frac{(\Delta \ln S_i - \mathbf{M})' V^{-1} (\Delta \ln S_i - \mathbf{M})}{2} \right),
\]

where

\[
\mathbf{M} = (\mathbf{u} - \Sigma/2 - \lambda \cdot E(Z - 1)) \cdot \Delta t + \text{diag}(k_1, \ldots, k_J) \cdot \mathbf{a},
\]

\[
\mathbf{V} = \Sigma \cdot \Delta t + \begin{pmatrix} k_1 \cdot \gamma_1^1 & k_1 \cdot k_2 \cdot \gamma_{1,2} & \ldots & k_1 \cdot k_J \cdot \gamma_{1,J} \\
 k_2 \cdot k_1 \cdot \gamma_{1,2} & k_1 \cdot \gamma_2^2 & \ldots & k_2 \cdot k_J \cdot \gamma_{2,J} \\
 \vdots & \vdots & \ddots & \vdots \\
 k_J \cdot k_1 \cdot \gamma_{1,J} & k_J \cdot k_2 \cdot \gamma_{2,J} & \ldots & k_J \cdot \gamma_{2,J} \cdot \gamma_{2,J}^2 \\
\end{pmatrix}.
\]

The vector \( (k_1, \ldots, k_J) \) linked to \( k \) is obtained by means of (4), and \( E(Z - 1) \) is given in (6).
Proof: Appendix.

Here $\Delta \mathbf{Y}_t$ degenerates to a vector of dependent Bernoulli distributed random variables with ordinary moments derived from (8), i.e. $\Delta \mathbf{Y}_t \sim \text{Be}(\mathbf{h}, \omega)$. Next, we define multi BRCs by

**Definition 2:** A multi barrier reverse convertible linked to $\mathbf{S}_t$ is a derivative security on $\mathbf{S}_t$ issued at time 0 with payoff $V_T$ at maturity $T \geq 0$ given by

$$V_T = c \cdot D + L(\mathbf{S}_T) \cdot 1_A + D \cdot (1 - 1_A),$$

where

$$L(\mathbf{S}_T) = \min\{L_1 \cdot S_{1,T}, \ldots, L_J \cdot S_{J,T}\},$$

$$A = \left( \bigcup_j (S_{j,T} < D / L_j) \right) \cap \left( \bigcup_j (\inf\{S_{j,t} \mid 0 \leq t \leq T\} \leq B_j) \right),$$

and $c$, $D$, $L_1, \ldots, L_J$, and $B_1, \ldots, B_J$ are positive constants.

The constants $L_1, \ldots, L_J$ and $B_1, \ldots, B_J$ and represent the numbers of stocks delivered at maturity and the barrier levels for each underlying. An important feature of this investment is the worst-of-characteristic captured by $L(\cdot)$, i.e. a buyer of a multi BRC receives the worst performing stocks at maturity, if there is a barrier event and if at least one stock $j$ is below $D / L_j$. Otherwise, the buyer receives the nominal value $D$. Hence, the underlying which touches the barrier has not to be the same as the ultimately delivered one. As for single BRCs, there is also a coupon payment. Pricing multi BRCs can be performed by means of

**Lemma 2:** Under Assumptions 2 and 3, the price $V_0$ of a multi barrier reverse convertible at the issuing time 0 can be expressed as

$$V_0 = \exp(-r \cdot T) \cdot c \cdot D + \exp(-r \cdot T) \times \lim_{n \to \infty} E_0^G (L(\mathbf{S}_0^*) \cdot 1_{A^*} + D \cdot (1 - 1_{A^*})), $$

where $t_0 = 0$, $t_n = T$, $\Delta t = t_i - t_{i-1} = T / n$, and

$$\mathbf{S}_0^* = \mathbf{S}_0,$$

$$\Delta \ln \mathbf{S}_0^* = \ln \mathbf{S}_0^* \cdot \ln \mathbf{S}_{-1}^* = \mathbf{s} + \text{diag}(z_{1,i}, \ldots, z_{J,i}) \cdot \mathbf{y}_i,$$

$$L(\mathbf{S}_0^*) = \min\{L_1 \cdot S_{1,t_1}^*, \ldots, L_J \cdot S_{J,t_1}^*\}.$$
Here \( s_i, z_{ij}, \ldots, z_{jj} \), and \( y_i \) are mutually and serially independent random variables with distributions

\[
\begin{align*}
\mathbf{s}_i &\sim N((\text{diag}(r,...,r)) - \Sigma/2 - \lambda \cdot E(\mathbf{Z} - 1)) \cdot \Delta t, \Sigma \cdot \Delta t), \\
(z_{ij}, \ldots, z_{jj}) &\sim N(\mathbf{a}, \Gamma), \\
y_i &\sim Bc(\mathbf{h}_m),
\end{align*}
\]

for all \( i = 1, \ldots, n \), where \( \mathbf{h}_m \) is given in (8) and \( E(\mathbf{Z} - 1) \) is given in (6).

The proof of Lemma 2 is straightforward and hence omitted. A multi BRC can be valued at the issuing time by generating simulated samples of \( \{S_n^i\}_{i=0,1,\ldots} \) for a given \( \theta \) using the current observations \( S_0 \) of the underlying stocks as starting values, similar to Lemma 1. Dependent binary variables can be generated using conditional distributions.

### 4 Empirical Implications

The influence of jumps on pricing single BRCs is shown by changing the values of \( \sigma^2 \), \( \lambda \), and \( \gamma^2 \) while keeping the instantaneous variance \( d \text{var} \ln S_t/dt \) of the return on the stock constant. By applying the law of total variance on compound Poisson distributions, we obtain

\[
d \text{var} \ln S_t = (\sigma^2 + \lambda \cdot \gamma^2 + \lambda \cdot \alpha^2) dt.
\]

Using different combinations – capturing pure diffusions and jump-processes varying from low to high frequencies with large to small moves – we calculate spreads of single BRCs for different maturities which are shown in Exhibit 1.

As the results indicate, yield spreads generated by diffusion processes tend to zero when the maturity is short, since there is no possibility of sudden drops in the stock price within a short period of time. As a consequence, diffusion processes will be likely to underestimate the risk premium associated with BRCs when the maturity is short. However, this is not the case when jumps are incorporated – even when the mean number of jumps is relatively small. As the average number of jumps grows and the variance of the jump-size decreases, yield spreads first increase, but after reaching a certain point they become similar to that of diffusions, since jumps cause a
smaller movement. Simultaneously, as the maturity becomes larger, jump-diffusions allow the stock price to regenerate in case of a barrier event; resulting in lower yield spreads than for pure diffusions in general. Thus, if jumps are incorporated, pricing error will tend to increase when the maturity period is short-term and it will tend to decrease for longer-term maturities. With regard to the worst-of-characteristic of multi BRCs, this effect is likely to be more intensive for multi-asset investments, since jumps tend to occur more often.

**Exhibit 1: Term Structure of Yield Spreads**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>$\sigma^2$</th>
<th>$\lambda$</th>
<th>$\gamma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>.</td>
<td>0</td>
<td>0.1</td>
<td>3</td>
</tr>
<tr>
<td>+</td>
<td>0</td>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>$x$</td>
<td>0</td>
<td>10</td>
<td>0.03</td>
</tr>
</tbody>
</table>

**Explanation:** Exhibit 1 shows the term structure of the yield spreads associated with zero coupon single BRCs for different values of the parameter vector as listed above. The number of observations used for simulated samples ranges from 1 to 250, depending on the given maturity, and the number of replications is 200,000 respectively. Furthermore $D = 100, L = 1, B = 40, S_0 = 90, r = 0.05, \alpha = 0,$ and $\Delta t = 1/250.$

By applying the law of total cumulance of Brillinger (1969), the third central moment $\mu_3(\ln S_t)$ of $\ln S_t$ results in

$$\mu_3(\ln S_t) = (\lambda \cdot \alpha^3 + 3 \cdot \lambda \cdot \alpha \cdot \gamma^2) \cdot t,$$

which indicates the possibility of skewness for the distribution of stock returns. Similarly, we obtain for the fourth central moment $\mu_4(\ln S_t)$ of $\ln S_t$

$$\mu_4(\ln S_t) = 3 \cdot \lambda \cdot \gamma^4 \cdot t + 3 \cdot (\sigma^2 \cdot t + \lambda \cdot \gamma^2 \cdot t)^2.$$
if $\alpha = 0$, indicating heavier tails for the distribution of stock returns. However both skewness and kurtosis approach to that of Gaussian values with the time scale over which returns are computed, such that jumps become less important in the long-run.

5 Conclusion

This paper proposes a simple pricing framework for single and multi-asset BRCs that allows for large, sudden changes in prices at the time scale of interest, fits empirical properties of asset returns, and incorporates dependencies between the underlying stocks. As our results show, neglecting the possibility of sudden drops can result in underestimation of yield spreads for BRCs when the maturity is short, while resulting in lower spreads than for pure diffusion-based models when the maturity becomes larger. As a consequence, pricing error will tend to increase within a short-term time horizon and it will tend to decrease for longer-term maturities when jumps are incorporated. This effect is likely to be larger in case of multi-asset BRCs, since jumps tend to occur more often. However, jumps become less important in the long-run.

Appendix

Proof of Proposition 1

Applying Ito’s lemma on (1) yields

$$d\ln S_t = \left(\mu - \sigma^2/2 - \lambda \cdot E(Z - 1)\right) dt + \sigma dW_t + \ln Z dY_t.$$  \hspace{1cm} (9)

Given $k = 0$ number of jumps between $t_i$ and $t_{i-1}$, i.e. $\Delta Y_i = 0$, the distribution of $\ln S_{t_i}$ conditioning on $\ln S_{t_{i-1}}$ is

$$\ln S_{t_i} | \ln S_{t_{i-1}}, k = 0 \sim N(m(0),v(0)),$$

resulting in the probability density $f(\theta, \Delta \ln S_{t_i}, 0)$. If $\Delta Y_i = 1$, we obtain

$$\ln S_{t_i} | \ln S_{t_{i-1}}, k = 1 \sim N(m(1),v(1)),$$

and the density is $f(\theta, \Delta \ln S_{t_i}, 1)$. Since $P(\Delta Y_i = 0) = 1 - \lambda \cdot \Delta t + o(\Delta t)$, $P(\Delta Y_i = 1) = \lambda \cdot \Delta t + o(\Delta t)$, and $P(\Delta Y_i > 1) = o(\Delta t)$, we have

$$g(\theta, \Delta \ln S_{t_i}) = p(0) \cdot f(\theta, \Delta \ln S_{t_i}, 0) + p(1) \cdot f(\theta, \Delta \ln S_{t_i}, 1) + o(\Delta t),$$

from which we obtain Proposition 1. ■
Proof of Lemma 1

Define

\[ X_T = L \cdot S_T \cdot 1_A + D \cdot (1 - 1_A), \]
\[ X_T^\gamma = L \cdot S_T^\gamma \cdot 1_A^\gamma + D \cdot (1 - 1_A^\gamma). \]

Let \( \xi \) and \( \xi^\gamma \) be stopping times satisfying

\[ \xi = \inf(t | S_t \leq B, t \geq 0), \]
\[ \xi^\gamma = \inf(t_i | S^\gamma_{t_i} \leq B, i = 1,2,\ldots). \]

Then

\[ E^Q_0(X_T) = E^Q_0(X_T \cdot 1_{(\xi \leq T)}) + E^Q_0(X_T \cdot 1_{(\xi > T)}) \]
\[ = E^Q_0(X_T | \xi \leq T) \cdot Q(\xi \leq T) + E^Q_0(X_T | \xi > T) \cdot (1 - Q(\xi \leq T)) \]

and

\[ E^Q_0(X_T^\gamma) = E^Q_0(X_T^\gamma | \xi^\gamma \leq T) \cdot Q(\xi^\gamma \leq T) + E^Q_0(X_T^\gamma | \xi^\gamma > T) \cdot (1 - Q(\xi^\gamma \leq T)). \]

Furthermore, it follows from (9) that under the risk-neutral probability measure \( Q \)

\[ \Delta \ln S_{t_i} = s_i + \sum_{j=0}^{a_i} z_{i,j}, \]

where \( a_i \) is a random variable following the Poisson distribution with mean \( \lambda \cdot \Delta t \) and

\[ z_{i,j} \sim N(\alpha, \gamma^2), \]

such that

\[ Q(\xi \leq T) = Q(\xi^\gamma \leq T) + o(\Delta t) \]

and

\[ E^Q_0(X_T | \xi \leq T) = E^Q_0(X_T^\gamma | \xi^\gamma \leq T) + o(1) \]

yields

\[ E^Q_0(X_T) = E^Q_0(X_T^\gamma) + o(\Delta t). \]
Proof of Proposition 2

We need to show that \( p \) given in (7) is the joint probability mass function of \( \Delta Y_t = Y_t - Y_{t-1} \). Due to Proposition 1, the components \( p_k \) of \( p \) can be written as

\[
p_k = P \left( \bigcap_{j=1}^{J} (\Delta Y_{j,k} = k_j) \right) = E \left( \prod_{j=1}^{J} \Delta Y_{j,k} \cdot (1 - \Delta Y_{j,k})^{1-k_j} \right),
\]

where \( k = 1, \ldots, 2^J \), such that

\[
p = E \left( \frac{1 - \Delta Y_{j,k}}{\Delta Y_{j,k}} \right) \otimes \left( \frac{1 - \Delta Y_{j-1,k}}{\Delta Y_{j-1,k}} \right) \otimes \cdots \otimes \left( \frac{1 - \Delta Y_{1,k}}{\Delta Y_{1,k}} \right).
\]

Using

\[
\begin{pmatrix}
1 - \Delta Y_{j,k} \\
\Delta Y_{j,k}
\end{pmatrix} = \begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
\Delta Y_{j,k}
\end{pmatrix}
\]

yields

\[
p = \begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix}^{\otimes J} \cdot E \left( \frac{1}{\Delta Y_{j,k}} \otimes \frac{1}{\Delta Y_{j-1,k}} \otimes \cdots \otimes \frac{1}{\Delta Y_{1,k}} \right).
\]

And due to

\[
h = E \left( \frac{1}{Y_{j,1}} \otimes \frac{1}{Y_{j-1,1}} \otimes \cdots \otimes \frac{1}{Y_{1,1}} \right),
\]

Proposition 2 is established. ■

References


