Robust Investment Decisions
and The Value of Waiting to Invest

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Abstract. When investors and firms doubt that their model is a precise description of their decision problem, they are said to be ambiguity averse. We solve a firm’s investment problem in the case where investors and the firm are ambiguity averse about the growth rate of the project’s value. We use a robust method in which it is taken into account that the model may be wrong and, hence, alternative models are considered. In this setting, we provide explicit solutions both in a simple model, where only the project value is stochastic, as well as a model in which also the investment cost is stochastic. With ambiguity aversion the firm invests earlier than in the standard model without ambiguity. Furthermore, in contrast to standard models, ambiguity aversion implies the threshold value of the project, where investment takes place, is decreasing in volatility.

Keywords: Real options, Ambiguity, Robustness

EFM classification: 430
1 Introduction

A fundamental question in corporate finance is whether a firm should exploit an investment opportunity. Furthermore, as firms often have discrepancy in the timing of the investment this element must be taken into account. The issue of finding the optimal time for undertaking the investment has been addressed by several papers since the seminal paper by McDonald and Siegel (1986). They base their analysis on the option like approach—leading to the so-called real options analysis—and a key assumption is that all parameters in their model are known to investors. However, it can be difficult to provide precise estimates of the parameters in practice. Our paper contributes to the optimal investment literature by considering the investment problem when a firm takes parameter uncertainty into account. We demonstrate that this significantly impacts the firm’s investment decision.

One strand of real options literature relaxes the assumption of complete information by considering incomplete information models with updating of beliefs, see e.g. Decamps, Mariotti, and Villeneuve (2003). In this kind of models, McDonald and Siegel (1986) is taken as the starting point, but with the friction that the investor does not have complete information about the parameters in the model. Instead, the investor has as prior probability measure over the states of the nature and as more observations occur over time, the investor uses data to update the probability distribution of the parameters. However, one problem with this approach is that in principle one needs an infinite amount of data to reduce the variance of the parameters sufficiently. Consequently, the optimal investment decision depends on time as well as the underlying state variable. This feature makes it very hard to derive explicit solutions. That is, one would need to rely solely on numerical methods.

A different approach takes as a starting point that the economic agent does not trust the reference model he employs in his analysis. Importantly, the agent is averse against this kind of uncertainty. An early example of this is the Ellsberg paradox by Ellsberg (1961) with the famous urn experiments. This setting is known as Knightian uncertainty in which ambiguity aversion is present. That is, the investor does not trust the probability measure employed in the model and is averse from this lack of knowledge. Similar to risk aversion, the investor can be more or less ambiguity averse. Ambiguity aversion has been modeled in three different ways in the literature: Smooth preferences, the multiple prior approach, or the multiplier approach. Smooth preferences is a framework that considers the preferences of an investor and uses a concave function of all the models the investor considers possible. In this framework, ambiguity aversion is similar to risk aversion, since it is also a measure for the level of concavity in the function, see e.g. Klibanoff, Marinacci,
and Mukerji (2005). Unfortunately, the smooth preference approach is difficult to apply in continuous-time models as explained in e.g. Hansen and Sargent (2009).

In both the multiple prior approach and the multiplier approach the economic agent has a reference parameter as a starting point. For example, in a real options setting one can think of an estimate of the expected growth rate of the project’s value as the reference parameter. However, the agent (i.e. the firm) worries that this estimator is not correct (or has a low precision). Hence, the firm fears that the project’s value can evolve very differently than what is predicted. In the multiple prior setting, the worst outcome is chosen, and the model is completed as without ambiguity aversion with the important adjustment that the reference parameter is substituted with the worst outcome as a fixed parameter. Hence, the employed reference parameter only depends on the space of possible outcomes. In a real options setting this method is used by Nishimura and Ozaki (2007) and Trojanowska and Kort (2010). Both papers use cash flow as the underlying variable and they consider the growth rate as the parameter estimated with high uncertainty. In particular, the growth rate is assumed to lie within an interval where the boundaries cannot change over time. Thus, the worst possible outcome is constant (the lowest possible growth rate) and the model can be solved with dynamic programming as in Dixit and Pindyck (1994, Chapter 6). In the standard real options setting, the value of the option to invest has characteristics similar to a call option. In particular, the value is an increasing convex function of volatility, which induces investors to choose more uncertain projects, see e.g. Dixit and Pindyck (1994). However, in the multiple priors setting, Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) show that the value of the option is no longer a monotonic increasing function of volatility. Related to the lack of information, the investor will invest for a higher level of cash flow, if the interval is wider, i.e. if the worst outcome gets worse.

In the multiplier approach—also known as the robust decision making approach—the worst possible outcome is also chosen, but there is an opposite working penalty for choosing a parameter. Hence, the parameter is chosen endogenously, see e.g. Hansen and Sargent (2008) for an introduction to robust methods. In a continuous-time framework Anderson, Hansen, and Sargent (2003) show how to derive a robust Hamilton-Jacobi-Bellman (HJB) equation. The robust HJB-equation is similar to the standard HJM-equation except for extra terms taking the above measuring penalty into account. The penalty is measured as the relative entropy between the reference measure and other probability measures considered. This robustness framework has been used in financial economic to address problems in asset allocation and asset pricing, see e.g. Epstein and Schneider (2008); Chen and Epstein (2002); Maenhout (2004).

The present paper is, as far as the authors know, the first to use the multiplier approach
in a real options setting. To derive the optimal time to undertake an investment our model employs a set-up similar to the one in McDonald and Siegel (1986). For a start, we focus on a setting in which the underlying variable is the value of the project. That is, the project value is uncertain in the future and, in particular, it has an expected growth rate which we consider to be estimated with low precision so that the firm wants to make a robust investment decision taking this into account. Subsequently, we also address the investment problem when the investment cost is a state variable. Since we use the robust HJB-equation, we end up with a partial differential equation that is significantly different than the Euler-differential equation from the standard problem without ambiguity aversion and from the literature using the multiple priors approach. In technical terms, the present paper has two main results. First, we are able to derive the explicit solution to the robust differential equation. Second, using this solution we can derive the explicit value of the option to invest together with the optimal investment threshold. In economic terms, we find that the threshold value of the project—at which the investment is undertaken—has a functional form similar to the one in the non-ambiguity aversion problem as well as the multiple prior approach. However, our results reveal that ambiguity aversion enters in a more complicated manner and, therefore, it has multiple effects. In particular, the threshold value of the project is not a monotonic increasing function of volatility, since an increase in volatility can decrease the expected growth in value. Furthermore, when ambiguity aversion increases, we show that the threshold value converges to a limit. If the growth rate of the project value is low enough compared to volatility, the firm employs the simple NPV rule. In contrast, if the growth rate is high enough the firm will not invest until the project’s NPV is at a level strictly higher than 0, thus violating the simple NPV rule.

The remainder of the paper is organized as follows. Section 2 describes the simple model where only the value of the project is used as the underlying variable. Here, we derive explicit formulas for the value of investment and investment threshold. Section 3 introduces a stochastic investment cost. We derive explicit formulas for the value of investment and the investment threshold which is now measured by the relative pay-off. Section 4 concludes and a number of technical results and proofs are postponed to the appendix.

2 Simple model with fixed investment cost

Consider a firm with access to an investment opportunity. If the firm undertakes the investment it pays the investment cost $I$. For now, we let $I$ be fixed. In return, the firm receives the value of project, denoted $V$. Let $(\Omega, \mathcal{F})$ be a measure space and suppose the
firm (or the investor) has a reference probability measure $\mathbb{P}$ in which he starts his modeling of the project. Under the reference measure, the value of the project is assumed to follow a geometric Brownian motion with dynamics

$$dV_t = V_t (\mu dt + \sigma dB_t),$$

where $B$ is a Brownian motion under $\mathbb{P}$ and we have the filtration $\mathcal{F}_t = \sigma(B_s|0 \leq s \leq t)$. As mentioned in the Introduction, the investor is aware that he does not know the true model, but only some approximation due to for example parameter uncertainty. Nishimura and Ozaki (2007) and Trojanowska and Kort (2010) use the worst possible state for the dynamics of the value of the project, hence the solution is only dependent on the space of possible parameters and the only used the worst case parameter. Following the papers by Anderson et al. (2003) and Maenhout (2004) a method is used, where the investor takes all possible outcomes into consideration, and chooses the worst model in a combination on the decrease in drift and a penalty by choosing another measure than the reference measure. In the alternative models we have that

$$d\tilde{B}_t = dB_t + u_t dt$$

is a $\tilde{B}$ standard Brownian motion under the alternative measure $\tilde{\mathbb{P}}$ with filtration $\tilde{\mathcal{F}}_t$. Hence the dynamics of the value under the alternative measure is

$$dV_t = V_t (\mu - \sigma u_t)dt + \sigma \tilde{B}_t.$$

The investors problem is to find the optimal time of investment, which will be modelled as the optimal stopping time. Denote $\mathcal{T}$ the set of stopping times. The traditional problem without ambiguity is then

$$F(V,t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}} \left[ e^{-\delta(t-\tau)} (V_\tau - I) | \mathcal{F}_t \right]$$

In the model, the investor is ambiguity averse, and he wants to guard him against the worst alternative measure that is reasonable to his reference measure. There is a penalty from using another measure than the reference measure $\mathbb{P}$, which is giving by a subjective parameter of ambiguity aversion $\Psi$ and the relative entropy of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. If we first elave out the penalty the robust investment problem is

$$\tilde{F}(V,t) = \sup_{\tau \in \mathcal{T}} \inf_u \mathbb{E}^{\tilde{\mathbb{P}}} \left[ e^{\delta(t-\tau)} (V_\tau - I) | \tilde{\mathcal{F}}_t \right]$$

To solve the optimal stopping problem we use dynamic programming, in which we solve the robust Hamilton-Jacobi-Bellman (HJB) equation developed in Anderson et al. (2003).
\[ 0 = \sup_{\tau} \inf_{u} \left[ \tilde{F}_V V (\mu - \sigma u) + \frac{1}{2} \sigma^2 V^2 \tilde{F}_{VV} - \delta \tilde{F} + \frac{1}{2} \Psi^{-1} u^2 \right] \]

(3)

where subscripts are the partial derivatives of the value function. The last term is from the penalty of using another model than the reference model. The first order condition with respect to \( u \) is given by

\[ -\sigma \tilde{F}_V V + \Psi^{-1} u = 0 \iff u^* = \sigma V \tilde{F}_V \Psi \]

which is inserted back into the HJB equation

\[ 0 = \sup_{\tau} \left[ \tilde{F}_V V (\mu - \sigma^2 V \tilde{F}_V \Psi) + \frac{1}{2} \sigma^2 V^2 \tilde{F}_{VV} - \delta \tilde{F} + \frac{1}{2} \sigma^2 V^2 \tilde{F}_V^2 \Psi \right] \]

\[ = \sup_{\tau} \left[ \frac{1}{2} \sigma^2 V^2 \tilde{F}_{VV} + \mu V \tilde{F}_V - \delta \tilde{F} - \frac{1}{2} \sigma^2 V^2 \tilde{F}_V^2 \Psi \right] \]

(4)

We use the trick by Maenhout (2004) we assume that \( \Psi \) state dependent, i.e \( \Psi(V, t) \), and in order to induce homogeneity in the model we assume that \( \Psi(V, t) = \theta \tilde{F}(V, t) \), \( \theta > 0 \).

Our differential equation can now be written

\[ 0 = \frac{1}{2} \sigma^2 V^2 \tilde{F}_{VV} + \mu V \tilde{F}_V - \delta \tilde{F} - \frac{1}{2} \sigma^2 V^2 \theta \tilde{F}(V)^{-1} \tilde{F}_V^2 . \]

(5)

In addition, we have the usual absorbing condition, value matching condition and smooth pasting condition

\[ \lim_{V \to 0} \tilde{F}(V) = 0 \]

(6)

\[ \tilde{F}(V^*) = V^* - I \]

(7)

\[ \tilde{F}'(V^*) = 1 . \]

(8)

That is, if the value of the project is 0, so should the value of the option to invest in this project be zero. In contrast, if the project value increases to the level \( V^* \) the option is exercised. At this level, the value of the option equals the value of the project minus the investment cost. The final condition is the condition for optimality of the exercise boundary, \( V^* \).

The ODE in (5) differs from the standard one commonly seen in the literature—e.g. Dixit and Pindyck (1994) and Nishimura and Ozaki (2007). In particular, the last term makes the ODE nonlinear and different from the Euler equation. However, it is possible to find a closed form solution. With the above boundary conditions, this solution collapses to the same structure seen in the standard non-ambiguous framework. Therefore, we highlight this solution below.
Lemma 1. Assume $\theta \neq 1$. Then the general solution to (5) can be written on the form

$$\tilde{F}(V) = (y_1(V) + y_2(V))^{\frac{1}{1-\theta}}$$

where $y_1$ and $y_2$ are two linear independent functions of $V$.

The next corollary is the solution for the differential equation for $\theta = 1$.

Corollary 1. Assume $\theta = 1$. Then the solution to equation (5) can be written on the form:

$$\tilde{F}(V) = C_1 V^{\beta_1} \exp \left[ C_2 V^{\beta_2} \right]$$

To derive the value of the option to invest, we need to consider conditions (6)–(8), for all three cases of $\theta$. We also have the following assumption:

Assumption 1. The optimal investment threshold $V^*$ is a continuous function of $\theta$.

The assumption is not restrictive, since it does make sense that the investment threshold should have jumps as a function of the ambiguity parameter, and we will use the limit of $V^*$ when $\theta$ converges to 1. The threshold will also depend on the sign of $2\mu - \sigma^2$, since this is the relative mean of the value process.

Theorem 2.1. Let the investor have ambiguity aversion $\theta$. Then the value of the project can be written as

$$F(V) = A_1 V^{\beta_1}$$

where

$$\beta_1 = \frac{-(\mu - \frac{1}{2} \sigma^2) + \sqrt{(\mu - \frac{1}{2} \sigma^2)^2 + 2(1-\theta)\sigma^2 \delta}}{\sigma^2 (1-\theta)}$$

if $\theta \neq 1$ and $\theta \leq 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta}$, and

$$A_1 = (V^* - I)(V^*)^{-\beta_1}$$

If $\theta = 1$ then

$$\beta = \frac{\delta}{\mu - \frac{1}{2} \sigma^2}.$$

The optimal level of the project value at which investment takes place is

$$V^* = \frac{\beta_1}{\beta_1 - 1} I$$

If $\theta > 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta}$ then the value of the project is zero, which is also the case if $V^* < I$. 
Proof. Assume that \( \theta < 1 \). Then from Lemma 1 we have that \( \hat{\beta} > 0 \) and \( \hat{\beta}_2 < 0 \) and \( 1/(1 - \theta) > 0 \). From the conditions (6)–(8) we can set \( C_2 = 0 \) and the \( \tilde{F} \) can be written

\[
\tilde{F}(V) = (C_1 V^{\hat{\beta}_1})^{1/(1-\theta)}
= A_1 V^{\hat{\beta}_1}
\]

where

\[
\hat{\beta}_1 = -\left(\mu - \frac{1}{2} \sigma^2\right) + \sqrt{\left(\mu - \frac{1}{2} \sigma^2\right)^2 + 2(1-\theta)\sigma^2 \delta}
\]

\[
\sigma^2(1-\theta)
\]

Since we are interested in \( \hat{\beta}_1/(1-\theta) \) we study the associated quadratic equation

\[
\hat{\beta}^2 + \left(\frac{2\mu}{\sigma^2} - 1\right) \frac{1}{1-\theta} \hat{\beta} - \frac{(1-\theta) 2\delta}{1-\theta^2 \sigma^2} = 0,
\]

i.e.

\[
(1-\theta) \frac{\sigma^2}{2} \hat{\beta}^2 + \left(\mu - \frac{\sigma^2}{2}\right) \frac{1}{\hat{\beta} - \delta} = 0.
\]

Since \( (1-\theta) \) we have a parabola with upward turning branches. Moreover, as \( \delta > \mu \), we get \( Q(0) = -\delta < 0 \) and \( Q(1) = -\theta \frac{a^2}{2} - (\delta - \mu) < 0 \). From this it follows that the positive root \( \beta \triangleq \hat{\beta}_1/(1-\theta) > 1 \). The value matching condition yields

\[
F(V^*) = A_1(V^*)^{\hat{\beta}_1} \triangleq V^* - I,
\]

i.e.

\[
A_1 = (V^* - I)(V^*)^{-\hat{\beta}_1}.
\]

The smooth pasting condition yields

\[
F'(V^*) = A_1 \hat{\beta}_1 (V^*)^{\hat{\beta}_1 - 1} \triangleq 1,
\]

i.e.

\[
V^* = \frac{\hat{\beta}_1}{\beta_1 - 1} I,
\]

as asserted.

Denote

\[
C = \frac{\beta_1}{\beta_1 - 1} = \frac{-\left(\frac{2\mu}{\sigma^2} - 1\right) + \left[\left(\frac{2\mu}{\sigma^2} - 1\right) + 4(1-\theta) \frac{2\delta}{\sigma^2}\right]^{1/2}}{-\left(\frac{2\mu}{\sigma^2} - 1\right) + \left[\left(\frac{2\mu}{\sigma^2} - 1\right) + 4(1-\theta) \frac{2\delta}{\sigma^2}\right]^{1/2} - 2(1-\theta)}
\]

7
If we let $\theta \to 1$ we have that

$$\lim_{\theta \to 1^-} C(\theta) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We use l'Hopital’s rule and we get $\lim_{\theta \to 1^-} C(\theta) = \frac{2\delta}{2\delta - 2\mu + \sigma^2}$. From the proof of Corollary 1 we have that when $\theta = 1$ that

$$\tilde{F}(V) = C_1 V^{\beta_1} \exp\left[ C_2 V^{\beta_2} \right]$$

where $\beta_1 = \frac{2\delta}{2\mu - \sigma^2}$ and $\beta_2 = 1 - \frac{2\mu}{\sigma^2}$ and we have that

$$\frac{\beta_1}{\beta_1 - 1} = \frac{2\delta}{2\delta - 2\mu - \sigma^2}$$

and so for the case $\theta = 1$ we set $C_2 = 0$. With $1 < \theta \leq 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta}$ we can again set $C_2 = 0$ due to continuous $V^*(\theta)$. For the case $\theta = 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta}$, we also set $C_2 = 0$.

When $\theta > 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta}$ we have to set both $C_1$ and $C_2$ equal to zero, otherwise the function $u(V)$ will oscillate and become negative.

Recall that the actual penalty of changing the measure was $\Psi$ and therefor that we can set $\tilde{F} \equiv 0$ for the case $\theta > 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta}$.

From the proof of Theorem 2.1 we can see, that the investor is concerned with his ambiguity aversion and the sign of $2\mu - \sigma^2$. If $2\mu - \sigma^2 < 0$ then the value of the project will be zero for a $\theta < 1$. If $2\mu - \sigma^2 = 0$ then the value fo the project is zero from $\theta = 1$, since the optimal threshold is $I$. For a small enough $\sigma$ we will have that $V^* > I$ for all $\theta$ where the value of the project is positive. This may seem odd, but even in the case of no volatility and $\sigma = 0$, which makes it a deterministic, is the optimal threshold also higher than $I$, $V^* = \frac{\delta}{\delta - \mu}$, see Dixit and Pindyck (1994, Section 5.1).

### 2.1 Comparative statics

In this section we show how the investment decision is changed with ambiguity aversion. Generel the investor will exercise his investment option for a lower value of $V$, since he would like to eliminate the uncertainty about the drift of the value $\mu$. The effect is more pronounced for a higher $\theta$. In the original case without an unambiguous investor the value of the option is convex increasing in $V$ due to the option like behaviour. With an ambiguous investor we sohve the same convexity in the value function. From the dynamic in $V$ have a term of $-\sigma u$ in the drift, and when the volatility becomes high enough this will dominate the reference drift $\mu$, hence the drift will become negative, and the investor will exercise his option earlier.
The parameters of the base case are inspired by the original article McDonald and Siegel (1986) and given in table 1. In figure 1 we have the value of the project, thus the value of the option to invest as a function of the initial value of $V$. We see that introducing ambiguity aversion lowers the value of the option, since the investor is unsure about the dynamics of the option, and the value of waiting is then lowered with ambiguity aversion. This effect is stronger for a higher $\theta$, hence the investor will invest even for a smaller $V$.

In upper figure in Figure 2 we have the investment threshold as a function of the investors ambiguity aversion is the threshold is clearly decreasing, hence the investor will invest for a lower $V$ in order to eliminate the uncertainty about the drift $\mu$, and the more ambiguity averse the earlier does the investor want to eliminate this uncertainty.

Table 1: Parameters for the base case.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\theta$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>2%</td>
<td>0.2</td>
<td>0.3</td>
<td>10</td>
</tr>
</tbody>
</table>

Figure 1: The value of the project as a function of the value of $V$. The green line is with ambiguity aversion.
Figure 2: The investment threshold as a function of first ambiguity parameter $\theta$ and second the volatility. The blue line is without ambiguity.
Figure 3: The investment threshold as a function of both volatility and ambiguity aversion.
3 Extension to stochastic investment cost

If we now have that the cost of investment is also stochastic, then in the case with no ambiguity MS use homogeneity in the problem, which it is not intuitively clear that we have here. They use that they can reduce the number of processes to only one, the relative value of the project with respect to price and the investment threshold is then again a fixed barrier. Since there are now two stochastic processes we write the value process with subscripts on the parameters:

$$dV_t = V_t (\mu_V dt + \sigma_V dB^1_t)$$  (12)

We assume that there is a correlation between $V$ and $I$ denoted $\rho$, and the price process can be written

$$dI_t = I_t \left( \mu_I dt + \sigma_I \left( \rho dB^1_t + \sqrt{1-\rho^2} dB^2_t \right) \right)$$  (13)

where $B^1$ and $B^2$ are two independent Brownian motions in the reference measure $\mathbb{P}$. In this extended model, the investor is doubtful about both processes, and wants to guard him against alternative models in both processes, and we have the alternative Brownian motions

$$dB_t = dB^1_t + u_{1t} dt, \quad dB^2_t = dB^2_t + u_{2t} dt$$

which is inserted into the equations (12) and (13) and

$$dV_t = V_t \left( (\mu - \sigma_V u_{1t}) dt + \sigma_V dB^1_t \right)$$  (14)

$$dI_t = I_t \left( (\mu_I - \sigma_I (\rho u_{1t} + \sqrt{1-\rho^2} u_{2t})) dt + \sigma_I \left( \rho dB^1_t + \sqrt{1-\rho^2} dB^2_t \right) \right)$$  (15)

Let $\bar{F}(V, I)$ be the alternative value of the option to invest, and we have that the robust HJB equation can be written

$$0 = \left[ \bar{F}_V V (\mu_V - \sigma_V u_1) + \bar{F}_I I (\mu_I - \sigma_I (\rho u_1 + \sqrt{1-\rho^2} u_2)) + \frac{1}{2} \bar{F}_{VV} V^2 \sigma_V^2 + \frac{1}{2} \bar{F}_{II} I^2 \sigma_I^2 - \delta \bar{F} + V I \sigma_V \sigma_I \rho \bar{F}_{VI} + \frac{1}{2} tr(\Psi^{-1} u^\top u) \right]$$  (16)

where $tr$ is the trace of a matrix. We have that the penalty and controls are denoted

$$\Psi = \begin{pmatrix} \Psi_V & 0 \\ 0 & \Psi_I \end{pmatrix} \quad \text{and} \quad u = (u_1, u_2)^\top$$

Hence we have that
We will be able to set \( F(V, I) = I f(v) \) where \( v_t = V_t/I_t \) and use the homogeneity of the problem as in McDonald and Siegel (1986) to reduce the problem. The result is stated below with the proof in the appendix.

**Theorem 3.1.** Let the investor have ambiguity aversion parameters \( \theta_V \) and \( \theta_I \). Then the value of the project can be written

\[
F(V, I) = I f(v)
\]

where \( B \) is a constant and \( f(v) \) is a function depending on \( v = V/I \). The optimal investment threshold is

\[
v^* = \frac{\beta}{\beta - 1}
\]

### 3.1 Comparative Statics

In this section we have comparative statics of the results as in section 2.1. The effect of ambiguity aversion is generally the same as in the simple model, but in this model, the correlation between the two processes have a significant effect. This is due to the somewhat spillover effect from the control process \( u_1 \) but also due to the general correlation in the shocks. We have that ambiguity aversion again lowers the investment threshold, here measured in the relative value of the payoff and the price, but there are more subtle with two processes due to the correlation. In the upper figure of Figure 4 we have the value of the option as a function ambiguity aversion, where we assume that parameter for ambiguity aversion is the same in both processes. We have the same picture as in Figure 1, which was also expected, where the investment threshold lies lower then in the original model. The value is lower than without ambiguity aversion, since the value of waiting is smaller than in the original model. In the bottom figure we have the investment threshold \( v^* \) as a function of the volatility in the payoff process \( \sigma_V \) for three cases of correlation depending on the sign of correlation, from above we have negative, then positive and at the bottom positive. Recall that the process in consideration is \( v = V/I \), where the drift also depends on the volatility parameters \( \sigma_V \) and \( \sigma_I \). When the volatility changes there is both an option-like effect, which increases the value of \( v^* \), but there is also a drift effect, when the correlation is non-zero. When the correlation is negative the drift is increased with volatility, which always increases the threshold \( v^* \). With a positive correlation the drift-effect is negative in \( v \), hence \( v^* \) will be decreasing, but for a high enough \( \sigma_V \) the
Figure 4: The value of the project as a function of $V$, where the green line is with ambiguity aversion. Below is the investment threshold $v^*$ as a function of the volatility in the value for three different correlations from above: Negative, zero and positive.

The option-effect will dominate and $v^*$ will increase.

In Figure 5 we have the investment threshold as a function of both ambiguity parameters, and we can see that the value is decreasing in both parameters. The rate of decrease seems to be higher for the parameter $\theta_V$, but the reason is somewhat unclear: It could just be due to the way the processes are written that there is a higher effect in $\theta_V$'s, since this appears in both processes, but this could easily be checked by changing the way the processes are written and write the payoff process with both Brownian motions and we would probably have the higher effect in $\theta_I$.

In Figure 6 we have the investment threshold depending on both volatility parameters $\sigma_V$ and $\sigma_I$ for the three cases of correlation between the two processes (the case of positive correlation is not shown in this version of the paper due to file size restriction).
4 Conclusion

This paper examines the problem of an investment decision of a sure payoff for a either fixed or stochastic price. In the paper the investor is ambiguity averse, does not trust his reference model and is unsure about the value of waiting to invest. This lowers the value of the option and the investor will exercise earlier. The effect is higher the more ambiguity averse the investor and he will exercise for an even lower threshold. We also have that the value of waiting is no longer monotonically increasing in volatility, since it decreases the drift of the payoff. This result is in contradiction with the original model, where the value is monotonically increasing in a convex way. In the model, where the investor receives a sure payoff at time of investment the effect is quite clear and the investor eliminates all uncertainty by investing. When the price of investment is also stochastic we introduce ambiguity aversion the investment threshold, measured as the payoff relative to the price, is also lowered compared to the original model, but there are many effects in the value due to correlation and the two volatility parameters.
Figure 6: The investment threshold $v^*$ as a function of the volatilities for the three cases of correlation: Positive, zero and negative
A Proofs

A.1 Proof from section 2

Proof of Lemma 1. The structure of the proof is to transform the ODE in (5) to a differential equation with recognizable solution, and then transform this solution back to the initial problem. First we rewrite (5)

\[ 0 = \frac{1}{2} \sigma^2 V^2 \left( \tilde{F}_{VV} - \theta F(V)^{-1} \tilde{F}_V^2 \right) + \mu V \tilde{F}_V - \delta \tilde{F}, \]  

and abstracting from \( F = 0 \) we get

\[ 0 = \frac{\sigma^2}{2} V^2 \left( \frac{\tilde{F}_{VV} - \tilde{F}_V^2}{F^2(V)} + (1 - \theta) \left( \frac{\tilde{F}_V}{F} \right)^2 \right) + \mu V \frac{\tilde{F}_V}{F} - \delta. \]  

Let \( g(V) = \frac{\tilde{F}_V}{\tilde{F}}. \) Then \( g' = \frac{\tilde{F}_V g - \tilde{F}_V^2}{\tilde{F}^2} \) and (20) becomes

\[ g'(V) = \frac{2\delta}{\sigma^2 V^2} + \frac{-2\mu}{\sigma^2 V} g(V) + \frac{-(1 - \theta)}{q_2(V)} g^2(V), \]  

which we recognize as a Ricatti equation. Therefore, consider the transformation \( h(V) = q_2(V) g(V) \) yielding

\[ h'(V) = g_0(V) g_2(V) + \left( \frac{q_1(V) + \frac{q_2(V)}{q_2(V)}}{\frac{S(V)}{R(V)}} \right) h(V) + h^2(V). \]  

Finally, let \( u \) satisfy

\[ h(V) = \frac{u'(V)}{u(V)} \]  

then

\[ h'(V) = \frac{-u(V) u''(V) + (u'(V))^2}{u^2(V)}, \]  

wherewithal

\[ \frac{-u(V) u''(V) + (u'(V))^2}{u^2(V)} = S(V) + R(V) \frac{-u'(V)}{u(V)} + \left( \frac{u'(V)}{u(V)} \right)^2. \]  

Thus,

\[ 0 = u''(V) - R(V) u'(V) + S(V) u(V) \]  

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and inserting for \( R \) and \( S \) we get

\[
0 = u''(V) + \frac{2\mu}{\sigma^2} V u'(V) - (1 - \theta) \frac{2\delta}{\sigma^2} V^2 u(V),
\]

hence

\[
0 = V^2 u''(V) + \frac{2\mu}{\sigma^2} V u'(V) - (1 - \theta) \frac{2\delta}{\sigma^2} V^2 u(V),
\]

which we recognize as a second order Euler differential equation. By setting \( V = e^t \) we get an linear second order differential equation.

\[
u''(t) + \frac{2\mu}{\sigma^2} u'(t) - (1 - \theta) \frac{2\delta}{\sigma^2} u(t) = 0
\]

We have the characteristic function

\[
\beta^2 + \left( \frac{2\mu}{\sigma^2} - 1 \right) \beta - (1 - \theta) \frac{2\delta}{\sigma^2} = 0.
\]

where

\[
\hat{\beta} = \frac{1}{2} \left[ - \left( \frac{2\mu}{\sigma^2} - 1 \right) \pm \sqrt{\left( \frac{2\mu}{\sigma^2} - 1 \right)^2 + (1 - \theta) \frac{8\delta}{\sigma^2}} \right]
\]

If the solutions to the characteristic equation are complex, we write \( \hat{\beta} = \psi \pm i\phi \). The solution of \( u \) with respect to \( t \) then depends on the roots of the characteristic function, which depends mainly on \( \theta \). Denote \( K = 1 + \left( \frac{2\mu}{\sigma^2} - 1 \right)^2 \frac{\sigma^2}{8\delta} \). The solutions can be found in Spiegel and Liu (1999).

<table>
<thead>
<tr>
<th>Condition</th>
<th>( u(t) )</th>
<th>( u(V) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \theta &lt; K )</td>
<td>( C_1 e^{\hat{\beta}t} + C_2 e^{\hat{\beta}^2t} )</td>
<td>( C_1 V^{\hat{\beta}} + C_2 V^{\hat{\beta}^2} )</td>
</tr>
<tr>
<td>( \theta = K )</td>
<td>( C_1 e^{\hat{\beta}t} + C_2 e^{\hat{\beta}^2t} )</td>
<td>( C_1 V^{\hat{\beta}} + C_2 \log(V)V^{\hat{\beta}} )</td>
</tr>
<tr>
<td>( \theta &gt; K )</td>
<td>( e^{\psi t} (C_1 \cos(\phi t) + C_2 \sin(\phi t)) )</td>
<td>( V^\psi (C_1 \cos(\log V) + C_2 \sin(\log V)) )</td>
</tr>
</tbody>
</table>

Table 2: Solutions of the differential with respect to \( t \) and \( V \).

Hence all solutions is a linear combination of two linear independent functions, \( y_1 \) and \( y_2 \). Now recall that

\[
g(V) = \frac{1}{1 - \theta} \frac{u'(V)}{u(V)}
\]

but we also have

\[
g(V) = \frac{F'(V)}{F(V)}
\]

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i.e.
\[
\frac{\tilde{F}'(V)}{\tilde{F}(V)} = \frac{1}{1 - \theta} \frac{u'(V)}{u(V)}
\]
hence it follows that
\[
\tilde{F}(V) = \tilde{C}_3 u(V)^{\frac{1}{1 - \theta}},
\]
where \(\tilde{C}_3\) is an integration constant to be determined. Using the expression for \(u\) we obtain
\[
\tilde{F}(V) = \tilde{C}_3 \left( \tilde{C}_1 y_1(V) + \tilde{C}_2 y_2(V) \right)^{\frac{1}{1 - \theta}},
\]
(31)
which we rewrite as
\[
\tilde{F}(V) = (C_1 y_1(V) + C_2 y_2(V))^{\frac{1}{1 - \theta}},
\]
(33)
as asserted in equation (9)

Proof of Corollary 1. We assume that \(\tilde{F}(V) \neq 0\). The differential equation is now written
\[
0 = \frac{1}{2} \sigma^2 V^2 \left( t \frac{\tilde{F}''}{\tilde{F}} - \frac{\tilde{F}^2}{\tilde{F}} \right) + \mu V \tilde{F}_V - \delta \tilde{F}
\]
\[
= \frac{1}{2} \sigma^2 V^2 \left( \frac{\tilde{F}''}{\tilde{F}} - \left( \frac{\tilde{F}_V}{\tilde{F}} \right)^2 \right) + \mu V \tilde{F}_V - \delta
\]
We define
\[
g(V) = \frac{\tilde{F}_V}{\tilde{F}} \quad g_V = \frac{\tilde{F}_{VV}}{\tilde{F}} - \left( \frac{\tilde{F}_V}{\tilde{F}} \right)^2
\]
Thus the differential equation is written
\[
g_V + \frac{2\mu}{\sigma^2} V^{-1} g = \frac{2\delta}{\sigma^2} V^{-2}
\]
Set \(G(V) = \frac{2\mu}{\sigma^2} \int V^{-1} dV = \frac{2\mu}{\sigma^2} \log(V)\) and the solution to \(g\) is
\[
g(V) = \exp \left[ -\frac{2\mu}{\sigma^2} \log(V) \right] \left[ \frac{2\delta}{\sigma^2} \int V^{-2} \exp \left[ \frac{2\mu}{\sigma^2} \right] dV \right]
\]
\[
= \frac{2\delta}{\sigma^2} V^{-1} + C_1 V^{-\frac{2\mu}{\sigma^2}}
\]
To find \(\tilde{F}\) we set
\[
P(V) = \int \left( -\frac{2\delta}{2\mu - \sigma^2} V^{-1} - C_1 V^{1 - \frac{2\mu}{\sigma^2}} \right) dV
\]
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\begin{align*}
&= -\frac{2\delta}{2\mu - \sigma^2} \log(V) - \frac{C_1\sigma^2}{\sigma^2 - 2\mu} V^{1 - \frac{2\mu}{2\mu - \sigma^2}} \\
\text{and the solution for } \tilde{F} \text{ is} \\
\tilde{F}(V) = C_2 \exp \left[ -P(V) \right] = C_2 \exp \left[ - \left( \log(V) - \frac{2\delta}{2\mu - \sigma^2} - \frac{C_1\sigma^2}{\sigma^2 - 2\mu} V^{1 - \frac{2\mu}{2\mu - \sigma^2}} \right) \right] \\
&= C_1 V^{\frac{2\delta}{2\mu - \sigma^2}} \exp \left[ C_2 V^{1 - \frac{2\mu}{2\mu - \sigma^2}} \right]
\end{align*}

A.2 Proofs from section 3

Proof of Theorem 3.1. The two first order conditions with respect of the controls \( u \) are

\begin{align*}
- \tilde{F}_V V \sigma_V - \rho \sigma_I \tilde{F}_I V + \Psi_V^{-1} u_1 &= 0 \iff u_1 = \Psi_V \left( \tilde{F}_V V \sigma_V + \rho \sigma_I \tilde{F}_I \right) \\
- \tilde{F}_I \sqrt{1 - \rho^2} I + \Psi_I^{-1} u_2 &= 0 \iff u_2 = \Psi_I \tilde{F}_I I \sqrt{1 - \rho^2} I
\end{align*}

These are inserted back into the HJB equation to obtain a partial differential equation

\begin{align}
0 &= \tilde{F}_V \left( \mu V - \sigma_V \Psi_V \left( \tilde{F}_V V \sigma_V + \rho \sigma_I \tilde{F}_I \right) \right) \\
&+ \tilde{F}_I \left( \mu I - \sigma_I \left( \rho \Psi_V \left( \tilde{F}_V V \sigma_V + \rho \sigma_I \tilde{F}_I \right) + (1 - \rho^2) \Psi_I \tilde{F}_I I \sigma_I \right) \right) \\
&+ \frac{1}{2} \tilde{F}_{VV} V^2 \sigma_V^2 + \frac{1}{2} \tilde{F}_{II} I^2 \sigma_I^2 - \delta \tilde{F} + V I \rho \sigma_V \sigma_I \tilde{F}_V I \\
&+ \frac{1}{2} \Psi_V \left( \tilde{F}_V V \sigma_V + \rho \sigma_I \tilde{F}_I \right)^2 + \frac{1}{2} \Psi_I \tilde{F}_I^2 I^2 \left( 1 - \rho^2 \right) \sigma_I^2 
\end{align}

(34)

If we guess a solution on the form

\[ \tilde{F}(V, I) = I f(v) \]

with \( v = V/I \). Then the partial derivatives of \( \tilde{F} \) are

\[ \tilde{F}_V = f'(v), \quad \tilde{F}_I = f(v) - v f'(v), \quad \tilde{F}_{VV} = f''(v), \quad \tilde{F}_{II} = v^2 f''(v) \]

These are inserted into equation (34) and for simplicity we leave out the variables in \( f \)
\[ 0 = \mu V f' + \mu I (f - v f') - \delta I f + \frac{1}{2} \sigma_V^2 V^2 f''/I + \frac{1}{2} \sigma_I^2 I^2 v^2 f''/I \\
+ \rho \sigma_V \sigma_I V I (f''/I) - \frac{1}{2} \sigma_V^2 V f^2 \Psi_V - \frac{1}{2} \rho^2 \sigma_I^2 I I^2 (f - v f')^2 \Psi_V \\
- \rho \sigma_V \sigma_I V I f' (f - v f') \Psi_V - \frac{1}{2} \sigma_I^2 (1 - \rho^2) I^2 (f - v f')^2 \Psi_I \tag{35} \]

As in the simple model we set the penalties to be state dependent:

\[ \Psi_V = \frac{\theta_V}{F(V, I)} = \frac{\theta_V}{I f(v)}, \quad \Psi_I = \frac{\theta_I}{F(V, I)} = \frac{\theta_I}{I f(v)} \]

and insert these into (35) and we can divide by \( I \) to get

\[ 0 = \frac{1}{2} f'' \left( \sigma_V^2 + \sigma_I^2 - 2 \rho \sigma_V \sigma_I \right) v^2 \\
+ f' \left( \mu_V - \mu_I + \rho^2 \sigma_V^2 \theta_V - \rho \sigma_V \sigma_I \theta_V + \sigma_I^2 (1 - \rho^2) \theta_I \right) v \\
+ f \left( -\delta + \mu_I - \frac{1}{2} \rho^2 \sigma_I^2 \theta_V - \frac{1}{2} \sigma_I^2 (1 - \rho^2) \theta_I \right) \\
- \frac{1}{2} f^2 f^{-1} \left( \sigma_V^2 \theta_V + \sigma_I^2 (1 - \rho^2) \theta_I + \rho^2 \sigma_I^2 \theta_V - 2 \rho \sigma_V \sigma_I \theta_V \right) v^2 \tag{36} \]
References


