Valuation of CMS Spread Options with Nonzero Strike Rate

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Abstract

A generalized lognormal distribution is used to approximate the distribution of the difference between two CMS rates. Pricing models for CMS spread options with nonzero strike rates are then derived under the multifactor LIBOR market model and are shown to be analytically tractable for practical implementation. The models are shown to be robustly accurate in comparison with Monte Carlo simulation.

Keywords: CMS Spread Options, CMS Steepeners, CMS Range Accruals, BGM Model

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1 Introduction

The constant maturity swap (CMS) rate is quoted by the majority of brokers in the pricing of interest rate swaps, and this rate's reliable closing quotes are routinely reported in major financial quotation systems, such as Reuters or Bloomberg. The CMS rate has thus become one of the most widely used interest rate indexes in financial markets; based on this index, many exotic CMS rate derivatives have been developed to allow investors to meet hedging or investment objectives. In this article, we focus on the valuation of CMS spread options with nonzero strike rates.

CMS spread options are options on the difference between two CMS rates (e.g., 10-year CMS rate minus 2-year CMS rate). CMS spread options can help market practitioners to hedge against risks that are dependent on whether the spread between two interest rates is above or below a specified level, or within or outside a specific range on a specific future date. In addition, CMS spread options can be used to enhance profits from a relative change in the various ranges of the yield curve. For example, in December 2000, the spread between the 10-year and 2-year US dollar CMS rates was about 20 basis points, which was significantly narrower and increased dramatically to 200 basis points by December 2001 (see Figure 1). Investors would be able to generate considerable returns if they could take an accurate view in advance of the steepness of the yield curve during such a period.

As indicated in Sawyer (2005), trading volume of CMS spread options reached $30 billion in 2005 and has increased ever since. The most widely-traded CMS spread options are CMS range accruals and CMS steepeners. Range accruals pay a high fixed rate coupon if the CMS spread is above or below a particular barrier or remains within a pre-specified range for every day of the coupon period. Steepeners pay a high coupon in the first few years, after which investors are subsequently paid with a coupon based on the spread between two CMS rates, multiplied by a specified leverage ratio. Thus, the steeper the yield curve the greater the payoff for investors in CMS steepeners.

In the past two decades, several articles have examined interest rate spread options. Based on an extended Cox, Ingersoll and Ross (CIR, 1985) model, Longstaff (1990) derives

Figure 1: The spread between the 10-year and 2-year CMS rates.

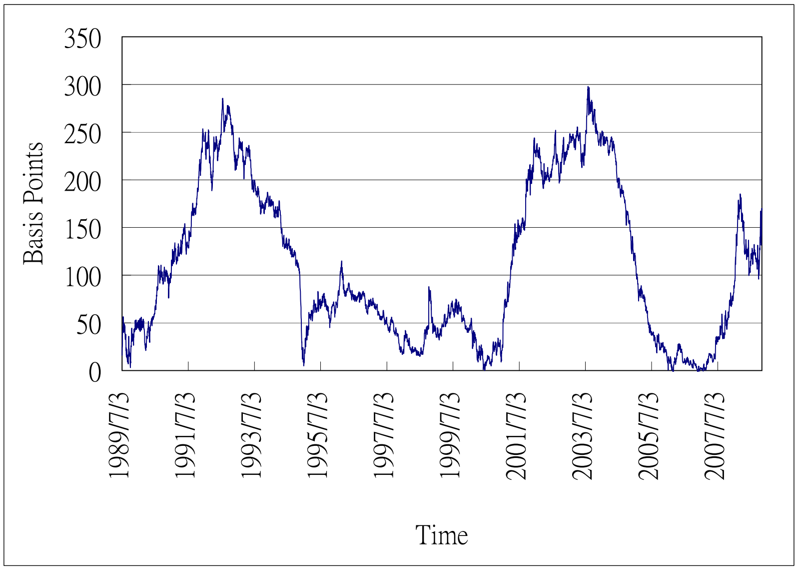

Figure 1 depicts the time series data of the spread in basis points between the 10-year and 2-year CMS rates from 1989/7/3 to 2008/10/31.

a closed-form pricing model for European yield spread options and examines the empirical implications of the model using market data. Fu (1996) observes that to price yield spread options, one should employ a multivariate model to capture imperfect correlations for underlying interest rates of differing maturities, thus pricing interest rate spread options within a two-factor Heath, Jarrow and Moton (HJM, 1992) model. Wu and Chen (2009) adopt a multifactor LIBOR market model to examine three types of interest rate spread options with zero strike rates: LIBOR vs. LIBOR; LIBOR vs. CMS; and CMS vs. CMS, all of which are widely traded in the marketplace or embedded in structured notes, such as range accruals and steepeners. However, the setting of a zero strike rate in Wu and Chen (2009) restricts the capability to price the (more frequent) case with a nonzero strike rate. Therefore, the main purpose of this article is to provide a more general pricing formula for CMS spread options that can handle the case of nonzero strike rates.

In this paper, we adopt a multifactor LIBOR market model (LMM) as the central model to specify the behavior of forward CMS rates.[1] As indicated in Wu and Chen

[1]The LMM and the swap market model, presented by Jamshidian (1997), are not compatible in that a swap rate and a LIBOR rate cannot be lognormally distributed under the same measure. Thus, it is problematic to choose either of the two models as a pricing foundation. Brace, Dun and Barton (1998)

(2009), the forward swap rate, the sum of forward LIBOR rates with lognormal distributions, can be well approximated by a lognormal distribution. In this way, the difficulty in pricing CMS spread options is similar to finding the probability distribution of the difference between two lognormal random variables. However, the difference between two lognormal random variables cannot be approximated directly by a lognormal random variable, due to its potentially negative value and negative skewness. Therefore, we employ a generalized family of lognormal distributions to approximate the probability distribution of the difference.[2] The resulting pricing formulas are shown to be significantly accurate as compared with Monte Carlo simulation.

In practice, the prices of CMS spread options, such as CMS range accruals and steepeners, are computed based on Monte Carlo simulation, which is time-consuming and inefficient. For example, financial institutions issuing hundreds of CMS range accruals might find it difficult, based on Monte Carlo simulation, to provide daily price quotations for customers. Our pricing formulas provide close prices to those computed from Monte Carlo simulation while taking much less time, and thus provide market practitioners with a new, efficient and time-saving approach to offering almost instantly quoted prices to clients, the daily mark-to-market of trading books and managing risks of trading positions.

This paper proceeds as follows: Section 2 reviews the LMM and its several implementation techniques; then uses a generalized family of lognormal distributions to approximate the distribution of the difference between two lognormal random variables. Section 3 provides approximate pricing formulas for CMS spread options. Section 4 provides a calibration procedure and examines the accuracy of the approximate formulas via Monte Carlo simulation. Section 5 presents the results and conclusion.

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suggest adoption of the LMM as the central model, due to its mathematical tractability; we follow this suggestion.

[2]For the details regarding the generalized family of lognormal distributions, please refer to Johnson, Kotz and Balakrishnan (1994).

2 The Model

In Section 2, we review the LMM and its several implemention techniques for the model. We also introduce a lognormalization approach for swap rates under the LMM. Finally, we use the generalized family of lognormal distributions to approximate the distribution of the spread between two swap rates.

2.1 The LIBOR Market Model

The LMM is developed by Brace, Gatarek and Musiela (BGM, 1997), Musiela and Rutkowski (1997), and Miltersen, Sandmann and Sondermann (1997), and has been extensively employed in practice to price interest rate derivatives due to its many merits. For example, the LMM specifies the dynamics of market-observable forward LIBOR rates rather than the abstract rate specified in the traditional interest rate model. The cap and floor pricing formulas derived within the LMM framework are the Black formula, which has been widely used in practice. The above two advantages make the parameter calibration of the LMM easier. In addition, most interest rate products can be priced within the LMM framework such that interest rate risks can be managed consistently and efficiently. In this subsection, we briefly review the LMM and introduce techniques for its practical implementation.

Consider that trading takes place continuously over an interval $[0,\mathcal{T}]$, $0 < \mathcal{T} < \infty$. Uncertainty is described by a filtered spot martingale probability space $\left(\Omega, F, \mathcal{Q}, \{\mathcal{F}_t\}_{t\in[0,\mathcal{T}]}\right)$ and a d-dimensional independent standard Brownian motion $Z(t) = (Z_1(t), Z_2(t), ..., Z_d(t))$ is defined on the probability space. The flow of information accruing to all agents in the economy is represented by the filtration $\{\mathcal{F}_t\}_{t\in[0,\mathcal{T}]}$, which satisfies the usual hypotheses.[3] Note that $\mathcal{Q}$ denotes the spot martingale probability measure. We define the notations as follows:

[3]The filtration $\{\mathcal{F}_t\}_{t\in[0,\mathcal{T}]}$ is right continuous, and $\mathcal{F}_0$ contains all the $\mathcal{Q}$-null sets of $\mathcal{F}$.

$B(t,T) \;=\;$ the time t price of a zero-coupon bond paying one dollar at time T.

$L(t,T) \;=\;$ the forward LIBOR rate contracted at time t and applied to the period $[T,\, T+\delta]$ with $0 \le t \le T \le T+\delta \le \mathcal{T}$.

$\mathcal{Q}^T \;=\;$ the forward martingale measure with respect to the numéraire $B(\cdot, T)$.

The relationship between $L(t,T)$ and $B(t,T)$ can be expressed as follows:

$$L(t,T) = \big(B(t,T) - B(t,T+\delta)\big)/\delta B(t,T+\delta). \tag{1}$$

By following the approach of BGM (1997), the LMM is constructed based on the arbitrage-free conditions in HJM (1992). We briefly specify their results as follows.

Proposition 1. The LIBOR Rate Dynamics under Measure $\mathcal{Q}$

Under the spot martingale measure $\mathcal{Q}$, the forward LIBOR rate $L(t,T)$ follows the following stochastic process:

$$dL(t,T) = L(t,T)\gamma(t,T)\cdot\sigma_B(t,T+\delta)dt + L(t,T)\gamma(t,T)\cdot dZ(t), \tag{2}$$

where $0 \le t \le T \le \mathcal{T}$, $\sigma_B(t,\cdot)$ is defined as follows:

$$\sigma_B(t,T) = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor \delta^{-1}(T-t)\rfloor} \frac{\delta L(t,T-j\delta)}{1+\delta L(t,T-j\delta)}\gamma(t,T-j\delta) & t\in[0,T-\delta] \\ & \&\ T-\delta>0, \\ 0 & \text{otherwise}, \end{cases} \tag{3}$$

where δ is some designated length of time,[4] *$\lfloor \delta^{-1}(T-t)\rfloor$ denotes the greatest integer that is less than $\delta^{-1}(T-t)$ and the deterministic function $\gamma : R_+^2 \to R^m$ is bounded and piecewise continuous.*

Note that $\{\sigma_B(t,T)\}_{t\in[0,T]}$ stands for the volatility process of the bond price $B(t,T)$ according to the derivation process in BGM (1997). Moreover, equation (2) is a stochastic process of a forward LIBOR rate under the spot martingale measure $\mathcal{Q}$. As to pricing interest rate derivatives, it is useful to know the stochastic processes of forward LIBOR rates under a forward martingale measure. The following proposition provides a general

[4] For the ease of computing equation (3), we may fix δ (for example, $\delta = 0.5$).

rule under which the dynamics of forward LIBOR rates are changed following a change in the underlying measure:

Proposition 2. The Drift Adjustment Technique in Different Measures

The dynamics of a forward LIBOR rate $L(t,T)$ *under an arbitrary forward martingale measure* $\mathcal{Q}^S$ *is given as follows:*

$$dL(t,T) = L(t,T)\gamma(t,T)\cdot\Big(\sigma_B(t,T+\delta)-\sigma_B(t,S)\Big)dt + L(t,T)\gamma(t,T)\cdot dZ(t) \tag{4}$$

where $0 \le t \le \min(S,T)$.[5]

According to the definition of the bond volatility process in (3), $\{\sigma_B(t,\cdot)\}_{t\in[0,\cdot]}$ in (2) and (4) are stochastic rather than deterministic. Therefore, the stochastic differential equations (2) and (4) are not solvable, and the distribution of $L(T,T)$ is unknown. To solve this problem, we present a technique that freezes the calendar time of the process $\{L(t,T-j\delta)\}_{t\in[0,T-j\delta]}$ in (3) at its initial time 0, and the resulting process is defined as follows:

$$\bar{\sigma}_B^0(t,T) = \begin{cases} \displaystyle\sum_{j=1}^{\lfloor \delta^{-1}(T-t)\rfloor} \frac{\delta L(0,T-j\delta)}{1+\delta L(0,T-j\delta)}\gamma(t,T-j\delta), & t\in[0,T-\delta] \\ & \& \; T-\delta>0, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $0 \le t \le T \le \mathcal{T}$. It is worth noting that the process $\{\bar{\sigma}_B^0(t,T)\}_{t\in[0,T]}$ is deterministic. By substituting $\bar{\sigma}_B^0(t,T+\delta)$ for $\sigma_B(t,T+\delta)$, the drift and volatility terms in (2) and (4) will be deterministic, so we can solve (2) and (4) and find the approximate distribution of $L(T,T)$ to be lognormally distributed.

This technique was first employed by BGM (1997) for pricing interest rate swaptions, developed further in Brace, Dun and Barton (1998), and formalized by Brace and Womersley (2000). This approximation has been shown to be significantly accurate and is widely used in practice. We present the result in the following proposition.

[5] We employ $Z(t)$ to denote an independent d-dimensional standard Brownian motion under an arbitrary measure.

Proposition 3. The Lognormalized LIBOR Market Model

The dynamics of a lognormalized forward LIBOR rate $L(t,T)$ under an arbitrary forward martingale measure $\mathcal{Q}^S$ is given by:

$$dL(t,T) = L(t,T)\Delta_0(t,T;S)dt + L(t,T)\gamma(t,T)\cdot dZ(t), \tag{6}$$

where

$$\Delta_0(t,T;S) = \gamma(t,T)\cdot\Big(\bar{\sigma}_P^0(t,T+\delta) - \bar{\sigma}_P^0(t,S)\Big) \tag{7}$$

and $0 \leq t \leq \min(S,T)$.

2.2 An Approximate Distribution of a Swap Rate

This subsection defines a forward swap rate in terms of forward LIBOR rates and presents an approach toward finding an approximate distribution of the swap rate under the LMM framework. Define an n-year forward swap rate at time t with reset dates $\{T_0, T_1, ..., T_{N-1}\}$ and payment dates $\{T_1, T_2, ..., T_N\}$ as follows:

$$S_n(t,T) = \sum_{i=0}^{N-1} w_{n,i}(t)L(t,T_i), \quad 0 \leq t \leq T = T_0, \tag{8}$$

where the year fraction δ is a constant and the number of payment dates $N = n/\delta$ with $\delta = T_i - T_{i-1}, i = 1, 2, ..., N$, and

$$w_{n,i}(t) = \frac{P(t,T_{i+1})}{\sum_{j=0}^{N-1} P(t,T_{j+1})}. \tag{9}$$

Equations (8) and (9) indicate that a swap rate is roughly a weighted average of LIBOR rates. Moreover, LIBOR rates under the LMM framework are approximately lognormally distributed. Therefore, the distribution of a swap rate is roughly a weighted average of lognormal distributions.

Brigo and Mercurio (2001) indicate that empirical studies have shown the variability of $w_{n,i}$ to be small compared to the variability of forward LIBOR rates.[6] Therefore, we can freeze the value of the processes $w_{n,i}(t)$ to its initial values $w_{n,i}(0)$ and obtain

$$S_n(t,T) \cong \sum_{i=0}^{N-1} w_{n,i}(0)L(t,T_i). \tag{10}$$

[6] See also Brace and Womersley (2000) for the proof of low variability.

Note that $S_n(T,T)$ is a weighted average of lognormally distributed variables and that its distribution is unknown. Although $S_n(T,T)$ is not a lognormal distribution, it can be well approximated by a lognormal distribution with the correct first two moments.[7] The accuracy of this technique is examined in Mitchell (1968). Furthermore, many areas of science verify the high degree of accuracy of the lognormal approximation for the sum of lognormal random variables (see e.g., Aitchison and Brown (1957), Crow and Shimizu (1988), Levy (1992), and Limpert Stahel and Abbt (2001)). In addition, we provide detailed empirical results in Section 4 to show the robust accuracy of our pricing formulas derived based on moment matching approximation.

Based on the studies cited above, we assume that $\ln S_n(T,T)$ follows a normal distribution with mean α and variance β^2. The moment generating function for $\ln S_n(T,T)$ is given by

$$\mathrm{G}_{\ln S}(h) = \mathrm{E}[S_n(T,T)^h] = \exp\left(\alpha h + \frac{1}{2}\beta^2 h^2\right). \tag{11}$$

Taking $h = 1$ and $h = 2$ in (11), we obtain the following two conditions to solve for α and β^2:[8]

$$\alpha = 2\ln\mathrm{E}[S_n(T,T)] - \frac{1}{2}\ln\mathrm{E}[S_n(T,T)^2], \tag{12}$$

$$\beta^2 = \ln\mathrm{E}[S_n(T,T)^2] - 2\ln\mathrm{E}[S_n(T,T)]. \tag{13}$$

2.3 The Approximation Distribution of the Difference between Two Lognormal Distributions

This subsection presents a generalized family of lognormal distributions and uses these distributions to price CMS spread options with nonzero strike rates. Assume that CMS_m and CMS_n denote, respectively, an m-year and an n-year CMS rates following lognormal distributions, and K is a nonnegative constant. The problem is how to compute the

[7] As indicated by the empirical studies in Brigo and Mercurio (2001), forward swap rates obtained from lognormal forward LIBOR rates are not far from being lognormal under the relevant measure.

[8] $\mathrm{E}[S_n(t,T)]$ and $\mathrm{E}[S_n(t,T)^2]$ are computable.

expectation given as follows:

$$\mathrm{E}\big[(X-K)^{+}\big], \tag{14}$$

where $X = \mathrm{CMS}_m - \mathrm{CMS}_n$ and $(a)^{+} = \mathrm{Max(a,0)}$.

As set forth in Subsection 2.2, the sum of lognormal random variables can be well approximated by a suitable lognormal random variable. However, the difference between two lognormal random variables, X, cannot be approximated directly by a lognormal random variable due to its potentially negative value and negative skewness. Therefore, we employ a generalized family of lognormal distributions to approximate the distribution of the difference.

The generalized family of lognormal distributions includes four types: regular, shifted, negative and negative-shifted. Their probability density functions are given as follows:

(a) Regular Lognormal Distribution:

$$f(y) = \frac{1}{\xi y\sqrt{2\pi}} \exp\Big(-\frac{1}{2\xi^2}\big(\log(y)-\mu\big)^2\Big), \quad y>0, \tag{15}$$

(b) Shifted Lognormal Distribution:

$$f(y) = \frac{1}{\xi (y-\tau)\sqrt{2\pi}} \exp\Big(-\frac{1}{2\xi^2}\big(\log(y-\tau)-\mu\big)^2\Big), \quad y>\tau, \tag{16}$$

(c) Negative Lognormal Distribution:

$$f(y) = \frac{-1}{\xi y\sqrt{2\pi}} \exp\Big(-\frac{1}{2\xi^2}\big(\log(-y)-\mu\big)^2\Big), \quad y<0, \tag{17}$$

(d) Negative-Shifted Lognormal Distribution:

$$f(y) = \frac{1}{\xi (-y-\tau)\sqrt{2\pi}} \exp\Big(-\frac{1}{2\xi^2}\big(\log(-y-\tau)-\mu\big)^2\Big), \quad y<-\tau, \tag{18}$$

where τ, μ and ξ denote, respectively, the location, scale and shape parameters. By observing these four distributions, the relation among them is that if a random variable U has a regular lognormal distribution, the random variable $U+\tau$ has a shifted lognormal distribution; $-U$ has a negative lognormal distribution; $-(U+\tau)$ has a negative-shifted lognormal distribution.

Denote, respectively, the first three moments of the general lognormal random variables Y by $M_1(\tau, \mu, \xi)$, $M_2(\tau, \mu, \xi)$ and $M_3(\tau, \mu, \xi)$, which are functions of τ, μ and ξ, and can be computed by the following proposition for each type.

Proposition 4. *(1) If Y has a regular lognormal distribution, then its first three moments in terms of the parameters τ, μ and ξ can be computed as follows:*

$$M_1(\tau, \mu, \xi) = \exp\{\mu + \frac{1}{2}\xi^2\}$$

$$M_2(\tau, \mu, \xi) = \exp\{2\mu + 2\xi^2\}$$

$$M_3(\tau, \mu, \xi) = \exp\{3\mu + \frac{9}{2}\xi^2\}.$$

(2) If Y has a shifted lognormal distribution, then its first three moments can be computed as follows:

$$M_1(\tau, \mu, \xi) = \tau + \exp\{\mu + \frac{1}{2}\xi^2\}$$

$$M_2(\tau, \mu, \xi) = \tau^2 + 2\tau \exp\{\mu + \frac{1}{2}\xi^2\} + \exp\{2\mu + 2\xi^2\}$$

$$M_3(\tau, \mu, \xi) = \tau^3 + 3\tau^2 \exp\{\mu + \frac{1}{2}\xi^2\} + 3\tau \exp\{2\mu + 2\xi^2\} + \exp\{3\mu + \frac{9}{2}\xi^2\}.$$

(3) If Y has a negative lognormal distribution, then its first three moments are computed as follows:

$$M_1(\tau, \mu, \xi) = -\exp\{\mu + \frac{1}{2}\xi^2\}$$

$$M_2(\tau, \mu, \xi) = \exp\{2\mu + 2\xi^2\}$$

$$M_3(\tau, \mu, \xi) = -\exp\{3\mu + \frac{9}{2}\xi^2\}.$$

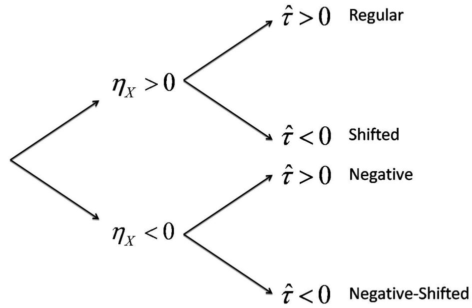
(4) If Y has a negative-shifted lognormal distribution, then its first three moments are computed as follows:

$$M_1(\tau, \mu, \xi) = -\tau - \exp\{\mu + \frac{1}{2}\xi^2\}$$

$$M_2(\tau, \mu, \xi) = \tau^2 + 2\tau \exp\{\mu + \frac{1}{2}\xi^2\} + \exp\{2\mu + 2\xi^2\}$$

$$M_3(\tau, \mu, \xi) = -\tau^3 - 3\tau^2 \exp\{\mu + \frac{1}{2}\xi^2\} - 3\tau \exp\{2\mu + 2\xi^2\} - \exp\{3\mu + \frac{9}{2}\xi^2\}.$$

Figure 2: The Determinant Procedure of the Generalized Lognormal Distribution



In addition, despite the fact that the distribution of X is unknown, the first three moments of X, namely $\mathrm{E}X$, $\mathrm{E}X^2$ and $\mathrm{E}X^3$, are computable. The parameters of the general lognormal random variable Y (which is used to approximate X) are obtained by the moment matching method, namely, solving the following system of equations:

$$\begin{aligned} M_1(\tau,\mu,\xi) &= EX \\ M_2(\tau,\mu,\xi) &= EX^2 \qquad \Rightarrow \qquad \hat{\tau},\ \hat{\mu},\ \hat{\xi}. \\ M_3(\tau,\mu,\xi) &= EX^3 \end{aligned} \tag{19}$$

The remaining question is how to choose an appropriate distribution from the generalized family of lognormal distributions. The determinant criterion of the approximating distribution $\hat{Y}$ for X depends on $\hat{\tau}$, computed from (19), and the skewness of X, which is defined as follows:

$$\eta_X = \frac{\mathrm{E}\big[X - \mathrm{E}X\big]^3}{\big(\mathrm{E}\big[X - \mathrm{E}X\big]^2\big)^{1.5}}. \tag{20}$$

The determinant procedure is specified as follows. If X has a positive skewness, namely $\eta_X > 0$, then we use the shifted lognormal random variable Y to compute (19) and obtain $\hat{\tau}$. Next, if $\hat{\tau} \geq 0$, then $\hat{Y}$ follows the regular lognormal distribution; otherwise, $\hat{Y}$ follows the shifted distribution. Otherwise, if X has a negative skewness, namely $\eta_X < 0$, we use the negative-shifted Y to compute (19) and obtain $\hat{\tau}$. Next, if $\hat{\tau} \geq 0$, then $\hat{Y}$ follows the negative lognormal distribution; otherwise, $\hat{Y}$ follows the negative-shifted distribution. The procedure is depicted in Figure 2.

After determining the approximating distribution $\hat{Y}$ (one of the four types: (15) $\sim$ (18)) and its parameters (computed from (19)), equation (14) can be approximated by

$$\mathrm{E}\big[(X-K)^+\big] \approx \mathrm{E}\big[(\hat{Y}-K)^+\big]. \tag{21}$$

Equation (21) can be computed analytically since the probability distribution of $\hat{Y}$ is known. We use this technique to price CMS spread options in the next section.

3 Valuation of CMS Spread Options

CMS spread options with nonzero strike rates within the LMM framework are priced in this section. The resulting pricing formula is capable of pricing the other three popular interest rate spread options: the difference between a CMS rate and a LIBOR rate; a LIBOR rate and a CMS rate; and a LIBOR rate and another LIBOR rate.

Consider a generalized CMS spread option on the difference between an m-year CMS rate and an n-year CMS rate, whose final payoff is specifically given as follows:

$$C_1(T) = \big(S_m(T,T) - S_n(T,T) - K\big)^+,$$

where K is a nonnegative strike rate and the definition of the swap rates is presented in (8).

CMS spread options are usually embedded in the CMS range accruals and the CMS steepeners, which are very popular in the structured notes market. CMS spread options are traded by investors who wish to take a position on future relative changes in various ranges of the yield curve; they can be tailored to hedge risk that depends upon whether the difference between two interest rates is above or below a specified level, or within or outside a specified range on a specific future date. In addition, CMS spread options can also be used as ancillary instruments for a two-way constant maturity swap. For example, an end-user can employ a two-way constant maturity swap to capitalize on anticipated yield curve movements while purchasing CMS spread options to eliminate downside risk.

Based on the martingale pricing method, the price of a CMS spread option can be computed under the forward martingale probability measure $\mathcal{Q}^T$ as follows:

$$C_1(0) = B(0,T)\mathrm{E}^{\mathcal{Q}^T}\Big[\big(S_m(T,T) - S_n(T,T) - K\big)^+\Big], \tag{22}$$

which cannot be analytically dervied since the probability distribution of the difference, $S_m(T,T) - S_n(T,T)$, is unknown. However, the approximate pricing formula of the CMS spread option can be computed based on the method introduced in Section 2. Let $X = S_m(T,T) - S_n(T,T)$, then compute $\mathrm{E}^{\mathcal{Q}^T}[X]$, $\mathrm{E}^{\mathcal{Q}^T}[X^2]$ and $\mathrm{E}^{\mathcal{Q}^T}[X^3]$, which are presented in Appendix A. Via the determinant criterion of the approximating distribution, we can find an appropriate approximating distribution, $\hat{Y}$, for X, then the pricing formula of the CMS spread option can be computed under the approximating distribution. The pricing formula within each type of the approximating distributions is derived in the following theorem. The proof is given in Appendix B.

Theorem 1. *The pricing formulas of CMS spread options within each type of the approximating distributions are given as follows.*

(i) If $\hat{Y}$ follows a regular lognormal distribution,

$$C_1(0) = B(0,T)\left(E^{\mathcal{Q}^T}[X]N\left(\frac{\ln\left(\frac{E^{\mathcal{Q}^T}[X]}{K}\right) + \frac{1}{2}\xi^2}{\xi}\right) - K \times N\left(\frac{\ln\left(\frac{E^{\mathcal{Q}^T}[X]}{K}\right) - \frac{1}{2}\xi^2}{\xi}\right)\right), \tag{23}$$

$$\textit{where} \quad \xi = \sqrt{\ln\left(\frac{E^{\mathcal{Q}^T}[X^2]}{\left(E^{\mathcal{Q}^T}[X]\right)^2}\right)}.$$

(ii) If $\hat{Y}$ follows a shifted-regular lognormal distribution,

$$C_1(0) = B(0,T)\left(\left(E^{\mathcal{Q}^T}[X]-\tau\right)N\left(\frac{\ln\left(\frac{E^{\mathcal{Q}^T}[X]-\tau}{K-\tau}\right)+\frac{1}{2}\xi^2}{\xi}\right)\right.$$
$$\left.-(K-\tau)N\left(\frac{\ln\left(\frac{E^{\mathcal{Q}^T}[X]-\tau}{K-\tau}\right)-\frac{1}{2}\xi^2}{\xi}\right)\right), \quad (24)$$

$$\textit{where} \quad \xi = \sqrt{\ln\left(\frac{E^{\mathcal{Q}^T}[(X-\tau)^2]}{\left(E^{\mathcal{Q}^T}[X]-\tau\right)^2}\right)}.$$

(iii) If $\hat{Y}$ follows a negative lognormal distribution,

$$C_1(0) = B(0,T)\left(E^{\mathcal{Q}^T}[X]N\left(-\frac{\ln\left(\frac{E^{\mathcal{Q}^T}[X]}{K}\right)+\frac{1}{2}\xi^2}{\xi}\right)\right.$$
$$\left.-K\times N\left(-\frac{\ln\left(\frac{E^{\mathcal{Q}^T}[X]}{K}\right)-\frac{1}{2}\xi^2}{\xi}\right)\right), \quad (25)$$

$$\textit{where} \quad \xi = \sqrt{\ln\left(\frac{E^{\mathcal{Q}^T}[X^2]}{\left(E^{\mathcal{Q}^T}[X]\right)^2}\right)}.$$

(iv) If $\hat{Y}$ follows a negative-shifted lognormal distribution,

$$
\begin{aligned}
C_1(0) =& B(0,T)\left(\left(E^{\mathcal{Q}^T}[X]+\tau\right)N\left(-\frac{\ln\left(\dfrac{E^{\mathcal{Q}^T}[X]+\tau}{K+\tau}\right)+\frac{1}{2}\xi^2}{\xi}\right)\right.\\
&\left.-(K+\tau)N\left(-\frac{\ln\left(\dfrac{E^{\mathcal{Q}^T}[X]+\tau}{K+\tau}\right)-\frac{1}{2}\xi^2}{\xi}\right)\right),
\end{aligned}
\tag{26}
$$

$$
\textit{where} \quad \xi = \sqrt{\ln\left(\frac{E^{\mathcal{Q}^T}[(X+\tau)^2]}{\left(E^{\mathcal{Q}^T}[X]+\tau\right)^2}\right)}.
$$

$\mathrm{E}^{\mathcal{Q}^T}[X]$, $\mathrm{E}^{\mathcal{Q}^T}[X^2]$ *and* $\mathrm{E}^{\mathcal{Q}^T}[X^3]$, *in (23)* $\sim$ *(26), are computed in Appendix A.*

The resulting pricing formulas for CMS spread options, (23) $\sim$ (26), somewhat resemble the Black formula and thus provide familiarity to end-users in their employment. In addition, all the parameters needed in the pricing formulas can be easily extracted from market data, which makes the pricing formulas more tractable and feasible for practitioners. The parameter calibration procedure is presented in Subsection 4.1.

Occasionally, interest rate spread options embedded in structured notes are options on the difference between CMS rates and LIBOR rates. These options are special cases of (22) since a LIBOR rate is a one-period swap rate. For example, an interest rate spread option on the difference between an m-year CMS rate and a 6-month LIBOR rate, denoted by $L_{6m}(T,T)$, can be specified as follows:

$$
\begin{aligned}
C_2(T) &= (S_m(T) - L_{6m}(T,T) - K)^+ \\
&= (S_m(T) - S_{0.5}(T) - K)^+,
\end{aligned}
$$

which is a CMS spread option on the difference between an m-year CMS rate and a 0.5-year CMS rate. This option can also be priced by using the pricing formulas in Theorem

1. Moreover, the following interest rate spread options can be priced via Theorem 1 as well:

$$C_3(T) = (L_{6m}(T,T)) - S_n(T) - K)^+$$
$$= (S_{0.5}(T) - S_n(T) - K)^+,$$

and

$$C_4(T) = (L_{6m}(T,T)) - L_{3m}(T,T) - K)^+$$
$$= (S_{0.5}(T) - S_{0.25}(T) - K)^+.$$

Therefore, the pricing formulas in Theorem 1 are capable of pricing many popular interest rate spread options.

We derive the pricing formulas for CMS spread options under various approximation techniques. Without these formulas, CMS spread options must be computed based on Monte Carlo simulation, which is well-known to be time-consuming and inefficient. As shown in the next section, our pricing formulas provide prices close to those computed from Monte Carlo simulation, while consuming much less time. This feature is a vital advantage and provides market practitioners with a new, efficient and time-saving approach to offer almost instantly quoted prices to clients, the daily mark-to-market of trading books, and managing risks of trading positions.

4 Parameter Calibration and Numerical Examples

This section first introduces a calibration method for the parameters in the LMM and then presents numerical examples to demonstrate the implementation of the pricing formulas derived in Section 3. Finally, we examine their accuracy by comparing them with Monte Carlo simulation.

4.1 Parameter Calibration

The Black formulas of caps and floors are extensively employed for price quotations in market practice. Since the cap and floor pricing formulas within the LMM framework are, in fact, the Black formulas, model volatilities can be extracted directly from quoted implied volatilities for cap (floor) prices. However, the correlation matrix of forward LIBOR rates cannot be extracted from quotations of cap prices since the standard pricing formula of caplets involves only a single forward LIBOR rate. In practice, two approaches can be employed to calibrate correlations between LIBOR rates. The first is presented by Rebonato (1999), who applies a historical correlation matrix to engage in calibration and the second is based on price quotations of swaptions.[9] Both approaches are tractable and widely-used in the marketplace.

In this paper, we adopt the Rebonato (1999) approach to engage in a simultaneous calibration of the LMM to the volatilities and correlation matrix of forward LIBOR rates.[10] We assume that there are n forward LIBOR rates in the m-factor framework. The steps to calibrate the parameters are given as follows.

First, we assume that each forward LIBOR rate, $L(\cdot, t_i)$, has a constant instantaneous volatility, namely for $i = 1, ..., n$, $\gamma(\cdot, t_i) = v_i$. This setting is presented in Table 1.[11] Thus, if the market-quoted volatility for t_1-year cap is ζ_1, then $v_1 = \zeta_1$. Next, for $i = 2, ..., n$, if the t_i-year cap is ζ_i, then $v_i = \zeta_i^2 t_i^2 - \zeta_{i-1}^2 t_{i-1}^2$.

Table 1: Instantaneous Volatilities of $L(t, \cdot)$

Instant. Total Vol.	Time t $\in (t_0, t_1]$	$(t_1, t_2]$	$(t_2, t_3]$	$\cdots$	$(t_{n-2}, t_{n-1}]$
Fwd Rate: $L(t, t_1)$	v_1	Dead	Dead	$\cdots$	Dead
$L(t, t_2)$	v_2	v_2	Dead	$\cdots$	Dead
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$L(t, t_n)$	v_n	v_n	v_n	$\cdots$	v_n

Second, we use the historical data of forward LIBOR rates to derive a market correlation matrix Σ. Σ is an n-rank ($n \geq d$), positive-definite and symmetric matrix and can

[9] For this approach, please refer to Brigo and Mercurio (2001) for details.

[10] Since caps and swaptions are actively traded financial instruments, price inconsistency between the two products is almost impossible. Thus, a calibration based only on cap data is not unreasonable. In addition, even if end-users adopt the other calibration approach, our pricing formulas remain workable.

[11] For other assumptions of volatility structures, please refer to Brigo and Mercurio (2001).

be written as

$$\Sigma = H\Gamma H,$$

where H is a real orthogonal matrix and Γ is a diagonal matrix. Let $A \equiv H\Gamma^{1/2}$ and thus, $\Sigma = AA'$. In this way, we can find an d-rank ($d \leq n$) matrix B such that $\Sigma^B = BB'$ is an approximate correlation matrix for Σ.

The advantage of finding B is that we may replace the n-dimensional original Brownian motion $dZ(t)$ with $BdW(t)$ where $dW(t)$ is a d-dimensional Brownian motion. In other words, we change the market correlation structure

$$dZ(t)dZ(t)' = \Sigma dt$$

to an approximate correlation structure

$$BdW(t)(BdW(t))' = BdW(t)dW(t)'B' = BB'dt = \Sigma^B dt.$$

The remaining problem is how to find a suitable matrix B. Rebonato (1999) proposes a method — assume that the ik-th element of B for $i = 1, 2, ..., n$ is specified as follows:

$$b_{i,k} = \begin{cases} \cos\theta_{i,k} \ \Pi_{j=1}^{k-1} \sin\theta_{i,j} & \text{if } k = 1, 2, ..., d-1, \\ \Pi_{j=1}^{k-1} \sin\theta_{i,j} & \text{if } k = d. \end{cases}$$

Thus, $\Sigma^B = BB'$ is a function of $\Theta = \left\{\theta_{i,k}\right\}_{i=1,...,n;k=1,...,d-1}$. We obtain optimal solution $\hat{\Theta}$ by solving the following optimization problem

$$\min_{\Theta} \sum_{i,j=1}^{n} |\Sigma_{i,j}^B - \Sigma_{i,j}|^2, \tag{27}$$

where $\Sigma_{i,j}$ is the ij-th element of Σ and $\Sigma_{i,j}^B$ is the ij-th element of Σ^B, specifically defined as follows:

$$\Sigma_{i,j}^B = \sum_{k=1}^{d} b_{i,k} b_{j,k}.$$

By substituting $\hat{\Theta}$ into B, we obtain optimal matrix $\hat{B}$ such that $\hat{\Sigma}^B(= \hat{B}\hat{B}')$ is an approximate correlation matrix for Σ.

Third, we use $\hat{B}$ to distribute the instantaneous total volatility, v_i, to each Brownian motion without changing the amount of the instantaneous total volatility.[1] That is,

$$v_i(\hat{b}_{i-k+1,1}, \hat{b}_{i-k+1,2}, \ldots, \hat{b}_{i-k+1,d}) = (\gamma_1(t, t_i), \gamma_2(t, t_i), \ldots, \gamma_d(t, t_i))$$

where $t_{k-1} \leq t < t_k$, $k = 1, ..., i$ and $i = 1, 2, ..., n$.

The above procedure is a general calibration method with no constraint on choosing the number of factors, d. The number of random shocks, d, may depend on the maturity range of interest rates involved in the financial products considered.[12] For example, we may use a three-factor model, i.e., $d = 3$, to capture the shift and twist of the entire yield curve. The first two random shocks can be interpreted, respectively, as the short-term and long-term factors causing a shift in various maturity ranges on the yield curve. The correlation between the short-term and long-term interest rates is specified by the third random shock. In accordance with this feature, the numerical examples in the following section are based on a three-factor model.

4.2 Numerical Examples

This subsection provides numerical examples to examine the accuracy of derived pricing formulas of CMS spread options via comparison with Monte Carlo simulation. We price three types of CMS spread options, 30-year CMS rate vs. 2-year CMS rate; 30-year CMS rate vs. 10-year CMS rate; and 10-year CMS rate vs. 2-year CMS rate. In each case, we consider three times to maturity, 1, 3 and 5 years, and three strike rates, 20, 40 and 60 basis points. In addition, the notional principal is assumed to be $1 and the simulations are based on 10,000 paths.

These CMS spread options are priced quarterly on the dates for the recent three years, namely, 2009/09/01, 2009/06/01, 2009/03/02, 2008/12/01, 2008/09/01, 2008/06/02, 2008/03/03, 2007/12/03, 2007/09/03, 2007/06/01, 2007/03/01, 2006/12/01 and 06/09/01. Market data for these dates are obtained from the Datastream database and are omitted to conserve space.

Results from these numerical examples are presented in Table 2. Prices expressed in basis points are shown in the form (A, B), where A denotes the value computed from Theorem 1 and B the value from Monte Carlo simulation. The standard error of Monte Carlo simulation for each case is about 0.5 basis points, which is omitted to save space.

[12]For more details regarding the performance of one- and multi-factor models, please refer to Driessen, Klaassen and Melenberg (2003) and Rebonato (1999).

The CMS-spread-option price, A, for each case listed in Table 2 is computed via one of the four formulas presented in Theorem 1. The determinant procedure, based on the parameters $\hat{\tau}$ and η_X, is shown in Figure 2 and, empirically, it depends on the shape of the term structure of interest rates. As the term structure is significantly upward-sloping, a formula with a shifted lognormal distribution is chosen to price CMS spread options; otherwise, a negative-shifted formula is employed. For example, the negative-shifted formula is used on date 2008/06/02, while the shifted formula is adopted on date 2006/09/01.

By observing these numerical examples, the pricing bias is not greater than one basis point, which shows that the approximate formulas are sufficiently and robustly accurate. Therefore, the derived pricing formulas yield prices close to Monte Carlo simulation but taking much less time. With this efficiency advantage, the pricing formulas are recommended for practical implementation.

5 Conclusion

This article uses the generalized lognormal distribution to approximate the distribution of the difference between two CMS rates, then derives the pricing formulas for the CMS spread options within the multifactor LMM. As compared with traditional interest rate models, the LIBOR rate specified in the LMM is market-observable and the model parameters can be easily extracted from market data. Therefore, the pricing models are feasible and tractable for implementation in practice. We employ Monte Carlo simulation to examine the accuracy of the pricing formulas by using past three-year market data. The results show that our pricing formulas are sufficiently and robustly accurate.

Table 2: Numerical Examples of CMS spread Options

Date	Type	TM/K	20 bps	40 bps	60 bps
2009/09/01	30y -2y	1	(190.9, 190.4)	(172.5, 172.2)	(154.3, 154.3)
		3	(128.7, 128.7)	(112.9, 112.2)	(97.5, 97.4)
		5	(103.6, 104.4)	(89.1, 89.8)	(75.0, 75.4)
	30y -10y	1	(27.0, 27.0)	(13.3, 13.4)	(3.6, 3.7)
		3	(26.3, 26.0)	(13.5, 13.4)	(4.0, 4.1)
		5	(30.1, 30.3)	(16.5, 16.5)	(5.4, 5.9)
	10y -2y	1	(152.2, 151.7)	(133.8, 133.8)	(115.7, 114.9)
		3	(91.4, 90.7)	(75.8, 75.1)	(60.7, 59.9)
		5	(62.1, 62.4)	(48.4, 48.8)	(35.5, 36.0)
2009/06/01	30y -2y	1	(192.0, 192.4)	(173.1, 173.2)	(154.4, 154.4)
		3	(90.5, 90.5)	(76.1, 76.1)	(62.3, 62.2)
		5	(56.4, 56.7)	(44.2, 44.2)	(33.0, 33.1)
	30y -10y	1	(13.7, 13.7)	(4.1, 4.1)	(0.3, 0.3)
		3	(8.5, 8.4)	(1.8, 1.8)	(0.0, 0.1)
		5	(7.0, 6.9)	(0.8, 0.8)	(0.0, 0.1)
	10y -2y	1	(171.0, 171.0)	(151.8, 151.7)	(133.0, 132.9)
		3	(77.4, 77.7)	(62.3, 61.9)	(48.0, 47.8)
		5	(42.3, 42.2)	(30.2, 30.1)	(19.4, 19.7)

* TM, K and bps denote the time to maturity, the strike rate and basis points, respectively. "30y-2y" means the CMS spread options on the difference between a 30-year CMS and a 2-year CMS, others are defined accordingly. The prices expressed in basis points are presented in the form (A, B), where A denotes the value computed from Theorem 1 and B the value from Monte Carlo simulation. The standard error of Monte Carlo simulation in each case is about 0.5 basis points, which is omitted to save space.

Table 2: Numerical Examples of CMS spread Options (Continued)

Date	Type	TM/K	20 bps	40 bps	60 bps
2009/03/02	30y -2y	1	(122.9, 123.0)	(104.4, 104.4)	(86.5, 86.5)
		3	(63.5, 63.5)	(49.4, 49.4)	(36.3, 36.3)
		5	(45.9, 46.7)	(33.4, 34.2)	(22.2, 23.1)
	30y -10y	1	(4.4, 4.4)	(0.2, 0.2)	(0.0, 0.0)
		3	(3.7, 3.7)	(0.0, 0.0)	(0.0, 0.0)
		5	(3.6, 3.9)	(0.0, 0.0)	(0.0, 0.0)
	10y -2y	1	(115.0, 115.0)	(95.8, 95.8)	(77.4, 77.4)
		3	(56.7, 56.3)	(42.0, 41.9)	(28.5, 28.0)
		5	(35.6, 35.6)	(23.3, 23.1)	(12.8, 13.7)
2008/12/01	30y -2y	1	(69.8, 69.8)	(54.2, 54.1)	(39.8, 39.8)
		3	(51.8, 51.8)	(38.9, 38.6)	(27.0, 27.3)
		5	(48.8, 49.7)	(36.1, 36.9)	(24.3, 24.9)
	30y -10y	1	(1.5, 1.5)	(0.0, 0.0)	(0.0, 0.0)
		3	(6.0, 6.1)	(0.2, 0.2)	(0.0, 0.0)
		5	(16.2, 16.9)	(4.1, 4.9)	(0.0, 0.1)
	10y -2y	1	(73.7, 73.5)	(57.8, 54.7)	(43.2, 42.4)
		3	(38.5, 38.3)	(25.6, 25.9)	(14.3, 15.0)
		5	(23.1, 23.9)	(12.4, 13.1)	(4.1, 4.9)
2008/09/01	30y -2y	1	(85.4, 85.4)	(68.1, 68.1)	(52.0, 52.1)
		3	(66.2, 66.2)	(52.4, 52.4)	(39.6, 39.5)
		5	(58.8, 58.8)	(46.1, 46.0)	(40.2, 39.3)
	30y -10y	1	(12.4, 12.4)	(2.8, 2.8)	(0.1, 0.1)
		3	(13.1, 13.1)	(3.7, 3.7)	(0.1, 0.1)
		5	(12.9, 13.1)	(3.2, 3.3)	(0.6, 0.5)
	10y -2y	1	(59.8, 59.9)	(43.5, 43.5)	(29.4, 29.0)
		3	(42.9, 42.9)	(30.3, 29.9)	(19.5, 19.0)
		5	(35.9, 35.5)	(24.8, 23.9)	(19.9, 20.3)

Table 2: The Numerical Examples of CMS spread Options (Continued)

Date	Type	TM/K	20 bps	40 bps	60 bps
2008/06/02	30y -2y	1	(105.4, 105.6)	(87.6, 88.0)	(70.6, 70.6)
		3	(78.8, 78.8)	(64.3, 64.3)	(50.7, 50.7)
		5	(65.5, 65.7)	(52.5, 52.5)	(46.3, 45.4)
	30y -10y	1	(20.2, 20.2)	(7.3, 7.4)	(1.0, 1.1)
		3	(17.5, 17.5)	(6.5, 6.6)	(0.7, 0.7)
		5	(14.4, 14.5)	(4.2, 4.3)	(1.2, 1.1)
	10y -2y	1	(69.7, 69.7)	(52.8, 52.6)	(37.6, 37.6)
		3	(49.7, 48.9)	(36.3, 36.4)	(24.5, 24.3)
		5	(40.6, 40.1)	(28.8, 28.2)	(23.5, 22.7)
2008/03/03	30y -2y	1	(203.7, 204.1)	(184.4, 184.6)	(165.2, 165.2)
		3	(134.9, 135.2)	(118.8, 118.8)	(103.1, 103.1)
		5	(100.1, 100.7)	(85.9, 86.2)	(79.1, 78.9)
	30y -10y	1	(35.8, 36.1)	(19.8, 19.9)	(7.4, 7.4)
		3	(26.5, 26.2)	(13.7, 13.7)	(4.4, 4.4)
		5	(20.6, 20.9)	(9.2, 9.2)	(4.7, 4.3)
	10y -2y	1	(151.4, 151.4)	(132.0, 132.0)	(112.8, 112.8)
		3	(97.5, 96.9)	(81.6, 81.1)	(66.4, 66.1)
		5	(69.2, 69.0)	(55.5, 55.1)	(48.9, 48.1)
2007/12/03	30y -2y	1	(116.8, 116.8)	(97.8, 97.8)	(79.3, 79.3)
		3	(83.0, 83.2)	(68.1, 68.0)	(54.0, 53.9)
		5	(63.8, 63.8)	(50.9, 50.8)	(44.8, 44.3)
	30y -10y	1	(18.4, 18.4)	(5.7, 5.7)	(0.5, 0.5)
		3	(15.5, 15.5)	(5.0, 5.0)	(0.3, 0.3)
		5	(12.8, 12.9)	(3.1, 3.3)	(0.6, 0.5)
	10y -2y	1	(81.9, 81.9)	(63.5, 63.5)	(46.4, 46.4)
		3	(55.8, 55.7)	(41.9, 41.5)	(29.6, 29.3)
		5	(40.6, 40.5)	(29.1, 28.9)	(23.9, 23.3)

Table 2: Numerical Examples of CMS spread Options (Continued)

Date	Type	TM/K	20 bps	40 bps	60 bps
2007/09/03	30y -2y	1	(64.9, 64.9)	(48.3, 48.3)	(33.6, 33.6)
		3	(57.0, 57.0)	(43.4, 43.4)	(31.4, 31.3)
		5	(47.0, 46.7)	(35.4, 35.5)	(30.0, 29.4)
	30y -10y	1	(10.0, 10.0)	(1.6, 1.6)	(0.0, 0.0)
		3	(10.5, 10.5)	(2.2, 2.2)	(0.0, 0.1)
		5	(9.5, 9.5)	(1.5, 1.5)	(0.2, 0.2)
	10y -2y	1	(40.9, 40.9)	(26.3, 26.2)	(15.1, 15.3)
		3	(35.5, 35.3)	(23.6, 23.2)	(14.2, 14.3)
		5	(27.7, 27.4)	(18.1, 18.0)	(14.1, 14.3)
2007/06/01	30y -2y	1	(25.0, 24.9)	(13.7, 13.7)	(6.5, 6.5)
		3	(27.4, 27.4)	(17.0, 16.8)	(9.7, 9.6)
		5	(27.4, 27.3)	(17.7, 17.6)	(10.6, 10.4)
	30y -10y	1	(2.9, 2.8)	(0.1, 0.1)	(0.0, 0.0)
		3	(4.0, 3.9)	(0.4, 0.3)	(0.0, 0.0)
		5	(3.4, 3.3)	(0.2, 0.2)	(0.0, 0.0)
	10y -2y	1	(12.9, 12.9)	(5.4, 5.4)	(1.9, 1.8)
		3	(15.0, 14.9)	(7.5, 7.4)	(3.3, 3.3)
		5	(15.8, 15.6)	(8.5, 8.3)	(4.2, 4.1)
2007/03/01	30y -2y	1	(31.9, 31.9)	(18.8, 18.8)	(9.8, 9.7)
		3	(34.5, 34.5)	(22.9, 22.6)	(14.1, 13.9)
		5	(32.2, 32.2)	(21.9, 21.8)	(13.9, 13.8)
	30y -10y	1	(3.8, 3.7)	(0.2, 0.2)	(0.0, 0.0)
		3	(5.3, 5.3)	(0.6, 0.6)	(0.0, 0.0)
		5	(4.9, 4.9)	(0.4, 0.4)	(0.0, 0.0)
	10y -2y	1	(17.9, 17.8)	(8.5, 8.4)	(3.4, 3.4)
		3	(19.9, 19.7)	(11.0, 10.8)	(5.4, 5.3)
		5	(18.6, 18.5)	(10.7, 10.6)	(5.5, 5.4)

Table 2: Numerical Examples of CMS spread Options (Continued)

Date	Type	TM/K	20 bps	40 bps	60 bps
2006/12/01	30y -2y	1	(35.0, 34.9)	(21.9, 21.5)	(12.4, 12.4)
		3	(38.3, 38.3)	(27.0, 26.6)	(16.7, 16.7)
		5	(39.7, 39.7)	(28.8, 28.3)	(19.9, 19.8)
	30y -10y	1	(5.4, 5.3)	(0.5, 0.5)	(0.0, 0.0)
		3	(7.8, 7.7)	(1.5, 1.4)	(0.1, 0.1)
		5	(8.4, 8.4)	(1.6, 1.6)	(0.1, 0.1)
	10y -2y	1	(19.2, 19.2)	(9.9, 9.9)	(4.5, 4.5)
		3	(21.5, 21.5)	(12.7, 12.5)	(6.9, 6.8)
		5	(22.1, 22.0)	(13.9, 13.8)	(8.2, 8.3)
2006/09/01	30y -2y	1	(33.1, 33.1)	(20.5, 20.5)	(11.6, 11.1)
		3	(32.1, 32.0)	(21.7, 21.6)	(13.8, 13.6)
		5	(34.8, 34.8)	(24.8, 24.6)	(16.8, 16.8)
	30y -10y	1	(3.9, 3.9)	(0.3, 0.3)	(0.0, 0.0)
		3	(6.2, 6.1)	(1.1, 1.0)	(0.1, 0.1)
		5	(7.1, 7.0)	(1.3, 1.3)	(0.1, 0.1)
	10y -2y	1	(19.5, 19.5)	(10.0, 9.9)	(4.5, 4.5)
		3	(17.7, 17.5)	(10.0, 10.0)	(5.3, 5.2)
		5	(19.2, 19.1)	(11.8, 11.7)	(6.8, 6.7)

Appendix A: $E^{\mathcal{Q}^T}[X]$, $E^{\mathcal{Q}^T}[X^2]$ and $E^{\mathcal{Q}^T}[X^3]$

By applying Proposition 3, the dynamics of $\{L(t,T_i)\}$ under probability measure $\mathcal{Q}^T$ is given as follows:

$$\frac{dL(t,T_i)}{L(t,T_i)} = \Delta_0(t,T_i;T)dt + \gamma(t,T_i)\cdot dZ(t). \tag{B.1}$$

Therefore, the solution of the stochastic differential equation (B.1) is computed as follows:

$$L(T,T_i) = L(0,T_i)\exp\left(\int_0^T \left(\Delta_0(u,T_i;T) - \frac{1}{2}\|\gamma(u,T_i)\|^2\right)du + \int_0^T \gamma(u,T_i)\cdot dZ(u)\right). \tag{B.2}$$

Based on the martingale pricing method, the price of the CMS spread options specified in (26) can be computed under the forward probability measure $\mathcal{Q}^T$ as follows:

$$C(0) = \mathrm{E}^{\mathcal{Q}^T}\left[(X-K)^+\right], \tag{B.3}$$

where $X = S_m(T) - S_n(T)$, and $S_m(T)$ and $S_n(T)$ are defined in (8).

$$\begin{aligned}
\mathrm{E}^{\mathcal{Q}^T}[X] &= \mathrm{E}^{\mathcal{Q}^T}[S_m(T)] - \mathrm{E}^{\mathcal{Q}^T}[S_n(T)] \\
&= \sum_{i=0}^{M-1} w_{m,i}(0)\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)\right] - \sum_{j=0}^{N-1} w_{n,j}(0)\mathrm{E}^{\mathcal{Q}^T}\left[L_j(T)\right] \\
&= \sum_{i=0}^{M-1} w_{m,i}(0)L_i(0)\exp\left(\int_0^T \Delta_0(u,T_i;T)du\right) \\
&\qquad - \sum_{j=0}^{N-1} w_{n,j}(0)L_j(0)\exp\left(\int_0^T \Delta_0(u,T_j;T)du\right).
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}^{\mathcal{Q}^T}[X^2] &= \mathrm{E}^{\mathcal{Q}^T}[S_m(T)^2] + \mathrm{E}^{\mathcal{Q}^T}[S_n(T)^2] - 2\mathrm{E}^{\mathcal{Q}^T}[S_m(T)S_n(T)] \\
&= \mathrm{E}^{\mathcal{Q}^T}\left[\left(\sum_{i=0}^{M-1} w_{m,i}(0)L_i(T)\right)^2\right] + \mathrm{E}^{\mathcal{Q}^T}\left[\left(\sum_{j=0}^{N-1} w_{n,j}(0)L_j(T)\right)^2\right] \\
&\quad - 2\mathrm{E}^{\mathcal{Q}^T}\left[\left(\sum_{i=0}^{M-1} w_{m,i}(0)L_i(T)\right)\left(\sum_{j=0}^{N-1} w_{n,j}(0)L_i(T)\right)\right] \\
&= \sum_{i=0}^{M-1}\sum_{k=0}^{M-1} w_{m,i}w_{m,k}\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_k(T)\right] + \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} w_{n,j}w_{n,k}\mathrm{E}^{\mathcal{Q}^T}\left[L_j(T)L_k(T)\right] \\
&\quad - 2\sum_{i=0}^{M-1}\sum_{j=0}^{N-1} w_{m,i}w_{n,j}\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_j(T)\right],
\end{aligned}$$

where

$$\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_j(T)\right] = L_i(0)L_j(0)\exp\left(\int_0^T \Phi_0(u,T_i,T_j;T)du\right),$$

$$\Phi_0(u,T_i,T_j;T) = \Delta_0(u,T_i;T) + \Delta_0(u,T_j;T) + \gamma(u,T_i)\cdot\gamma(u,T_j).$$

$$\begin{aligned}
\mathrm{E}^{\mathcal{Q}^T}[X^3] =& \mathrm{E}^{\mathcal{Q}^T}[S_m(T)^3] - 3\mathrm{E}^{\mathcal{Q}^T}[S_m(T)^2S_n(T)] + 3\mathrm{E}^{\mathcal{Q}^T}[S_m(T)S_n(T)^2] - \mathrm{E}^{\mathcal{Q}^T}[S_n(T)^3] \\
=& \sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\sum_{l=0}^{M-1} w_{m,i}w_{m,k}w_{m,l}\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_k(T)L_l(T)\right] \\
&- 3\sum_{i=0}^{M-1}\sum_{k=0}^{M-1}\sum_{j=0}^{N-1} w_{m,i}w_{m,k}w_{n,j}\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_k(T)L_j(T)\right] \\
&+ 3\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} w_{m,i}w_{n,j}w_{n,k}\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_j(T)L_k(T)\right] \\
&- \sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} w_{n,j}w_{n,k}w_{n,l}\mathrm{E}^{\mathcal{Q}^T}\left[L_j(T)L_k(T)L_l(T)\right],
\end{aligned}$$

where

$$\mathrm{E}^{\mathcal{Q}^T}\left[L_i(T)L_j(T)L_k(T)\right] = L_i(0)L_j(0)L_k(0)\exp\left(\int_0^T \Psi_0(u,T_i,T_j,T_k;T)du\right),$$

$$\begin{aligned}
\Psi_0(u,T_i,T_j,T_k;T) =& \Delta_0(u,T_i;T) + \Delta_0(u,T_j;T) + \Delta_0(u,T_k;T) \\
&+ \gamma(u,T_i)\cdot\gamma(u,T_j) + \gamma(u,T_i)\cdot\gamma(u,T_k) + \gamma(u,T_j)\cdot\gamma(u,T_k).
\end{aligned}$$

Appendix B: Proof of Theorem 1

Let X denote the difference between two CMS rates, namely, $X = S_m(T,T) - S_n(T,T)$. The CMS spread option can be computed as follows:

$$C(0) = B(0,T)\mathrm{E}^{\mathcal{Q}^T}\left[\left(X-K\right)^+\right], \tag{B.1}$$

$$= B(0,T)\Big(\mathrm{E}^{\mathcal{Q}^T}\left[XI_{\{X\geq K\}}\right] - K\mathrm{E}^{\mathcal{Q}^T}\left[I_{\{X\geq K\}}\right]\Big), \tag{B.2}$$

where $I_{\{.\}}$ is an indicator function and X follows one of the four types of generalized lognormal distributions. We first consider that X follows a regular lognormal distribution.

Assume that X has a regular lognormal distribution, namely $\ln(X) \sim N(\mu, \xi^2)$, where μ and ξ can be computed in terms of $E^{\mathcal{Q}^T}[X]$ and $E^{\mathcal{Q}^T}[X^2]$ as follows:

$$\begin{aligned}\mu =& 2\ln\left(E^{\mathcal{Q}^T}[X]\right) - \frac{1}{2}\ln\left(E^{\mathcal{Q}^T}[X^2]\right),\\ \xi^2 =& \ln\left(E^{\mathcal{Q}^T}[X^2]\right) - 2\ln\left(E^{\mathcal{Q}^T}[X]\right).\end{aligned}$$

The second expectation of (B.2) is computed as follows:

$$\begin{aligned}\mathrm{E}^{\mathcal{Q}^T}\left[I_{\{X\geq K\}}\right] &= P\big(\ln(X) \geq \ln(K)\big)\\ &= N\left(\frac{\ln\left(\mathrm{E}^{\mathcal{Q}^T}[X]/K\right) - \frac{1}{2}\xi^2}{\xi}\right).\end{aligned} \tag{B.3}$$

The first expectation of (B.2) is computed as follows:

$$\begin{aligned}\mathrm{E}^{\mathcal{Q}^T}\left[XI_{\{X\geq K\}}\right] &= \mathrm{E}^{\mathcal{Q}^T}\left[\mathrm{e}^{\ln(X)}I_{\{\ln(X)\geq\ln(K)\}}\right]\\ &= \mathrm{E}^{\mathcal{Q}^T}\left[\mathrm{e}^{\xi\Phi+\mu}I_{\{\xi\Phi+\mu\geq\ln(K)\}}\right] \qquad (\text{where } \Phi\sim N(0,1)),\\ &= \mathrm{e}^{\mu}\int_{\frac{\ln(K)-\mu}{\xi}}^{\infty}\mathrm{e}^{\xi\phi}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}\phi^2}d\phi\\ &= \mathrm{e}^{\mu+\frac{1}{2}\xi^2}\int_{\frac{\ln(K)-\mu}{\xi}}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}(\phi-\xi)^2}d\phi\\ &= \mathrm{e}^{\mu+\frac{1}{2}\xi^2}\int_{\frac{\ln(K)-\mu-\xi^2}{\xi}}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\frac{-1}{2}w^2}dw \qquad (\text{let } W = \Phi - \xi)\\ &= \mathrm{e}^{\mu+\frac{1}{2}\xi^2}N\left(\frac{\mu+\xi^2-\ln(K)}{\xi}\right)\\ &= E^{\mathcal{Q}^T}[X]N\left(\frac{\ln\left(E^{\mathcal{Q}^T}[X]/K\right)+\frac{1}{2}\xi^2}{\xi}\right).\end{aligned} \tag{B.4}$$

Equations (B.2), (B.3) and (B.4) lead to (23).

Second, let $X = U + \tau$, which has a shifted lognormal distribution, where τ is a constant and U is a regular lognormal distribution with $\ln(U) \sim N(\mu, \xi^2)$. (B.1) can be rewritten as follows:

$$\begin{aligned}C(0) &= B(0,T)\mathrm{E}^{\mathcal{Q}^T}\left[\left(X-K\right)^+\right],\\ &= B(0,T)\mathrm{E}^{\mathcal{Q}^T}\left[\left(U-(K-\tau)\right)^+\right],\end{aligned} \tag{B.5}$$

By replacing K in (B.1) by $K+\tau$, the CMS spread option can be derived and is given in (24).

Third, let X have a negative lognormal distribution, namely $X=-U$, where U is a regular lognormal distribution. (B.1) can be rewritten as follows:

$$C(0) = B(0,T)\mathrm{E}^{\mathcal{Q}^T}\Big[\big(-U-K\big)^+\Big], \tag{B.6}$$

$$= B(0,T)\Big(-\mathrm{E}^{\mathcal{Q}^T}\Big[UI_{\{U\leq -K\}}\Big] - K\mathrm{E}^{\mathcal{Q}^T}\Big[I_{\{U\leq -K\}}\Big]\Big), \tag{B.7}$$

(B.7) can be computed similarly to the first and second expectation of (B.2), which leads to (25).

Fourth, let X have a negative-shifted lognormal distribution, namely $X=-(U+\tau)$, where U is a regular lognormal distribution and τ is a constant. (B.1) can be rewritten as follows:

$$C(0) = B(0,T)\mathrm{E}^{\mathcal{Q}^T}\Big[\big(-U-(K+\tau)\big)^+\Big], \tag{B.8}$$

By replacing K in (B.6) by $K+\tau$, the CMS spread option can be derived and is given in (26).

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