The valuation of vulnerable multi-asset options and hedging of credit risks

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January 13, 2011

Abstract

This paper develops a method for the change of probability measures when the component processes of the multidimensional Brownian motion are correlated and applies the method to derive formulae for pricing vulnerable options on the maximum or minimum of n underlying assets. Based on the proposed pricing formulae, we further develop a delta-like strategy to hedge the default risk of vulnerable multi-asset options. Real-time analysis shows that the hedging performance is satisfactory when we adopt the proposed hedging strategy to hedge the credit risk induced by the bankruptcy of Lehman Brothers Holdings.

Key words: Vulnerable Option; Rainbow Option; Default Risk; Delta Hedging

JEL classification: G13

EFM classification codes: 450

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1 Introduction

Options on multiple assets, including maximum options, minimum options, spread options, and quanto options, are widely traded in financial markets. Many structured financial products traded in over-the-counter markets link their payoffs with multi-asset options as well. Accordingly, the valuation and hedge of multi-asset options are of both theoretical and practical interests. Margrabe (1978) analyzes European options on exchanging one asset for another. Stulz (1982) develops formulae for European options on the maximum or minimum of two assets, whereas Johnson (1987) extends the results to the case of several assets. However, these formulae cannot be applied to value structured financial products and over-the-counter options without modifications, since these securities are exposed to potential credit risk due to the possibility of their issuers being unable to make the necessary payments at the maturity date.

A number of approaches have been developed to incorporate the impact of credit risk on the value of derivatives. These approaches can be divided into two major categories. The first group determines the event of default and recovery rate based on the evolution of the asset value of the issuer, and thus are called firm value models. The literature that employs firm value models to price contingent claims subject to default risk includes Merton (1974), Black and Cox (1976), Johson and Stulz (1987), Cooper and Mello (1991), Klein (1996), Klein and Inglis (2001), Episcopos (2004), and Baule, Entrop, and Wikens (2008), to name a few. Since the recovery rate depends on the firm's value in default, the firm value models are very intuitive.

The second group, called intensity models, specifies an exogenous default process to trigger default. Although the intensity models allow greater flexibility in the timing of a default event, they often assume the recovery rate to be exogenous. It indicates that the recovery rate is independent of the asset value of the issuer. Examples of intensity models include

Duffie and Huang (1996), Jarrow and Turnbull (1995), and Jarrow and Yu (2001), to name a few.

Klein (1996) provides a closed-form method to price vulnerable Black-Scholes options by employing the firm value model. The model in Klein (1996) not only allows for correlation between the asset of the issuer and the asset underlying the option, but also allows for the proportion of nominal claims paid out in default to depend on the asset value of the issuer. Another advantage of the model in Klein (1996) is ease for empirical calibration. This is because Klein sets the Brownian motions driving the values of the issuer's asset and the asset underlying the option to be correlated and thus enables each price process to consist only of a one-dimensional Brownian motion. This advantage will benefit more under the case of n underlying assets. Nevertheless, the derivative investigated in Klein (1996) is a vulnerable option on a single risky asset. In this paper we extend the model used in Klein (1996) and develop formulae to value vulnerable options on n multiple assets.

In the literature, Johnson (1987) uses a trick from Margrabe (1987) that treats a call option as an option on exchanging one asset for another to value a default-free option on the maximum or minimum of n assets. Since the method cannot be applied to a vulnerable derivative, we turn to price vulnerable multi-asset options by using the Martingale pricing theorem. We note that the multidimensional Girsanov theorem plays an important role in the change of probability measures when applying the Martingale pricing theorem to value multi-asset options. However, the multidimensional Girsanov theorem is not appropriate for adoption in our framework since the $(n+1)$ Brownian motions are correlated. Thus, this paper develops a method for the change of probability measures, which is suitable even when the component processes of the multidimensional Brownian motion are correlated, and proposes closed-form formulae to value vulnerable maximum options and vulnerable minimum options by this method.

Based on the proposed closed-form formulae, the second contribution of this research is to develop a delta-like strategy for hedging the default risk of vulnerable multi-asset options. During the financial crisis in late 2008, several major institutions included Lehman Brothers and Merrill Lynch either failed or were acquired by other institutions. Hedging the credit risk of over-the-counter securities thus becomes an important issue for investors. Although credit derivatives provide convenient ways to hedge default risks, it is difficult for an individual investor to acquire a suitable position. Moreover, the hedging effectiveness of credit derivatives is not always satisfactory, because the credit index underlying the credit derivatives is usually composed of a basket rather than by the issuer of vulnerable derivatives only. Recognizing that the value change in the issuer's asset is an important source of default risks, a more ideal scheme is to offset this risk by an instrument, such as the stock of the counterparty, that is highly correlated with the asset value of the option's issuer. Thus, this research develops a delta-like strategy that enables us to hedge the credit risk of vulnerable options by directly selling short the stocks of the issuer. We also conduct real-time analysis to investigate the performance of the proposed strategy for hedging the credit risk induced by the bankruptcy of Lehman Brothers Holdings. The hedging effectiveness of the proposed strategy is satisfactory.

The remaining parts of this paper are arranged as follows. We begin by introducing the theoretical framework in the next section and develop a method for the change of probability measures when the component processes of the multidimensional Brownian motion are correlated. By applying this method, we propose pricing formulae for valuing vulnerable maximum options and vulnerable minimum options, and provide a numerical analysis. Finally, we develop a strategy for hedging the credit risks of vulnerable options based on the proposed pricing formulae, and conduct real-time analysis to measure the hedging performance of the proposed strategy. Concluding remarks are given in the last section.

2 Theoretical framework

This paper extends the model of Klein (1996) and provides a closed-form formula to value a vulnerable rainbow option exposed to n risky assets and credit risks of the option's issuer. The assumptions and rules of settlement used throughout this paper are summarized in this section. They are identical to those of Klein (1996) under the setting of $n = 1$.

Assumption 1. Denote the time t prices of the n assets underlying a vulnerable rainbow option as $S_{1,t}, S_{2,t}, \cdots, S_{n,t}$. The risk-neutral processes for $S_1, S_2, \cdots, S_n$ are given by:

$$dS_i = S_i \left(rdt + \sigma_i dW_i\right), \qquad \forall\ i = 1, 2, \cdots, n, \tag{1}$$

where r denotes the risk-free rate, σ_i $(i = 1, 2, \cdots, n)$ is the instantaneous volatility of S_i, and dW_i $(i = 1, 2, \cdots, n)$ is the standard Wiener process defined in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{Q})$. On the other hand, denote the writer of the vulnerable option as firm XYZ, and let the time t value of XYZ's assets be V_t. We follow the setting of Klein (1996) to assume the market value of XYZ's assets, V_t, including the short position in the option being valued. Under the risk-neutral measure, the market value of XYZ's assets follows the diffusion process:

$$dV = V \left(rdt + \sigma_V dW_V\right), \tag{2}$$

where σ_V is the instantaneous volatility of XYZ's assets. Moreover, the prices of the n underlying assets, S_i $(i = 1, 2, \cdots, n)$, and the asset value of XYZ, i.e., V, are correlated with a covariance matrix given by:

$$\Theta \equiv \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1n}\sigma_1\sigma_n & \rho_{1V}\sigma_1\sigma_V \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2n}\sigma_2\sigma_n & \rho_{2V}\sigma_2\sigma_V \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1n}\sigma_1\sigma_n & \rho_{2n}\sigma_2\sigma_n & \cdots & \sigma_n^2 & \rho_{nV}\sigma_n\sigma_V \\ \rho_{1V}\sigma_1\sigma_V & \rho_{2V}\sigma_2\sigma_V & \cdots & \rho_{nV}\sigma_n\sigma_V & \sigma_V^2 \end{bmatrix}, \tag{3}$$

where ρ_{ij} $(i, j = 1, 2, \cdots, n)$ denotes the correlation coefficient between assets i and j, and ρ_{iV} is the correlation coefficient between asset i and XYZ.

Assumption 2. The writer of the rainbow option, i.e., firm XYZ, is risky. Specifically, the firm has debt outstanding, D, with maturity $T_D > T$, where T denotes the expiration date of the rainbow option being valued. It follows that a credit loss occurs if V_T is less than a specific amount D^* when the option matures. As pointed out in Klein (1996), D^* may be less than D due to the possibility that the option's writer continues operating even when V_T is less than D. Furthermore, assume that the total deadweight cost associated with bankruptcy expressed as a percentage of XYZ's asset value is α, and all claims on firm XYZ are of equal priority. It implies that the amount received by the holders of the options is only the proportion $(1 - \alpha)V_T/D$ of the nominal claim in the event of a credit loss.

Assumption 3. Markets are perfect and frictionless. There are neither transaction costs nor taxes, and securities are traded in continuous time.

Assumption 4. The continuous risk-free rate r is constant. Accordingly, the time t value of the riskless bond B_t is defined by:

$$dB_t = rB_t dt,$$

or equivalently $B_t = e^{rt}$ with $B_0 = 1$.

We note that the dynamic process displayed in Equation (1) is easy for empirical calibration since each price process consists only of a one-dimensional Brownian motion. Unlike Equation (1), the dynamic process of the assets underlying an n-asset derivative is usually set to be composed of a d-dimensional Brownian motion in the literature. To illustrate:

$$d\tilde{S}_i = \tilde{S}_i \left(rdt + \sum_{j=1}^{d} \tilde{\sigma}_{ij} d\tilde{W}_j \right), \qquad \forall\ i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, d,$$

where $d > 1$, the volatility matrix σ_{ij} $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, d)$ is an adapted process, and $d\tilde{W}_i$ $(i = 1, 2, \cdots, d)$ is a standard Wiener process. Clearly, it is difficult to estimate all the parameters σ_{ij} $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, d)$ simultaneously. Compared with the setting in the literature, the price process as in Equation (1) has the advantage in estimating parameters and thus is expected to facilitate additional empirical studies of multi-asset option prices.

Based on the framework and rules of settlement described in Assumptions 1-4, the final payoff from a vulnerable call option on the maximum of n assets can be represented as:

$$C_T^{\max} = \begin{cases} (\max S_{i,T} - K), & \text{if } \max S_{i,T} > K \text{ and } V_T > D^*, \\ (1-\alpha)\frac{V_T}{D}(\max S_{i,T} - K), & \text{if } \max S_{i,T} > K \text{ and } V_T < D^*, \\ 0, & \text{if } \max S_{i,T} < K, \end{cases} \tag{4}$$

where $i = 1, 2, \cdots, n$. For a vulnerable call option on the minimum of n assets, the final payoff received by an investor of the option is:

$$C_T^{\min} = \begin{cases} (\min S_{i,T} - K), & \text{if } \min S_{i,T} > K \text{ and } V_T > D^*, \\ (1-\alpha)\frac{V_T}{D}(\min S_{i,T} - K), & \text{if } \min S_{i,T} > K \text{ and } V_T < D^*, \\ 0, & \text{if } \min S_{i,T} < K. \end{cases} \tag{5}$$

3 Pricing vulnerable rainbow options

This research investigates the valuation of a vulnerable n-asset rainbow option by using the martingale pricing theorem. When applying this pricing theorem, the multidimensional Girsanov theorem plays an important role in the change of probability measures. As shown on page 225 in Shreve (2004), the conclusion of the multidimensional Girsanov theorem implies that the component processes of the multidimensional Brownian motion are independent.

Please note that the price of the option being valued depends not only on the prices of the n underlying assets, but also on the asset value of XYZ. According to Assumption 1, the $(n+1)$ Brownian motions, i.e., $(W_{1,t}, W_{2,t}, \cdots, W_{n,t}, W_{V,t})$, are correlated under $\mathcal{Q}$. It leads that we cannot apply the multidimensional Girsanov theorem without modifications under our framework, although the settings of price dynamics as in Equations (1) and (2) have advantages in estimating parameters. Thus, we develop the following lemma which is suitable for the change of probability measures even when the components of the multidimensional Brownian motion are correlated.

Lemma 1. Let each $W_{i,t}$ $(i = 1, 2, \cdots, k)$ be a Brownian motion and these Brownian motions, $W_{1,t}$, $W_{2,t}$, $\cdots$, $W_{k,t}$, are correlated. Specifically, the correlation coefficient for any two Brownian motions, $W_{i,t}$ and $W_{j,t}$, is given by $\rho_{ij}t$. Denote $\mathbf{X} = [x_1, x_2, \cdots, x_k]^\top$, where $x_i = W_{i,t}/\sqrt{t}$. We note that $\mathbf{X}$ follows a k-dimensional multivariate standard normal distribution with a correlation matrix given by:

$$\mathbf{\Gamma} \equiv \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1k} \\ \rho_{12} & 1 & \cdots & \rho_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1k} & \rho_{2k} & \cdots & 1 \end{bmatrix}.$$

Moreover, set $\mathbf{H} = [h_1, h_2, \cdots, h_k]$, where each h_i $(i = 1, 2, \cdots, k)$ is $\mathcal{F}_0$-measurable, and define:

$$\frac{d\mathcal{R}}{d\mathcal{Q}} = e^{\mathbf{H}\mathbf{X}\sqrt{t} - \frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t}. \tag{6}$$

We have:

$$\begin{aligned} \mathsf{E}^{\mathcal{Q}}\left[\frac{d\mathcal{R}}{d\mathcal{Q}}\,\mathbf{1}_{(\psi)} \mid \mathcal{F}_\mathbf{0}\right] &= \int\int\cdots\int_\psi (2\pi)^{\frac{-N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{X}^\top\mathbf{\Gamma}^{-1}\mathbf{X}}\ \frac{d\mathcal{R}}{d\mathcal{Q}}\ dx_1 dx_2 \cdots dx_k \\ &= \int\int\cdots\int_\psi (2\pi)^{\frac{-N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{V}^\top\mathbf{\Gamma}^{-1}\mathbf{V}} dv_1 dv_2 \cdots dv_k \\ &= \mathsf{E}^{\mathcal{R}}\left[\mathbf{1}_{(\psi)} \mid \mathcal{F}_0\right], \end{aligned}$$

where $\mathsf{E}^{\mathcal{Q}}[\cdot \mid \mathcal{F}_0]$ and $\mathsf{E}^{\mathcal{R}}[\cdot \mid \mathcal{F}_0]$ denote the expectations conditional on $\mathcal{F}_0$ under measures $\mathcal{Q}$ and $\mathcal{R}$, respectively,

$$\mathbf{V} = [v_1, v_2, \cdots, v_k]^\top,$$

and

$$v_i = x_i - \sum_{j=1}^{k} \rho_{ij} h_j \sqrt{t}, \qquad \forall\ i = 1, 2, \cdots, k. \tag{7}$$

Define $W_{i,t}^{\mathcal{R}} = v_i \sqrt{t}$. Based on the definition of x_i, i.e., $x_i = W_{i,t}/\sqrt{t}$, and Equation (7), the relationship between $W_{i,t}$ and $W_{i,t}^{\mathcal{R}}$ can be represented as:

$$W_{i,t}^{\mathcal{R}} = W_{i,t} - \sum_{j=1}^{k} \rho_{ij} h_j t, \qquad \forall\ i = 1, 2, \cdots, k. \tag{8}$$

According to Equation (8), each $W_{i,t}^{\mathcal{R}}$ $(i = 1, 2, \cdots, k)$ is a Brownian motion.

Proof: See Appendix 1.

Based on Lemma 1, the value of a vulnerable path-independent option on multiple assets can be calculated easily. The current value of a vulnerable maximum call on n underlying assets can be computed by discounting the expectation of the final payoff displayed in Equation (4). It is:

$$C_0^{\max} = e^{-rT} \mathsf{E}^{Q} \left\{ \mathsf{E}_1^{max} + \mathsf{E}_2^{max} + \mathsf{E}_3^{max} + \mathsf{E}_4^{max} \mid \mathcal{F}_0 \right\}, \tag{9}$$

where

$$\mathsf{E}_1^{max} = \sum_{i=1}^{n} S_{i,T} \mathbf{1}_{(S_{i,T} > K)} \left(\prod_{j=1,2,\cdots,n; j \neq i} \mathbf{1}_{(S_{i,T} > S_{j,T})} \right) \mathbf{1}_{(V_T > D^*)},$$

$$\mathsf{E}_2^{max} = (1-\alpha) \frac{V_T}{D} \sum_{i=1}^{n} S_{i,T} \mathbf{1}_{(S_{i,T} > K)} \left(\prod_{j=1,2,\cdots,n; j \neq i} \mathbf{1}_{(S_{i,T} > S_{j,T})} \right) \mathbf{1}_{(V_T < D^*)},$$

$$\mathsf{E}_3^{max} = -K \left[\mathbf{1}_{(V_T > D^*)} - \left(\prod_{i=1,2,\cdots,n} \mathbf{1}_{(S_{i,T} < K)} \right) \mathbf{1}_{(V_T > D^*)} \right],$$

and

$$\mathsf{E}_4^{max} = -K(1-\alpha) \frac{V_T}{D} \left[\mathbf{1}_{(V_T < D^*)} - \left(\prod_{i=1,2,\cdots,n} \mathbf{1}_{(S_{i,T} < K)} \right) \mathbf{1}_{(V_T < D^*)} \right].$$

By applying Lemma 1, the vulnerable rainbow call on the Maximum can be priced by Proposition 1.

Proposition 1. Suppose that Assumptions (1)-(4) hold and the final payoff of a call option on the maximum of n assets is given by Equation (4). Based on Lemma 1, this vulnerable maximum call can be priced as follows:

$$\begin{aligned} C_0^{\max} =& \sum_{i=1}^{n} S_{i,0} N_{n+1}\left(\mathcal{A}_1^{(i)}; \mathcal{J}^{(i)}\right) \\ &+ \sum_{i=1}^{n} (1-\alpha)\frac{V_0}{D} e^{(r+\rho_{iV}\sigma_i\sigma_V)T} S_{i,0} N_{n+1}\left(\widetilde{\mathcal{A}}_1^{(i)}; \widetilde{\mathcal{J}}^{(i)}\right) \\ &- K e^{-rT}\left[N(c_2(V,D^*)) - N_{n+1}\left(\mathcal{A}_2; \mathcal{M}\right)\right] \\ &- (1-\alpha)\frac{V_0}{D} K \left[N(-\widetilde{c}_2(V,D^*)) - N_{n+1}\left(\widetilde{\mathcal{A}}_2; \widetilde{\mathcal{M}}\right)\right], \end{aligned} \tag{10}$$

where $\mathcal{A}_1^{(i)}$ and $\widetilde{\mathcal{A}}_1^{(i)}$ denote the ith row of $\mathcal{A}_1$ and $\widetilde{\mathcal{A}}_1$, respectively,

$$\mathcal{A}_1 = \begin{bmatrix} d_1(S_1,K) & a_1(S_1,S_2) & a_1(S_1,S_3) & \cdots & a_1(S_1,S_n) & c_1(V,D^*,S_1) \\ d_1(S_2,K) & a_1(S_2,S_1) & a_1(S_2,S_3) & \cdots & a_1(S_2,S_n) & c_1(V,D^*,S_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_1(S_n,K) & a_1(S_n,S_1) & a_1(S_n,S_2) & \cdots & a_1(S_n,S_{n-1}) & c_1(V,D^*,S_n) \end{bmatrix},$$

$$\widetilde{\mathcal{A}}_1 = \begin{bmatrix} \widetilde{d}_1(S_1,K) & \widetilde{a}_1(S_1,S_2) & \widetilde{a}_1(S_1,S_3) & \cdots & \widetilde{a}_1(S_1,S_n) & -\widetilde{c}_1(V,D^*,S_1) \\ \widetilde{d}_1(S_2,K) & \widetilde{a}_1(S_2,S_1) & \widetilde{a}_1(S_2,S_3) & \cdots & \widetilde{a}_1(S_2,S_n) & -\widetilde{c}_1(V,D^*,S_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{d}_1(S_n,K) & \widetilde{a}_1(S_n,S_1) & \widetilde{a}_1(S_n,S_2) & \cdots & \widetilde{a}_1(S_n,S_{n-1}) & -\widetilde{c}_1(V,D^*,S_n) \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} -d_2(S_1,K) & -d_2(S_2,K) & -d_2(S_3,K) & \cdots & -d_2(S_n,K) & c_2(V,D^*) \end{bmatrix},$$

and

$$\widetilde{\mathcal{A}}_2 = \begin{bmatrix} -\widetilde{d}_2(S_1,K) & -\widetilde{d}_2(S_2,K) & -\widetilde{d}_2(S_3,K) & \cdots - \widetilde{d}_2(S_n,K) & -\widetilde{c}_2(V,D^*) \end{bmatrix}.$$

Moreover, $N_{n+1}(.;\Phi)$ represents the cumulative probability of an $(n+1)$-dimensional multivariate normal distribution with mean vector 0 and covariance matrix Φ. For $i, j = 1, 2, \cdots, n$, the parameters in Equation (10) are defined as follows:

$$d_1(S_i, K) = \frac{\ln(S_{i,0}/K) + (r + 0.5\sigma_i^2)T}{\sigma_i\sqrt{T}},$$

$$a_1(S_i, S_j) = \frac{\ln(S_{i,0}/S_{j,0}) + 0.5\sigma_{ij}^2 T}{\sigma_{ij}\sqrt{T}},$$

$$c_1(V, D^*, S_i) = \frac{\ln(V_0/D^*) + (r - 0.5\sigma_V^2 + \rho_{iV}\sigma_i\sigma_V)T}{\sigma_V\sqrt{T}},$$

$$\widetilde{d}_1(S_i, K) = \frac{\ln(S_{i,0}/K) + (r + 0.5\sigma_i^2 + \rho_{iV}\sigma_i\sigma_V)T}{\sigma_i\sqrt{T}},$$

$$\widetilde{a}_1(S_i, S_j) = \frac{\ln(S_{i,0}/S_{j0}) + (0.5\sigma_{ij}^2 + \rho_{V,ij}\sigma_V\sigma_{ij})T}{\sigma_{ij}\sqrt{T}},$$

$$\widetilde{c}_1(V, D^*, S_i) = \frac{\ln(V_0/D^*) + (r + 0.5\sigma_V^2 + \rho_{iV}\sigma_i\sigma_V)T}{\sigma_V\sqrt{T}},$$

$$c_2(V, D^*) = \frac{\ln(V_0/D^*) + (r - 0.5\sigma_V^2)T}{\sigma_V\sqrt{T}},$$

$$d_2(S_i, K) = \frac{\ln(S_{i,0}/K) + (r - 0.5\sigma_i^2)T}{\sigma_i\sqrt{T}},$$

$$\widetilde{c}_2(V, D^*) = \frac{\ln(V_0/D^*) + (r + 0.5\sigma_V^2)T}{\sigma_V\sqrt{T}},$$

and

$$\widetilde{d}_2(S_i, K) = \frac{\ln(S_{i,0}/K) + (r - 0.5\sigma_i^2 + \rho_{iV}\sigma_i\sigma_V)T}{\sigma_i\sqrt{T}},$$

where

$$\sigma_{ij} = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j},$$

and

$$\rho_{V,ij} = \frac{\rho_{iV}\sigma_i - \rho_{jV}\sigma_j}{\sigma_{ij}}.$$

The covariance matrix $\mathcal{J}^{(i)}$ $(i = 1, 2, \cdots, n)$ displayed in Equation (10) denotes the sub-matrix obtained from $\mathbf{J}^{(i)}$ by deleting its $(i+1)$th row and $(i+1)$th column, where:

$$\mathbf{J}^{(i)} = \begin{bmatrix} 1 & \rho_{i,i1} & \rho_{i,i2} & \rho_{i,i3} & \cdots & \rho_{i,in} & \rho_{iV} \\ \rho_{i,i1} & 1 & \rho_{i1,i2} & \rho_{i1,i3} & \cdots & \rho_{i1,in} & \rho_{V,i1} \\ \rho_{i,i2} & \rho_{i1,i2} & 1 & \rho_{i2,i3} & \cdots & \rho_{i2,in} & \rho_{V,i2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{i,in} & \rho_{i1,in} & \rho_{i2,in} & \rho_{i3,in} & \cdots & 1 & \rho_{V,in} \\ \rho_{iV} & \rho_{V,i1} & \rho_{V,i2} & \rho_{V,i3} & \cdots & \rho_{V,in} & 1 \end{bmatrix},$$

$$\rho_{i,ij} \equiv \rho_{ij,i} = \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}},$$

and

$$\rho_{ij,ik} \equiv \rho_{ik,ij} = \frac{\sigma_i^2 - \rho_{ik}\sigma_i\sigma_k - \rho_{ij}\sigma_i\sigma_j + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}}.$$

Similarly, the covariance matrix $\widetilde{\mathcal{J}}^{(i)}$ $(i = 1, 2, \cdots, n)$ displayed in Equation (10) denotes the sub-matrix obtained from $\widetilde{\mathbf{J}}^{(i)}$ by deleting its $(i+1)$th row and $(i+1)$th column, where:

$$\widetilde{\mathbf{J}}^{(i)} = \begin{bmatrix} 1 & \rho_{i,i1} & \rho_{i,i2} & \rho_{i,i3} & \cdots & \rho_{i,in} & -\rho_{iV} \\ \rho_{i,i1} & 1 & \rho_{i1,i2} & \rho_{i1,i3} & \cdots & \rho_{i1,in} & -\rho_{V,i1} \\ \rho_{i,i2} & \rho_{i1,i2} & 1 & \rho_{i2,i3} & \cdots & \rho_{i2,in} & -\rho_{V,i2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{i,in} & \rho_{i1,in} & \rho_{i2,in} & \rho_{i3,in} & \cdots & 1 & -\rho_{V,in} \\ -\rho_{iV} & -\rho_{V,i1} & -\rho_{V,i2} & -\rho_{V,i3} & \cdots & -\rho_{V,in} & 1 \end{bmatrix}.$$

There are still two covariance matrices, $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, in Equation (10), which are given by:

$$\mathcal{M} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1n} & -\rho_{1V} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2n} & -\rho_{2V} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1n} & \rho_{2n} & \rho_{3n} & \rho_{4n} & \cdots & 1 & -\rho_{nV} \\ -\rho_{1V} & -\rho_{2V} & -\rho_{3V} & -\rho_{4V} & \cdots & -\rho_{nV} & 1 \end{bmatrix},$$

and

$$\widetilde{\mathcal{M}} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1n} & \rho_{1V} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2n} & \rho_{2V} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1n} & \rho_{2n} & \rho_{3n} & \rho_{4n} & \cdots & 1 & \rho_{nV} \\ \rho_{1V} & \rho_{2V} & \rho_{3V} & \rho_{4V} & \cdots & \rho_{nV} & 1 \end{bmatrix}.$$

In the literature, Johnson (1987) develops a closed-form formula to price maximum options. We note that the pricing formula displayed in Proposition 1 reduces to that in Johnson (1987) when the default risk is trivial. Moreover, when the number of the underlying assets, n, is set to be one, our pricing formula is identical to that of vulnerable Black-Schole options proposed by Klein (1996).

The current value of vulnerable minimum calls on n underlying assets can be similarly calculated by discounting the expectation of the final payoff displayed in Equation (5). It is:

$$C_0^{\min} = e^{-rT}\mathsf{E}^Q\left\{\mathsf{E}_1^{min} + \mathsf{E}_2^{min} + \mathsf{E}_3^{min} + \mathsf{E}_4^{min} \mid \mathcal{F}_0\right\}, \tag{11}$$

where

$$\mathsf{E}_1^{min} = \sum_{i=1}^{n} S_{i,T}\mathbf{1}_{(S_{i,T}>K)}\left(\prod_{j=1,2,\cdots,n;j\neq i}\mathbf{1}_{(S_{i,T}<S_{j,T})}\right)\mathbf{1}_{(V_T>D^*)},$$

$$\mathsf{E}_2^{min} = (1-\alpha)\frac{V_T}{D}\sum_{i=1}^{n} S_{i,T}\mathbf{1}_{(S_{i,T}>K)}\left(\prod_{j=1,2,\cdots,n;j\neq i}\mathbf{1}_{(S_{i,T}<S_{j,T})}\right)\mathbf{1}_{(V_T<D^*)},$$

$$\mathsf{E}_3^{min} = -K\left[\left(\prod_{i=1,2,\cdots,n}\mathbf{1}_{(S_{i,T}>K)}\right)\mathbf{1}_{(V_T>D^*)}\right],$$

and

$$\mathsf{E}_4^{min} = -K(1-\alpha)\frac{V_T}{D}\left[\left(\prod_{i=1,2,\cdots,n}\mathbf{1}_{(S_{i,T}>K)}\right)\mathbf{1}_{(V_T<D^*)}\right].$$

By applying Lemma 1, the vulnerable rainbow call on the minimum can be priced by the following Proposition 2.

Proposition 2. Suppose that Assumptions (1)-(4) hold and the final payoff of a call option on the minimum of n assets is given by Equation (5). Based on Lemma 1, this vulnerable minimum call can be priced as follows:

$$\begin{aligned} C_0^{\min} =& \sum_{i=1}^{n} S_{i,0} N_{n+1}\left(\mathcal{B}_1^{(i)}; \mathcal{G}^{(i)}\right) \\ &+ \sum_{i=1}^{n} (1-\alpha)\frac{V_0}{D} e^{(r+\rho_{iV}\sigma_i\sigma_V)T} S_{i,0} N_{n+1}\left(\widetilde{\mathcal{B}}_1^{(i)}; \widetilde{\mathcal{G}}^{(i)}\right) \\ &- K e^{-rT} N_{n+1}\left(\mathcal{B}_2; \mathcal{Z}\right) \\ &- (1-\alpha)\frac{V_0}{D} K N_{n+1}\left(\widetilde{\mathcal{B}}_2; \widetilde{\mathcal{Z}}\right), \end{aligned} \tag{12}$$

where $\mathcal{B}_1^{(i)}$ and $\widetilde{\mathcal{B}}_1^{(i)}$ denote the ith row of $\mathcal{B}_1$ and $\widetilde{\mathcal{B}}_1$, respectively,

$$\mathcal{B}_1 = \begin{bmatrix} d_1(S_1, K) & -a_1(S_1, S_2) & -a_1(S_1, S_3) & \cdots & -a_1(S_1, S_n) & c_1(V, D^*, S_1) \\ d_1(S_2, K) & -a_1(S_2, S_1) & -a_1(S_2, S_3) & \cdots & -a_1(S_2, S_n) & c_1(V, D^*, S_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_1(S_n, K) & -a_1(S_n, S_1) & -a_1(S_n, S_2) & \cdots & -a_1(S_n, S_{n-1}) & c_1(V, D^*, S_n) \end{bmatrix},$$

$$\widetilde{\mathcal{B}}_1 = \begin{bmatrix} \widetilde{d}_1(S_1, K) & -\widetilde{a}_1(S_1, S_2) & -\widetilde{a}_1(S_1, S_3) & \cdots & -\widetilde{a}_1(S_1, S_n) & -\widetilde{c}_1(V, D^*, S_1) \\ \widetilde{d}_1(S_2, K) & -\widetilde{a}_1(S_2, S_1) & -\widetilde{a}_1(S_2, S_3) & \cdots & -\widetilde{a}_1(S_2, S_n) & -\widetilde{c}_1(V, D^*, S_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \widetilde{d}_1(S_n, K) & -\widetilde{a}_1(S_n, S_1) & -\widetilde{a}_1(S_n, S_2) & \cdots & -\widetilde{a}_1(S_n, S_{n-1}) & -\widetilde{c}_1(V, D^*, S_n) \end{bmatrix},$$

$$\mathcal{B}_2 = \begin{bmatrix} d_2(S_1, K) & d_2(S_2, K) & d_2(S_3, K) & \cdots & d_2(S_n, K) & c_2(V, D^*) \end{bmatrix},$$

and

$$\widetilde{\mathcal{B}}_2 = \begin{bmatrix} \widetilde{d}_2(S_1, K) & \widetilde{d}_2(S_2, K) & \widetilde{d}_2(S_3, K) & \cdots \widetilde{d}_2(S_n, K) & -\widetilde{c}_2(V, D^*) \end{bmatrix}.$$

Similarly, the covariance matrices $\mathcal{G}^{(i)}$ and $\widetilde{\mathcal{G}}^{(i)}$ $(i = 1, 2, \cdots, n)$ displayed in Equation (12) denote the sub-matrices obtained from $\mathbf{G}^{(i)}$ and $\widetilde{\mathbf{G}}^{(i)}$ by deleting the corresponding $(i+1)$th row and $(i+1)$th column, respectively, where:

$$\mathbf{G}^{(i)} = \begin{bmatrix} 1 & -\rho_{i,i1} & -\rho_{i,i2} & -\rho_{i,i3} & \cdots & -\rho_{i,in} & \rho_{iV} \\ -\rho_{i,i1} & 1 & \rho_{i1,i2} & \rho_{i1,i3} & \cdots & \rho_{i1,in} & -\rho_{V,i1} \\ -\rho_{i,i2} & \rho_{i1,i2} & 1 & \rho_{i2,i3} & \cdots & \rho_{i2,in} & -\rho_{V,i2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\rho_{i,in} & \rho_{i1,in} & \rho_{i2,in} & \rho_{i3,in} & \cdots & 1 & -\rho_{V,in} \\ \rho_{iV} & -\rho_{V,i1} & -\rho_{V,i2} & -\rho_{V,i3} & \cdots & -\rho_{V,in} & 1 \end{bmatrix},$$

and

$$\widetilde{\mathbf{G}}^{(i)} = \begin{bmatrix} 1 & -\rho_{i,i1} & -\rho_{i,i2} & -\rho_{i,i3} & \cdots & -\rho_{i,in} & -\rho_{iV} \\ -\rho_{i,i1} & 1 & \rho_{i1,i2} & \rho_{i1,i3} & \cdots & \rho_{i1,in} & \rho_{V,i1} \\ -\rho_{i,i2} & \rho_{i1,i2} & 1 & \rho_{i2,i3} & \cdots & \rho_{i2,in} & \rho_{V,i2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\rho_{i,in} & \rho_{i1,in} & \rho_{i2,in} & \rho_{i3,in} & \cdots & 1 & \rho_{V,in} \\ -\rho_{iV} & \rho_{V,i1} & \rho_{V,i2} & \rho_{V,i3} & \cdots & \rho_{V,in} & 1 \end{bmatrix}.$$

The remaining two covariance matrices, $\mathcal{Z}$ and $\widetilde{\mathcal{Z}}$, in Equation (12) are given by:

$$\mathcal{Z} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1n} & \rho_{1V} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2n} & \rho_{2V} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1n} & \rho_{2n} & \rho_{3n} & \rho_{4n} & \cdots & 1 & \rho_{nV} \\ \rho_{1V} & \rho_{2V} & \rho_{3V} & \rho_{4V} & \cdots & \rho_{nV} & 1 \end{bmatrix},$$

and

$$\widetilde{\mathcal{Z}} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \rho_{14} & \cdots & \rho_{1n} & -\rho_{1V} \\ \rho_{12} & 1 & \rho_{23} & \rho_{24} & \cdots & \rho_{2n} & -\rho_{2V} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1n} & \rho_{2n} & \rho_{3n} & \rho_{4n} & \cdots & 1 & -\rho_{nV} \\ -\rho_{1V} & -\rho_{2V} & -\rho_{3V} & -\rho_{4V} & \cdots & -\rho_{nV} & 1 \end{bmatrix}.$$

According to Proposition 2, it is also observed that the pricing formula of vulnerable minimum options reduces to that of minimum options proposed by Johnson (1987) when the default risk is trivial.

4 Numerical analyses

To gauge the impact of credit risks on the option's price, Exhibit 1 displays the theoretical values of vulnerable call options on the maximum of two hypothetical stocks and compares the vulnerable prices to the corresponding default-free values. We apply Proposition 1 to calculate the values of vulnerable options, whereas the corresponding default-free prices are based on the formula proposed by Johnson (1987). The parameters of the Base Case in Exhibit 1 are given by: $r = 0.04833$, $T = 0.3333$, $K = 40$, $S_1 = S_2 = 40$, $\sigma_1 = \sigma_2 = \sigma_V = 0.3$, $V_0 = 5$, $D = D^* = 5$, $\alpha = 0$, $\rho_{12} = \rho_{1V} = 0.5$, and $\rho_{2V} = 0$. To ensure the correlation matrix is positive semi-definite for various combinations of ρ_{12} and ρ_{1V}, we set the value of ρ_{2V} in the Base Case to be zero. All parameters in the Base Case are identical to the settings in the numerical analyses of Klein (1996) except for the values of S_2, σ_2, ρ_{12}, and ρ_{2V}. These parameters are not required when pricing one-asset vulnerable options as done in Klein (1996).

Similar to the properties of the default-free counterpart, Exhibit 1 indicates that the value of the vulnerable call grows with the time to maturity T and the volatility of the underlying assets σ_1. On the contrary, the strike price K and the correlation between the two underlying stocks ρ_{12} induce negative impacts on vulnerable prices.

Unlike default-free options, the V/D ratio, deadweight cost α, volatility of XYZ's assets σ_V, and correlations ρ_{1V} and ρ_{2V} indeed influence the values of vulnerable options. As expected, the price of a vulnerable option is never greater than its corresponding default-free value. However, it grows as the V/D ratio increases. When the V/D ratio reaches 2.0, almost all vulnerable values in our Exhibit 1 converge to their default-free values. We also note that an issuer with a specified V/D ratio may bring different credit risks on different-maturity options. To illustrate, the issuer with a V/D ratio of 1.2 can be viewed as a riskless writer when the option being issued is a 30-day call, because the vulnerable value reaches its default-free price in the case of $V/D = 1.2$ and $T = 0.833$. However, when the written option is a four-month or seven-month call, the issuer is no longer a riskless firm for investors. It follows that the V/D ratio of XYZ required by risk-aversion investors is relatively high when the option holds long maturity.

We also observe that the impact of the asset volatility σ_V on vulnerable prices depends on the V/D ratio. Exhibit 1 exhibits that the value of σ_V does not affect vulnerable values very much when the V/D ratio equals 2.0 in our numerical analysis. However, the impact of σ_V grows as the V/D ratio declines. As shown in Panel B of Exhibit 1, the vulnerable value under the case of $V/D = 1.0$ decreases from \$4.508 to \$4.349 when the value of σ_V increases from 20% to 40%. It implies that the volatility of XYZ's assets σ_V indeed plays an important role when the V/D ratio of an issuer is not large enough.

Exhibit 1 also reveals that the values of ρ_{1V} and ρ_{2V} both have positive impacts on vulnerable prices. This is because a larger value of either ρ_{1V} or ρ_{2V} implies a smaller probability of default when the vulnerable option is in-the-money at maturity. As expected, the values of ρ_{1V} and ρ_{2V} have no impacts on vulnerable values when the issuer is riskless or the V/D ratio is large enough.

5 Hedging the credit risk

Hedging the credit risk of over-the-counter securities is an important issue for investors. However, an individual investor is not able to acquire a suitable position easily in a credit default swap or credit derivative. Recognizing that the value change in the issuer's asset, ΔV, is an important source of default risk, an ideal scheme is to offset this risk by an instrument, such as the stock of firm XYZ, that is highly correlated with the asset value of XYZ. It motivates us to develop a delta-like strategy that enables investors to hedge the credit risk by selling short XYZ stock based on the pricing formulae offered in Propositions 1 and 2.

Define $\Delta_V \equiv \Delta C/\Delta V$ as the rate of change of the option price with respect to the asset value of XYZ. Once the pricing formulae of options displayed in (10) and (12) are at hand, the value of Δ_V can be calculated by:

$$\Delta_V = \frac{C(V+h) - C(V)}{h}, \tag{13}$$

where h is a small value, and $C(V)$ and $C(V+h)$ are the values of vulnerable calls given the asset values are V and $V+h$, respectively. For a vulnerable n-asset maximum call, $C(\cdot)$ can be priced by (10), whereas (12) can be used to value vulnerable n-asset minimum calls.

According to the definition of Δ_V, a position with zero Δ_V has no credit risk. It follows that the credit risk of a vulnerable option with non-zero Δ_V is able to be eliminated as long as a new position that offsets the value of Δ_V is added. One of the best candidates to construct the new position is XYZ stock, since it is highly dependent on the asset value of the underlying firm XYZ. Accordingly, the hedging of credit risks can be implemented by selling short stocks of the option's issuer to make the whole position Δ_V neutral. The hedging scheme is named Δ_V-neutral strategy or Δ_V strategy.

Two properties concerning Δ_V that may influence the performance of Δ_V strategy are worth highlighting. Exhibit 2 plots the impact that comes from the maximum of the two underlying stocks on the value of Δ_V calculated by the hypothetical vulnerable maximum call investigated in Exhibit 1. All parameters in Exhibit 2 are identical to the Base Case defined in Exhibit 1, except for the value of S_1 and the strike price. The strike price here is set to be \$60. We observe that the change in the value of Δ_V is very slight when the call is deep-in-the-money or deep-out-of-the-money. While the call is at-the-money or near at-the-money, the value of Δ_V changes rapidly. The implies that the hedging effectiveness of Δ_V strategy may be better when the call is deep-in-the-money or deep-out-of-the-money. Exhibit 3 reveals the negative impact of XYZ's asset value V on the value of Δ_V. Similarly, all parameters in Exhibit 3 are the same as those in the Base Case, except for the value of V. It indicates that the more worthless XYZ's asset is, the more credit risk will be embedded in the vulnerable option.

Before identifying the number of shares to short, the relationship between the stock price and the asset value of the issuer needs to be clarified. Ronn and Verma (1986) and Duan (1994) suggest that the time t equity value, E_t, can be viewed as a call option and written as:

$$E_t = V_t N(d_1(V_t, D)) - De^{-r(T_D - t)} N(d_1(V_t, D) - \sigma_V \sqrt{T_D - t}), \tag{14}$$

where

$$d_1(V_t, D) = \frac{\ln (V_t/D) + (r + 0.5\sigma_V^2)(T_D - t)}{\sigma_V \sqrt{T_D - t}}. \tag{15}$$

Clearly, the relationship between the equity price and asset value behind the above model is:

$$\Delta_E \equiv \frac{\Delta E}{\Delta V} = N(d_1(V_t, D)), \tag{16}$$

where T_D is the maturity of debts. In the literature, many empirical studies set T_D as the time of the next audit. Recognizing that Δ_E represents the rate of change of the equity value with respect to the asset value of XYZ, and Δ_V exhibits the rate of change of the option price with respect to the asset value of XYZ, the credit risk of vulnerable options can be offset by shorting κ shares of XYZ stock, where κ satisfies:

$$\Delta_V dV = \kappa \Delta_E dV. \tag{17}$$

In other words, Δ_V-neutral strategy can be easily implemented by shorting κ shares of stock XYZ, where $\kappa = \Delta_V / \Delta_E$, once the values of Δ_V and Δ_E are at hand.

It is worth noting that the values of Δ_V and Δ_E depend on the asset value V and asset volatility σ_V, which cannot be observed directly from market information. Duan (1994) develops a maximum likelihood method to estimate the unobserved asset value V and asset volatility σ_V from the observed equity value E. As asset value V follows a lognormal process, the one-period transition density of V is characterized by:

$$\ln \frac{V_{t+1}}{V_t} \sim N(\mu_V, \sigma_V^2),$$

where $N(\mu_V, \sigma_V^2)$ denotes a normal distribution with mean μ_V and variance σ_V^2. Subsequently, the corresponding log-likelihood function for a sample of unobserved V_t $(t = 1, 2, \cdots, m)$ can be written as:

$$\begin{aligned} &L_V\left(V_t, t = 1, 2, \cdots, m; \mu_V, \sigma_V\right) \\ &= -\frac{m-1}{2}\ln(2\pi) - \frac{m-1}{2}\ln\sigma_V^2 - \frac{1}{2\sigma_V^2}\sum_{t=2}^{m}\left[\ln\frac{V_t}{V_{t-1}} - \mu_V\right]^2. \end{aligned} \tag{18}$$

Herein, the equity model displayed in (14) allows us to transform the unobserved sample of asset values to the observed sample of equity values, and enables us to rewrite the log-likelihood function for the observed sample of equity values as:

$$\begin{aligned} L\left(E_t, t = 1, 2, \cdots, m; \mu_V, \sigma_V\right) = &-\frac{m-1}{2}\ln(2\pi) - \frac{m-1}{2}\ln\sigma_V^2 \\ &- \sum_{t=2}^{m}\ln\left(N(d_1(\widehat{V}_t(\sigma_V), D^*))\right) - \frac{1}{2\sigma_V^2}\sum_{t=2}^{m}\left[\ln\frac{\widehat{V}_t(\sigma_V)}{\widehat{V}_{t-1}(\sigma_V)} - \mu_V\right]^2, \end{aligned} \tag{19}$$

where the asset value estimate, $\widehat{V}_t(\sigma_V)$, is the unique solution to (14) for a given σ_V and observed equity value E_t. By maximizing the objective function defined in (19), the estimates of V_t $(t = 1, 2, \cdots, m)$ and σ_V can now be obtained by market information.

6 Measuring hedging performance

On September 15, 2008, Lehman Brothers filed for bankruptcy protection, helping trigger the financial crisis of 2008-2010. As the biggest investment bank collapses since 1990, the credit risk induced by it bankruptcy could be a representative lesson to be investigated. This research thus conducts real-time analysis to explore the hedging performance of the proposed Δ_V strategy during this bankrupt event.

This research specifically investigates the hedging effectiveness of the proposed Δ_V strategy for a hypothetical two-asset maximum call issued by Lehman Brothers Holdings (LEH, hereafter) in LEH's bankrupt event. The two underlying assets of the hypothetic call are stocks of General Electric Corporation (GE, hereafter) and Bank of America Corporation (BAC, hereafter). The data thus comprise the daily closing prices of LEH, BAC, and GE stocks, as well as the debt and shares outstanding of LEH in its quarterly financial statements. All data are listed on the website of Yahoo Finance. Since LEH was delisted from the New York Stock Exchange on September 17, 2008, the data used in the real-time study range from September 4, 2007 to September 16, 2008. In particular, we start the in-sample estimation for LEH's asset value V and asset volatility σ_V on September 4, 2007, and reserve T days to conduct an out-of-sample hedging comparison for a T-day hypothetic call.

Before estimating the parameters in the equity model, the setting of T_D displayed in (14) should be clarified. Ronn and Verma (1986) and Duan (1994) apply the option pricing theorem to price equity value E as in Equation (14), since the equity of a limited liability company can be viewed as a call option. Based on the option pricing theorem, the value of Δ_E defined in (16) stands for the probability that firm value V_{T_D} is greater than the debt at time T_D under the probability measure $\mathcal{R}$. In practice, the settlement day of derivatives is usually two days after its maturity. Since the real concern of investors should be the default probability on settlement day, we set T_D to be $T + 2$ days when estimating the value of Δ_E

and parameters in (14).

Exhibit 4 displays the historical prices of LEH stock. We observe that LEH's stock price dropped from \$56.46 (September 4, 2007) to \$0.3 (September 16, 2008). Moreover, there are at least four price jumps around the time at which LEH went bankrupt. On September 9, 11, and 15 of 2008, LEH's stock price dropped 45%, 42%, and 94%, respectively, whereas it increased 43% on September 16, 2008. Exhibit 5 plots the two underlying stock prices, BAC and GE, and shows that the stock prices of BAC are more volatile than that of GE in mid-2008.

The following explains the operation of the proposed strategy in detail. For ease of reference, denote the $(T+1)$th day before September 16, 2008 as time 0, the Tth day before September 16, 2008 as time 1, $\cdots$, and September 16, 2008 as time T, where T is the time to maturity of the hypothetical vulnerable option to be hedged. For an investor with a T-day vulnerable maximum call, the Δ_V strategy involves shorting κ shares of LEH stock at time 0, rebalancing the hedged position in the following T days, and buying back all the short-sold LEH stock at time T. In particular, let κ_t be the total LEH shares required to be short at time t. The number of LEH shares sold to rebalance the hedged position at t is:

$$\beta_t = \begin{cases} \kappa_0, & \text{when } \ t = 0, \\ \kappa_t - \kappa_{t-1}, & \text{when } \ t = 1, 2, \cdots, T-1. \end{cases} \tag{20}$$

Incorporating the cost arising from buying the full short position of LEH stock back at the end of the hedge, the cumulated cash inflow resulting from Δ_V strategy is as follows:

$$CF_T = \sum_{t=0}^{T-1} \beta_t E_t e^{r(T-t)} - \kappa_{T-1} E_T. \tag{21}$$

Based on Assumption 2, the holder of a vulnerable option receives only the proportion $(1-\alpha)V_T/D$ of the nominal claim in the event of a credit loss. Accordingly, the total maturity value received by an option holder who carries out Δ_V strategy, $MV_T(\Delta_V)$, is:

$$MV_T(\Delta_V) = (1-\alpha)\frac{V_T}{D}\text{Max}\left[\text{Max}(S_{1,T}, S_{2,T}) - K, 0\right] + CF_T, \tag{22}$$

whereas the maturity value of a vulnerable option received by investors without hedging, MV_T, is:

$$MV_T = (1-\alpha)\frac{V_T}{D}\text{Max}\left[\text{Max}(S_{1,T}, S_{2,T}) - K, 0\right], \tag{23}$$

where the time T value of LEH's assets, V_T, can be estimated by (19).

Exhibit 6 displays the real-time hedging performance of Δ_V-neutral strategy for the vulnerable maximum calls with various times to maturity T and strike prices K. As mentioned above, we estimate the asset value V and asset volatility σ_V with the data beginning on September 4, 2007, and reserve T days to conduct an out-of-sample hedging comparison. It implies that all hypothetic calls mature at the last day of our dataset, i.e., September 16, 2008. The is the reason why options with different maturities have the same maturity values in our Exhibit 6.

As shown in Exhibit 6, the maturity value of a 10-day default-free call with a strike price of \$15 is \$14.550. Investing the corresponding \$15 10-day vulnerable call without hedging for default risk receives only \$11.200 in LEH's bankrupt event, whereas the amount received by an investor who takes Δ_V strategy is \$14.483, which is very close to the default-free value \$14.550. There are similar results observed in the hedging performance of 20-day and 30-day calls. The holder of a \$15 20-day vulnerable call receives \$14.156 as long as the proposed hedging strategy is adopted, while the hedging result of a \$15 30-day vulnerable call is \$14.429. Both of the two values are very close to the maturity value of the corresponding default-free option, i.e., \$14.550, as well.

We also note that the characteristic of Δ_V is very different when the option is at-the-money. As mentioned in Exhibit 2, the rate of change in Δ_V with respect to the maximum price of the underlying stocks changes rapidly when the option is at-the-money, and thus the hedging performance of strategies depending on Δ_V may be poor for an at-the-money option. This conjecture is clearly borne out in Exhibit 6. Recognizing that the maximum price of the two underlying stocks on September 16, 2008 is \$29.55, the option with a strike price of \$30 can be viewed as an at-the-money option. We observe that the hedging performance of Δ_V strategy is more satisfactory when the vulnerable option is in-the-money or out-of-the-money.

To further highlight the hedging scheme of Δ_V strategy, the associated stock prices, total shares sold, and the estimated V/D ratio of LEH from hedging the \$25 20-day maximum call are plotted in Exhibits 7-9. Exhibit 7 indicates that the maximum of the two underlying stocks comes from BAC starting from August 20, 2008. We also observe that the stock price of LEH starts tumbling dramatically after September 8, 2008. Exhibit 8 shows the associated number of total shares sold, κ_t. Based on Equation (17), the value of κ_t is affected by the values of Δ_V and Δ_E. The bigger the value of Δ_V, the greater the value of κ_t. Oppositely, the smaller the value of Δ_E, the greater the value of κ_t. We also note that the impact of LEH's asset value, V, on the values of Δ_E is positive, but its impacts on the value of Δ_V are negative, as shown in Exhibit 3, as long as the option investigated is not default-free. Consequently, the total shares sold for hedging increase as the market value of LEH's asset drops. The phenomenon stands out very clearly in Exhibits 7 and 8. The number of shares sold required by Δ_V strategy is mainly dominated by the stock price of LEH, which is highly correlated with the market value of LEH's assets. Particularly, the number of shares sold increases obviously once LEH's stock price drops evidently, and vice versa.

We also note that the value of the vulnerable call, C, has a positive impact on the value of Δ_V, and thus total shares sold for hedging grow as the value of the vulnerable option increases. This phenomenon is also borne out in Exhibits 7 and 8. The stock price of LEH is observed to increase from \$15.17 to \$16.20 on September 5, 2008, however, the corresponding number of shares sold for implementing Δ_V strategy does not decrease. This is because the maximum stock price, i.e., BAC, increases from \$30.6 to \$32.23, and results in both the call price and the value of Δ_V appreciating.

Shortly before 1 a.m. Monday morning (New York time) on September 15, 2008, LEH announced it would file for bankruptcy protection citing bank debt of \$613 billion, \$155 billion in bond debt, and assets worth \$639 billion. This implied that the V/D ratio proclaimed by LEH was 83.20% before the opening of the New York Stock Exchange on September 15, 2008. Exhibit 9 displays the estimated V/D ratio of LEH from hedging the \$25 20-day maximum call. As September 12, 2008 is the last trading day before September 15, 2008, we note that the estimated V/D ratio for LEH is 81.42% on September 12, 2008, which is very close to the announcement of LEH.

7 Conclusions

This paper develops a method for the change of probability measures when the component processes of the multidimensional Brownian motion are correlated. We note that in order to model the correlations between assets, many studies set the price process of the assets underlying an n-asset derivative to be composed of a d-dimensional Brownian motion. Unlike the setting in the literature, the method herein allows all Brownian motions to be correlated and thus enables each price process to consist only of one-dimensional Brownian motion. Since this setting for price dynamics has advantages for empirical calibration, it is expected to facilitate additional empirical studies for the prices of multi-asset options.

We further apply the method to propose pricing formulae for vulnerable options on the maximum or minimum of n underlying assets. In addition to vulnerable maximum options and vulnerable minimum options, we note that other vulnerable path-independent options on multiple assets can be valued easily by using this method. We also develop a delta-like strategy to hedge the default risk of vulnerable multi-asset options based on the proposed pricing formulae and investigate hedging effectiveness by applying the proposed strategy to hedge the default risk induced by the bankruptcy of Lehman Brothers Holdings. The hedging performance is satisfactory.

One potential extension of the current paper is to incorporate the risk due to a change in the volatility of the issuer's asset, σ_V, into the hedging of credit risks. That will be left for our future study.

Appendix

Proof of Lemma 1

Given that $\mathbf{X} = [x_1, x_2, \cdots, x_k]^\top$ follows a k-dimensional multivariate standard normal distribution, $\mathbf{H} = [h_1, h_2, \cdots, h_k]$ is $\mathcal{F}_0$-measurable, and

$$\frac{d\mathcal{R}}{d\mathcal{Q}} = e^{\mathbf{HX}\sqrt{t} - \frac{1}{2}\mathbf{H\Gamma H}^\top t},$$

the expectation of the random variable $d\mathcal{R}/d\mathcal{Q}$ when event ψ occurs can be represented as:

$$\begin{aligned} & E^{\mathcal{Q}}\left[e^{\mathbf{HX}\sqrt{t} - \frac{1}{2}\mathbf{H\Gamma H}^\top t}\,\mathbf{1}_{(\psi)} \mid \mathcal{F}_0\right] \\ &= \int\int\cdots\int_\psi (2\pi)^{\frac{-N}{2}}\,|\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{X}^\top\mathbf{\Gamma}^{-1}\mathbf{X}}\left(e^{\mathbf{HX}\sqrt{t} - \frac{1}{2}\mathbf{H\Gamma H}^\top t}\right) dx_1 dx_2 \cdots dx_k \\ &= \int\int\cdots\int_\psi \mathcal{Z}\; dx_1 dx_2 \cdots dx_k, \end{aligned} \tag{A.1}$$

where

$$\mathcal{Z} \equiv (2\pi)^{\frac{-N}{2}}\,|\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{X}^\top\mathbf{\Gamma}^{-1}\mathbf{X}}\left(e^{\mathbf{HX}\sqrt{t} - \frac{1}{2}\mathbf{H\Gamma H}^\top t}\right).$$

Assume that $\mathbf{V} = [v_1, v_2, \cdots, v_k]^\top$ and

$$v_i = x_i - \sum_{j=1}^{k} \rho_{ij} h_j \sqrt{t}, \qquad \forall\; i = 1, 2, \cdots, k,$$

the relationship between $\mathbf{X}$ and $\mathbf{V}$ can be written as:

$$\mathbf{X} = \mathbf{V} + \mathbf{\Gamma H}^\top \sqrt{t}, \tag{$A.2$}$$

or

$$\mathbf{X}^\top = \mathbf{V}^\top + \mathbf{H\Gamma}^\top \sqrt{t}. \tag{$A.3$}$$

According to Equations (A.2) and (A.3), the term $\mathcal{Z}$ in (A.1) can be displayed as:

$$\begin{aligned}
\mathcal{Z} &\equiv (2\pi)^{\frac{-N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{X}^\top\mathbf{\Gamma}^{-1}\mathbf{X}} e^{\mathbf{H}\mathbf{X}\sqrt{t}-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t} \\
&= (2\pi)^{-\frac{N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{-\frac{1}{2}\left(\mathbf{V}^\top+\mathbf{H}\mathbf{\Gamma}^\top\sqrt{t}\right)\mathbf{\Gamma}^{-1}\left(\mathbf{V}+\mathbf{\Gamma}\mathbf{H}^\top\sqrt{t}\right)+\mathbf{H}\sqrt{t}\left(\mathbf{V}+\mathbf{\Gamma}\mathbf{H}^\top\sqrt{t}\right)-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t} \\
&= (2\pi)^{-\frac{N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{-\frac{1}{2}\mathbf{V}^\top\mathbf{\Gamma}^{-1}\mathbf{V}-\frac{1}{2}\mathbf{V}^\top\mathbf{H}^\top\sqrt{t}-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{\Gamma}^{-1}\mathbf{V}\sqrt{t}-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{\Gamma}^{-1}\mathbf{\Gamma}\mathbf{H}^\top t+\mathbf{H}\mathbf{V}\sqrt{t}+\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t} \\
&= (2\pi)^{-\frac{N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{-\frac{1}{2}\mathbf{V}^\top\mathbf{\Gamma}^{-1}\mathbf{V}-\frac{1}{2}(\mathbf{H}\mathbf{V})^\top\sqrt{t}-\frac{1}{2}\mathbf{H}\mathbf{V}\sqrt{t}-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t+\mathbf{H}\mathbf{V}\sqrt{t}+\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t} \\
&= (2\pi)^{-\frac{N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{V}^\top\mathbf{\Gamma}^{-1}\mathbf{V}}. \qquad \text{(A.4)}
\end{aligned}$$

Based on the result in Equation (A.4), Equation (A.1) can be rewritten as:

$$\begin{aligned}
E^{\mathcal{Q}}\left[e^{\mathbf{H}\mathbf{X}\sqrt{t}-\frac{1}{2}\mathbf{H}\mathbf{\Gamma}\mathbf{H}^\top t}\ \mathbf{1}_{(\psi)} \mid \mathcal{F}_0\right] &= \int\int\cdots\int_{\psi}(2\pi)^{-\frac{N}{2}}\ |\mathbf{\Gamma}|^{\frac{-1}{2}} e^{\frac{-1}{2}\mathbf{V}^\top\mathbf{\Gamma}^{-1}\mathbf{V}} dv_1 dv_2\cdots dv_k \\
&= E^{\mathcal{R}}\left[\mathbf{1}_{(\psi)} \mid \mathcal{F}_0\right].
\end{aligned}$$

The proof of Lemma 1 is complete.

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Exhibit 1

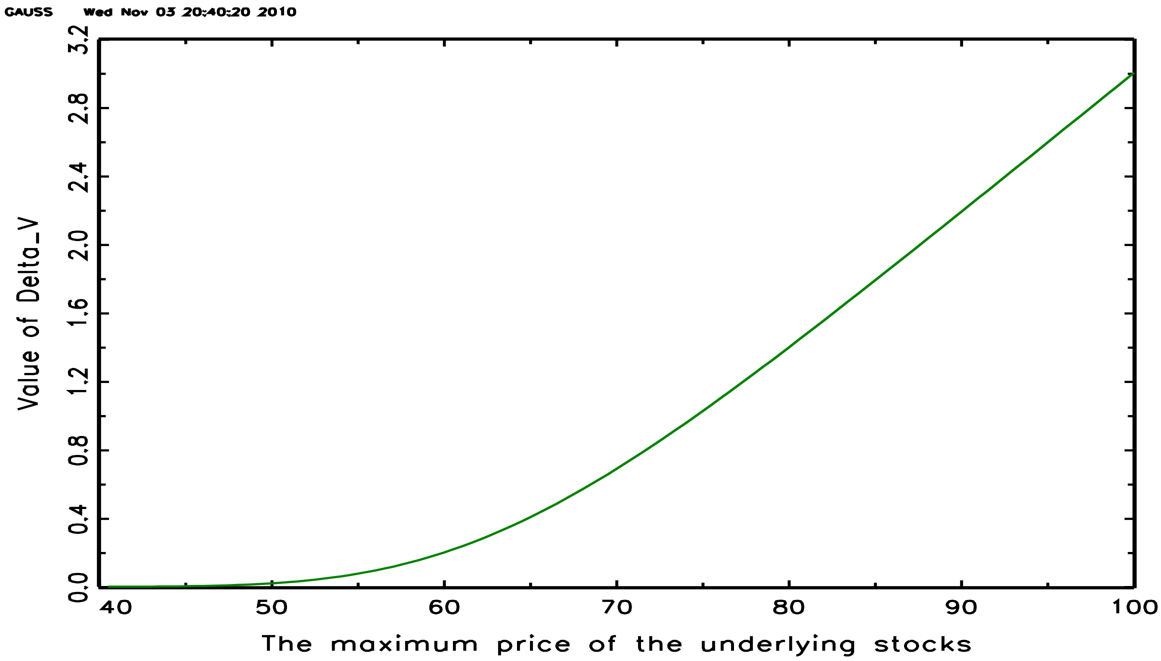
Values of vulnerable two-asset maximum call options

			Default-free Value	Panel A：V/D =1			Panel B：α =0		
				α =0	α =0.5	α =1	V/D =1.0	V/D =1.2	V/D =2.0
T	0.0833		2.198	2.146	1.744	1.343	2.146	2.198	2.198
	0.3333	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
	0.5833		6.334	5.969	4.934	3.899	5.969	6.226	6.334
σ_1	0.2		3.996	3.794	3.051	2.308	3.794	3.959	3.996
	0.3	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
	0.4		5.397	5.178	4.327	3.477	5.178	5.358	5.397
σ_V	0.2		4.639	4.508	3.740	2.972	4.508	4.632	4.639
	0.3	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
	0.4		4.639	4.349	3.554	2.759	4.349	4.550	4.638
K	30		13.257	12.552	9.970	7.387	12.552	13.126	13.257
	40	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
	50		0.766	0.737	0.629	0.520	0.737	0.760	0.766
ρ_{12}	-0.5		5.732	5.495	4.546	3.596	5.495	5.694	5.732
	0		5.274	5.041	4.142	3.242	5.041	5.233	5.274
	0.5	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
ρ_{1V}	-0.5		4.639	4.227	3.020	1.813	4.227	4.536	4.639
	0		4.639	4.352	3.342	2.331	4.352	4.583	4.639
	0.5	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
ρ_{2V}	0	(Base Case)	4.639	4.429	3.638	2.847	4.429	4.601	4.639
	0.5		4.639	4.529	3.968	3.407	4.529	4.628	4.639

Note：*The parameters used in the Base Case are given by: r = 0.04833, T = 0.3333, K = 40,* $S_1 = S_2 = 40$, $\sigma_1 = \sigma_2 = \sigma_V = 0.3$, $V_0 = 5$, $D = D^* = 5$, $\alpha = 0$, $\rho_{12} = \rho_{1V} = 0.5$, *and* $\rho_{2V} = 0$. *The default-free option price is calculated by the formula proposed in Johnson (1987).*

Exhibit 2

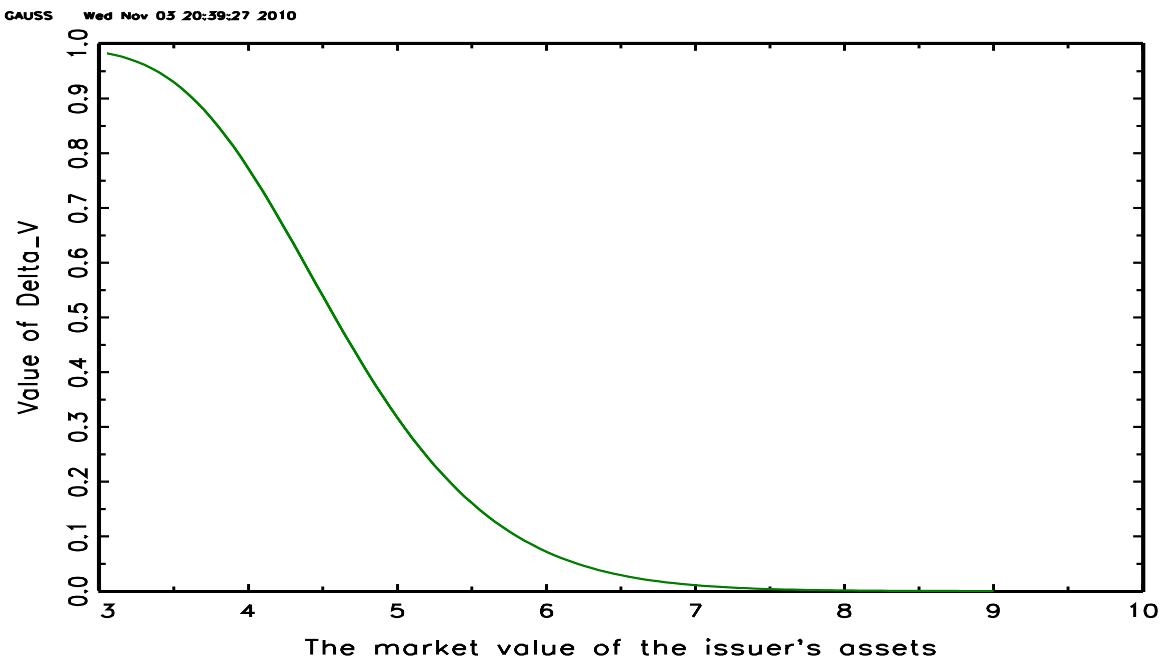
The influence that comes from the maximum of the two underlying stocks, S_1, on the value of Δ_V



Notes: The parameters are given by: $r = 0.04833$, $T = 0.3333$, $K = 60$, $S_2 = 40$, $\sigma_1 = \sigma_2 = \sigma_V = 0.3$, $V_0 = 5$, $D = D^* = 5$, $\alpha = 0$, $\rho_{12} = \rho_{1V} = 0.5$, *and* $\rho_{2V} = 0$. *All parameters are identical to those of the Base Case defined in Exhibit 1, except for the strike price and the value of* S_1.

Exhibit 3

The influence that comes from the value of XYZ's assets on the value of Δ_V



Notes: *The parameters are given by:* $r = 0.04833$, $T = 0.3333$, $K = 40$, $S_1 = S_2 = 40$, $\sigma_1 = \sigma_2 = \sigma_V = 0.3$, $D = D^* = 5$, $\alpha = 0$, $\rho_{12} = \rho_{1V} = 0.5$, *and* $\rho_{2V} = 0$. *All parameters are identical to those of the Base Case defined in Exhibit 1, except for the asset value of XYZ, i.e.,* V_0.

Exhibit 4

The stock price of LEH ranging from September 4, 2007 to September 16, 2008

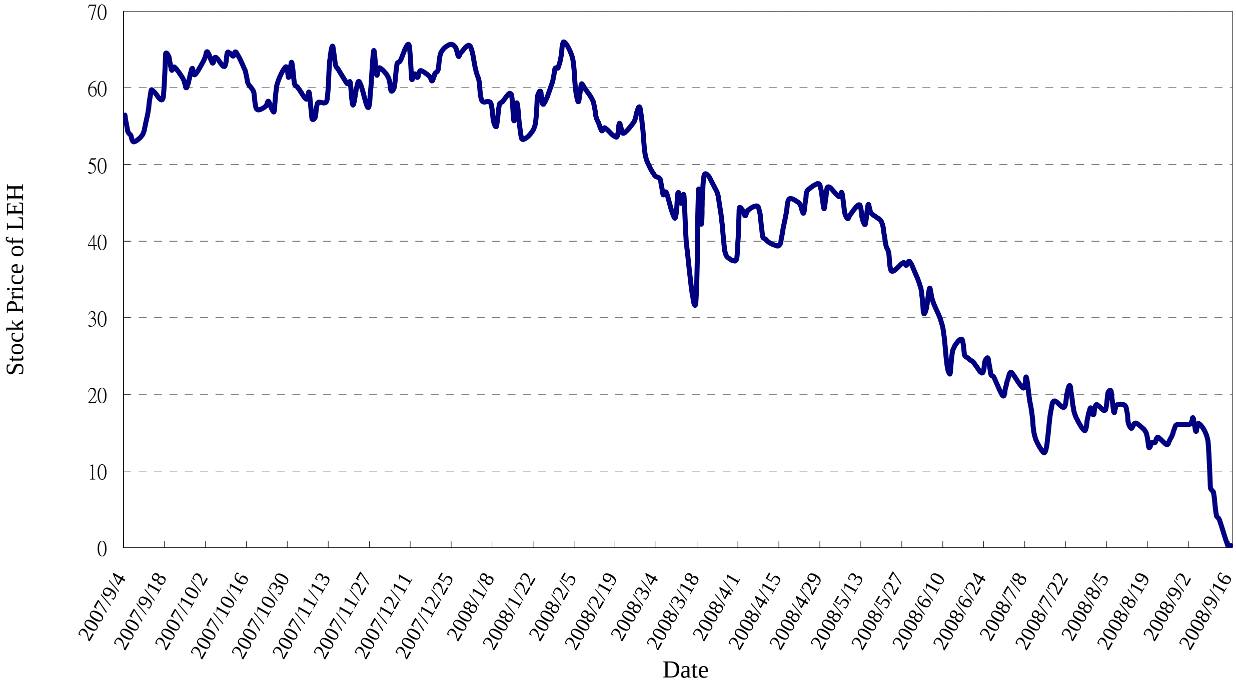


Exhibit 5

The stock prices of the two underlying assets ranging from September 4, 2007 to September 16, 2008

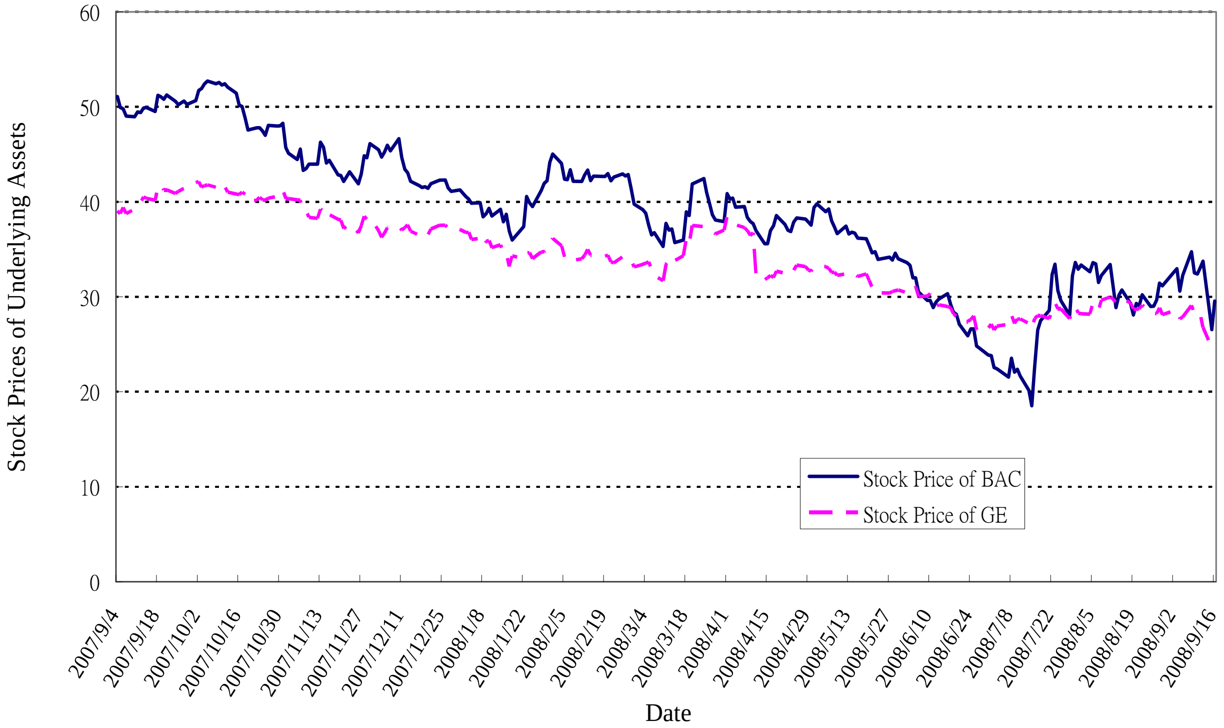


Exhibit 6

Hedging effectiveness of Δ_V -neutral hedging strategy

Strike	Maturity value from default-free	10-day vulnerable option		20-day vulnerable option		30-day vulnerable option	
		Maturity value with Δ_V hedging	Maturity value without hedging	Maturity value with Δ_V hedging	Maturity value without hedging	Maturity value with Δ_V hedging	Maturity value without hedging
15	14.550	14.483	11.200	14.156	11.086	14.429	11.020
20	9.550	9.759	7.351	9.619	7.276	9.736	7.233
25	4.550	5.034	3.502	5.079	3.467	5.040	3.446
30	0.000	0.616	0.000	0.735	0.000	0.590	0.000
35	0.000	0.038	0.000	0.057	0.000	- 0.034	0.000

Note: (1) The data used in the real-time study range from September 4, 2007 to September 16, 2008. In particular, we start the in-sample estimation for the LEH asset value V and asset volatility σ_V on September 4, 2007, and reserve T days to conduct an out-of-sample hedging comparison for a T-day hypothetic call. Thus, all hypothetic calls mature on the last day of our dataset, i.e., September 16, 2008. (2) The maturity value without hedging is calculated by Equation (23), whereas the total maturity value received by the holder of a vulnerable option who carries out Δ_V strategy is computed based on Equation (22).

Exhibit 7

The stock prices of LEH, BAC, and GE

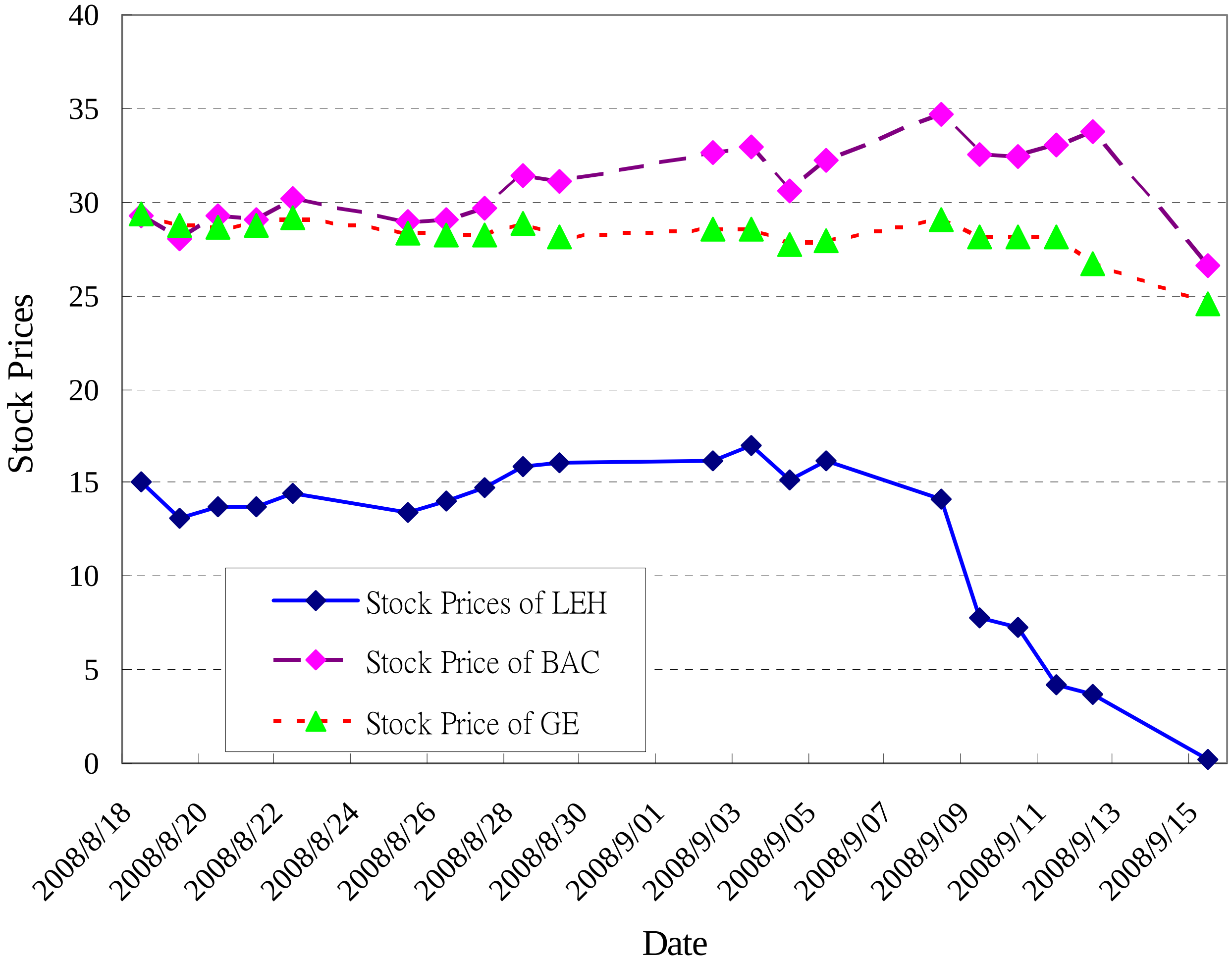


Exhibit 8

Shares sold for hedging the maximum call option

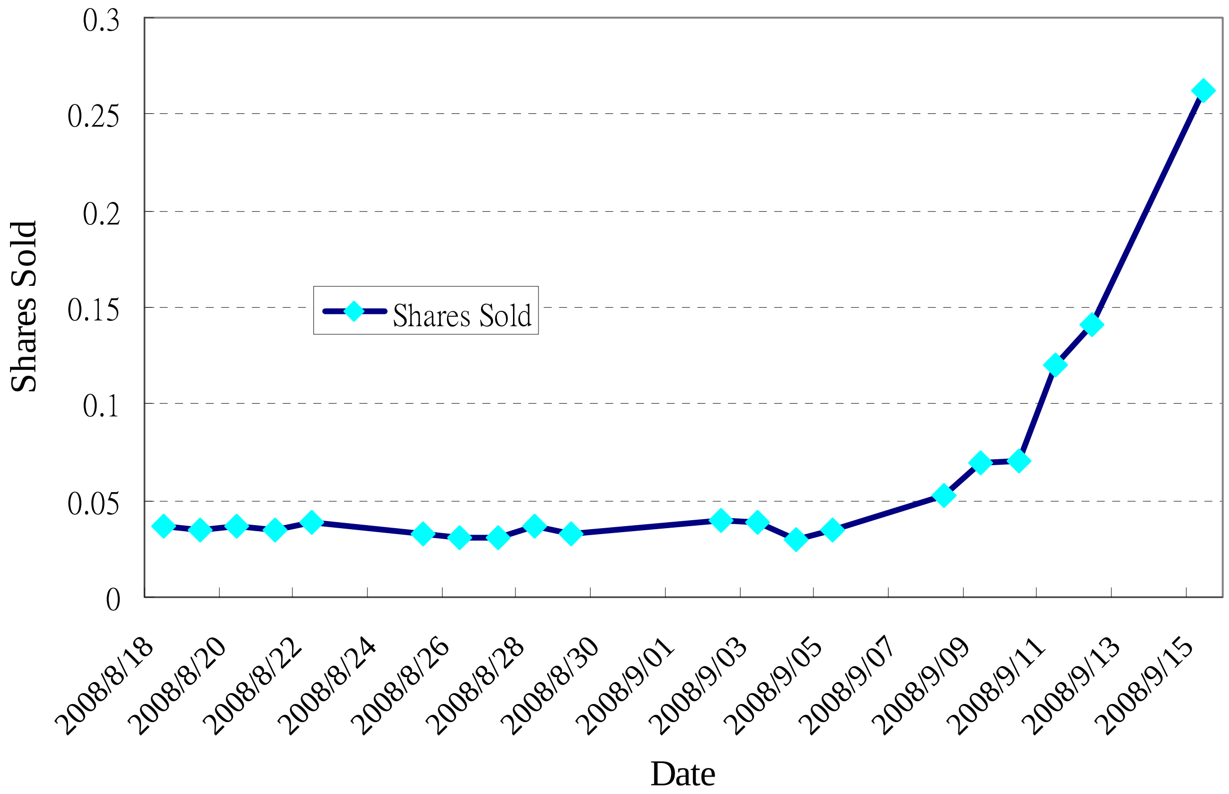


Exhibit 9

The estimated V/D ratio of LEH