Can Market Risk Perception Drive to Inefficient Prices? Theory and Evidence

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November 10, 2010

Abstract

This work presents an asset pricing model of informed investors with constant absolute-risk aversion (CARA) utility functions who trade with liquidity investors when prices and dividends are normally distributed. Adopting a competitive rational expectation equilibrium perspective, we find that the model shows two types of unique linear equilibrium price: the informationally semi-strong efficient price, similar to the original model of Campbell and Kyle (1993), and the completely informationally inefficient prices. We argue that the former Pareto dominates (is dominated by) the latter in the presence of low (high) market risk perception measured by risk aversion and market microstructure variance. The estimates of the model using real data confirm our theoretical findings. The S&P 500 Index is informationally efficient during 1871–2009, and inefficient in the sub-period 1995–2000.

JEL: C13; G12; G15

Keywords: Asset Pricing Theory, Efficient Market Hypothesis, State Space Models, Dot-Com Bubble.

*I thank Stefano Lovo (HEC, Paris) and Tommaso Proietti (University of Rome Tor Vergata) for helpful comments. I am in debt with Roberto Monte (University of Rome Tor Vergata) who supervised me and encouraged me to study this field while working for a PhD.

1 Introduction

This work presents a model that uses a rational expectation equilibrium perspective to show that financial markets are efficient when informed rational investors have a *low* market risk perception, and are inefficient when investors' risk perception is *high*. The work is innovative because the result is obtained without the need to invoke in the model short-sale constraints (Harrison and Kreps, 1978), subjective prior (Tirole, 1982), *ex ante* inefficiency (Allen, Morris, and Postlewaite, 1993), bounded rational investors or individual irrationality (DeLong *et al.*, 1990), diverse belief (Kurz, 1994, 1997, and 2007), overconfidence (Scheinkman and Xiong, 2003), or heterogeneous investors (Gulko, 2005), which is normally done.

In the finance literature the issue of efficiency versus inefficiency of prices, and the assumption of rational versus irrational markets, have been thoroughly investigated, theoretically and empirically, since the efficient markets hypothesis formulated by Fama (1970).[1] Indeed, they are still open to criticism. The hypothesis is that financial prices efficiently incorporate all public and private information and that prices can be regarded as the optimal estimates of true investment value at all times. In turn, it incorporates the notion of rational investors, who are able to use all the available information to solve their investment problems, maximizing the expected utility of consumption.[2]

The model assumes the structure of the economy used in the celebrated work by Campbell and Kyle (for convenience C.K.) (1993). There are two type of investors who trade one risky asset: the informed risk-averse investors, who choose the optimal mumber of shares of the asset that solve the expected-utility problem, and some noise traders who come to the market for liquidity reasons. Investors observe the dividend process and have private information regarding the dividend growth rate. The price is written as the fundamental value of public and private information with a discount term, accounting for investors' risk aversion, and a linear term expressing the sensitivity of the price to supply shocks. According to this setting we depart from the *ex ante* assumption that the equilibrium price is fully informative as in C.K. Our model lets the informed investors determine their optimal demands by postulating an equilibrium price that is not *a priori* informative. Thus investors solve the investment problem and determine their asset's demand that make price optimal. This procedure let us obtain several candidate equilibrium prices that always include the fully informative one. We select from the candidate equilibrium prices the one with the highest utility for the informed investors and we consider it as the equilibrium price of the model. Finally, the equilibrium price depends on market parameters.

Our main result is that inefficient prices are determined by the investors' market risk perception. In case of *high* risk aversion and/or *high* market microstructure volatility, the efficient equilibrium price is Pareto dominated by the inefficient one. The inefficiency is observed when neither public information nor private information are correctly revealed in the asset price. As a consequence of this inefficiency, investors ask a higher risk premium with respect to the efficient price. This reaction is an observed anomaly in real markets called the equity premium. Moreover, when a stock price does not equal the present value of future expected dividend streams, and its changes in price should not be attributable to news about dividends or discount factors, the higher discount in price leads to high variability in price dynamics. This is the observed excess volatility in real stock prices.

We estimate the model using real financial market data. We compare the estimates when the model assumes the efficient price versus the inefficient price. The S&P 500 Index reflects the fundamental values, and it validates the efficient market hypothesis during the long period

[1] The term was originally coined in an unpublished working paper by Harry Roberts (1967), whereas the history of the efficient market hypothesis begins with Cardano in 1564 as reported by Sewell (2008).

[2] Note that having agents with rational expectations implies that on average the population (even if no one person does) updates its expectations appropriately.

1871–2009, while in the sub-period 1995–2000, characterized by high market volatility, we obtain an opposite result. The likelihood ratio test, used to compare the fit of the two models, rejects at 1% the null hypothesis of the efficient market. Furthermore, the estimates seem to confirm the theoretical results that focus on the role of market volatility and investors' risk aversion to determine inefficient prices.

The paper is organized as follows. Section 2 reviews the literature on efficient markets, bubbles, and empirical anomalies. The economy of the model is spelled out in Section 3. In Section 4 we derive the rational expectation equilibria of the full model, the efficient price model (Model A), the inefficient price model (Model B), and the utility criteria used to compare different candidate equilibrium prices. Section 5 calibrates the model and shows the theoretical results. Section 6 provides the estimates of the model and shows that real data are close to our theoretical results. Section 7 concludes.

2 Literature Review

The works of Grossman (1976), Grossman and Stiglitz (1976), and later Kyle (1985, 1989), C.K. (1993), Wang (1994) are the first to include the efficient market hypothesis and the presence of informed investors and noise traders in a contest of competitive markets to analyze the role of information in price dynamics.[3] In these models the rational informed investors are those active traders who know everything and their trading is perfect. Noise traders model the irrationality in the market assuming that such investors do not collect information and whose trading activity is informative for others. In such models, prices can deviate from fundamental values due to the action of the noise traders and the desire of the rational agents to exploit them as much as possible. Equilibrium prices are achieved in the Walrasian auctioneer scenario: each agent solves her optimization problem treating the market-clearing price in that period as parametric, the auctioneer announces a price and receives from all market participants what their demand/supply would be at that price. The auctioneer then determines the excess demand at that price and keeps announcing prices until a price is arrived at that sets excess demand to zero.[4] This rational expectation equilibrium perspective has great success in capturing the dynamics and informativeness of asset prices, mainly because of the easy tractability of the equilibrium price. The main result of these models is that there exists a unique equilibrium price in the semi-strong form. Further, noise traders play a key role in clearing the market and avoiding breakdown of the market.[5]

Despite these important theoretical results, and some critiques for the induced "schizophrenia" of the informed investors (Hellwig, 1980; Kyle, 1989; Back, 1992), these models do not capture some important market empirical anomalies (see Siegel, 2002 for an extensive review of all anomalies). The financial literature of last twenty years reviewed these apparent anomalies, taking the efficient markets hypothesis as a benchmark. They include the equity premium puzzle (Mehra and Prescott, 1985), the excess volatility in stock returns and price-dividend ratios (Grossman and Shiller, 1981; LeRoy and Porter, 1981; Shiller, 1981), the predictability of stock returns (Poterba and Summers, 1988; Fama and French, 1989; Campbell and Shiller, 1988). According to Shiller (1998) these anomalies suggest that the underlying principles of rational behavior, and the efficient markets hypothesis, are not entirely correct and that we need to look at other models of human behavior.

Economic theory defines the asset price bubble as any situation in which the price of an

[3]See Brunnermeier, 2001 for an extended literature review.

[4]See the book by Hens and Schenk-Hoppe (2009) for a wide discussion of different theoretical, analytical, and empirical techniques that explain the market dynamics of asset prices.

[5]Without the presence of noise traders, the no-trade theorem of Milgrom and Stokey (1982) applies and there is no exchange.

asset deviates from its fundamental value. Hence an asset price bubble is an inefficient price. On the other hand, the presence of bubbles and market anomalies discussed above have not been investigated together. This is due to the empirical difficulties related with the measure of expected dividends that have yet to be realized. For this reason economists sought to determine the conditions that allow a bubble to arise. Tirole (1982) adds the assumption of infinite trading by finitely many traders.[6] Different works demonstrate the validity of these assumptions and show how a bubble arises when the assumptions are relaxed. Harrison and Kreps (1978) assume that traders start with different initial beliefs, De Long *et al.* (1990) exploit the possibility of trading with rational and irrational traders, Tirole (1985) shows different examples of the presence of bubbles with infinite traders, and Townsend (1980) describes a model with inefficient allocated resources as a reason for having money (interpreted as a bubble).

Literature on financial bubbles shows that differences of opinion among investors and short sales constraints are sufficient to generate a price bubble. See Miller (1977), Harrison and Kreps (1978), Chen, Hong, and Stein (2002), Scheinkman and Xiong (2003), and the extensive empirical works confirming this results such as Lamont and Thaler (2003) and Ofek and Richardson (2003). These works stand in contrast to the rational bubble literature (Blanchard and Watson, 1982; Allen, Morris, and Postlewaite, 1993) in which these two ingredients are not crucial to generating a rational bubble in a finite or infinite horizon setting. We are in line with these latter works when we show inefficient prices without assuming short-sale constraints or investors' subjective prior, but we depart from these works when we do not assume *ex ante* inefficiency or the finite horizon.

The seminal work of C.K. (1993) shows that the price of a risky asset can deviate from the expected present value based on public and private information by a discount term, accounting for investors' risk aversion, and a linear term expressing the sensitivity of the price to the supply shocks. According to this setting they estimate the efficient equilibrium price on real market data. Their main findings suggest that prices are noise affected, risk aversion is captured by the constant term, and the S&P 500 reflects the price written in this linear form.

Our theoretical findings are similar to Monte *et al.* (2009) who show, under asymmetric information, that markets admit equilibria with inefficient prices. They show that a *low* investors' perception of market risk induces investors to trade as perfect competitors, and consequently informationally efficient equilibria are achieved, while a *high* investors' perception of market risk leads investors to prefer strategic trading, and informationally inefficient equilibria result. This work does not assume any asymmetry in the information structure, and any difference among informed or uninformed investors. As a consequence, we interpret the presence in the market of inefficient prices not in terms of perfect or imperfect competition but only as a possible explanation of some market anomalies rationally determined by the informed investors.

3 The Economy

Consider an economy composed of completely informed risk-averse rational investors and noise traders exchanging a risky asset. Assume that dividend and price are normally distributed and that changes in the *level* of dividend and stock price have constant variance. As a consequence

[6] The assumption of efficient allocation guarantees that any investor who agrees to buy an overvalued (undervalued) asset must believe that he will benefit from reselling it at a future time. The assumption of common priors is discussed in Morris (1995). In case of finitely many traders, bubbles simply do not arise because the last traders refuse to purchase an overvalued asset knowing there will be not any opportunity to resell it.

the variance of percentage returns and dividend growth rate increases (decreases) when the level of price and dividends decreases (increases). Let us de-trend dividend and stock prices by an exponential growth trend ξ obtained by market data as the dividend growth mean. Therefore we have

$$D(t) \equiv D^u(t)e^{-\xi t} \qquad\qquad P(t) \equiv P^u(t)e^{-\xi t} \tag{1}$$

where the variables D^u and P^u are the observed dividend and price of our stock, and $D(t)$ and $P(t)$ are the de-trended dividend and price. We assume that changes in D_t and P_t are homoskedastic and normally distributed. We also assume that both have one unit root, but a particular combination of levels of prices and dividends is stationary, that is we assume they are cointegrated.

Let r be the time-invariant risk-free interest rate. Then a permanent one-dollar change in the de-trended dividend has a discounted value of $1/(r-\xi)$ dollars, such that the cointegrating stationary vector is $D(t) - (r - \xi)P(t)$. We call the unconditional mean of this variable $\gamma = E[D(t) - (r - \xi)P(t)]$. As in C.K. the investors expect the price can be decomposed into the sum of a “fundamental” value, a constant term, and a noise term:

$$P(t) = \frac{\gamma}{(r-\xi)} + V(t) + \Theta(t) \tag{2}$$

$$= p_0 + V(t) + \Theta(t) \tag{3}$$

where p_0 captures the constant risk premium per share of stock demanded by risk-averse informed investors, $V(t)$ represents the expected future dividend (i.e., public information) and the nondividend component (interpreted as the investors' private information), and $\Theta(t)$ is the noise component. This price's form is convenient because (i) it is linear, (ii) it does not require any assumption about the discount rate, and (iii) the noise trading component, given by a random supply of the stock, captures the presence of liquidity traders. Finally, note that noise trading influences the stock price because the informed investors are risk-averse and ask a risk premium that is captured by the constant term. A special case of Equation (2) is our benchmark case (Model A) that is the efficient price derived in the work of C.K. in which the fundamental value $V(t)$ is the present expected value of dividend and nondividend discounted at the risk-free rate r.

Dividend structure. There are continuous dividend announcements to the market. The dividend (de-trended) is the sum of permanent and temporary components, independently distributed and not directly observed by the informed investors:

$$D(t) = D_0(t) + D_1(t).$$

The permanent component is a brownian motion process, and the temporary component is a mean reverting process, a continuous-time AR(1), given by

$$dD_0(t) = \alpha_I I(t) + \sigma_0 dw_0(t), \qquad dD_1(t) = -\alpha_D D_1(t)dt + \sigma_D dw_D(t) \tag{4}$$

where $dw_0(t)$ and $dw_D(t)$ are two standard independent brownian motions, σ_0 and σ_D constitute the innovations in $D_0(t)$ and $D_1(t)$, and the quantities σ_0^2 and σ_D^2 are the innovation variance of $D_0(t)$ and $D_1(t)$ respectively. The idea that dividends have a private hidden information content is an old one (see Lintner, 1956; Miller and Modigliani, 1961; Watts, 1973) and it has been tested empirically by several works (Shiller, 1981; DeAngelo *et al.*, 1992). The parameter α_I captures this hidden private information content in the dividend process and is useful for scaling the unit of I. $I(t)$ measures how much $D_0(t)$ is expected to increase in the future. The positive parameter α_D measures the speed mean reversion of the transitory component and $-\alpha_D D_1(t)$ is the expected growth rate of dividends.[7]

[7]The scaling parameter does not change the final results (see C.K., Appendix A).

Information structure. Informed investors receive private information $I(t)$ (a private signal) about the asset price. It is convenient to interpret it as the unknown part of the dividend process, i.e., the "nondividend information" component.[8] The information is modeled as a mean reverting process:

$$dI(t) = -\alpha_I I(t) + \rho_I \sigma_0 \, dw_0(t) + (2\rho_I - \rho_I^2)^{1/2} \sigma_0 \, dw_I(t) \tag{5}$$

in which $dw_I(t)$ is a standard brownian motion independent of $dw_0(t)$, and σ_I constitutes the innovation in $I(t)$. Investors receive new information about the traded stock captured by the two random components of the process, measured by the standard deviations σ_0 and σ_I. The parameter α_I captures the mean-reverting speed at which the new information is updated into the price. As α_I increases (decreases), private information decays faster (slower) and it is short lived in the price dynamics. The correlation structure between $dD(t)$ and $dI(t)$ is given by $\chi = -\frac{\rho_I}{\sqrt{2\rho_I}}$, which ensures that

$$E\left\{ I(t+s) \mid D[-\infty, t] \right\} = 0, s \geq 0 \tag{6}$$

and the history of the dividend process cannot forecast the future of $I(t)$. A technical condition is imposed on the parameter $0 \leq \rho_I \equiv \sigma_I^2 / 2\sigma_0^2 \leq 2$ to guarantee that D does not forecast I, that is the variables $D(t)$ and $I(t)$ are independently distributed. C.K. show that Equation (6) uniquely determines the diffusion term in the $D(t)$, $I(t)$ processes.

Noise trading. Following the noisy rational expectational models,[9] we assume that the total amount of risky asset supply is $1 + \Theta(t)$. The process $\Theta(t)$ models the deviation of the current risky asset supply from its long-run stationary level normalized to 1. This assumption implies that noise traders—those passive players buying and selling either for liquidity reasons or as tax-related trading—have inelastic demand of $1 - \Theta(t)$ shares of the stock at time t, such that $\Theta(t)$ is the number of remaining shares available to the market, i.e., to the informed investors. The noise process has a non-null mean reverting dynamics

$$d\Theta(t) = -\alpha_\Theta \Theta(t) \, dt + \sigma_\Theta \, dw_\Theta(t), \tag{7}$$

in which $dw_\Theta(t)$ is a standard Brownian motion independent by $[dw_{D_0}(t),\, dw_{D_1}(t),\, dw_I(t)]$, the positive parameter α_Θ is the constant mean speed of reversion of the process $\Theta(t)$ towards its long-run null level, and σ_Θ is the variance of market noise. The stochastic supply of the risky asset in the aggregate market makes the market incomplete.

The informed investors' information set at time t is given by the (public) dividend and (private) nondividend processes

$$\mathfrak{F}(t) \;\equiv\; \sigma[D_0(s), D_1(s), I(t), \Theta(t), P(t);\, s \leq t] \;=\; \sigma[D(t), I(t), P(t);\, s \leq t],$$

where the σ-field of $\mathfrak{F}(t)$ is generated by two observed variables, $D(t)$ and $P(t)$, and the latent variable $I(t)$ is extracted with the Kalman-Filter procedure. Finally, note that informed investors are able to extract the missing variable $\Theta(t)$ from price and dividend, therefore they have complete information in equilibrium.

[8] C.K. defines $I(t)$ as the measurement error on the transitory component. The information process is defined as $I(t) \equiv \hat{D}_1(t) - D_1(t) = D_0(t) - \hat{D}_0(t)$, where $[\hat{D}_0(t); \hat{D}_1(t)]$ are the investors' estimates of $D_0(t)$ and $D_1(t)$ respectively.

[9] See Diamond and Verrecchia, 1981; C.K., 1993; Wang, 1993; He and Wang, 1995; Vives, 1995; Foster and Viswanathan, 1996; Brunnermeier, 2001; Allen Morris Shin, 2006.

CARA-Utility. We assume that informed investors have a constant absolute-risk aversion (CARA) utility function

$$u[t, c(t)] = -e^{-[\beta t + \varphi c(t)]}, \tag{8}$$

where β is the time-impatience parameter and φ is the coefficient of the absolute risk aversion. The use of the CARA utility function, and the assumption of normality of dividend and stock prices, let the expected future dividends be discounted at the riskless rate of interest. This is equivalent to saying that an increase in the expected future dividend, given by a higher value of investors' private information, is captured by a change in the variable $V(t)$ in Equation (2), while an increase (decrease) in the informed investors' risk aversion causes a lower (higher) value of the constant term. Investors choose consumption and inventory of risky assets to maximize their utility given the information set

$$\max_{\Psi(t),\, c(t)} \mathbf{E}\left[-\int_{t=0}^{+\infty} u[t, c(t)]\, dt\, |\mathfrak{F}(t)\right]$$

The use of CARA preferences implies that the investors' optimal asset demand, and thus the optimal equilibrium price, are independent of their wealth distribution as well as the level of aggregate wealth. This greatly simplifies our optimization problem.[10] The model has a closed-form solution according to that preference.

4 Equilibrium

In this section we solve for the equilibrium of the economy described in Section 3. We use the rational expectation equilibrium perspective developed by Lucas (1972), Green (1973), Grossman (1976), and Kreps (1977). Informed investors conjecture the form of the equilibrium price and maximize the expected utility functions subject to the budget dynamic constraint and conditioned on their own private information, as well as the information that the equilibrium prices generate. Market clearing is imposed to verify the conjectured price function. We remark that the investors' optimization problem determines the optimal demand at the stock price conditioned on the conjecture price function. Hence the equilibrium price is optimal because it is the price that make the optimal demand.

Following the equilibrium price described in (2), the investors write the price's form linearly depending on the state variables of the economy:

$$P(t) = p_0 + p_{D_0} D_0(t) + p_{D_1} D_1(t) + p_I I(t) + \Theta(t) \tag{9}$$

in which p_0 is assumed to be negative because it is the discount on price given by the investors' risk aversion, the state variables $[D_0(t), D_1(t)]$ account for the observed dividend $D(t)$, $I(t)$ is the hidden stationary private information, and $\Theta(t)$ is the aggregate supply shock of the stock. The variance of the stock price is given by $\sigma_P = p_{D_0}^2 \sigma_{D_0} + p_{D_1}^2 \sigma_{D_1} + p_I^2 \sigma_I + \sigma_\Theta$. If we assume a constant variance of price this implies that the variance of percentage of returns increases as the price of the stock decreases and vice versa. This phenomenon has been studied since the works of Black (1976) and Nelson (1987). Our benchmark case Model A assumes a constant variance of price because the price coefficients are constant. Model B will not assume a constant variance of price because it lets free the price coefficients.

Investment opportunity. Given the processes of $[D(t), I(t), \Theta(t)]$ described in Equations (4–7), the stock price follows the process

$$dP(t) = [-p_{D_1} \alpha_D D_1(t) + (p_{D_0} \alpha_I - p_I \alpha_I) I(t) - \alpha_\Theta \Theta(t)]\, dt + H\, \mathbf{dw(t)} \tag{10}$$

[10]We know that the CARA utility allows negative consumption and negative wealth, so we prefer to avoid these problems by not imposing non-negative constraints.

where $H = \{p_{D_0}\sigma_0 + p_I \rho\, \sigma_0,\ p_{D_1}\sigma_D,\ p_I\sqrt{2\rho_I - \rho^2}\sigma_0,\ \sigma_\Theta\}$ and $\mathbf{dw(t)}$ is the (1×4) vector of brownian motions. Our investment opportunity is given by $Q(t)$, the instantaneous excess return on one share of risky asset, governed by the process

$$dQ(t) = [D(t) - rP(t)]\, dt + dP(t) \tag{11}$$

where the risk-free rate r is assumed to be constant. According to Wang (1993), $dQ(t)$ is interpreted as the return on a zero-wealth portfolio long one share of stock fully financed by borrowing at the risk-free rate. Q gives the undiscounted cash flow from the zero-wealth portfolio.

Given the process (10), $Q(t)$ satisfies the stochastic differential equation

$$\begin{aligned} dQ(t) &= [D(t) - rP(t)]\, dt + dP(t) \\ &= [e_0 + e_{D_0}D_0(t) + e_{D_1}D_1(t) + e_I I(t) + e_\Theta \Theta(t)]\, dt + H\, \mathbf{dw(t)} \end{aligned} \tag{12}$$

where $e_0 = -rp_0$, $e_{D_0} = 1 - rp_{D_0}$, $e_{D_1} = 1 - p_{D_1}(r - \alpha_D)$, and $e_\Theta = -(r + \alpha_\Theta)$. The conditional expectation of the excess return on one share of stock is $E_t[dQ] = [e_0 + e_{D_0}D_0 + e_{D_1}D_1 + e_I I + e_\Theta\Theta]\, dt$. The expected excess return is affected by all the state variables of the economy, while σ_Θ, the market microstructure variance, directly influences the price volatility but does not affect the investment opportunity. As in Wang (1993), the level of aggregate stock supply affects dQ because it determines the total risk exposure of the economy.

The optimization problem. Let us denote by $\Psi(t)$ the holding of the risky asset at time t: that is, the investors' inventory. The investors' wealth $W(t)$ has the following dynamics:

$$dW(t) = [rW(t) - c(t)]\, dt + \Psi(t)\, dQ(t), \tag{13}$$

where $c(t)$ is the investor's consumption policy and dQ is given by Equation (12). The investors maximize the expected value of their exponential utility over the infinite time horizon, subject to the wealth dynamics and given the information set at time t, by controlling their inventory $\Psi(t)$ and their consumption $c(t)$. The investor's optimization problem is

$$\begin{aligned} &\max_{\Psi(t),\, c(t)} \mathbf{E}\left[-\int_{t=0}^{+\infty} e^{-[\beta t + \varphi c(t)]}\, dt \,|\mathfrak{F}(t) \right] \\ s.t.\quad & dW(t) = [rW(t) - c(t)]\, dt + \Psi(t)\, dQ(t) \end{aligned} \tag{14}$$

where $\mathbf{E}[\,\cdot\,|\mathfrak{F}(t)]$ is the conditional expectation operator given the information set $\mathfrak{F}(t)$. Let $V(Z, W, t)$[11] be the value function, where (Z, W) are the state variables moving the investment opportunities and $Z = (1,\, D_0,\, D_1,\, I,\, \Theta)^\top$. The variables of the economy can be written in compact form as

$$dZ(t) = AZ(t)\, dt + B^{1/2}\, dw(t) \tag{15}$$

where

$$A \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_I & 0 \\ 0 & 0 & -\alpha_D & 0 & 0 \\ 0 & 0 & 0 & -\alpha_I & 0 \\ 0 & 0 & 0 & 0 & -\alpha_\Theta \end{pmatrix}, \quad B^{1/2} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sigma_0 & 0 & 0 & 0 \\ 0 & \sigma_D & 0 & 0 \\ -\rho_I\sigma_0 & 0 & (2\rho_I - \rho_I^2)^{1/2}\sigma_0 & 0 \\ 0 & 0 & 0 & \sigma_\Theta \end{pmatrix}. \tag{16}$$

[11] We assume that $V(Z, W, t)$ is twice differentiable in each of the state variables.

As in C.K. we consider the $Z(t)$ process as a Vector Autoregression (VAR). The value function $J(Z, W, t)$ satisfies the Bellman equation

$$\begin{aligned} 0 &= \max_{\Psi(t),\, c(t)} \left\{ -\int_{t=0}^{+\infty} e^{-[\beta t + \varphi c(t)]} + E\left[dJ(Z, W, t)\right] ds \right\} \\ dZ(t) &= AZ(t)\, dt + B^{1/2}\mathbf{dw(t)} \\ dW(t) &= [rW(t) - c(t) + \Psi(t)SZ(t)]\, dt + \Psi(t)T^{1/2}\mathbf{dw(t)} \\ 0 &= \lim_{s\to\infty} E\left[J(Z, W, t+s)\right]. \end{aligned} \tag{17}$$

The solution of this problem is given in Theorem (1).

Theorem 1 *We solve the investors' optimization problem conjecturing the following form of the value function*

$$J(Z, W, t) = -e^{-\beta t - r\varphi W + \Phi(Z) - \lambda}, \tag{18}$$

where $\Phi(Z) = \frac{1}{2}Z^\top LZ$*. The optimal share of the stock is a linear function of* $Z(t)$

$$\tilde{\Psi}(t) = -\frac{T^{1/2}\left(B^{1/2}\right)^\top L - S}{r\varphi T} Z(t) \tag{19}$$

and the optimal consumption is given by

$$\tilde{c}(t) = \frac{\frac{1}{2}Z^\top(t)LZ^\top(t) + r\varphi W(t) + \lambda - \ln(r)}{\varphi}, \tag{20}$$

where $L \equiv (l_{i,j})_{i,j=1}^5$ *is a symmetric real matrix and* λ *is a real number satisfying*

$$r[1 + \lambda - \log(r)] - \beta - \frac{1}{2}\mathbf{tr}\left[\left(B^{1/2}\right)^\top LB^{1/2}\right] = 0. \tag{21}$$

The investors' demand and consumption are optimal when the matrix L *has the coefficients that are solutions of the algebraic Riccati equation*

$$LUL - LX - X^\top L - Y = 0, \tag{22}$$

for

$$\begin{aligned} U &\equiv B^{1/2}\left[TI_4 - \left(T^{1/2}\right)^\top T^{1/2}\right]\left(B^{1/2}\right)^\top \\ X &\equiv T\left(A - \tfrac{1}{2}rI_5\right) - B^{1/2}\left(B^{1/2}\right)^\top S \\ Y &\equiv S^\top S \end{aligned} \tag{23}$$

and I_n *is the identity matrix with dimensions* n*.*

Proof. 1 *See Appendix A.*

Market clearing. Market clearing is the condition that ensures that the conjectured price in Equation (9) is the equilibrium price. The condition implies that investors' demand is constrained to respond to the stochastic risky asset supply. In other words, when the market clears investors' demand must sum to $1 + \Theta$. Thus

$$\tilde{\Psi}(t) = 1 + \Theta(t). \tag{24}$$

According to Equation (24) the coefficients of $\tilde{\Psi}(t)$ satisfy the equalities

$$\psi_0 = 1, \quad \psi_{D_0} = 0, \quad \psi_{D_1} = 0, \quad \psi_{D_I} = 0, \quad \psi_\Theta = 1 \tag{25}$$

that we impose in our optimization problem (19–23) to determine the coefficients $[p_0, p_{D_0}, p_{D_1}, p_I]$. Note that Equation (23) verifies that the conjectured form of the equilibrium price (9) is optimal. On this account the investors' optimization problem should be more appropriately interpreted as the determination of the risky asset price that makes the rational investors' equilibrium demand for the risky asset optimal.

Theorem (1) and market clearing condition (24) determine two types of candidate equilibrium prices: the informationally semi-strong efficient price and the inefficient prices. We obtain two types of candidates because we assume the price coefficients are misspecified when investors solve the optimization problem. The efficient candidate equilibrium price is our benchmark case and is in line with the efficient market hypothesis. We call this efficient price Model A. The inefficient candidate equilibrium prices are those that deviate from the efficient one and in which public and private information are incorrectly reflected in the market price. We call these inefficient prices Model B.

4.1 Model A: Efficient Equilibrium Price

Our benchmark case is the efficient equilibrium price; i.e., the fundamental value $V(t)$ is the present expected value of dividend and nondividend discounted at the risk-free rate r.

Proposition 1 *The economy defined in Equations (4–8) has a stationary rational expectations equilibrium in which price is efficient:*

$$\begin{aligned}\tilde{P}(t) &= \tilde{V}(t) + \tilde{p}_0 + \Theta(t) \\ &= \tilde{p}_0 + \tilde{p}_{D_0} D_0(t) + \tilde{p}_{D_1} D_1(t) + \tilde{p}_I I(t) + \Theta(t)\end{aligned} \tag{26}$$

where

$$\tilde{V}(t) = E_t \int_{s=0}^{\infty} e^{-rs} D^u(t+s)\, ds = E_t \int_{s=0}^{\infty} e^{-(r-\xi)s} D(t+s)\, ds$$

and price $\tilde{P}(t)$ has the following coefficients:

$$\tilde{p}_0 = -\left(\frac{[(r-\xi+\alpha_I)^2 - 2(r-\xi)\alpha_I \rho_I]\sigma_0^2}{(r-\xi)^2 (r-\xi+\alpha_I)^2} + \frac{\sigma_D^2}{(r-\xi+\alpha_D)^2} \right) \frac{r}{r-\xi} \varphi \tag{27}$$

$$\tilde{p}_{D_0} \equiv \frac{1}{r-\xi}, \qquad \tilde{p}_{D_1} \equiv \frac{1}{r-\xi+\alpha_D}, \qquad \tilde{p}_I \equiv \frac{1}{r-\xi} - \frac{1}{r-\xi+\alpha_I} \tag{28}$$

Proof. 2 *See Appendix B.*

The constant term is obtained when informed investors maximize their objective function (14) using the price coefficients in the form of Equation (27). Note that $\tilde{p}_0$ depends on the fundamental risk parameters (α_D, α_I, ρ_I, σ_0, σ_D) and on the investors' risk aversion φ that increases the expected return on the stock by increasing risk premium with a higher negative term. As shown in C.K. and Wang (1993), this is a simple discount on the price to account for the increasing discount rate. Note that it is possible to extract from (27) the algebraic form of γ given in Equation (2) and a measure of investors' risk aversion once we estimate Model A using real data (see Section 6).

4.2 Model B: Inefficient Equilibrium Prices

Economic theory defines an asset price bubble as an asset price deviating from its fundamental value, that is the discounted expected value of all dividends yielded by the asset. A bubble is an inefficient price. Model B shows the equilibrium prices, which are not the present value of the expected discounted dividend and nondividend flow. These equilibria arise from the informed investors' optimal demand (19) and market clearing condition (24) when we let the price coefficients be misspecified.

Proposition 2 *The economy defined in Equations (4–8) has a stationary rational expectations equilibrium in which the price has the following forms:*

$$\widehat{P} = \widehat{p}_0 + \widehat{p}_{D_0} D_0(t) + \widehat{p}_{D_1} D_1(t) + \widehat{p}_I I(t) + \Theta(t) \tag{29}$$

in which at least one of the price coefficients is not defined as in Equation (27).

Proof. 3 *Proof of Proposition 2 is possible only via numerical solution due to the high complexity of matrices (23). See Section 5.*

4.3 Utility Function

Model A and Model B describe two types of equilibria: the semi-strong efficient equilibrium price and the inefficient prices. For any given set of exogenous parameters we have a multiple candidate equilibrium, which corresponds to Model A or Model B. This is due to the high nonlinearities of Equations (19–23). We remark that each candidate is a mathematical solution of the infinite-horizon optimization problem and only one is the equilibrium price that maximizes the investors' utility. We Pareto rank all these candidate equilibrium prices according to the utility criteria and we consider the one with the highest utility as the equilibrium price. Here we show the form of the value function (18) when Theorem (1) is verified:

$$J(Z, L, \lambda, t) = \lambda + \frac{1}{2} l_{1,1} + f[D_0(t), D_1(t), I(t), \Theta(t)] \tag{30}$$

where the constant term $\lambda + \frac{1}{2} l_{1,1}$ is different for each candidate equilibrium price and represents the "essential utility part" of the value function. We replace the investors' expected utility with the "essential utility part" to choose among different equilibrium candidates the equilibrium price, thus

$$\tilde{J}(L, \lambda) \equiv \lambda + \frac{1}{2} l_{1,1} \tag{31}$$

Since we are in a suitable neighborhood of the origin of the Euclidean space of the states of the economy (D_0, D_1, I, Θ) we have

$$ZLZ \simeq l_{1,1},$$

such that it follows that

$$-e^{-\beta t - r\varphi Y + \frac{1}{2} Z^\top LZ - \lambda} \simeq \beta t + r\varphi Y + \tilde{J}(L, \lambda).$$

We conclude that the higher is the "essential utility part" of the expected utility, the higher is the expected utility itself in the considered neighborhood.

5 Numerical Solutions

This section shows with numerical routines the candidate equilibrium prices and the correspondent "essential utility part." We fix the exogenous parameters of the model as follows:

$$r = 0.05; \quad \xi = 0.011; \quad \beta = 0.30; \quad \varphi = 0.50;$$
$$\alpha_D = 0.50; \quad \alpha_I = 0.40; \quad \alpha_\Theta = 0.05;$$
$$\sigma_0 = 0.50; \quad \sigma_D = 0.10; \quad \sigma_I = 0.40; \quad \sigma_\Theta = 0.50.$$

Table 1 shows different candidate equilibrium prices for these parameters. The candidate equilibrium price with the highest "essential utility part" is the equilibrium price. In Table 1, the equilibrium price obtained via numerical solution is the same equilibrium derived algebraically in Model A:

$$\tilde{p}_0 = -\,91.773, \qquad \tilde{p}_{D_0} = \frac{1}{r-\xi} = 25.641,$$
$$\tilde{p}_{D_1} = \frac{1}{r-\xi+\alpha_D} = 1.855, \qquad \tilde{p}_I = \frac{\alpha_I}{(r-\xi)(r-\xi+\alpha_I)} = 18.446.$$

We observe that this efficient price is the one giving the highest utility for the informed investors with respect to the other inefficient candidate equilibrium prices. On this account we conclude that the efficient equilibrium price Pareto dominates the other candidates, characterized by some extent of inefficiency. Note that the constant term $\tilde{p}_0 = -91.773$ is negative as suggested by theory, which subtracts a constant term from the price to account for the expected return on the stock for risk-averse informed investors, instead of increasing the discount rate. Note what are the other candidate equilibrium prices. These candidate prices are inefficient because they have a lower (higher) value of p_I or p_{D_0} with respect to the full-informative equilibrium price. As a consequence of this inefficiency they have a higher (lower) value of p_0. This is correct from the investors' point of view because investors ask a higher (lower) discount term as a compensation for inefficiency and price moves downward (upward) as a consequence of this higher (lower) discount.[12]

Table 2 shows the candidate equilibrium prices when there is a positive change in investors' risk aversion, i.e., φ is unity.[13] Among the candidate equilibrium prices, the one with the highest "essential utility part" ($\lambda + \frac{1}{2}l_{1,1} = 43.96$) is an inefficient price with the following coefficients:

$$\widehat{p_0} = -2664.632, \quad \widehat{p}_{D_0} = -89.311, \quad \widehat{p}_{D_1} = 1.855, \quad \widehat{p_I} = -13.384$$

in which $\widehat{p}_{D_0} = -89.311$ and $\widehat{p_I} = -13.384$, the coefficients associated with the permanent component of dividend and with private information, deviate from their efficient values given by $\tilde{p}_{D_0} = \frac{1}{r-\xi} = 25.641$ and $\tilde{p}_I = \frac{\alpha_I}{(r-\xi)(r-\xi+\alpha_I)} = 18.446$. We interpret these negative coefficients as the investors' willingness to anti-correlate the private and public information flows with price. As a consequence of this inefficiency, investors ask a higher discount: $\widehat{p_0} = -2664.632$ as a compensation for the inefficient price.

Table 3 confirms the previous result when market microstructure variance increases, i.e., σ_Θ is unity. The equilibrium price obtained as a solution of the investment optimization problem for the informed investors has the following coefficients:

$$\widehat{p_0} = -465.202, \quad \widehat{p}_{D_0} = -80.445, \quad \widehat{p}_{D_1} = 1.855, \quad \widehat{p_I} = -87.639$$

in which the coefficients of the permanent component of dividend process $\widehat{p}_{D_0} = -80.445$ and of private information $\widehat{p_I} = -87.639$ deviate from the efficient values. We remark that a negative value of p_{D_0} means that the permanent part of dividend is anticorrelated with price without implying that price moves downward when dividend increases. The coefficients measure only the price reaction to dividend and nondividend news. The movement of price is driven by the constant part. The inefficient price described above has a negative constant term that is five times the discount requested in the benchmark case. This higher discount term leads us to conclude that the above price has a downward dynamics as a consequence of the market microstructure volatility increase.

[12] The direction of price movement is directly linked with the value of p_0 so a higher negative discount implies a fall in the price dynamics, and a lower value implies a price increase.

[13] We double the risk aversion parameter with respect to Table 1.

According to numerical results, an increase (decrease) in the informed investors' risk perception determines an informational inefficient (efficient) equilibrium price. We measure an increase in the market risk perception with an increase in the investor's risk aversion and in the market microstructure volatility. These two parameters determine inefficient prices and as a consequence of this inefficiency investors demand a higher discount for the risky asset. We interpret this result as the need of the informed investors to protect themselves from the high risk perception with a noninformative price, and noninformative prices are the optimal response when there is high market risk perception. Finally, we consider this inefficient equilibrium price a rational bubble, that is a price that optimally solves the infinite-horizon investors' optimization problem and deviates from the correspondent fundamental value.

Our inefficient equilibrium price captures some "market anomalies" in a real financial market. When price is not equal to the present value of future expected dividend streams, and changes in price are not attributed to news about dividends or discount factors, we observe a higher constant term, that is a higher discount requested by the informed investors. This phenomenon is observed in real financial markets under the name of the equity premium puzzle. Secondly, when price is inefficient we observe higher price coefficients so the variance of price

$$\sigma_P = p_{D_0}^2 \sigma_{D_0} + p_{D_1}^2 \sigma_{D_1} + p_I^2 \sigma_I + \sigma_\Theta$$

is higher with respect to the variance of the efficient price. This phenomenon is called excess volatility and it arises from fears of risk that, in our model, are measured by the investors' risk aversion and the market noise volatility.

Figure 1 shows the "essential utility part" of the efficient (Model A) and inefficient (Model B) prices as functions of risk aversion and noise volatility. For $\varphi = 0.1$ and $\sigma_\Theta = 0.1$ the equilibrium price is the efficient one as confirmed by the red area. Increasing noise volatility up to $\varphi = 3.0$, the efficient one is still Pareto dominating the inefficient one. For $\varphi = 0.5$ we observe that the equilibrium price depends on noise volatility's value. Thus, an increase in the investors' risk aversion and/or noise volatility leads the inefficient price to having the highest utility. Further, risk aversion has a stronger effect on determining the inefficient equilibrium price with respect to noise volatility.

Robustness check. Numerical results are obtained letting the machine search for mathematical solutions using the Newtonian method. We initialize the research letting the price coefficients and each element of Equation (23), which guarantee the solution of the optimization problem, range over $(-10; 10)$. All equations and starting values are real, thus our search is only for real roots. We control our results by expanding the range and using the method of secant. We obtain the same results. We change the exogenous parameters of the model without a significant change in the result. Tables are available upon request. Finally, we prove that only risk aversion and noise volatility determine the inefficiency of the equilibrium price. This implies that neither dividend (σ_0, σ_D) nor information variances (σ_I) affect the final result.

6 Econometric Method and Empirical Results

6.1 Preliminary Data Analysis

We estimate the model described in previous sections on annual U.S. time-series data for real stock prices and dividends. Stock market data are taken from Shiller (2000) and are similar to the dataset used by C.K. and Campbell and Shiller (1987, 1988). The dataset consists of monthly stock price, the corresponding dividend data, and a price index during the period

January 1871–December 2009.[14] Our real stock price is the Standard and Poors Composite Stock Price Index multiplied by the CPI-U (Consumer Price Index-All Urban Consumers) in June 2010, and divided by the corresponding CPI. We apply the same procedure to the corresponding S&P Stock Price dividend per share to obtain a real dividend series. The procedure is used in Shiller (2000) to allow raw nominal data to be real.

We call the real unadjusted price and dividend P_t^u and D_t^u, to distinguish them from the de-trended real price and dividend, P_t and D_t, in the manner of Equation (1), in which we choose $\xi = 0.0115$ as the average mean dividend growth rate over the sample. The de-trended operation aims to remove exponential growth from the *ex ante* mean of data, without forcing data to revert to a trend line, and to get rid of exponential growth from the variance of data, a rescaling effect similar to a logarithm transformation. Finally, we normalize D_t and P_t such that the mean of price equals one by dividing each time series with the mean of price. Figure 1 plots the de-trended real price and dividend $\times$ 10.

Table 4 presents the main statistics of the data. We consider January of each year as our annual data point to avoid problems with time aggregation. Table 5 presents the results of ADF, PP, and KPPS tested for stationarity both for P_t and D_t, and for P_t^u and D_t^u. For comparison we compute the test for $\ln(P_t^u)$ and $\ln(D_t^u)$. Unit root tests show that price and dividend have unit roots in the first level. The results for ADF, with five lags, and PP test for P_t, P_t^u, and $\ln(P_t^u)$ do not always reject the null hypothesis of nonstationarity, while KPPS always rejects stationarity. The ADF test for D_t, D_t^u, and $\ln(D_t^u)$ does not reject the null in presence of a trend at the 5% level. KPSS confirms the unit root of the dividend series in the level. The PP test accounts for autocorrelation of the error term and shows that the dividend series is stationary with trend.[15] In light of the results above, we assume a unit root in the stock price and dividend time series, in line with C.K. and Campbell and Shiller (1987, 1988). Finally note that the de-trended operation given by the sample mean growth rate of dividends does not have any effect on the unit root assumption.

Table 6 reports other time series properties of data such as the sample correlation of the integrated process ΔP_t, ΔD_t until the fifth lags and, in the bottom of the table, the sample standard deviations of ΔP_t and ΔD_t. The correlation analysis suggests that price and dividend have a mean reverting component: this is due to the positive first autocorrelation (0.14 and 0.22, respectively) while the other autocorrelations are all negative.[16] This can support the rejection of the null hypothesis in the unit root test for dividends. An interesting result is given by the cross correlation between the dividend change from the end of one year to the end of the next, ΔD_t, and the corresponding price change, ΔP_t. A very low value appears at the contemporaneous level (0.03) and at the third level (0.06), but a high correlation (0.44) between ΔD_t and ΔP_{t-1}. It means that only price change between P_t and P_{t-2}, the difference between one year and the two previous years, helps to explain the change in actual dividend. These findings support the idea that an hidden variable called "private information" I_t might help to explain the relationship between price and dividend.

This work assumes that price and dividend each have one unit root, so they are integrated

[14] The dataset was retrieved from Robert Shiller's website at http://www.econ.yale.edu/shiller/. More details on the dataset are available in Shiller's book (2000). In this note we report his analysis: monthly dividend data are computed from the S&P four-quarter tools for the quarters since 1926, with linear interpolation to monthly figures. Dividend and earnings data before 1926 are from Cowles and associates, interpolated from annual data. Stock price data are monthly averages of daily closing prices through January 2010. The CPI-U (Consumer Price Index-All Urban Consumers) published by the U.S. Bureau of Labor Statistics begins in 1913; for years before 1913 it is spliced to the CPI Warren and Pearson's price index, by multiplying it by the ratio of the indexes in January 1913. December 1999 and January 2000 values for the CPI-U are extrapolated.

[15] Assuming a unit root in dividends implies that noise does not help to explain the stock price volatility (as first shown by Kleidon, 1986; Marsh and Merton, 1986; and Timmermann, 1996).

[16] The autocorrelation sample in 1871–1988 supports the mean reverting component only for dividends, as C.K. show in their work.

processes of order one $I(1)$, and that a linear combination is stationary [they are $I(0)$]. We test the existence of cointegration with the Engle-Granger two-step method (where the null is no cointegration, so the residual is a random walk). Table 7 shows the regression result and the ADF test on the residuals. We estimate $D_t = 0.073 + 0.012P_t$ using the heteroskedasticity-robust standard error, thus two-thirds of the dividend's mean are explained by the constant that accounts for the unconditional expected excess return per share of stock demanded by risk-averse investors. The ADF test rejects the unit root hypothesis at 5%. We conclude that there exists a linear combination of P_t and D_t that is $I(0)$ so price and dividend are cointegrated. In our model the coefficient regressor (0.012) equals $(r - \xi)$, the interest rate less the dividend growth rate ($\xi = 0.011$), hence the implied interest rate is 2.4%, a very low level justified by the high constant. If we regress dividend on price without a constant we estimate $D_t = 0.022P_t$, which implies an interest rate of 3.3%. If we reverse the estimate we obtain $P_t = -3.113 + 57.003D_t$, such that $(r - \xi) = 0.017$ and $r = 2.9\%$. This value is very close to the mean rate of return on our stock index (3.4%). We fix in our model $r = 3\%$ and for comparison we also fix $r = 6\%$. Finally we let r be free and we estimate it.

6.2 Estimates of the Model

Our model is set up in continuous time but we estimate it using discrete-time data. We estimate it via an exact discrete analog according to the procedure originally introduced by Bergstrom (1966, 1983) and recently discussed in McCrorie (2009). This is in contrast to C.K. who estimate a discrete-time transformation of their original model showing that the stacked vector of point-sampled and time-averaged transformation of the continuous-time variables follows a discrete-time AR(1) process, with a transition matrix that is related to the underlying continuous-time parameters. We find it more suitable to estimate the model using the state space approach. Price and dividend are two observed variables, the measurement equation of the state space model, and $Z(t)$ is the vector of state variables representing the transition equation. We write the state space dynamics in compact form as

$$\begin{aligned} Y_t &= C(\mu)Z_t \\ dZ_t &= A(\theta)Z_t dt + B^{1/2}(\theta)\ \mathbf{dw(t)} \end{aligned} \tag{32}$$

where the vector $Y_t = [P_t, D_t]$, the matrix $C(\mu)$ contains the price coefficients $[p_0, p_{D_0}, p_{D_1}, p_I, 1]$, $\{A(\theta), B(\theta)\}$ are matrices containing the unknown parameters $\theta = [\alpha_D, \alpha_I, \alpha_\Theta, \sigma_0, \sigma_D, \sigma_I, \sigma_\Theta]$ to estimate, and $\mathbf{dw(t)}$ is the vector of independent brownian motions.

Appendix B reports the exact discrete matrices of our continuous time model and the likelihood function. We use the Kalman filter to extract the hidden information and to compute the estimates for any variables. Our data are nonstationary hence we use a noninformative (diffuse) prior distribution for the corresponding parameters under a Bayesian paradigm. We initialize the filter assuming that the unconditional mean of each state variables is zero and a arbitrarily large covariance matrix as suggested by de Jong (1991).[17]

Data 1871–2009. Table 8 shows the Maximum Log-Likelihood (ML) values for model A, where we impose the informationally efficient form as given in Equation (27), and Model B, in which price is assumed informationally inefficient. Each row of the table presents different assumptions about the interest rate, while the columns report the absence or the presence of market noise. The first result is that Model A has higher MLs for any assumption about interest rate even when we assume there are no noise traders in the market. Secondly, noise traders strongly improve the goodness of this model, as in C.K., when we assume interest rate equals 3% and 6%, but not when the model estimates r (ML = 837.62).

[17] Our variance covariance matrix is a diagonal matrix with 10^6 on the diagonal.

Table 8 shows that assuming full noise and interest rate equals 3% we estimate an ML equals 834.88 for Model A and ML equals 830.65 for Model B. This great difference supports the goodness of Model A with respect to Model B. A similar result is confirmed when we assume $r = 6\%$ or when we estimate it separately. Table 9 shows the estimated price coefficients; the two results are straightforward. The constant term p_0 is negative as shown by the theoretical model to account for the risk aversion of the investors. The estimate of the interest rate equals 3.09% (with ML = 831.62), which is in line with the interest rate implied by the estimated regression given in the previous section.

According to our estimates we conclude that the S&P 500 Index reflected the fundamental values, and the efficient-market-hypothesis prevailed during the long period 1871–2009. It is important to emphasize that we are aware that during in the long period there were at least two structural breaks that are not considered in the estimates of the model. These estimates were important to confirm the efficiency of the S&P 500 Index, to have a benchmark for following estimates, and to compare our results with those of C.K.

Dot-Com Bubble Our main goal of this work is to verify whether theoretical results given in Section 5 have some correspondence on real financial data. We take monthly prices and dividends of the S&P 500 Index during 1995–2000 in order to have 72 data points in our estimation procedure. Table 10 shows unit root tests values at 5% level: price and dividend have unit roots in level and they are stationary if differenced by order one. In the bottom of Table 10 we test the cointegration using Johansen's methodology. The values of λmax $= 30.55\,(15.67)$ and λtrace $= 31.79\,(19.96)$ reject the null hypothesis at 5% level of zero cointegrating vector and they accept the hypothesis of one vector of cointegration. We estimate $D_t = 0.049 + 0.0017 P_t$ using the heteroskedasticity-robust standard error.[18] The average dividend growth rate over the five-year period is low: $\xi = 0.0011$. Hence the coefficient on P_t equals $(r - \xi)$, the interest rate minus the trend growth rate, so the implied interest rate is $r = 0.0028$. As C.K. have shown, this low interest rate value is justified by the high value of λ. When we impose no constant in the regression we find $r = 0.015$. On the other hand, to take account of the real interest rate, which was not so low during the Dot-Com bubble, we find it suitable to assume in our estimations different values of the interest rate: $r = 1.5\%, 3\%$, or we let it run free in the model and we estimate it. Finally, we assume there is noise in the market when we estimate Model A or Model B and we carefully compute the maximum of the log-likelihood function, changing the initial values of the Kalman Filter and the ones in the optimization routine. However, the overall result does not change.

Table 11 shows that the Maximum Log-Likelihood of Model B is always higher than that of Model A. We know the model with more parameters (Model B) will always fit at least as well (have a greater log-likelihood) as Model A. Whether it fits significantly better and should thus be preferred can be determined by deriving the probability or p-value of the obtained difference given by $\mathrm{LR} = -2(\mathrm{ML}_A - \mathrm{ML}_B)$. The LR-test used to compare the fitting ability of the two models rejects at 1% the null hypothesis of Model A, in favor of Model B, for any assumptions of the interest rate. Finally, note that with the assumption of $r = 1.5\%$ we have the best fitting ability when Model A is assumed, as shown by ML$=540.96$, but this is much lower with respect to ML$=636.39$ when we estimate Model B.

Table 12 reports the price coefficients of Model A and Model B for different assumptions about interest rate. We still find evidence of the negative term of p_0 as suggested by theory. On the other hand, the estimate of r when we let it run free is very low ($r = 0.002$). This is in line with the preliminary analysis but it is not as reasonable as when we consider the results when r is assumed equal to 0.015.

[18] Recall that the constant term equals λ, the unconditional expected excess return per share of stock demanded by risk-averse investors.

The estimates of the parameters given in Table 13b are in line with our theoretical results. We have direct support from real data about what theory says about inefficient prices for three reasons. The constant term of the estimate of Model B is lower ($p_0 = -0.061$) with respect to the constant term of Model A ($p_0 = -0.177$); that is, risk-averse investors demand a lower risk premium and as a consequence price increases. This is in line with the positive asset bubble that Figure 2 shows during the period 1995–2000.

Secondly, we observe that the permanent component of the dividend process is correctly estimated ($p_{D_0} = 71.942$ and $p_{D_0} = 71.944$ in Model A and Model B, respectively) while investors estimate a lower value of their private information ($p_I = 0.001$) and the permanent component of dividendd ($P_{D_1} = 1.089$). This is in line with our numerical results and conforms with the investors' behavior during the asset bubble: they know what are the real dividends and they do not trade according to their private information. Moreover, the low value of p_I is correctly estimated because private information decays very fast ($\alpha_I = 0.905$); that is, investors cannot exploit their private information for a long period.

Thirdly, we find that the inefficient price has the highest Maximum Log-Likelihood when investors estimate a high value of market microstructure volatility with respect to dividends and information: $\sigma_\Theta = 0.124$ (see Table 13b). This evidence supports our theoretical results according to which the market microstructure volatility plays a key role in determining inefficient prices.

Finally, we test that price estimated in Table 13b is inefficient as shown by the differences between $\widehat{p}_{D_1}$ with $\tilde{p}_{D_1}$ and $\widehat{p}_I$ with $\tilde{p}_I$.

7 Conclusion

The main result of this paper is that, still adopting a competitive rational equilibrium perspective, there exist two types of equilibrium price solutions of the optimal investment problem: the semi-strong efficient one, in which price reflects fundamental values, and the inefficient ones. The equilibrium price, defined as the one with highest utility for the informed investors, is determined by investors' risk perception. The efficient price Pareto dominates (are dominated by) the inefficient ones when risk aversion is *low* (*high*) and/or when noise volatility is *low* (*high*). According to our findings the risk averse investors, who maximize their investment opportunities and formulate consistent beliefs according with their public and private information, rationally determined an asset bubble when they perceived high market risk. We proved this result numerically. Moreover, when the equilibrium price is inefficient we observe some market anomalies such as higher risk premium and the excess volatility, common in real markets.

The second result of this work is that using real data, the estimates of the model seem to confirm our theoretical findings. The S&P 500 Index reflects the fundamental values, given by dividends and hidden private information, during the long period 1871–2009. On the other hand, it has not supported the market efficiency hypothesis in the sub-period 1995–2000. The higher maximum log-likelihood of the model under inefficient price assumption leads us to reject, using the likelihood ratio test, the null hypothesis of an efficient market at 1%. The result is confirmed under different assumptions of interest rate. Furthermore, the estimated inefficient price shows that investors demand a lower risk premium, which according to our theory leads to a price increase; investors estimate a very fast decay of their private information on price, which leads to lower sensitivity of their private information on the price dynamics; and there is a high value of the market microstructure volatility, one of the main driving factors of inefficient prices according to our theory. These three stylized facts are discussed in our theoretical model.

In conclusion, the model sheds light on the role of market risk perception, such as the risk aversion and the market microstructure volatility, to determined efficient or inefficient prices. Further research in this direction might help to understand whether market risk perception uniquely characterized the inefficients equilibrium prices. This is because the model does not uniquely identify whether risk aversion or market volatility caused the bubble in the 2000s; nor does it show whether these variables determine positive or negative bubbles. A second direction of research is to obtain a nontrivial situation where there are two informed investors and two optimal equilibrium prices: an efficient one that is strictly preferred by the former investor, and inefficient one preferred by the latter, with no one that is strictly preferred by both. We aim to reproduce and to study the prisoner dilemma game applied to financial markets.

Appendix A. Solution to Investors's Optimization Problem

Proof of Theorem 1. We solve the investor's optimization problem showing that the conjectured value function (18) is the investor's objective function (14). We assume the value function has the following form

$$J(Z,W,t) = -e^{-(\beta t + r\varphi W - \Phi(Z) + \lambda)},$$

where $\Phi(Z) = \frac{1}{2} Z^\top L Z$, $Z = (1,\, D_0,\, D_1,\, I,\, \Theta)^\top$ is defined as the (5x1) vector of the state variables and $L \equiv (l_{j,k})_{j,k=1}^5$. The dynamics of $Z(t)$ are written in compact form as in equation (46). We write the excess return per one share of stock in terms of the state vector $P(t) = \bar{P} Z(t)$, such that

$$\begin{aligned} dQ =& (D(t) - (r-\xi)P(t))dt + dP(t) = (D(t) - (r-\xi)\bar{P}Z(t))dt + \bar{P}AZ(t)dt + \bar{P}B^{1/2}dw(t) \\ =& SZ(t)dt + T^{1/2}dw(t) \end{aligned}$$

where

$$S(t) \equiv M - \bar{P}(r-\xi) + \bar{P}A, \qquad M \equiv (0,1,1,0,0), \qquad \bar{P} \equiv (p_0, p_{D_0}, p_{D_1}, p_I, 1), \qquad T^{1/2} \equiv \bar{P}B^{1/2}$$

We need to prove that

(i) the function (14) is a solution to the Bellman equation

$$\partial_t J(Z,W,t) + \max_{\Psi,c}\{\mathcal{G}J(Z,W,t) - e^{-(\beta t + \varphi c)}\} = 0, \tag{33}$$

where $\mathcal{G}$ is the infinitesimal generator of the diffusion process $(Z(t), W(t))$;

(ii) the control $(\mathring{\Psi}(t), \mathring{c}(t))$ satisfies

$$(\mathring{\Psi}(t), \mathring{c}(t)) \in \arg\max\{\mathcal{G}J(\mathring{Z}(t), \mathring{W}(t), t) - e^{-(\beta t + \varphi \mathring{c}(t))}\} = 0,$$

where $(\mathring{Z}(t), \mathring{W}(t))$ is a solution to

$$\begin{aligned} dZ(t) =& AZ(t)dt + B^{1/2}dw(t) \\ dW(t) =& (rW(t) - c(t) + \Psi(t)SZ(t))dt + \Psi(t)T^{1/2}dw(t) \end{aligned} \tag{34}$$

corresponding to the choice of $(\mathring{\Psi}(t), \mathring{c}(t))$;

(ii) the trasversality condition

$$\lim_{T\to+\infty} \mathbf{E}_{Z,W,t}\left[J(t+T, \mathring{Z}(t+T), \mathring{W}(t+T))\right] = 0, \tag{35}$$

where $(\mathring{Z}(t), \mathring{W}(t))$ is a solution to (34) corresponding to the choice of$(\mathring{\Psi}(t), \mathring{c}(t))$, holds true.

To show that $J(Z,W,t) = -e^{-(\beta t + \Phi(Z) + \varphi r W + \lambda)}$ is a solution of (33), we start by determining the operator $\mathcal{G}$. A straightforward computation yields

$$\begin{aligned} \mathcal{G} \equiv \tfrac{1}{2} \textstyle\sum_{j,k=1}^5 B_{j,k}\, \partial^2_{Z_j,Z_k} + \Psi \sum_{j=1}^5 T^{1/2} \left(B^{1/2}\right)_j^\top \partial^2_{W,Z_j} + \tfrac{1}{2}\Psi^2\, T\, \partial^2_{W,W} \\ + \textstyle\sum_{j=1}^5 (AZ)_j\, \partial_{Z_j} + (rW - c - \Psi SZ)\, \partial_W. \end{aligned} \tag{36}$$

On the other hand, using J as a shorthand for $J(Z,W,t)$, we have

$$\partial_{Z_j} J = -(Z^\top L)_j J, \qquad \partial_W J = -r\varphi J, \qquad \partial^2_{Z_j,Z_k} J = \left(LZZ^\top L - L\right)_{j,k} J,$$
$$\partial^2_{Z_j,W} J = r\varphi (Z^\top L)_j\, J, \qquad \partial^2_{W,W} J = r^2\varphi^2\, J.$$

Therefore we can write

$$\begin{aligned}\mathcal{G}J = \tfrac{1}{2}\left(\textstyle\sum_{j,k=1}^5 B_{j,k}\,\left(LZZ^\top L - L\right)_{j,k}\right) J + \tfrac{1}{2} r^2\varphi^2 T\Psi^2\, J +\\ r\varphi\left(\textstyle\sum_{j=1}^5 \left(T^{1/2}\left(B^{1/2}\right)^\top\right)_j (Z^\top L)_j\right)\Psi J - \left(\textstyle\sum_{j=1}^5 (AZ)_j\,(Z^\top L)_j\right) J - r\varphi\,(rW - c - SZ\Psi)\, J.\end{aligned} \tag{37}$$

Now, thanks to the properties of the trace functional, we have

$$\sum_{j,k=1}^5 B_{j,k}\left(LZZ^\top L - L\right)_{j,k} J = \mathbf{tr}\left((B^{1/2})^\top (LZZ^\top L - L) B^{1/2}\right) = Z^\top LBLZ - \mathbf{tr}\left(\left(B^{1/2}\right)^\top LB^{1/2}\right). \tag{38}$$

Moreover,

$$\sum_{j=1}^5 \left(T^{1/2}\left(B^{1/2}\right)^\top\right)_j (Z^\top L)_j = T^{1/2}\left(B^{1/2}\right)^\top LZ, \tag{39}$$

and

$$\sum_{j=1}^5 (AZ)_j\,(Z^\top L)_j = Z^\top LAZ. \tag{40}$$

Hence, combining (37) with (38)-(40), it follows

$$\begin{aligned}\mathcal{G}J = \tfrac{1}{2} Z^\top LBLZ - \tfrac{1}{2}\mathbf{tr}\left(\left(B^{1/2}\right)^\top LB_u^{1/2}\right) J + r\varphi T^{1/2}\left(B^{1/2}\right)^\top LZ\,\Psi J + \tfrac{1}{2} r^2\varphi^2 T\,\Psi^2 J\\ - Z^\top LAZ\, J - r\varphi\,(rW - c - SZ\Psi)\; J.\end{aligned} \tag{41}$$

The latter, on account of

$$\partial_t J = -\beta J, \qquad Z^\top LAZ = \frac{1}{2}(Z^\top A^\top LZ + Z^\top LAZ)$$

we rewrite Equation (33) in the form

$$\begin{aligned}&\left(-\beta + \frac{1}{2} Z^\top LBLZ - \frac{1}{2}(Z^\top A^\top LZ + Z^\top LAZ) - \frac{1}{2}\mathbf{tr}\left(\left(B^{1/2}\right)^\top LB_u^{1/2}\right) - \varphi r^2 W\right) J\\ &+ \max_{\Psi,c}\{r\varphi((T^{1/2}\left(B^{1/2}\right)^\top L + S)Z\Psi + \frac{1}{2} r\varphi T\Psi^2)J + \varphi rcJ - e^{-(\beta t + \varphi c)}\}\\ &= 0.\end{aligned} \tag{42}$$

Hence, setting

$$J \equiv J(t,Z,W,\Psi) \equiv \left((T^{1/2}\left(B^{1/2}\right)^\top L + S)Z\Psi + \frac{1}{2} r\varphi T\Psi^2\right) J,$$

and

$$K \equiv K(t,Z,W,c) \equiv r\varphi cJ - e^{-(\beta t + \varphi c)},$$

we can write

$$\begin{aligned}&\max_{\Psi,c}\{r\varphi((T^{1/2}\left(B^{1/2}\right)^\top L + S)Z\Psi + \frac{1}{2} r\varphi T\Psi^2)J + \varphi rcJ - e^{-(\beta t + \varphi c)}\}\\ &= r\varphi\max_\Psi\{J(t,Z,W,\Psi)\} + \max_c\{K(t,Z,W,c)\}\end{aligned}$$

Maximizing J [resp. K] with respect to Ψ, the first conditions are

$$\frac{dJ}{d\Psi} =((T^{1/2}\left(B^{1/2}\right)^\top L+S)Z+r\varphi\Psi T)J=0 \qquad\qquad \frac{dK}{dc}=r\varphi J+\varphi e^{-(\beta t+\varphi c_u)}=0$$

that yields

$$\tilde{\Psi}=-\frac{(T^{1/2}\left(B^{1/2}\right)^\top L+S)}{r\varphi T}Z \qquad\qquad \tilde{c}=\frac{\frac{1}{2}Z^\top LZ+r\varphi W+\lambda-\log(r)}{\varphi}$$

which is the desired optimal strategy (19) and optimal consumption (20). Moreover, the second order condition

$$\frac{d^2J}{d\Psi^2}=r\varphi TJ\leq 0 \qquad\qquad \frac{d^2K}{dc^2}=-e^{-\varphi c}\leq 0$$

guarantees that Ψ is optimal for J and, similarly, c is optimal for K. As a consequence,

$$\max_{\Psi}\{J(t,Z,W,\Psi)\}=-\frac{1}{2r\varphi T}\left(Z^\top\left(LB^{1/2}(T^{1/2})^\top+S^\top\right)\left(T^{1/2}\left(B^{1/2}\right)^\top L+S\right)Z\right)J$$

and

$$\max_{c}\{K(t,Z,W,c)\}=r\left(\frac{1}{2}Z^\top LZ+r\varphi W+\lambda-\log(r)+1\right)J.$$

In light of what shown above, the Bellman equation (42) takes the form

$$\begin{aligned}
&\left(-\beta+\frac{1}{2}Z^\top LBLZ-\frac{1}{2}(Z^\top A^\top LZ+Z^\top LAZ)-\frac{1}{2}\mathbf{tr}\left(\left(B^{1/2}\right)^\top LB_u^{1/2}\right)-\varphi r^2W\right)J\\
&-\frac{1}{2T}\left(Z^\top(LB^{1/2}(T^{1/2})^\top+S^\top\right)\left(T^{1/2}\left(B^{1/2}\right)^\top L+S\right)ZJ\\
&+r\left(\frac{1}{2}Z^\top LZ+r\varphi W+\lambda-\log(r)+1\right)J=0
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{1}{2}Z^\top\left(LBL-\frac{1}{T}\left(L^\top B^{1/2}(T^{1/2})^\top+S^\top\right)\left(T^{1/2}\left(B^{1/2}\right)^\top L+S\right)-A^\top L-LA+rL\right)ZJ\\
&+\left(r\lambda+r(1-\log(r))-\beta-\frac{1}{2}\mathbf{tr}\left(\left(B^{1/2}\right)^\top LB^{1/2}\right)\right)J=0.
\end{aligned} \tag{43}$$

On the other hand,

$$\begin{aligned}
&LBL-\frac{1}{T}\left(LB^{1/2}(T^{1/2})^\top+S^\top\right)\left(T^{1/2}\left(B^{1/2}\right)^\top L+S\right)-A^\top L-LA+rL \qquad (44)\\
&=\frac{1}{T}\left(LB^{1/2}\left(TI_5-\left(T^{1/2}\right)^\top T^{1/2}\right)\left(B^{1/2}\right)^\top L\right)\\
&-\frac{1}{T}\left(L\left(B^{1/2}\left(T^{1/2}\right)^\top S+T\left(A-\frac{1}{2}rI_5\right)\right)+\left(S^\top T^{1/2}\left(B^{1/2}\right)^\top+T\left(A^\top-\frac{1}{2}rI_5\right)L\right)+S^\top S\right).
\end{aligned}$$

Therefore, combining (43) with (44), it clearly follows that $J(t,Z,W)$ is a solution of the Bellman equation (33), provided that the matrix L and the parameter λ are chosen to fulfill (22) and (21), respectively.

We are left with proving that the trasversality condition (35) holds true. To this goal, we apply Itô's formula to the identity

$$J(t+\Delta t, \mathring{Z}(t+\Delta t), \mathring{W}(t+\Delta t)) - J(t, \mathring{Z}(t), \mathring{W}(t)) = \int_t^{t+\Delta t} dJ(s, \mathring{Z}(s), \mathring{W}(s))$$

which allows to write

$$\begin{aligned} J(t+\Delta t, \mathring{Z}(t+\Delta t), \mathring{W}(t+\Delta t)) - J(t, \mathring{Z}(t), \mathring{W}(t)) = & \int_t^{t+\Delta t} (\partial_s J(s, \mathring{Z}(s), \mathring{W}(s)) + \mathcal{G}J(s, \mathring{Z}(s), \mathring{W}(s)))\, ds \\ & + \int_t^{t+\Delta t} B_{Z,W}^{1/2} \nabla_{Z,W} J(s, \mathring{Z}(s), \mathring{W}(s))\, d\tilde{w}(s), \end{aligned}$$

where $B_{Z,W}^{1/2}$ stands for the diffusion matrix of the process $(\mathring{Z}(s), \mathring{W}(s))$ and $\nabla_{Z,W}$ denotes the gradient operator in the state space of $(\mathring{Z}(s), \mathring{W}(s))$. On the other hand, since $J(t, Z, W)$ is a solution of the Bellman equation (17) and $(\mathring{Z}(s), \mathring{W}(s))$ corresponds to an optimal control, we have

$$\int_t^{t+\Delta t} (\partial_s J(s, \mathring{Z}(s), \mathring{W}(s)) + \mathcal{G}J(s, \mathring{Z}(s), \mathring{W}(s)))\, ds = \int_t^{t+\Delta t} e^{-(\beta s + \varphi \mathring{c}(s))}\, ds.$$

On account of the latter, applying the expectation operator on both the sides of (**??**), we obtain

$$\begin{aligned} & \frac{\mathbf{E}_{t,Z,W}\left[J(t+\Delta t, \mathring{Z}(t+\Delta t), \mathring{W}(t+\Delta t))\right] - \mathbf{E}_{t,Z,W}\left[J(t, \mathring{Z}(t), \mathring{W}(t))\right]}{\Delta t} \\ & = \frac{1}{\Delta t}\mathbf{E}_{t,Z,W}\left[\int_t^{t+\Delta t} e^{-(\beta s + \varphi \mathring{c}(s))}\, ds\right], \end{aligned}$$

and, passing to the limit as $\Delta t \to 0$, it follows

$$\frac{d\mathbf{E}_{t,Z,W}\left[J(t, \mathring{Z}(t), \mathring{W}(t))\right]}{dt} = \mathbf{E}_{t,Z,W}\left[e^{-(\beta t + \varphi \mathring{c}(t))}\right].$$

On the other hand, by virtue of c-first order condition,

$$e^{-(\beta t + \varphi \mathring{c}(t))} = -\beta J(t, \mathring{Z}(t), \mathring{W}(t)).$$

Hence, $\mathbf{E}_{t,Z,W}\left[J(t, \mathring{Z}(t), \mathring{W}(t))\right]$ satisfies the differential equation

$$\frac{d\mathbf{E}_{t,Z,W}\left[J(t, \mathring{Z}(t), \mathring{W}(t))\right]}{dt} = -\beta \mathbf{E}_{t,Z,W}\left[J(t, \mathring{Z}(t), \mathring{W}(t))\right],$$

and the desired trasversality condition clearly follows.

Appendix B. Proof of Proposition 1

We know that price is efficient when it is the expected future discounted dividends

$$P(t) = E\left[\int_{s=0}^{\infty} e^{-rs} D^u(t+s)ds \mid \mathfrak{F}_t\right] = E\left[\int_{s=0}^{\infty} e^{-(r-\xi)s} D(t+s)ds \mid \mathfrak{F}_t\right] \tag{45}$$

where the process $D(t)$, the continuos-time dividend yield of a risky asset, is defined as $D(t) = D_0(t) + D_1(t)$ and

$$\begin{aligned} dD_0(t) =& \alpha_I I(t) + \sigma_0 dw_0(t), \\ dD_1(t) =& -\alpha_D D_1(t)dt + \sigma_D dw_D(t), \\ dI(t) =& -\alpha_I I(t) + \rho_I \sigma_0 dw_0(t) + (2\rho_I - \rho_I^2)^{1/2}\sigma_0 dw_I(t) \end{aligned}$$

Let us rewrite the dividend and informative signals as a trivariate Ornstein-Uhlenbeck process with $Z = (D_0,\, D_1,\, I)^\top$

$$dZ(t) = A_1 Z(t)dt + B_1^{1/2}\mathbf{dw(t)} \tag{46}$$

where

$$A_1 \equiv \begin{pmatrix} 0 & 0 & \alpha_I \\ 0 & -\alpha_D & 0 \\ 0 & 0 & -\alpha_I \end{pmatrix}, \quad B_1^{1/2} \equiv \begin{pmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_D & 0 \\ -\rho_I\sigma_0 & 0 & (2\rho_I - \rho_I^2)^{1/2}\sigma_0 \end{pmatrix}, \quad \mathbf{dw(t)} \equiv \begin{pmatrix} dw_0 \\ dw_D \\ dw_I \end{pmatrix}. \tag{47}$$

Now $Z(t)$ can be expressed in an integral form as

$$Z(t+s) = \mathcal{B}(s)Z(t) + \int_{\tau=0}^{s} e^{A(s-\tau)} B^{1/2} dw(t+\tau),$$

where $\mathcal{B}(s) = e^{As}$. Solving differential equation $d\mathcal{B}(s)/ds = A\mathcal{B}(s)$, with boundary condition $\mathcal{B}(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ yields

$$\mathcal{B}(s) = \begin{pmatrix} 1 & 0 & 1 - e^{-\alpha_I s} \\ 0 & e^{\alpha_D s} & 0 \\ 0 & 0 & e^{-\alpha_I s} \end{pmatrix}.$$

Since $E(D(t) \mid \mathfrak{F}_t) = D(t)$ we have

$$E\left[D(t+s) \mid \mathfrak{F}_t\right] = D_0 + e^{\alpha_D s} D_1 + (1 - e^{-\alpha_I s}) E\left[I(t) \mid \mathfrak{F}_t\right],$$

thus

$$E\left[\int_{s=0}^{\infty} e^{-(r-\xi)s} D(t+s)ds \mid \mathfrak{F}_t\right] = E_t \int_{s=0}^{\infty} e^{-(r-\xi)s}\left\{D_0 + e^{\alpha_D s} D_1 + (1 - e^{-\alpha_I s}) E\left[I(t) \mid \mathfrak{F}_t\right]\right\} ds$$

$$= \frac{1}{r-\xi} D_0(t) + \frac{1}{r-\xi+\alpha_D} D_1(t) + \frac{1}{r-\xi} - \frac{1}{r-\xi+\alpha_I}\hat{I}(t),$$

and the coefficients are

$$\tilde{p}_{D_0} \equiv \frac{1}{r-\xi}, \qquad \tilde{p}_{D_1} \equiv \frac{1}{r-\xi+\alpha_D}, \qquad \tilde{p}_I \equiv \frac{1}{r-\xi} - \frac{1}{r-\xi+\alpha_I}$$

The constant term is obtained from Equation (19) imposing the efficient price described above and the market clearing condition (24)

$$\tilde{\Psi} = -\frac{T^{1/2}\left(B^{1/2}\right)^{\top} L - S}{r\varphi T} = 1$$

where

$$T^{1/2} = \frac{(r-\xi-\alpha_I(\rho-1))\sigma_0}{(r-\xi)(r-\xi+\alpha_I)}; \qquad B^{1/2} = 0; \qquad S = -(r-\xi)p_0$$

$$T = \left(-\frac{\alpha_I^2\left(-2+\rho_I\right)\rho_I\sigma_0^2}{(r-\xi)^2\left(r-\xi+\alpha_I\right)^2} + \left(\frac{1}{r-\xi} - \frac{\alpha_I\rho_I}{(r-\xi)\left(r-\xi+\alpha_I\right)}\right)^2\sigma_0^2 + \frac{\sigma_D^2}{\left(r-\xi+\alpha_D\right)^2} + \sigma_\Theta^2\right)$$

that implies

$$\tilde{p}_0 = -\left(\frac{((r-\xi+\alpha_I)^2 - 2(r-\xi)\alpha_I\rho_I)\sigma_0^2}{(r-\xi)^2(r-\xi+\alpha_I)^2} + \frac{\sigma_D^2}{(r-\xi+\alpha_D)^2}\right)\frac{r}{r-\xi}\varphi,$$

solution of the proof.

Appendix C. From Continuous Time Model to Discrete Time Data

Our model is in continuous-time while our data are discrete. We use Bergstrom's (1984, Thm. 3, p. 1167) solution to reformulate our model such that the discrete version, called the exact discrete solution, satisfies the discrete-time data. As underlined by McCrorie (2009), the exact discrete model differs from conventional discrete-time VAR models in that the coefficient matrix and the covariance matrix are functions of the exponential of a matrix. This leads to the problem of identifiability that it was discussed in Appendix A of C.K.'s work. The exact discrete form is obtained from the solution of the continuous-time model (46) given by

$$Z(t) = F(\theta)Z(t-1) + \epsilon_t \qquad (t = 1, ..., T), \tag{48}$$

where

$$F(\theta) = e^{A(\theta)}, \qquad (t = 1, ;T), \tag{49}$$

and the variance-covariance of the independent Gaussian white noise is the solution of the following integral

$$E(\epsilon_t\, \epsilon_t^\top) = \int_0^1 e^{rA(\theta)}\Sigma(\mu) \left(e^{rA(\theta)}\right)^\top dr = \Omega(\theta, \mu) \tag{50}$$

where we assume zero mean and covariance matrices: $E(\epsilon_t) = 0$, and $E(\epsilon_t \epsilon_s^\top) = 0$ with $(s \neq t)$.

The exact discrete form (48-49) gives the following matrices

$$F(\theta) = \begin{pmatrix} 1 & 1 - e^{\alpha_I} & 0 & 0 \\ 0 & e^{\alpha_I} & 0 & 0 \\ 0 & 0 & e^{\alpha_D} & 0 \\ 0 & 0 & 0 & e^{\alpha_\Theta} \end{pmatrix}$$

$$\Omega(\theta, \mu) = \begin{pmatrix} -\frac{\sigma_0^2(e^{-2\alpha_I}(-1+e^{\alpha_I})^2\rho - \alpha_I)}{\alpha_I} & -\frac{\sigma_0^2 e^{-2\alpha_I}(-1+e^{\alpha_I})^2\rho}{\alpha_I} & 0 & 0 \\ -\frac{\sigma_0^2 e^{-2\alpha_I}(-1+e^{\alpha_I})^2\rho}{\alpha_I} & -\frac{\sigma_0^2(-1+e^{-2\alpha_I})\rho}{\alpha_I} & 0 & 0 \\ 0 & 0 & -\frac{(-1+e^{-2\alpha_D})\sigma_D^2}{2\alpha_D} & 0 \\ 0 & 0 & 0 & -\frac{(-1+e^{-2\alpha_\Theta})\sigma_\Theta^2}{2\alpha_\Theta} \end{pmatrix} \tag{51}$$

that we use to estimate the parameters $\theta = [\alpha_I,\ \alpha_D,\ \alpha_\Theta,\ \sigma_0,\ \sigma_D,\ \sigma_\Theta, \rho]$ in our likelihood function. The exact discrete model (3)–(7) differs from conventional discrete-time VAR models in that the coefficient matrix and the covariance matrix are functions of the exponential of a matrix. We obtain the estimates of the unobservable state vector $Z(t)$ based on the information available to time $\Im(t)$, where $\Im(t)$ contains the observation y until $y(t)$. We use Kalman filter to obtain a recursive procedure for calculating the estimates of the parameters θ in the state vector. The Kalman filter procedure needs initial values. We use the method of diffuse prior such that we assume zero mean for the all state variable and for the variance-covariance an identity matrix with 10^6 on the diagonal.[19]

The likelihood function of our state-space time series is calculated using the Kalman filter technique. The joint density of $\{y_1, y_2, \ldots, y_T\}$ is $L = \prod_{t=1}^{T} p(y_t \mid \Im_{t-1})$ where $p(y_t \mid \Im_{t-1}) = N(\widehat{y}_{t|t-1}, f_{t|t-1})$ and $f_{t|t-1} = E(y_t - \widehat{y}_{t|t-1})(y_t - \widehat{y}_{t|t-1})^\top$. The log-likelihood is given by

$$ln\, L = -\frac{1}{2}\sum_{t=1}^{T} ln|f_{t|t-1}| - \frac{1}{2}\sum_{t=1}^{T}(y_t - \widehat{y}_{t|t-1})^\top f_{t|t-1}^{-1}(y_t - \widehat{y}_{t|t-1}). \tag{52}$$

and we estimate our parameter θ maximizing equation (52).

[19] Other methods to initialize the Kalman filter are taken in consideration but they do not change the finale estimations. See Harvey, 1989)

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Figures

Figure 1. Market Risk Perception and Utility of Equilibrium Price: Model A and Model B.

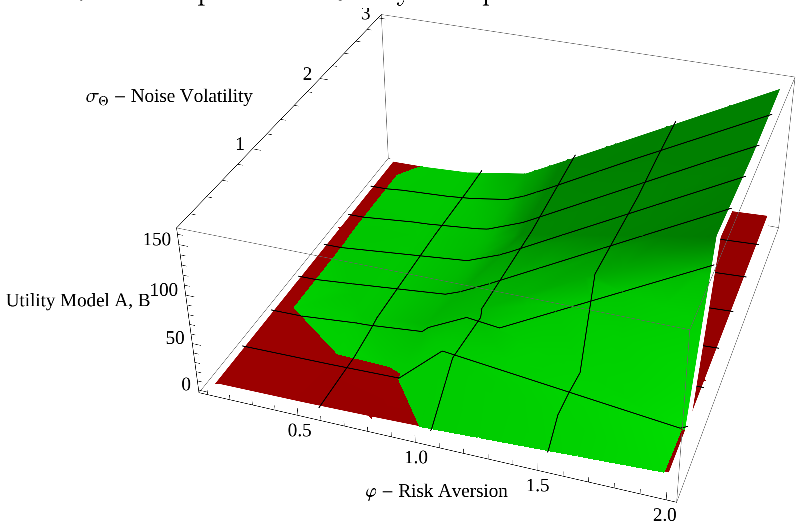
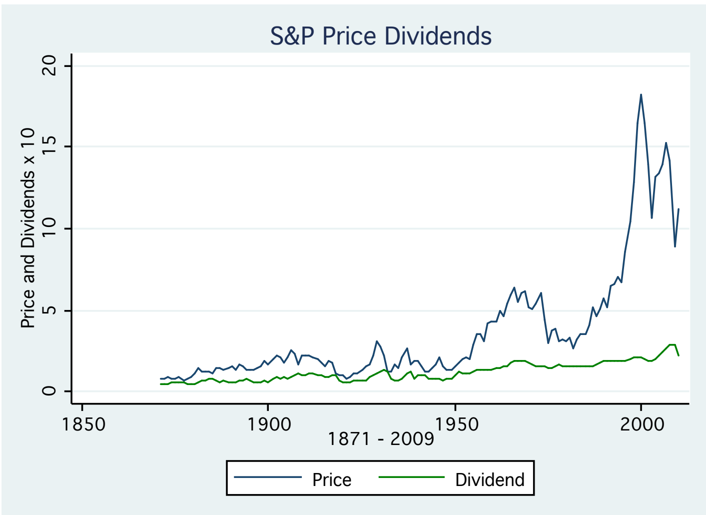


Figure 2. Stock Price and Dividend x10.



Tables

Parameter Set: $r = 0.05$;
$\beta = 0.30$; $\alpha_D = 0.50$; $\alpha_I = 0.10$; $\alpha_\Theta = 0.05$; $\sigma_0 = 0.50$; $\sigma_D = 0.10$; $\sigma_I = 1.00$; $\xi = 0.011$.

Table 1. Low risk aversion and noise volatility: $\sigma_\Theta = 0.50$; $\varphi = 0.50$.

		Candidate Equilibrium Price			
Equilibrium	Utility	p_0	p_{D_0}	p_{D_1}	p_I
Model A	2.90	-91.773	25.641	1.855	18.446
Model B	0.97	-94.218	25.641	1.855	-30.080
Model B	0.95	-93.148	25.641	1.855	-30.897
Model B	0.65	-82.902	25.641	1.855	51.713
Model B	0.62	-81.688	25.641	1.855	52.479
Model B	0.41	-93.635	25.641	1.855	5.775

Table 2. High risk aversion and low noise volatility: $\sigma_\Theta = 0.50$; $\varphi = 1.00$.

		Candidate Equilibrium Price			
Equilibrium	Utility	p_0	p_{D_0}	p_{D_1}	p_I
Model B	43.96	-2664.632	-89.311	1.855	-13.384
Model A	4.08	-183.546	25.641	1.855	18.446
Model B	3.87	-188.435	25.641	1.855	-30.080
Model B	3.79	-186.296	25.641	1.855	-30.897
Model B	3.16	-187.271	25.641	1.855	5.775
Model B	2.60	-165.804	25.641	1.855	51.713
Model B	2.50	-163.376	25.641	1.855	52.479

Table 3. Low risk aversion and high noise volatility: $\varphi = 0.50$; σ_Θ=1.00.

		Candidate Equilibrium Price			
Equilibrium	Utility	p_0	p_{D_0}	p_{D_1}	p_I
Model B	8.65	-465.202	-80.445	1.855	-87.639
Model B	8.39	-675.844	-89.311	1.855	-13.384
Model A	2.90	-91.773	25.641	1.855	18.446
Model B	1.39	-91.633	25.641	1.855	18.446
Model B	0.98	-94.733	25.641	1.855	-29.615
Model B	0.94	-92.594	25.641	1.855	-31.252
Model B	0.67	-83.506	25.641	1.855	51.274
Model B	0.42	-93.688	25.641	1.855	5.775

Table 4. Descriptive Statistics of data.

Variable	Mean	Std. Dev.	Skewness	Kurtosis
P_t	0.994	1.002	2.067	6.745
D_t	0.032	0.014	0.730	2.837
P_t^u	376.379	379.342	2.067	6.745
D_t^u	12.127	5.642	0.725	2.83
$ln(P_t^u)$	-0.362	0.798	0.650	2.607
$ln(D_t^u)$	-3.548	0.466	0.047	1.912

Table 5. Unit Root Test.

	ADF		PP		KPSS	
	drift	trend	drift	trend	drift	trend
P_t	0.29	-1.11	-0.99	-2.29	1.15*	0.26
D_t	-1.06	-3.46*	-1.04	-3.34*	1.65**	0.24
P_t^u	0.29	-1.11	-1.00	-2.28	1.15**	0.26
D_t^u	-1.06	-3.46*	-1.12	-3.34	1.65*	0.24
$ln(P_t^u)$	-0.82	-2.29	-1.08 -	-2.63	1.48*	0.20*
$ln(D_t^u)$	-2.04	-4.81**	-1.74	-4.07**	1.69**	0.09
Note: * rejects the at 5 %, ** rejects at 1%						

Table 6. Time Series Properties of the data.

Correlations:	ΔD_t	ΔP_t		ΔD_t	ΔP_t
ΔD_t	1.00	0.03	ΔP_t	0.03	1.00
ΔD_{t-1}	0.22	-0.00	ΔP_{t-1}	0.44	0.14
ΔD_{t-2}	-0.16	-0.00	ΔP_{t-2}	0.06	-0.11
ΔD_{t-3}	-0.17	-0.06	ΔP_{t-3}	-0.19	-0.09
ΔD_{t-4}	-0.15	-0.08	ΔP_{t-4}	-0.15	-0.16
ΔD_{t-5}	-0.11	-0.02	ΔP_{t-5}	-0.06	-0.19
$\sigma(\Delta D_t)$ =0.003, $\sigma(\Delta P_t) = 0.246$ $\sigma(\Delta D_t)/\sigma(\Delta P_t) = 0.013$					

Table 7. OLS Regression with heteroskedasticity-robust standard error.

	dividend
price	0.0126***
	(11.13)
_cons	0.0730***
	(18.81)
N	140
ADF-test residuals: -2.64*	
Portmanteau (Q) statistic = 448.82	
t statistics in parentheses	
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$	

Table 8. Mapping the Maximum Log-Likelihood function 1871-2009, Model A and Model B.

	Model A		Model B	
Interest rate assumption	NoNoise	Full Noise	NoNoise	Full Noise
3%	807.42	834.88	811.99	830.65
6%	810.82	833.15	808.49	831.47
Free r	814.75	831.62	814.96	828.92

Table 9. Data 1871-2009: Price Coefficient of Model A and Model B.

Assumption: Interest rate = 0.03, Full Noise

	p_0	p_{D_0}	p_I	p_{D_1}	ML
Model A	-0.001	68.965	65.980	2.985	834.00
Model B	-0.061	39.809	39.803	0.007	830.65

Assumption: Interest rate = 0.06, Full Noise

	p_0	p_{D_0}	p_I	p_{D_1}	ML
Model A	-0.001	22.471	20.657	1.814	833.15
Model B	-0.006	73.246	73.236	0.004	831.47

Assumptions: free r and free γ, Full Noise

	p_0	p_{D_0}	p_I	p_{D_1}	ML
Model A^a	-0.000	64.888	52.342	12.546	831.62
Model B^b	-0.016	4.963	4.892	0.071	828.92

Note: a) interest rate estimated is 0.0309

Note: b) interest rate estimated is 0.0686

Table 10. Preliminary Data Analysis 1995-2000

	ADF		PP		KPSS
	drift	trend	drift	trend	
Price	-1.39	-1.43	-1.41	-1.17	0.14
Dividend	-2.51	0.37	-2.06	2.05	0.26
D.Price	-5.70	-5.83	-7.06	-7.17	0.15
D.Dividend	-3.18	-4.54	-3.63	-4.72	0.17

Cointegrating regression: $D_t = 0.049 + 0.0017P_t + \epsilon_t$
Johansen test: rank(r) $= 1$: $\lambda max = 1.23$ (3.76 at 5%)
rank(r) $= 1$: $\lambda trace = 1.23$ (3.76 at 5%)

Table 11. Mapping the Maximum Log-Likelihood function 1995-2000, Model A and Model B.

Interest rate assumption	Model A Full Noise	Model B Full Noise	LR-test
1.5%	540.96	636.39	190.86***
3%	487.17	638.58	302.08***
Free r	489.31	638.43	298.83***

LR-test: -2 (Model A-Model B)
χ_3 = 7.82 (5% =*), 11.35 (1% =**), 16.27 (0.1% =***)

Table 12. Data 1995-2000.: Price's Coefficient of Model A and Model B.

Assumption: Interest rate = 0.015, Full Noise

	p_0	p_{D_0}	p_I	p_{D_1}	ML
Model A	-0.177	71.942	67.882	4.060	540.96
Model B	-0.061	71.944	0.001	1.089	636.39

Assumption: Interest rate = 0.03, Full Noise

	p_0	p_{D_0}	p_I	p_{D_1}	ML
Model A	-0.124	34.602	31.147	3.455	487.17
Model B	-0.430	34.604	0.002	0.002	638.58

Assumptions: free r and free γ, Full Noise

	p_0	p_{D_0}	p_I	p_{D_1}	ML
Model A[a]	-0.506	210.13	70.755	139.23	489.31
Model B[b]	-0.883	0.004	1089.196	118.025	638.43

Note: a) interest rate estimated is 0.006

Note: b) interest rate estimated is 0.002

Table 13a. Data 1995-2000. Parameter Estimation Model A.

Assumption: Interest rate = 0.015, Full Noise.
Maximum Log-Likelihood = 540.96

$\tilde{p}_0$	$\tilde{p}_{D_0}$	$\tilde{p}_I$	$\tilde{p}_{D_1}$
-0.177	71.942	67.882	4.060
α_D = 0.232 (0.11)	α_I= 0.232 (0.000)	α_Θ = 0.001 (0.000)	φ = 16.75
σ_D= 0.001 (0.02)	σ_Θ=0.001 (0.001)	ρ_I=1.299 (0.002)	λ = 0.002 (0.031)

Table 13b. Data 1995-2000. Parameter Estimation Model B.

Assumption: Interest rate = 0.03. Full Noise.
Maximum Log-Likelihood = 636.39

$\widehat{p}_0$	$\widehat{p}_{D_0}$	$\widehat{p}_I$	$\widehat{p}_{D_1}$
-0.061	71.944	0.001	1.089
$\tilde{p}_0$	$\tilde{p}_{D_0}$	$\tilde{p}_I$	$\tilde{p}_{D_1}$
-0.071	71.942	70.854	67.114
α_D = 0.001 (0.11)	α_I= 0.905 (0.000)	α_Θ = 0.001 (0.000)	
σ_D= 0.001 (0.004)	σ_Θ=0.124 (0.001)	ρ_I=0.001 (0.002)	λ = 0.001 (0.001)