Consistent valuation of project finance and LBO's using the flows-to-equity method

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Abstract

A common method of valuing the equity in leveraged transactions is the flows-to-equity method whereby the free cash flow available to equity holders is discounted at the cost of equity. This method uses a standard definition of equity free cash flow, but the cost of equity varies over time as leverage varies. Various formulas can be used to calculate the time-varying cost of equity, most of which are inconsistent with the assumptions underlying the free cash flow calculation. In this paper we show how to include correctly the following in the flows-to-equity method:

- A releveraging formula consistent with a fixed debt plan;
- A yield spread on debt which is fair compensation for default risk;
- The part of the yield spread which is "excessive";
- The expected cost of financial distress.

We show that each of these can have a significant effect on valuation and the value derived in a consistent way can differ substantially from that derived by more conventional procedures.

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1. Introduction

The general topic of this paper is the valuation of investments that have fixed debt plans. In other words, at the time of the valuation the future amount of debt is a function of time alone. The amount of debt is not expected to fluctuate with the future value of the investment. This type of situation arises in LBO's (Baldwin 2001a), project finance (Esty 1999), and other highly leveraged transactions (HLT's) where the future amortisation of the debt has been agreed at the time of the investment. Our focus is especially on valuing the equity in such investments directly through the "flows-to-equity" method, whereby the project's equity free cash flows are discounted at the cost of levered equity. This method focuses directly on the cash flows equity-holders will actually receive rather than valuing equity indirectly as the difference between total project value and debt, as under the standard adjusted present value approach.

As emphasized by Esty, the cost of equity is time varying in investments with fixed debt plans, since leverage and thus also the risk of equity changes over time as the debt plan unfolds. It is therefore necessary to use a time varying discount rate when using the flows-to-equity method in investments with fixed debt plans. The correct way to do this is to calculate an implied market value leverage ratio and releverage the cost of equity to reflect this changing leverage ratio at each future date.

We make three main points. First, the standard way of releveraging the cost of equity for use in the flows-to-equity approach is inconsistent with the assumption of a fixed debt plan. This inconsistency can lead to significant undervaluation of the equity. We show how to calculate the cost of equity consistent with an evolving fixed debt plan. Second, the most commonly used method of implementing the flows-to-equity method implicitly assumes that the entire debt spread results in negative NPV whereas, in reality, a large part of it is compensation for the risk of debt. We show how to incorporate this into the flows-to-equity method. Failure to do so also leads to undervaluation of the equity. Third, we show how to incorporate other effects of the financing plan in a way that is consistent with the present value of the tax saving from debt, including the present value of financial distress costs and debt that is expensive in the sense that its yield exceeds fair compensation for credit risk. We also discuss the more general issue of the conditions under which it is reasonable to assume a fixed debt plan.

The topic is important because the combination of a fixed debt plan and valuation using flows-to-equity is common in practice. There are several advantages which account for its popularity (Esty 1999, Baldwin 2001a). In particular, the flows-to-equity approach:

- can allow for time-varying debt, which is inconsistent with a constant WACC;
- can allow for time-varying effective tax rates in a simple way;
- can accommodate debt which is not issued at its fair price (including expensive or subsidised debt);
- can easily allow for several rounds of equity financing;
- focuses directly on the cash flows that accrue to equity-holders.

In order to implement the flows-to-equity method a formula is needed with which to releverage the cost of equity at each future date (Baldwin (2001b)). There are two such releveraging formulas in common use. One formula is consistent with debt being a constant proportion of the firm's market value, the assumption which underlies the WACC (see Miles-Ezzell (1980), "ME"). The formula for releveraging the cost of equity consistent with a constant leverage ratio (continuously adjusted) is:

$$R_E = R_U + (D/E)(R_U - R_D) \tag{1}$$

where R_E is the cost of equity, R_U is the unlevered cost of equity, R_D is the cost of debt, D is the market value of debt, and E is the market value of equity.

The other common releveraging formula is consistent with a fixed amount of debt (see Miller and Modigliani (1963) "MM"). The formula for releveraging the cost of equity consistent with this assumption is:

$$R_E = R_U + (D/E)(1-T)(R_U - R_D) \tag{2}$$

where T is the corporate tax rate. In this paper we ignore the effect of investor taxes.

In practice these two formulas are used in a variety of forms. For instance, both Esty (1999) and Baldwin (2001b) use a simplified version of (1). They use the CAPM, so the releveraging formula is expressed in terms of betas. They also assume that the debt is riskless, giving the formula:

$$\beta_E = \beta_U((E+D)/E) \tag{3}$$

where β_E is the equity beta, and β_U is the unlevered equity beta. In terms of discount rates, this is equivalent to:

$$R_E = R_U + (D/E)(R_U - R_F) \tag{4}$$

An important feature of all the above formulas is that none is based on a debt policy whereby the amount of debt is scheduled to change over time in a predetermined manner, the policy that will actually be pursued in a typical HLT. In some applications such inconsistency does not matter because the use of a releveraging formula inconsistent with the debt policy which will actually be pursued does not have a material effect on the valuation (see Cooper and Nyborg 2007). However, in HLT's the tax benefit of debt is a first order component of value. Therefore, treating this element of the valuation in a consistent way is important. In this paper we show how to do this and calculate the size of the resulting adjustment to the present value.

In addition to the issue of consistency between the releveraging formula and the leverage policy, there is another issue of consistency. This concerns the yield spread on the borrowing, the measure of equity free cash flow used, and the releveraging formula. The commonly used formula (3) assumes that debt is riskless. The use of equation (3) therefore implicitly assumes that the entire yield spread on the debt results in a negative NPV from borrowing. However, a large part of the yield spread in HLT's is simply compensation for the risk of debt. A valuation method which treats the entire yield spread as implying that debt is expensive will give an underestimate of value. We show how to deal with this correctly and the effect of this on present value.

There are two other value effects of leverage which can be significant in HLT's. The first is debt which is expensive, in the sense of having a yield that exceeds the fair yield required to compensate for credit risk. One benefit of the flows-to-equity approach is that, unlike other valuation approaches, it does not assume that debt is issued at a fair price (from the equity-holder's perspective). The other issue is the expected cost of financial distress, which can be substantial at the leverage ratios used in HLT's. We generalize the model to include negative NPV debt as well as costs of financial distress.

We illustrate our results using the realistic example studied by Esty (1999). Esty's example has the key features of project finance and LBO's: a relatively large amount of debt, relatively high margins on the debt, and a fixed debt plan. We show how to include in his valuation the four elements discussed above:

- A releveraging formula consistent with a fixed debt plan;
- A yield spread on debt which is fair compensation for default risk;
- The part of the yield spread which is "excessive";
- The expected cost of financial distress.

We show that each of these can have a significant effect on valuation and the value derived in a consistent way can differ substantially from that derived by more conventional procedures.

The paper is organised as follows. Section 2 provides a simple example of the incorrect valuation that may result from from using the flows-to-equity approach with the standard cost of equity formulas. In Section 3 we derive our basic releveraging formula and related results, assuming zero NPV debt and no financial distress costs, and show that this gives the correct answer in the simple example in Section 2. Section 4 extends the analysis to include negative NPV debt and costs of financial distress. Section 5 shows the size of the effects using a realistic numerical model, and Section 6 gives the conclusions.

2. Numerical example of incorrect valuations using standard formulas in the flows-to-equity method

As a motivation for the subsequent analysis, in this section we present an example that illustrates the misvaluation that may result in the flows-to-equity method when using the standard releveraging formulas given in the Introduction [equations (1), (2), and (4)]. We do this by comparing values calculated using the flows-to-equity method to the correct value as calculated from a standard two-step adjusted present value approach. The parameters are: Corporate tax rate, T: 35%; Yield on debt, Y: 5.00%; Riskfree rate, R_F : 3.00%; Unlevered cost of equity, R_U : 9.00%.

Table 1, Panel A, sets out the after tax operating cash flows, debt plan, and equity free tax cash flows. The project has an investment of 100 at time zero and gives rise to after-tax operating free cash flow of 20, 60, 45, 20 in the following years. The debt plan is to borrow 90 and pay it down according to the amortization schedule shown. The equity cash flows are the operating cash flows plus the tax saving from interest minus the debt flows. The initial equity value is equal to the unlevered value, V_U, plus the present value of the tax shield (PVTS), i.e., the project's adjusted net present value (APV). Thus, V_U is the project's net present value (NPV), calculated by discounting the operating cash flow at the unlevered cost of equity. The present value of the tax shield is calculated by discounting projected interest payments at the yield of the debt. Cooper and Nyborg (2008) show that this is consistent with no arbitrage, given certain assumptions about the default process for the debt (see Section 3).

Panel B of Table 1 computes the value of the investment using the standard implementation of the flows-to-equity method as laid out by Esty (1999). This is derived as follows. From Panel A one first inputs the equity cash flows and the debt plan. In the R_E column, one enters the releveraging formula to be used, in this case (4). In the PV equity column, one enters the equity value (ex cash flow) caculated assuming last period's equity value grows at R_E. For example, the PV equity at date 1 is $27.3335 \times 1.2876 - 7.075 = 28.1185$. The value of the equity is solved iteratively by choosing an initial end of period equity value (the first row in the fourth column) so that the sum of the discounted equity cash flows equals that equity value less the initial equity outflow.

As seen, the solution when using (4) as the releveraging formula, is 17.3335, which is 17.42% below the correct APV value as calculated in Panel A. The calculated value is below the unleveraged value, implying a negative value of PVTS. To illustrate the effect of using a different cost of equity formula, if the MM formula (2) is used as the releveraging formula it gives a computed equity value of 23.2178, which is 10.62% *above* the correct value. In contrast, using the ME formula (1) gives an equity value of 20.7949, which is 0.93% below the correct valuation. While this is a relatively small error, in other examples the error from using the ME formula is substantially larger.

Table 1: Example of valuation error in the standard implementation of the flows to equity method

Panel A: Free cash flows, debt plan, and benchmark adjusted present value

Year	Operating Cash Flow (FCFF)	Debt	Net Principal Repayment	Interest	Tax saving	Equity Cash Flow (FCFE)	Unlevered discount factor	Discount factor Tax shield
0	-100	90	-90	0.00	0.0000	-10.0000	1.0000	1.0000
1	20	80	10	4.50	1.5750	7.0750	0.9174	0.9524
2	60	30	50	4.00	1.4000	7.4000	0.8417	0.9070
3	45	0	30	1.50	0.5250	14.0250	0.7722	0.8638
4	20	0	0	0.00	0.0000	20.0000	0.7084	0.8227
V_U:	17.7662		**PVTS:**	3.2234		**APV equity:**	**20.9895**	

Panel B: Flows-to-equity valuation using Esty's (1999) method with Eq (4) as the releveraging formula

Year	Equity Cash Flow (ECF)	Debt	PV equity end period	Debt plus equity (V)	Leverage (D/V)	R_U	R_E	Discount Factor	Present Value of ECF
0	-10.000	90	27.3335	117.3335	0.767044	9.00%	28.76%	1.0000	-10.0000
1	7.075	80	28.1185	108.1185	0.739929	9.00%	26.07%	0.7767	5.4949
2	7.400	30	28.0492	58.0492	0.516803	9.00%	15.42%	0.6161	4.5588
3	14.025	0	18.3486	18.3486	0	9.00%	9.00%	0.5338	7.4860
4	20.000	0	0.0000	0.0000	0	9.00%	9.00%	0.4897	9.7938
							Sum (PV equity):		**17.3335**

This example illustrates the two central issues addressed by this paper: First, any flows-to-equity valuation method involves a releveraging formula for the cost of equity. Each formula makes implicit assumptions about the risk of PVTS, the debt strategy, and other factors which we discuss in section 4 below. For the valuation method to be legitimate these assumptions need to be consistent. Second, the correct valuation method is always that given by no-arbitrage valuation (see for example, Berk and DeMarzo 2007). The most transparent way to derive this value is to use APV. However, the flows-to-equity method is commonly used in practice. Therefore, it is important to know what flows-to-equity valuation procedure corresponds to the correct APV value when particular assumptions are made about the debt policy. The answer to this question is known for the specific cases involving a constant amount of debt or a constant proportion of debt (see, for instance, Cooper and Nyborg 2007). However, the correct procedure with a changing debt plan has not been derived and that is the purpose of this paper.

In the next section, we develop a releveraging formula which always results in the correct value using the flows-to-equity method (i.e. the same as the no-arbitrage value derived using the adjusted present value calculation). In the following section we then extend the formula to include other realistic features of highly leveraged transactions.

3. A releveraging formula assuming zero NPV debt and no distress costs

In this section we derive our basic results using a simplified model with a fixed debt plan, fairly priced debt, and no costs of financial distress. In the next section we allow for mispriced debt and costs of financial distress. Throughout, we consider a project funded with debt that will amortize according to a fixed schedule. The project has expected after tax unlevered cash flows of C(t). The debt face value at time t will be D(t). The promised yield on the debt is fixed at Y and the corporate tax rate is T.[2]

We assume that the discount rate for the unlevered flows is constant and equal to R_U. The unlevered value, $V_U(t)$, is calculated by discounting the unlevered free cash flows (after corporate taxes) at the unlevered discount rate:

$$V_U(t) = \sum_{i=1}^{\infty} C(t+i)/(1+R_U)^i \tag{5}$$

The fundamental APV relationship always gives the correct value, and we use it to derive the correct discount rate formulas:

$$V_L(t) = V_U(t) + PVTS(t) \tag{6}$$

where $V_L(t)$ is the levered value at time t, and $PVTS(t)$ is the present value at time t of the debt tax saving from that date onwards. All leverage-adjusted discount rates are derived from (6). The reason that particular formulas differ is because they make different assumptions about the size and risk of PVTS (see Cooper and Nyborg 2004).

The value of equity can be calculated from the APV formula as:

$$E(t) = V_L(t) - D(t) = V_U(t) + PVTS(t) - D(t) \tag{7}$$

However, the point of the flows-to-equity method is to obtain the equity value by discounting the equity free cash flow, which is defined as:

$$FCFE(t) = C(t) - D(t-1)Y(1-T) - \left(D(t-1) - D(t)\right) \tag{8}$$

where Y is the yield (or coupon) on the debt. The equity discount rate, $R_E(t)$, is defined implicitly as the rate required to give the correct value of the equity by discounting equity flows and values period-by-period:

$$E(t) = \frac{FCFE(t+1) + E(t+1)}{1 + R_E(t)} \tag{9}$$

where the equity values and equity free cash flow are given by (7) and (8). Hence a consistent flows-to-equity valuation procedure is the one that delivers an equity value from equation (9) which is the same as that calculated using equation (7).

There is one slightly unusual feature of this procedure which is worthy of note. Although the approach is standard, the definition of equity free cash flow (8) mixes the *expected* cash flow from operations with a *promised* debt payment. Since the promised debt yield is not equal to the expected cash flow on the debt the equity free cash flow given by (9) is not equal to the expected cash flow on equity. Nevertheless, this definition of equity free cash flow is used in the standard version of the flows-to-equity method. However, since the equity discount rate is used to discount this hybrid cash flow it is important to realise that the correct discount rate to use in the procedure is not exactly equal to the expected return on equity.

Implementation of the flows-to-equity method requires the calculation of $R_E(t)$ starting from the unlevered cost of capital, R_U . The relationship between them can be derived from equations (5)-(9) with one further assumption. The crucial extra ingredient is an assumption about the risk of PVTS. This is the fundamental difference between the ME approach and the MM approach. For a fixed debt plan, the relevant assumption is that the risk of the debt tax shield is the same as the risk of the debt, which is also the MM assumption. With a fixed debt plan and simplifying

[2] Although we treat the interest rates as fixed, the same approach can be used with variable rate debt.

assumptions regarding the treatment of tax losses, Cooper and Nyborg (2008) show that the value of the debt tax shield is given by:

$$PVTS(t) = \sum_{i=0}^{\infty} D(t+i)YT/(1+Y)^{i+1} \tag{10}$$

where Y is the promised yield on the debt. This assumes that (A1) the amount of debt at every future date is determined at time zero and will not change if the firm does not default, (A2) the debt is fairly priced, (A3) if the debt defaults there is zero recovery, and (A4) there are no costs of financial distress. Hence the only effect of debt on the total after-tax cash flow of the firm is through the debt tax shield.

Appendix 1 shows that that, with the assumptions (A1)-(A4), $R_E(t)$ is given by:

Result 1: (*Proof: Appendix 1*): With a fixed debt plan the equity discount rate is given by:

$$R_E(t) = R_U + \left[(D(t) - PVTS(t))/E(t)\right](R_U - Y) \tag{11}$$

The equity discount rate is subscripted with time because the key point of the flows-to-equity method is that the discount rate varies over time as the leverage ratio varies.

There are two differences between (11) and the standard formula (4). One is that the second term contains the spread over the debt yield (R_U-Y) rather than the spread over the riskless rate (R_U-R_F). This lowers the equity discount rate and therefore raises the estimated equity value. The reason is that the equity free cash flow has already had the full debt yield deducted from it. Ignoring this in the releveraging formula essentially double-counts the spread of the risky debt. The second difference is that the leverage in (11) is lowered by PVTS. This also reduces the equity discount rate and increases estimated equity value. The reason for the difference is that the tax shields arising from the fixed debt plan have a low level of risk and do not, therefore, increase the equity discount rate by as much as ME formulas like (4) assume.

To see the relationship between (10) and the standard MM and ME formulas, we define a variable which measures PVTS relative to its standard MM level:

$$\alpha(t) = PVTS(t)/TD(t) \tag{12}$$

(note that $\alpha(t)$ is not defined if *D(t) = 0*). Hence $\alpha(t)$ is the present value of debt tax shields divided by the tax shield that would arise from a fixed amount of permanent debt at the level D(t). In general for HLT's the value of $\alpha(t)$ will be less than one, because the level of debt will be expected to reduce over time. We can now restate (11) as:[3]

$$R_E(t) = R_U + \left[D(t)/E(t)\right]\left(1-\alpha(t)T\right)\left(R_U - Y\right) \tag{13}$$

If $\alpha(t)=1$, this collapses to the MM formula, (2), with the debt yield used as the cost of debt. If $\alpha(t)=0$ it collapses to the ME formula, (1), with the debt yield used as the cost of debt. Thus $\alpha(t)$ adjusts the releveraging formula to reflect the extent of the fixed debt plan. Using (13), we have a releveraging formula for the cost of equity that should be used in the flows-to-equity method. It is easily verified that using this formula in the example in Section 2 gives the same value as the standard APV procedure.

The intuition of the formula is that the ME formula applies whenever the risk of PVTS is the same as the risk of the operating cash flows (Cooper and Nyborg 2006). The MM formula applies to perpetual debt which generates PVTS with the same risk as the debt. In this case the variable $\alpha(t)$ is measuring the size of the PVTS resulting from the fixed debt plan as a proportion of that which would result from permanent debt.

From the perspective of implementation, a potential drawback with (11) and (13) is that they require the calculation of *PVTS*(*t*) at every date. Next, we show that $\alpha(t)$ can be related to the duration of the debt, so that it can be calculated directly without first calculating *PVTS*(*t*).

Define the conventional duration of the aggregate cash flows in the fixed debt plan by:

$$DUR(t) = \left[\sum_{i=1}^{\infty} iB(t+i)/(1+Y)^i\right]/D(t) \tag{14}$$

where $B(t+i)$ is the total cash flow going to the debt holders at time $(t+i)$:

$$B(t+i) = D(t+i-1)(1+Y) - D(t+i) \tag{15}$$

[3] The formula (13) applies if D(t)>0. If D(t)=0 the alternative formula (11) must be used because $\alpha(t)$ is not defined.

Result 2: (*Proof: Appendix 2*)

$$\alpha(t) = DUR(t)Y/(1+Y) \tag{16}$$

The expression for $\alpha(t)$ given by (16) can be used in (13) to get levered equity discount rates. In the special case of perpetual debt, duration is equal to $(1+Y)/Y$, and $\alpha = 1$. Thus, $\alpha(t)$ is the duration of the project's debt as a fraction of the duration of a perpetuity with the same yield. The importance of duration here is that it measures the effective maturity of the fixed debt plan, not that it measures the sensitivity of its value to interest rate changes.

4. Generalization of the model: Negative NPV debt and costs of distress

In the previous section we assumed that there are no costs of financial distress and that debt is priced to have zero NPV to the shareholders of the borrowing firms. However, Almeida and Philippon (2007) have shown that distress costs can have a substantial effect on the net benefit of debt. This effect is likely to be especially important for highly leveraged transactions, so in this section we incorporate the costs of financial distress into our valuation procedure. Furthermore, one of the stated benefits of the flows-to-equity approach is that it can handle debt which has an interest rate above a "fair" rate. We define a fair interest rate as the rate which would have a zero NPV to shareholders of the borrowing firm, excluding the financing side-effects and incorporate this into our valuation formula.

We use a simplified version of the model given in Almeida and Philippon (2007). Essentially, we extend their analysis to derive its implications for the flows-to-equity valuation method. We maintain the assumption (A1) of a fixed debt plan and replace assumptions (A2)-(A4) with more general assumptions: (A2') part of the debt spread exceeds fair compensation for default risk and therefore represents a loss of NPV to equity-holders, (A3') costs of financial distress are experienced only when debt defaults, (A4') in default the value of the firm falls by a fixed proportion of the face value of the debt, (A5) The marginal probability of default per period is constant. The justifications for the new assumptions are as follows. Assumptions (A2') and (A4') are simple generalisations. Almeida and Philippon provide a justification for (A3'). We base our assumption (A5) on the idea that HLT's are structured to match the maturity structure of debt to the profile of the underlying cash flows. One way of doing this would be to make the debt structure generate a constant marginal probability of default, which is what we assume.

We wish to value the firm from the perspective of the original equity-holders. The side-effects of financing now include the tax shield from debt, distress costs, and the effect of expensive debt. We assume that if default occurs distress costs are a fixed proportion of the face value of debt prior to default. The logic is that the firm value at default will be related to the amount of debt which has triggered default and the distress costs will be a proportion of the firm value. When expensive debt is issued we allow for its effect in the following way. The impact of the expensive debt on the equity-holders is the amount by which the promised yield exceeds the fair yield that would be required to compensate debt-holders for default risk. This loss of value occurs when the firm is solvent, but is zero in the default state.

We introduce some additional notation:

- Fair promised yield on debt from point of view of equity-holders: y
- Recovery rate in default per dollar face value of debt: ρ
- Financial distress cost per dollar face value of debt: ϕ

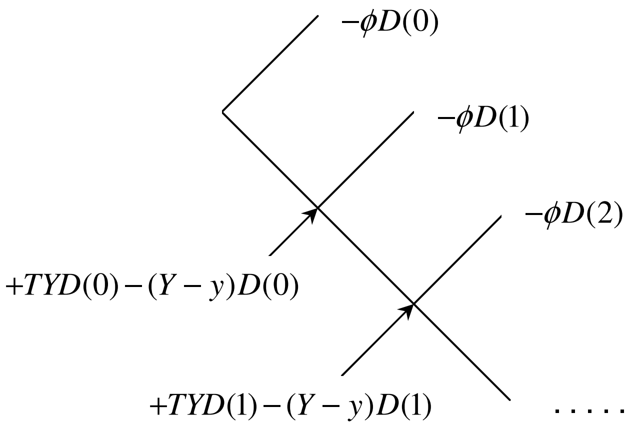
Table 2 shows these financing side-effects in a single-period version of the model. In order to calculate the APV value of the firm, these are the components we need to value.

Table 2: Financing side-effects in a single period version of the model

Component	**State**	
	Solvent	*Default*
Tax saving from debt	$+DYT$	
Distress cost		$-\phi D$
Loss to equity from overpriced debt	$-D(Y-y)$	
Total financing side-effects	$+DYT-D(Y-y)$	$-\phi D$

Figure 1 shows the evolution of the components of the adjusted present value in a multiperiod model. At the end of the first period there is a gain of $TYD(0)$ from the interest tax shield in the solvent state. This is offset by an excess cost of $(Y-y)D(0)$ if the debt is expensive. In the default state there is a cost represented by the amount $-\phi D(0)$.

Figure 1: Evolution of the APV components in the multiperiod model



To derive the equity discount rate using these assumptions, we start from the APV formula as before:

$$V_L(t)=V_U(t)+PVFS(t) \tag{17}$$

where PVFS(t) is the present value at time t in the solvent state of all future financing side-effects shown in Figure 1 (including the probability of distress costs at future dates). To determine PVFS we need a risk-adjusted probability to use in the valuation tree. As in the simple case, we derive the risk-adjusted probability from the condition for fairly-priced debt. Under the risk-neutral probability of default, q, this must have an expected return equal to the riskless rate. Fairly priced debt pays (1+y) per dollar of face value if it does not default and $\rho\left(1+y\right)$ if it does default. So:

$$(1-q)\left(1+y\right)+q(1+y)\rho=(1+R_F) \tag{18}$$

Solving for q gives:

$$q = \left(y - R_F\right) / \left[\left(1 + y\right)\left(1 - \rho\right)\right] \tag{19}$$

The components of the adjusted present value can be valued using this probability in conjunction with riskless discounting at R_F. A claim that pays \$1 in the solvent state and 0 in the default state is worth $(1-q)/\left(1+R_F\right)$ at the beginning of the period and \$1 in the default state is worth $q/\left(1+R_F\right)$. Thus, the loss from expensive debt of $D\left(Y-y\right)$ in the solvent state and 0 in the default state is worth $D\left(Y-y\right)(1-q)/\left(1+R_F\right)$ at the beginning of the period.

Using the risk-neutral valuation procedure, we can value all the APV components at time t:

$$\begin{aligned} PVFS(t) = & \sum_{i=0}^{\infty} D\left(t+i\right)\left(1-q\right)^{i+1} YT / \left(1+R_F\right)^{i+1} \\ & -\sum_{i=0}^{\infty} D\left(t+i\right)\left(1-q\right)^{i+1}\left(Y-y\right)/\left(1+R_F\right)^{i+1} \\ & -\sum_{i=0}^{\infty} \phi D\left(t+i\right)\left(1-q\right)^{i} q / \left(1+R_F\right)^{i+1} \\ = & \sum_{i=0}^{\infty} D\left(t+i\right) YT^* \left[\left(1-q\right)/\left(1+R_F\right)\right]^{i+1} \end{aligned} \tag{20}$$

Where:

$$T^* = T - (Y-y)/Y - \frac{q\phi}{(1-q)Y} \tag{21}$$

Equations (20) and (21) depend on q, which is defined by (19). We define:

$$c = \rho\left(y - R_F\right) / \left[\left(1-\rho\right)\left(1+R_F\right)\right] \tag{22}$$

$$\left(1+\gamma\right) = \left(1+y\right)/\left(1-c\right) \tag{23}$$

We then have:

$$PVFS(t) = \sum_{i=0}^{\infty} D\left(t+i\right) YT^* / \left(1+\gamma\right)^{i+1} \tag{24}$$

This differs from the simple case in two ways. First, it uses an adjusted tax rate, T^*, that includes the effects of negative NPV debt and costs of financial distress. Second, it uses an adjusted yield that allows for the effect of the recovery rate. Note that when

$\rho = 0$ then $\gamma = y$, so that the adjusted yield is equal to the fair yield and we "discount" the APV components at y, as in the simple case.

Using the same basic procedure as for Result 1, but with PVFS given by (24) instead of PVTS given by (10), we get:

Result 3: (Proof: Appendix 3)

$$R_E(t) = R_U + \frac{D(t)}{E(t)}\left[\left(1-\alpha^*(t)T^*\right)\left(R_U - Y\right) + \alpha^*(t)T^*(\gamma - Y) + (T - T^*)Y\right] \tag{25}$$

where[4]

$$\alpha^*(t) = \frac{PVFS(t)}{T^* D(t)} \tag{26}$$

This parallels (13), but there are two extra terms. One involves the difference between γ and Y and another the difference between T and T^*. The first term allows for the effect of debt which is "expensive" in the sense that the interest rate is above a fair interest rate. The use of T^* rather than T incorporates the effect of financial distress costs into the formula for the cost of equity.

Paralleling Result 2, we can eliminate *PVFS* (*t*) from the expression for $\alpha^*(t)$:

Result 4: (Proof: Appendix 4)

Using an interest rate of γ, define the conventional duration of the aggregate debt cash flows by:

$$d_\gamma(t) = \left[\sum_{i=1}^{\infty} iB_\gamma\left(t+i\right)/\left(1+\gamma\right)^i\right]/D\left(t\right) \tag{27}$$

where $B_\gamma(t+i)$ be the total cash flow going to the debt holders at time $\left(t+i\right)$:

$$B_\gamma\left(t+i\right) = D\left(t+i-1\right)\gamma + D\left(t+i-1\right) - D\left(t+i\right) \tag{28}$$

We have

$$\alpha(t) = d_\gamma(t)Y/(1+\gamma) \tag{29}$$

[4] As with Result 1 this formula applies if D(t)>0. If D(t)=0 the alternative formula $R_E(t) = R_U + (PVFS(t)/E(t))(\gamma - R_U)$ must be used because $\alpha^*(t)$ is not defined.

The reason why Y appears in (29) derives from the expression for $PVFS$(t) in (24), where Y also appears.

5. The size of the effects: Which adjustments matter most?

Esty (1999) develops an example using the flows-to-equity method. We will use this example to illustrate the potentially large differences in valuation generated when using the standard formulas for the cost of equity, as compared with the formulas we have developed above.

Table 3 presents Esty's calculation. The formulas and parameter values that are used are:

$$R_E(t) = R_F + \beta_E(t)P \tag{30}$$

where

- R_F is the riskfree rate, which is assumed to be 8%
- P is the risk premium, which is assumed to be 7.4%
- β_U is the unlevered asset beta, which is assumed to be 0.6
- β_E (t) is the beta of the equity in period t, which is calculated according to:

$$\beta_E(t) = \beta_U \frac{V_L(t)}{E(t)} = \frac{\beta_U}{1 - L(t)} \tag{31}$$

where:

$$L(t) = D(t)/V_L(t) \tag{32}$$

We show in Appendix 5 that this approach gives the same answer as using the capital cash flow (CCF) approach when debt is risk-free. In this approach, the free cash flows available to the combination of debt and equity are discounted at the unlevered cost of equity, thus implicitly discounting tax shields at the cost of unlevered equity (see Ruback 2002). Thus Esty's approach is correct whenever the CCF approach is correct (see Cooper and Nyborg (2007) for conditions when this holds). The problem is that the high leverage in the types of transaction we are considering means that debt is rarely risk-free. In fact the promised yield on the debt in Esty's example is 10%, which is well above the riskless interest rate. The spread on the debt must represent either a reward for risk or expensive debt, or both. We return to this below in the context of the example.

Table 3: Flows to equity valuation with R_E determined by Equation (4), Esty (1999) Exhibit 4.

Year	Equity Cash Flow (ECF)	Debt	PV equity end period	Debt plus equity	Leverage	Equity Beta using Eqn.(3)	RE	Discount Factor	Present Value of ECF
0	-300,000	0	406,688	406,688	0.000	0.60	12.44%	1.0000	-300,000
1	-170,000	700,000	627,280	1,327,280	0.527	1.27	17.39%	0.8894	-151,192
2	-254,349	1,300,000	990,743	2,290,743	0.568	1.39	18.27%	0.7576	-192,691
3	171,446	1,275,000	1,000,265	2,275,265	0.560	1.36	18.10%	0.6406	109,824
4	175,490	1,250,000	1,005,818	2,255,818	0.554	1.35	17.96%	0.5424	95,186
5	167,058	1,225,000	1,019,384	2,244,384	0.546	1.32	17.78%	0.4598	76,818
6	159,901	1,175,000	1,040,685	2,215,685	0.530	1.28	17.45%	0.3904	62,430
7	153,143	1,125,000	1,069,173	2,194,173	0.513	1.23	17.11%	0.3324	50,907
8	147,661	1,050,000	1,104,467	2,154,467	0.487	1.17	16.66%	0.2838	41,912
9	155,080	975,000	1,133,402	2,108,402	0.462	1.12	16.26%	0.2433	37,732
10	150,023	900,000	1,167,665	2,067,665	0.435	1.06	15.86%	0.2093	31,396
11	146,243	800,000	1,206,639	2,006,639	0.399	1.00	15.38%	0.1806	26,415
12	142,864	700,000	1,249,401	1,949,401	0.359	0.94	14.93%	0.1565	22,364
13	140,761	575,000	1,295,146	1,870,146	0.307	0.87	14.41%	0.1362	19,173
14	139,060	450,000	1,342,732	1,792,732	0.251	0.80	13.93%	0.1191	16,556
15	138,636	300,000	1,391,112	1,691,112	0.177	0.73	13.40%	0.1045	14,487
16	144,114	150,000	1,433,372	1,583,372	0.095	0.66	12.90%	0.0922	13,280
17	212,953	0	1,405,390	1,405,390	0.000	0.60	12.44%	0.0816	17,381
18	278,987	0	1,301,234	1,301,234	0.000	0.60	12.44%	0.0726	20,252
19	281,798	0	1,181,310	1,181,310	0.000	0.60	12.44%	0.0646	18,193
20	284,638	0	1,043,626	1,043,626	0.000	0.60	12.44%	0.0574	16,343
21	287,506	0	885,948	885,948	0.000	0.60	12.44%	0.0511	14,681
22	290,403	0	705,756	705,756	0.000	0.60	12.44%	0.0454	13,188
23	293,330	0	500,222	500,222	0.000	0.60	12.44%	0.0404	11,848
24	296,286	0	266,164	266,164	0.000	0.60	12.44%	0.0359	10,643
25	299,275	0	0	0				0.0319	9,561
								SUM	106,688

The procedure used in Table 3 is the same dynamic procedure as in Table1, Panel B. The leverage ratio changes over time and thus the cost of equity does too. The 4th column from the left gives the present value of the equity at the beginning of a period, say *t*. This is calculated by taking the value in the previous period, *t - 1*, multiplying it by 1 + R_E(t - 1) and subtracting the equity cash flow at *t*. Given E(t), we can then calculate in succession V_L(t), L(t), β_E (t), and R_E (t). The valuation is done in a spreadsheet by searching for the equity value at time 0 that gives an ex cash flow equity value of 0 in the final period. Since the tax shield is implicitly part of the equity cash flow, the valuation in Table 2 gives the adjusted net present value. As seen, the APV of the equity net of the initial investment of 300,000 is 106,688.

The project described in Table 3 generates cash flows for 25 years. The equity cash flows presented in the table are calculated by Esty based on operating cash flows (see Esty, 1999).[5] For Esty's calculation the way the equity cash flows are derived is not important, given the debt profile. However, in Table 4 we show a set of operating cash flows which are consistent with the equity cash flows and debt schedule in Table 3.[6]

Table 4: Operating and equity cash flows

Year	Operating Cash Flow (FCFF)	Debt	Net Principal Repayment	Interest	Tax saving	Equity Cash Flow (FCFE)
0	-300,000	0	0	0	0	-300,000
1	-870,000	700,000	-700,000	0	0	-170,000
2	-812,349	1,300,000	-600,000	70,000	28,000	-254,349
3	274,446	1,275,000	25,000	130,000	52,000	171,446
4	276,990	1,250,000	25,000	127,500	51,000	175,490
5	267,058	1,225,000	25,000	125,000	50,000	167,058
6	283,401	1,175,000	50,000	122,500	49,000	159,901
7	273,643	1,125,000	50,000	117,500	47,000	153,143
8	290,161	1,050,000	75,000	112,500	45,000	147,661
9	293,080	975,000	75,000	105,000	42,000	155,080
10	283,523	900,000	75,000	97,500	39,000	150,023
11	300,243	800,000	100,000	90,000	36,000	146,243
12	290,864	700,000	100,000	80,000	32,000	142,864
13	307,761	575,000	125,000	70,000	28,000	140,761
14	298,560	450,000	125,000	57,500	23,000	139,060
15	315,636	300,000	150,000	45,000	18,000	138,636
16	312,114	150,000	150,000	30,000	12,000	144,114
17	371,953	0	150,000	15,000	6,000	212,953
18	278,987	0	0	0	0	278,987
19	281,798	0	0	0	0	281,798
20	284,638	0	0	0	0	284,638
21	287,506	0	0	0	0	287,506
22	290,403	0	0	0	0	290,403
23	293,330	0	0	0	0	293,330
24	296,286	0	0	0	0	296,286
25	299,275	0	0	0	0	299,275

[5] Table 3 is based on Exhibit 4 in Esty (1999).

[6] These are different to Esty's operating cash flows because he does not include the tax saving from debt in his definition of the equity cash flow and his calculation implicitly assumes that interest is paid in advance. We have constructed a set of operating cash flows that are consistent with the standard method of switching between operating free cash flow and equity cash flow.

Table 5 values the same equity cash flows as in Table 3 but using our formula (13) for $R_E(t)$. Here we have used Y = 10%, consistent with what is assumed in Esty's example. The result is an NPV of 264,608, almost twice as much as using the cost of equity formula (4). The reason for this dramatic increase in value (APV) is that equation (4) overstates the cost of equity. For example, in Year 2, Esty estimates R_E to be 18.3%, whereas our formula yields an R_E of 14.5% -- a difference of 3.8%. During the period the debt is outstanding the average difference in $R_E(t)$ between the two procedures is 2.1%, with the difference being larger at short horizons than at long horizons.

Table 5: Flows to equity valuation with R_E determined using Equation (13)

Year	Equity Cash Flows (ECF)	Total Debt (Book Value)	PV equity end period	Debt plus equity	D/E	RU	RE using Eqn. (13)	Disc'nt Factor	Present Value of ECF
0	-300,000	0	564,608	564,608	0.000	12.44%	11.24%	1.0000	-300,000
1	-170,000	700,000	798,043	1,498,043	0.877	12.44%	13.64%	0.8990	-152,829
2	-254,349	1,300,000	1,161,265	2,461,265	1.119	12.44%	14.52%	0.7911	-201,209
3	171,446	1,275,000	1,158,453	2,433,453	1.101	12.44%	14.52%	0.6908	118,429
4	175,490	1,250,000	1,151,150	2,401,150	1.086	12.44%	14.53%	0.6032	105,854
5	167,058	1,225,000	1,151,302	2,376,302	1.064	12.44%	14.52%	0.5267	87,987
6	159,901	1,175,000	1,158,591	2,333,591	1.014	12.44%	14.46%	0.4599	73,539
7	153,143	1,125,000	1,172,928	2,297,928	0.959	12.44%	14.38%	0.4018	61,535
8	147,661	1,050,000	1,193,925	2,243,925	0.879	12.44%	14.24%	0.3513	51,874
9	155,080	975,000	1,208,913	2,183,913	0.807	12.44%	14.12%	0.3075	47,687
10	150,023	900,000	1,229,609	2,129,609	0.732	12.44%	13.99%	0.2694	40,424
11	146,243	800,000	1,255,436	2,055,436	0.637	12.44%	13.81%	0.2364	34,568
12	142,864	700,000	1,286,008	1,986,008	0.544	12.44%	13.64%	0.2077	29,670
13	140,761	575,000	1,320,601	1,895,601	0.435	12.44%	13.41%	0.1828	25,726
14	139,060	450,000	1,358,660	1,808,660	0.331	12.44%	13.19%	0.1611	22,409
15	138,636	300,000	1,399,270	1,699,270	0.214	12.44%	12.94%	0.1424	19,737
16	144,114	150,000	1,436,158	1,586,158	0.104	12.44%	12.69%	0.1261	18,167
17	212,953	0	1,405,390	1,405,390	0.000	12.44%	12.44%	0.1119	23,823
18	278,987	0	1,301,234	1,301,234	0.000	12.44%	12.44%	0.0995	27,757
19	281,798	0	1,181,310	1,181,310	0.000	12.44%	12.44%	0.0885	24,935
20	284,638	0	1,043,626	1,043,626	0.000	12.44%	12.44%	0.0787	22,399
21	287,506	0	885,948	885,948	0.000	12.44%	12.44%	0.0700	20,122
22	290,403	0	705,756	705,756	0.000	12.44%	12.44%	0.0622	18,076
23	293,330	0	500,222	500,222	0.000	12.44%	12.44%	0.0554	16,238
24	296,286	0	266,164	266,164	0.000	12.44%	12.44%	0.0492	14,587
25	299,275	0	0	0	0.000	12.44%	0.00%	0.0438	13,104
								SUM	264,608

Our results differ from Estys' for two reasons. First, equations (3) and (4), used by Esty, assume that the debt has a beta of zero, even though the interest rate on the debt is 2% above the risk-free rate. This inflates the estimated equity beta because all the risk of the project is assumed to be carried by the equity even though some risk is in fact borne by the debt. Second, equation (3) assumes that the risk of the tax shield is equal to the risk of the firm even though the debt policy is fixed. Both effects overstate the equity discount rate and thereby undervalue the value of the project.

If the debt in Esty's example is indeed risk-free, yet pays a 2% premium over the risk-free rate, the implicit assumption is that a portion of the project's APV is given to creditors. That the debt is underpriced in this way is, of course, possible in a real-life situation. As argued by Esty, one of the advantages of the flows-to-equity approach is that it looks at the valuation from shareholders' perspective, without assuming that the debt is priced fairly. In contrast the standard approach where one first values the firm and then subtracts the face value of debt to get the value of equity assumes that the debt is fairly priced. Esty's method implicitly assumes that the entire 2% interest premium represents debt underpricing (or, equivalently, an excessive interest rate).

We can make Esty's approach consistent with fairly priced debt by using equation (1) rather than equation (4) to set the equity discount rate. This makes the discount rate depend on the debt yield, Y, rather than the riskless rate. The reason that it is consistent with fairly priced debt is that the standard definition of equity cash flow deducts the full debt yield as a cost, as we have noted above. Alternatively, using the CAPM an interest rate of 10% gives an implied debt beta of approximately 2/7.4, or 0.27.[7] Working through Esty's example with this value and the formula for the equity beta being the standard one, i.e.,

$$\beta_E(t) = \frac{\beta_U - L(t)\beta_D}{1 - L(t)}, \tag{33}$$

[7] This calculation ignores the possibility of default. The promised interest rate on the debt is not equivalent to its expected rate of return. The actual expected rate of return of the debt is lower. Thus, the correct debt beta is also lower. If we assume it is 0.2, for example, the APV is would be 252,103.

we get an APV of 227,565. This is still too low, as compared with the valuation based on our formula. The source of the undervaluation is still an equity discount rate that is too high. For example, R_E (2) is now 15.3% - lower than before, but still too high compared with our estimate in Table 3.

The reason for the remaining difference can be seen by comparing equation (1) with equation (13). Equation (1) assumes that $\alpha(t) = 0$, which is equivalent to assuming that PVTS has a risk equal to the risk of the operating free cash flow rather than the risk of the debt. Since the method implicitly overestimates the risk of PVTS it underestimates the value of equity.

Table 6: Sensitivity of equity value to assumptions

(Basic parameters T = 0.4, Y=10%, R_F = 8%).

INPUTS	**Esty R_E**		**Generalized CN**				
R_E formula	**(3)**	**(1)**	**(13)**	**(13)**	**(13)**	**(13)**	**(13)**
Fair yield, y	0.080	0.100	0.100	0.090	0.100	0.100	0.100
Recovery rate, ρ	0.000	0.000	0.000	0.000	0.000	0.413	0.413
Distress cost, ϕ	0.000	0.000	0.000	0.000	0.165	0.000	0.165
OUTPUTS							
T*	0.400	0.400	0.400	0.300	0.369	0.400	0.347
Adjusted yield, γ	0.080	0.100	0.100	0.090	0.100	0.1145	0.1145
Equity value	106,688	227,565	264,608	208,086	243,311	241,652	207,919
Average R_E	15.90%	14.18%	13.76%	14.49%	14.02%	14.01%	14.45%

In Table 6 we show the sensitivity of the equity value to different assumptions. The first column is Esty's calculation, from Table 3 above. The second column is the Esty method with equity discount rate calculated using the debt yield rather than the riskless rate. As discussed above this makes a large difference and is the most important element of the calculation to treat consistently in this particular example. The other columns are calculated using a cost of equity given by equation (13) above. The first of these is the base case analyzed in Table 3. The next column shows the effect of assuming that half the debt spread is excessive, so the fair debt spread is 9%

rather than 10%. The final three columns show the effect of distress costs and debt recovery assumptions. The parameter values are from Almeida and Philippon (2007).

Each of the changes in assumption has a material effect on the valuation. In combination they give a wide range of values. The net effect of the correct assumptions in any particular situation will vary and cannot be predicted without using a consistent treatment of the equity discount rate.

6. Concluding Remarks

We have developed formulas for tax adjusted discount rates in highly levered transactions. Our formulas are best interpreted as being suitable for project finance or other structures where the amount of debt follows a predictable pattern, conditional on solvency. Our analysis is concerned with developing a consistent method for using the flows-to-equity method. This is nontrivial when the leverage ratio changes over time. The appropriate discount rate for equity flows varies over time with the duration of the debt. Using an example from Esty (1999), we show that the values we get with our approach for equity values and discount rates can be substantially different from those obtained from standard approaches.

We have extended the basic framework to allow for debt which has a higher than fair interest rate, distress costs, and recovery in default. The formulas in this general scenario parallel those in the simpler case, but involve modified tax and interest rates. These modifications depend on the extent to which yield spread on the debt is unfair, the level of distress costs, and recovery rates.

Although we focus on the flows-to-equity method, there are alternatives which can be used to value highly leveraged transactions. The WACC and capital cash flow approaches can be used to incorporate the tax benefit of debt directly in the DCF calculation (see Cooper and Nyborg (2007) for a review). Alternatively, adjusted present value (APV) can be used to separately calculate the tax benefit of the debt (Arzac 1996). We have shown the links between the three approaches and how all the features which the flows-to-equity method is designed to capture can also be included in the APV approach. In practice, implementing the flows-to-equity approach correctly is more complicated than using APV. Since the consistent version of the flows-to-equity approach is derived from the APV formula we believe that it is an open question as to whether the flows-to-equity method can achieve anything that APV cannot.

The analysis that we have discussed assumes that the debt will be run down to zero as the project matures. An obvious extension would be to run the leverage down to a target level and then assume that the leverage ratio stays constant from that point onwards. This could be accommodated in our analysis by switching the debt policy to a standard Miles-Ezzell policy from the time the leverage ratio drops to the target level. Alternatively, one could assume that the debt level will be increased if the investment is successful. In that case the valuation should include the option value of increasing the tax shield in those circumstances.

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Appendix 1: Proof of the relationship between $R_E(t)$ and R_U

From equations (5) - (7) in the main text:

$$(E(t)+D(t)-PVTS(t))(1+R_U)=E(t+1)+D(t+1)-PVTS(t+1)+C(t+1) \quad \text{(A1.1)}$$

From equations (8) and (9):

$$E(t)(1+R_E(t))=E(t+1)+C(t+1)+D(t+1)-D(t)[1+Y(1-T)] \quad \text{(A1.2)}$$

From equation (10):

$$(PVTS(t))(1+Y)=D(t)TY+PVTS(t+1) \quad \text{(A1.3)}$$

Taking equation (A1.1) plus (A1.3) minus (A1.2) gives:

$$E(t)R_U+D(t)R_U-PVTS(t)R_U-E(t)R_E(t)+PVTS(t)Y=D(t)Y \quad \text{(A1.4)}$$

Rearranging (A1.4) gives equation (11) of the main text.

Appendix 2: The relationship between $\alpha(t)$ and duration.

From (14) and (15):

$$D(t)DUR(t)=\sum_{i=1}^{\infty} iD(t+i-1)/(1+Y)^{i-1}-\sum_{i=1}^{\infty} iD(t+i)/(1+Y)^{i}$$

$$=\sum_{i=0}^{\infty}(i+1)D(t+i)/(1+Y)^{i}-\sum_{i=1}^{\infty} iD(t+1)/(1+Y)^{i}$$

$$=D(t)+\sum_{i=1}^{\infty} D(t+i)/(1+Y)^{i}$$

$$=\sum_{i=0}^{\infty} D(t+i)/(1+Y)^{i}$$

$$=PVTS(t)(1+Y)/YT \quad \text{(A2.1)}$$

where the last equality follows from (10). Hence:

$$DUR(t)=\left[PVTS(t)/TD(t)\right](1+Y)/Y=\alpha(t)(1+Y)/Y \quad \text{(A2.2)}$$

Appendix 3: Proof of the relationship between $R_E(t)$ and R_U in the general case.

The proof follows along the same lines as in Appendix 1 with PVTS replaced by PVFS. From equations (5) - (7) in the main text:

$$(E(t)+D(t)-PVFS(t))(1+R_U)=E(t+1)+D(t+1)-PVFS(t+1)+C(t+1) \quad \text{(A4.1)}$$

From equations (8) and (9):

$$E(t)(1+R_E(t))=E(t+1)+C(t+1)+D(t+1)-D(t)[1+Y(1-T)] \quad \text{(A4.2)}$$

From equation (24):

$$(PVFS(t))(1+\gamma)=D(t)T^{*}Y+PVFS(t+1) \quad \text{(A4.3)}$$

Taking equation (A4.1) plus (A4.3) minus (A4.2) gives:

$$E(t)R_U+D(t)R_U-PVFS(t)R_U-E(t)R_E(t)+PVFS(t)\gamma=D(t)Y-D(t)YT+D(t)YT^{*} \quad \text{(A4.4)}$$

Rearranging (A4.4) gives equation (25) of the main text, with $\alpha^{*}(t)$ given by (26).

Appendix 4: The relationship between $\alpha^{*}(t)$ and duration.

Making the same substitutions as in Appendix 3 allows us to follow the same line of argument as in Appendix 2 result given in the text. Note that we now use (24) instead of (10) at the final step.

Appendix 5: Proof of the equivalence between the Esty and the capital cash flow (CCF) methods when debt is riskless.

This follows along the lines of Appendix 1, setting Y=R_F. Under the CCF assumption the tax saving has the same risk as the operating free cash flow, so (A1.3) becomes:

$$(PVTS(t))(1+R_U)=D(t)TR_F+PVTS(t+1) \quad \text{(A1.3')}$$

Taking equation (A1.1) plus (A1.3') minus (A1.2) gives:

$$E(t)R_U+D(t)R_U-E(t)R_E(t)=D(t)R_F \quad \text{(A1.4')}$$

Rearranging (A1.4') gives equation (4) of the main text.