# The bond-stock mix: a new insight

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**Abstract**

This paper studies the impact of consumption on the bond-stock mix. Contrary to the existing literature, we allow for an investor who is more risk averse towards consumption than towards wealth. We obtain, compared to previous results, different behaviors for the speculative and hedging demands of stocks and of bonds in terms of investor's horizon, risk aversions and wealth. Our opportunity set is composed of two predictive random variables, the risk free rate and the market price of risk. As a consequence, we can also assess the impact on strategies of two types of variables, that is, a time preference related variable and a market price of risk linked variable.

*JEL Classification:* G13

*Keywords:* Bond-stock mix; consumption; risk aversion; investment horizon; wealth.

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## 1. Introduction

Canner *et al.* (1997) show that market investment practices contradict the blind mean variance framework. Indeed, the relative investment in stocks and bonds, depend, contrary to the results of the static framework, on the investor's appetite for risk. In addition, Brennan et al. (1997) show the need for funds managers to consider long investment horizon. Since bonds and stocks returns are predictable, Brennan et al. (1997) explain that the choice of a dynamic opportunity set for the bond-stock allocation is relevant. Their empirical illustration proves that a dynamic opportunity set modifies investment strategies subsequently and greatly improves portfolio returns. These two papers triggered various studies on the bond-stock mix in light of a dynamic opportunity set. Some articles focus on which extent a dynamic opportunity set can reconcile theoretical results with popular advice (Brennan and Xia, 2000; Munk et al., 2004; Lioui 2007). Another strand of the literature analyzes the bond-stock mix issue as part of a life-cycle investment strategy (Mauer et al., 2008; Munk and Sørensen, 2010). However, this investment life-cycle studies slightly depart from the original problem of Brennan et al. (1997) because they must consider labor income and retirement endowment impacts. Finally, some papers tackle the bond-stock issue with a dynamic opportunity set only linked to time preference, (Sørensen, 1999, Bajeux-Besnainou et al., 2003). Although this restricted setting is relevant to disentangle some of the motives behind the bond-stock choices, it tends to arbitrarily minor the impact of the equity market.

In all of the above cited articles the impact of consumption is not underlined. However, Wachter (2002) in the case of an investment in the equity market only demonstrates that consumption significantly modifies the horizon of the investor. Indeed, the life-cycle setting makes it hard to disentangle the role of consumption. Other frameworks that include consumption do not allow for a thorough study of the consumption effect, (Campbell and Viceira, 2001; Munk and Sørensen, 2004; Sangvinatsos and Wachter, 2005). Campbell and Viceira (2001) consider an infinitely lived investor and then do not study the impact of the investment horizon. Munk and Sørensen (2004) and Sangvinatsos and Wachter (2005) only include term structure factors in the opportunity set. Moreover, Sangvinatsos and Wachter (2005) study the issue of consuming and saving for terminal wealth separately. Finally, Munk and Sørensen (2004) and

Sangvinatsos and Wachter (2005) assume that investors have identical risk aversion towards consumption and wealth, which contradicts recent empirical evidence (Meyer and Meyer, 2005).

We assume that our dynamic opportunity set is composed of predictive factors related to both the bond and the equity markets. To fully assess the impact of consumption on the bond stock mix, we derive our allocation results in such a way that we can progressively analyze three embedded frameworks: i) a setting in which consumption or terminal wealth only is considered; ii) a setting in which both consumption and terminal wealth are acknowledged; iii) a setting in which consumption related risk aversion is greater than the wealth related one Meyer and Meyer (2005). To the best of our knowledge, the impact on allocation of that full description of the role of consumption has not been much highlighted in the literature.

Our framework demonstrates that taking into account a dynamic opportunity set and a realistic description of consumption risk aversion results in a bond-stock mix very different from a setting in which one objective is considered or consumption and wealth risk aversions are equal. We find that both the speculative and hedging components of the allocation are impacted by investor's wealth and horizon. Moreover, we show that this impact is tied to connections between the speculative and hedging motives, which, to the best of our knowledge, have not been previously underlined. Finally, we provide an illustration of the effect of investor's horizon, wealth and risk aversions.

The remainder of the paper is organized as follows. Section 2 presents our setting as well as the optimal solutions. Section 3 provides a numerical analysis. Section 4 concludes.

## 2. Setting and optimal solution

Our setting is restricted, for ease of exposition, to two state variables. We suppose that the dynamics of bond prices are governed by the short rate. Regarding the equity related opportunity set, we retain a mean-reverting market price of risk perfectly negatively correlated with the stock price.[1] Finally, to check the sensitivity of our results to the choice of the opportunity set, we envisage two embedded setting: one that includes only time preference; another one that adds the variations of the market price of risk.

[1] Wachter (2002) explains that, in that kind of modeling, the equity market price of risk can be understood as a stock dividend yield.

. We sum up our framework in the following assumptions:

**Assumption 1.** Following Sørensen (1999), Munk *et al*. (2004) and Munk and Sørensen (2004, 2010), we assume that the instantaneous short rate follows a mean-reverting process which dynamics is given by (Vasicek, 1977):

$$dr_t = \alpha[\theta - r_t]dt - \sigma_r dz_P(t), \qquad (1)$$

where $dz_P(t)$ is a standard Wiener process defined on a filtered probability space ($\Omega, F = \{F_t, 0 \leq t < \infty\}, P$) satisfying the usual hypotheses. The positive constants $\alpha, \theta$, and $\sigma_r$ stand for the speed of adjustment, the long term mean and the volatility of the short rate, respectively.

**Assumption 2.** The market price of this risk, $\lambda_P$, is constant (Sørensen, 1999; Campbell and Viceira, 2001; Munk and Sørensen, 2004). Therefore, the price of a zero-coupon bond maturing at time $T_P$ is governed by the following stochastic differential equation:

$$\frac{dP(t,T_P)}{P(t,T_P)} = [r_t + \sigma_P(t,T_P)\lambda_P]dt + \sigma_P(t,T_P)dz_P(t), \qquad (2)$$

where the zero-coupon bond volatility $\sigma_P(t,T_P)$ is deterministic and has an exponential form, i.e. $\sigma_P(t,T_P) \equiv \sigma_r D_\alpha(t,T_P)$, with $D_\alpha(t,T_P) = \dfrac{1-\exp(-\alpha(T_P - t))}{\alpha}$.

**Assumption 3.** The price of the stock index obeys to the following dynamics:

$$\frac{dS^{(i)}(t)}{S^{(i)}(t)} = \left[r_t + \sigma_S \lambda_S^{(i)}(t)\right]dt + \sigma_S dz_S(t),\ i \in \{a,b\}, \qquad (3)$$

where $\sigma_S$ is a positive constant and denotes the volatility of the stock index, and $dz_S(t)$ is a standard Wiener process defined on ($\Omega, F = \{F_t, 0 \leq t < \infty\}, P$). We further assume that the two Wiener processes are correlated, that is $\rho_{SP}dt \equiv dz_P(t)dz_S(t)$. We consider two cases (a) and (b). In case (a), the equity market price of risk is assumed to be constant, i.e. $\lambda_S^{(a)}(t) \equiv \lambda_S$. This case is in line with Brennan and Xia (2000) and Munk and Sørensen (2004, 2010). In case (b), the stock index market price of risk is supposed to follow a mean-reverting process (Wachter, 2002; Munk *et al.,* 2004), perfectly negatively correlated with the stock index price:

$$d\lambda_S^{(b)}(t) = \kappa\left[\overline{\lambda}_S - \lambda_S^{(b)}(t)\right]dt - \sigma_\lambda dz_S(t)\,, \tag{4}$$

where the constants $\kappa, \overline{\lambda}_S$, and $\sigma_\lambda$ denote the speed of adjustment, the long term mean and the volatility of the market price of risk, respectively.

**Assumption 4.** Investors can freely invest in two risky assets, namely the zero-coupon bond and the stock index, and in the instantaneously riskless asset.

In order to find optimal proportions invested in the zero-coupon bond and in the stock index,[2] which we denote, for each case (a) and (b), by $\pi_S^{(i)}(t), \pi_P^{(i)}(t)\ i \in \{a,b\}$, respectively, an investor having an investment horizon $T_I$ solves the following programs in cases (a) and (b):

$$J_t^{(i)} \equiv \max_{C_s^{(i)}, t\le s\le T_I, W_{T_I}^{(i)}} E\left[\int_t^{T_I} U_\beta\left(C_s^{(i)}\right)ds + U_\gamma\left(W_{T_I}^{(i)}\right)\middle| F_t\right], i \in \{a,b\} \tag{5}$$

$$s.t.\ E\left[\int_t^{T_I} M_s^{(i)}C_s^{(i)}ds + M_{T_I}^{(i)}W_{T_I}^{(i)}\middle| F_t\right] = W_t^{(i)}, i \in \{a,b\} \tag{6}$$

The satisfaction stemming from consumption, $C$, is modeled by the utility function $U_\beta(C) \equiv \omega\dfrac{C^{1-\beta}}{1-\beta}$, where[3] $\beta \ge 1$ represents the investor's constant relative risk aversion towards consumption. Similarly, $U_\gamma(W) \equiv [1-\omega]\dfrac{W^{1-\gamma}}{1-\gamma}$ is the utility function for describing the investor's satisfaction arising from (terminal) wealth, $W$, with $\gamma \ge 1$ denotes the investor's constant relative risk aversion towards wealth. $\omega \in [0,1]$ is a weighting factor. For ease of exposition, our analysis conforms to the empirical case $\gamma \le \beta$, Meyer and Meyer (2005).

The optimization program (5, 6) is cast into the martingale approach (Cox and Huang, 1989, 1991; Karatzas *et al.* 1987). $M_s^{(i)}, i \in \{a,b\}$ stands for the pricing kernel, which is a function of the short rate and the market prices of risk but need not to be further specified for our study.

As a first step to study optimal risky proportions, we derive the relative risk aversion of our investor. Since she is characterized by two different risk aversions, $\beta$ and $\gamma$, we define her relative risk

[2] Optimal proportion invested in the riskless asset is equal to $1 - \pi_S^{(i)}(t) - \pi_P^{(i)}(t), i \in \{a,b\}$.

[3] In line with most of the literature on asset allocation, we restrict our study for an investor less risk averse than the Bernoulli investor.s

aversion, $RRA^{(i)}(t), i \in \{a,b\}$, using the indirect utility function (Merton, 1969, 1971, 1973). It represents the total satisfaction obtained by the investor at optimum:

$$RRA^{(i)}(t) \equiv \frac{-W^{(i)}(t) J_{WW}^{(i)}(t)}{J_W^{(i)}(t)}, i \in \{a,b\}, \tag{7}$$

where $J_W^{(i)}(t)$ and $J_{WW}^{(i)}(t)$ stand for the first and second partial derivatives of $J^{(i)}(t)$ with respect to $W^{(i)}(t)$, respectively.

Relying on the results of Cox and Huang (1989, 1991) and Karatzas *et al.* (1987), Eq. (7) writes explicitly as follows.

**Proposition 1:**

*The relative risk aversion, $RRA(t)$, is given by:*

$$\frac{1}{RRA^{(i)}(t)} = \frac{1}{\gamma} + \left[\frac{1}{\beta} - \frac{1}{\gamma}\right] \pi_C^{(i)}(t), \tag{8}$$

*where $\pi_C^{(i)}(t)$ represents the share of current wealth destined to satisfy future consumption. It can be expressed in terms of optimal consumption, as well as the dynamic of the market as follows:*

$$\pi_C^{(i)}(t) = \left[1 + \left[\frac{1-\omega}{\omega}\right]^{\frac{1}{\gamma}} \frac{q_\gamma^{(i)}(t,T_I)}{Q_\beta^{(i)}(t,T_I)} \left[C_t^{(i)}\right]^{\left(\frac{\beta-\gamma}{\gamma}\right)}\right]^{-1}, i \in \{a,b\}, \tag{9}$$

*where $q_j^{(a)}(t,T_I) \equiv q^{(a)}(j,T_I,r(t))$ and $q_j^{(b)}(t,T_I) \equiv q^{(b)}(j,T_I,r(t),\lambda_S(t))$, $j \in \{\beta,\gamma\}$ are functions given in the appendix and $Q_\beta^{(i)}(t)$ is defined such that : $Q_\beta^{(i)}(t) \equiv \int_t^{T_I} q_\beta^{(i)}(t,u)du$, $i \in \{a,b\}$. Moreover, optimal consumption can be implicitly defined from the investor's wealth:*

$$W_t^{(i)} = C_t^{(i)} Q_\beta^{(i)}(t,T_I) + \left[\frac{1-\omega}{\omega}\right]^{\frac{1}{\gamma}} \left[C_t^{(i)}\right]^{\frac{\beta}{\gamma}} q_\gamma^{(i)}(t,T_I), i \in \{a,b\} \tag{10}$$

**Proof**. *available from the authors upon request.*

Proposition 1 deserves the following comments. First, when an investor considers only the satisfaction from terminal wealth, $\pi_C^{(i)}(t)$ is nil by definition and the relative risk aversion simply equals $\gamma$ and is independent of market conditions (Eq. (8)). Similarly, when $\beta = \gamma$ Eq (8) shows that the investor's relative risk aversion is also given by $\gamma$ . Second, since $q_{\gamma}^{(i)}(t,T_I), i \in \{a,b\}$ are positive functions independent of consumption and wealth (see Eqs (A.1) and (B.1) in the appendix), Eq. (10) demonstrates a strictly increasing relationship between consumption and wealth. Moreover, given that $\gamma < \beta$ , Eq. (9) shows that $\pi_C^{(i)}(t)$ is a decreasing function of consumption and therefore a decreasing function of wealth. As a result, Eq. (8) indicates that total relative risk aversion is a decreasing function of wealth. In the case $\gamma = \beta$ , we point out that $\pi_C^{(i)}(t)$ does no longer depend neither on consumption nor on terminal wealth. Finally, for a sufficient large amount of wealth, equations (9,10) demonstrate that the investor behaves as if she only cares about terminal wealth, since her saving for consumption, Eq. (9), is equal to zero. This results is emphasized in proposition two, which shows that investor's wealth influences optimal allocation in assets only through the proportion $\pi_C^{(i)}(t), i \in \{a,b\}$.

Having studied the relative risk aversion, we are now equipped to state our second proposition, which pertains to optimal proportions invested in the two risky assets.

**Proposition 2:**

*Optimal risky proportions are given in cases (a) and (b) by:*

$$\pi_S^{(a)}(t) = \frac{1}{RRA^{(a)}(t)} \frac{\lambda_S^{(a)} - \rho_{SP}\lambda_P}{\sigma_S\left[1-\rho_{SP}^2\right]}, \tag{11}$$

$$\pi_P^{(a)}(t) = \frac{1}{RRA^{(a)}(t)} \frac{\lambda_P - \rho_{SP}\lambda_S^{(a)}}{\sigma_P(t;T_P)\left[1-\rho_{SP}^2\right]} + \pi_C^{(a)}(t)\left[1-\frac{1}{\beta}\right]\frac{\sigma_P\left(t,T_{IP}^{(a)}\right)}{\sigma_P(t,T_P)} + \left[1-\pi_C^{(a)}(t)\right]\left[1-\frac{1}{\gamma}\right]\frac{\sigma_P(t;T_I)}{\sigma_P(t;T_P)} \tag{12}$$

*and,*

$$\begin{aligned}\pi_S^{(b)}(t) = {} & \frac{1}{RRA^{(b)}(t)} \frac{\lambda_S^{(b)}(t) - \rho_{SP}\lambda_P}{\sigma_S\left[1-\rho_{SP}^2\right]} + \pi_C^{(b)}(t)\frac{-\sigma_\lambda}{\sigma_S}\left[E_\beta\left(t,T_{IS}^{(b)}\right) + F_\beta\left(t,T_{IS}^{(b)}\right)\lambda_S^{(b)}(t)\right] \\ & + \left[1-\pi_C^{(b)}(t)\right]\frac{-\sigma_\lambda}{\sigma_S}\left[E_\gamma(t,T_I) + F_\gamma(t,T_I)\lambda_S^{(b)}(t)\right]\end{aligned} \tag{13}$$

$$\pi_P^{(b)}(t) = \frac{1}{RRA^{(b)}(t)} \frac{\lambda_P - \rho_{SP}\lambda_S^{(b)}(t)}{\sigma_P(t;T_P)\left[1-\rho_{SP}^2\right]} + \pi_C^{(b)}(t)\left[1-\frac{1}{\beta}\right]\frac{\sigma_P\left(t;T_{IP}^{(b)}\right)}{\sigma_P(t;T_P)} + \left[1-\pi_C^{(b)}(t)\right]\left[1-\frac{1}{\gamma}\right]\frac{\sigma_P(t;T_I)}{\sigma_P(t;T_P)} \tag{14}$$

*respectively.* $E_j(t,T_I) \equiv E(j,t,T_I), j \in \{\beta,\gamma\}$ *and* $F_j(t,T_I) \equiv F(j,t,T_I), j \in \{\beta,\gamma\}$ *are deterministic functions of time, and* $T_{IS}^{(b)}, T_{IP}^{(i)}, i \in \{a,b\}$ *are temporal horizons such that:* $T_{IS}^{(b)}, T_{IP}^{(i)} \le T_I$ *. All of these quantities are given in the appendix.*

**Proof**. *available from the authors upon request*.

The demand is divided into a mean-variance component, first term of Eq. (11-14), and a Merton-Breeden hedging demand (see Merton, 1971, 1973; Breeden, 1979), which is the sum of the remaining terms in Eq. (12-14). For the rest of the text, we denote "Merton-Breeden" by their initials M-B for ease of exposition.

First, as noted by Brennan and Xia (2000), in case (a), there is no M-B hedging demand for equity. This should come as no surprise since an individual can invest in a zero-coupon bond perfectly correlated with the interest rate, which is the only random variable of the opportunity set. Nevertheless, our setting differs from the one of Brennan and Xia (2000) since they solely consider terminal wealth. Indeed, Proposition 1 shows that $RRA^{(a)}(t)$ and $\pi_C^{(a)}(t)$ do depend on the random risk free rate and the investor's horizon, wealth and relative risk aversions.

Second, Eq. (13) shows that a M-B term arises for equity when its market price of risk is a random process of the opportunity set. Moreover, this M-B hedging demand pertains only to the stock's demand. Once again, this should come as no surprise since equity price and its market price of risk are perfectly (negatively) correlated in case (b). Nevertheless, the impacts on $\pi_C^{(i)}(t)$ of stochastic interest rates and stochastic market price of risk can not, in our empirically relevant setting, be disentangled. Indeed, as shown by Proposition 1, $RRA^{(b)}(t)$ and $\pi_C^{(b)}(t)$ are both function of the risk free rate and the market price of risk.

$\pi_C^{(i)}(t)$ is also useful to retrieve the M-B hedging demands linked to consumption and terminal wealth, respectively. Indeed, if we set $\pi_C^{(i)}(t) \equiv 1$ ( $\pi_C^{(i)}(t) \equiv 0$ ), the third (second) term of Eq. (12-14) vanishes out and the second (third) term can be identified as the consumption (terminal wealth) M-B hedging demand. This property shows that $T_{IS}^{(b)}, T_{IP}^{(b)}$ are temporal horizons linked to consumption as highlighted by Wachter (2002). Finally Eqs (B1, B9) in appendix B prove that these consumption related horizons are functions of both the market price of risk and the risk free rate. As a consequence, taking into account consumption further intricate the impact of the opportunity set, even when predictive variables are perfectly correlated with investment instruments.

## 3. Numerical illustration

We devote this section to various numerical analyses of our theoretical results. We consider different values for the parameters of risk aversions towards wealth and towards consumption, that is, $\gamma = 1,3,6$ and $\beta = 1,3,5,6,7,8,9,10,12$ with $\beta \geq \gamma$. Since one major aspect of this paper is to outline the impact of investor's horizon, we perform our numerical investigation for different investment horizons equal to 1, 5, 10 and 20 years, and for an initial wealth[4] of 100.

The base case parameters used in this section are provided in Table (1). The interest rate parameters are similar to those in Munk *et al.* (2004). The parameters of the stock index and its market price of risk are inferred from Wachter (2002). The correlation coefficient is taken from Brennan and Xia (2000). To focus on the investment horizon, the maturity of the bond maturity is set to 20 years and the current value of the risk free rate and the equity market price of risk are set equal to their long term means,[5] that is, $r(t)$=3.69% and $\lambda_S^{(a)} = \lambda_S^{(b)}(t) = 27.3\%$.

**Table 1: Base case parameters.** This table reports the base case parameters (in %) used for the different numerical illustrations.

| $\omega$ | $\sigma_S$ | $\alpha$ | $\theta$ | $\sigma_r$ | $\lambda_P$ | $\overline{\lambda}_S$ | $\rho_{SP}$ | $\kappa$ | $\sigma_\lambda$ |
|---|---|---|---|---|---|---|---|---|---|
| 50 | 15.1 | 3.95 | 3.69 | 2.237 | 27.47 | 27.3 | 23 | 27.1 | 6.55 |

Table (2) reports values of $\pi_C^{(i)}(t), i \in \{a,b\}$ for different levels of risk aversion and for different investment horizons. First, $\pi_C^{(i)}(t)$ is a decreasing function of $\beta$. This feature which may seem puzzling at first sight is actually due to the fact that $\beta$ represents the fear of the movements of consumption and not the fear of not consuming. In that sense, the more averse towards consumption the investor is, the less wealth she puts aside to satisfy future consumption because she optimally implements investment strategies that ensure stable flows of income used for intermediate consumption.

[4] We also investigate in the sequel the impact of wealth and demonstrate that it is significant for low levels of wealth.
[5] Our analysis can equally be achieved for other market conditions. However, we limit our presentation to the long-term mean market conditions for ease of exposition.

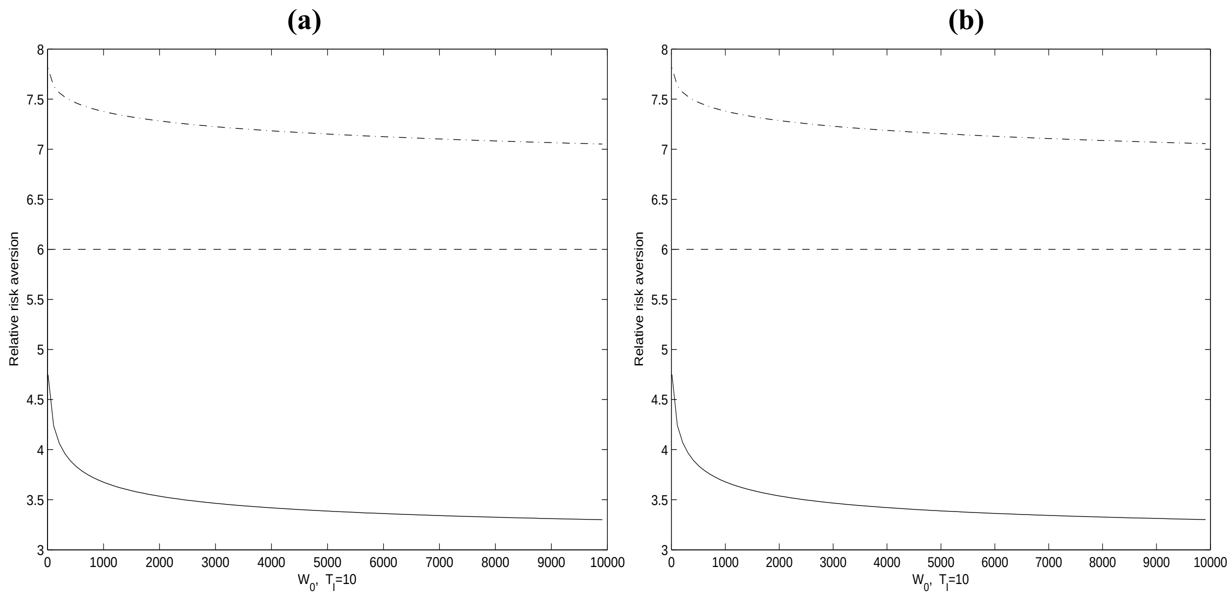
Vice versa, $\pi_C^{(i)}(t)$ is an increasing function of $\gamma$. Finally, $\pi_C^{(i)}(t)$ is an increasing function of the horizon: a long-term investor saves more money for future consumption.

**Table 2: $\pi_C^{(i)}(t)$.** This table reports the proportion of wealth to satisfy future consumption. The base case parameters are given in Table (1).

| | | Case (a) | | | | Case (b) | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\beta$ | T_I | | | | T_I | | | |
| | | **1** | **5** | **10** | **20** | **1** | **5** | **10** | **20** |
| 1 | 1 | 0.50 | 0.83 | 0.91 | 0.95 | 0.50 | 0.83 | 0.91 | 0.95 |
| | 3 | 0.04 | 0.19 | 0.33 | 0.48 | 0.04 | 0.19 | 0.33 | 0.47 |
| | 5 | 0.02 | 0.11 | 0.19 | 0.29 | 0.02 | 0.11 | 0.19 | 0.29 |
| | 7 | 0.02 | 0.09 | 0.15 | 0.23 | 0.02 | 0.09 | 0.15 | 0.22 |
| 3 | 3 | 0.51 | 0.85 | 0.93 | 0.97 | 0.51 | 0.85 | 0.93 | 0.97 |
| | 5 | 0.14 | 0.52 | 0.74 | 0.90 | 0.14 | 0.52 | 0.74 | 0.90 |
| | 7 | 0.07 | 0.30 | 0.51 | 0.75 | 0.07 | 0.30 | 0.51 | 0.75 |
| | 9 | 0.05 | 0.21 | 0.37 | 0.59 | 0.05 | 0.21 | 0.37 | 0.59 |
| 6 | 6 | 0.51 | 0.85 | 0.93 | 0.98 | 0.51 | 0.85 | 0.93 | 0.98 |
| | 8 | 0.26 | 0.69 | 0.86 | 0.95 | 0.26 | 0.70 | 0.86 | 0.95 |
| | 10 | 0.14 | 0.52 | 0.75 | 0.91 | 0.14 | 0.52 | 0.75 | 0.91 |
| | 12 | 0.10 | 0.39 | 0.63 | 0.85 | 0.10 | 0.39 | 0.63 | 0.85 |

Figures (1a) and (1b) exhibit a standard result from proposition 1. The relative risk aversion is a decreasing function of wealth when an investor is more risk averse towards consumption and is constant otherwise ($\beta = \gamma$).

**Figure 1. Relative risk aversion.** The figure displays the relative risk aversion in function of wealth. The base case parameters are given in Table (1), $\beta = 8, \gamma = 6$ dashed-dotted line; $\beta = \gamma = 6$ dashed line; $\beta = 5, \gamma = 3$ plain line;



Moreover, we can see that this decreasing pattern arises especially for low level of wealth. This fact justifies our choice of an amount of initial wealth equal to one hundred when we investigate the impact of investor's horizon and risk aversions.

Table (3) reports values for the relative risk aversion in both cases (a) and (b). First, the relative risk aversion is an increasing function of the investment horizon. Indeed, the longer the horizon, the more movements of state variables investors have to consider when making their decisions. This remarkable result will have important consequences on assets demand that will be highlighted in Tables (4) and (5). Second, when we take into account an additional risk in the opportunity set, case (b), the relative risk aversion tends to increase, particularly for high risk aversions and long investment horizons. This effect is not substantial, though.

**Table 3: Relative risk aversion.** The table reports values for the risk aversion parameter in both cases (a) and (b). The base case parameters are given in Table (1).

| | | Case (a) | | | | Case (b) | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\beta$ | T_I | | | | T_I | | | |
| | | 1 | 5 | 10 | 20 | 1 | 5 | 10 | 20 |
| 1 | 1 | 1,00 | 1,00 | 1,00 | 1,00 | 1,00 | 1,00 | 1,00 | 1,00 |
| | 3 | 1,03 | 1,15 | 1,28 | 1,47 | 1,03 | 1,15 | 1,28 | 1,46 |
| | 5 | 1,02 | 1,10 | 1,18 | 1,30 | 1,02 | 1,10 | 1,18 | 1,30 |
| | 7 | 1,02 | 1,08 | 1,15 | 1,24 | 1,02 | 1,08 | 1,15 | 1,24 |
| 3 | 3 | 3,00 | 3,00 | 3,00 | 3,00 | 3,00 | 3,00 | 3,00 | 3,00 |
| | 5 | 3,18 | 3,79 | 4,26 | 4,69 | 3,18 | 3,79 | 4,27 | 4,69 |
| | 7 | 3,12 | 3,63 | 4,24 | 5,24 | 3,12 | 3,63 | 4,24 | 5,25 |
| | 9 | 3,09 | 3,48 | 3,97 | 4,92 | 3,09 | 3,48 | 3,97 | 4,93 |
| 6 | 6 | 6,00 | 6,00 | 6,00 | 6,00 | 6,00 | 6,00 | 6,00 | 6,00 |
| | 8 | 6,41 | 7,26 | 7,64 | 7,87 | 6,41 | 7,26 | 7,64 | 7,88 |
| | 10 | 6,37 | 7,59 | 8,57 | 9,44 | 6,37 | 7,59 | 8,57 | 9,45 |
| | 12 | 6,30 | 7,47 | 8,74 | 10,41 | 6,30 | 7,47 | 8,75 | 10,43 |

Finally, for a given $\gamma$, the relative risk aversion increases then decreases with $\beta$. This humped shape is accentuated with the investment horizon. Nevertheless, the hump disappears for high risk aversion towards wealth ($\gamma = 6$) and long investment horizons ($T_I = 10,20$). This feature deserves the following comments. For any value of risk aversion towards consumption higher than the risk aversion towards wealth, the relative risk aversion increases with $\beta$. This stems simply from Eq. (8). Nevertheless, increasing $\beta$ also yields to a decrease of $\pi_C^{(i)}(t)$. The higher the $\beta$, the larger is the decreasing impact of $\pi_C^{(i)}(t)$ on the relative risk aversion. Hence, the hump is obtained. As shown in Table (2), the proportion of wealth to meet future consumption is relatively high for high wealth related risk aversion and long investment horizon. This feature lessens the decreasing impact of $\pi_C^{(i)}(t)$ and therefore the hump does no longer exist.

Table (4) reports the total allocation for stocks and bonds as well as the bond-stock ratio in both cases (a) and (b). As risk aversion towards wealth increases, the demand in bonds and in stocks decreases and the ratio increases. The more risk averse towards wealth an investor, the larger is her cash allocation. Besides, the bond demand increases with investment horizons.[6] This is due to the increasing hedging demand of long-term investors (Wachter, 2002; Brennan and Xia, 2000). Indeed, Table (5b) separates the total demand in a speculative and an M-B hedging components. We clearly notice that the bond hedging component increases along with the investor's horizon.

Moreover, the stock allocation decreases with the investment horizon. In case (a) where the stock demand is solely affected by the mean-variance component (see Table (5a)), the longer the investment horizon, the fewer speculative demand an investor has. Indeed, as shown by Eq. (11) horizon influences this demand through the relative risk aversion, which is increasing in investor's horizon (see Table (2)). In case (b) where a hedging component arises, the stock allocation still decreases with investment horizon since the additional hedging demand does not offset the sharp decrease in the speculative demand (Table 5a).

The stock and bond allocation exhibit a reverse humped shape with respect to the risk aversion towards consumption. Nevertheless, for long investment horizons (10 and 20 years), the stock demand is monotonic decreasing in $\beta$ and the bond demand is increasing in $\beta$ for large wealth related risk aversion ($\gamma = 6$). This reverse hump is actually explained by the results from Table (3).

For high degrees of risk aversions the time-preference hedging demand becomes important relative to the bond speculative demand (Table 5b). The increasing pattern in consumption related risk aversion is then

[6] However, for $\gamma = 1$, the bond allocation, for different values of $\beta > 1$, exhibits a decreasing then increasing pattern as investment horizon increases. This result is actually due to a small increase of the hedging component (Table 5b) that does not offset the decrease in the speculative demand. The discussion on the shape of the relative risk aversion, following Table 2, provides a convincing explanation.

explained by the increasing behavior of this M-B component as a function of risk aversion (Lioui and Poncet, 2001; Munk and Sørensen, 2004). As a consequence, the bond-stock ratio is in general an increasing function of consumption related-risk aversion because of time preference effect. Except for a small humped shaped effect when the time preference effect becomes small because of low degree in wealth related risk aversion.

**Table 4: The bond-stock mix.** The table reports the bond and stock allocations as well as the bond-stock ratio. The base case parameters are given in Table (1).

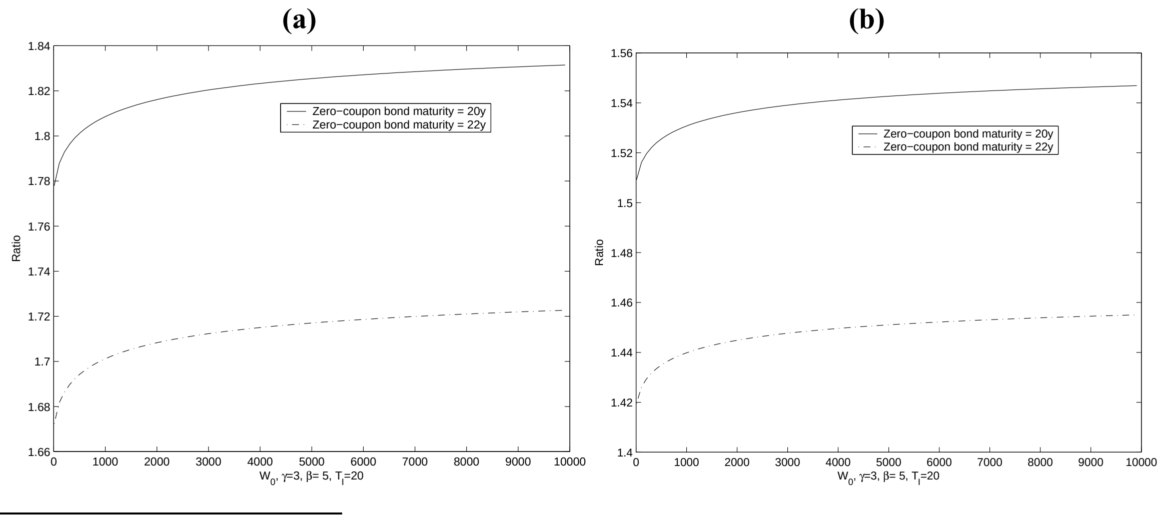
| | | | Case A | | | | Case B | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\beta$ | | T_I | | | | T_I | | | |
| | | | 1 | 5 | 10 | 20 | 1 | 5 | 10 | 20 |
| 1 | 1 | Stock | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 |
| | | Bond | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 |
| | | Ratio | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 |
| | 3 | Stock | 1.42 | 1.28 | 1.15 | 1.00 | 1.42 | 1.28 | 1.17 | 1.03 |
| | | Bond | 0.70 | 0.65 | 0.63 | 0.65 | 0.70 | 0.65 | 0.63 | 0.65 |
| | | Ratio | 0.49 | 0.51 | 0.55 | 0.65 | 0.49 | 0.51 | 0.54 | 0.62 |
| | 5 | Stock | 1.44 | 1.34 | 1.24 | 1.13 | 1.44 | 1.34 | 1.25 | 1.15 |
| | | Bond | 0.71 | 0.67 | 0.66 | 0.66 | 0.71 | 0.67 | 0.66 | 0.66 |
| | | Ratio | 0.49 | 0.50 | 0.53 | 0.59 | 0.49 | 0.50 | 0.53 | 0.58 |
| | 7 | Stock | 1.44 | 1.36 | 1.28 | 1.18 | 1.44 | 1.36 | 1.29 | 1.20 |
| | | Bond | 0.71 | 0.68 | 0.67 | 0.67 | 0.71 | 0.68 | 0.67 | 0.67 |
| | | Ratio | 0.49 | 0.50 | 0.52 | 0.57 | 0.49 | 0.50 | 0.52 | 0.56 |
| 3 | 3 | Stock | 0.49 | 0.49 | 0.49 | 0.49 | 0.50 | 0.53 | 0.55 | 0.56 |
| | | Bond | 0.28 | 0.37 | 0.45 | 0.57 | 0.28 | 0.37 | 0.45 | 0.57 |
| | | Ratio | 0.57 | 0.75 | 0.92 | 1.17 | 0.55 | 0.69 | 0.83 | 1.02 |
| | 5 | Stock | 0.46 | 0.39 | 0.34 | 0.31 | 0.48 | 0.43 | 0.40 | 0.37 |
| | | Bond | 0.27 | 0.36 | 0.45 | 0.56 | 0.27 | 0.36 | 0.45 | 0.56 |
| | | Ratio | 0.59 | 0.94 | 1.30 | 1.79 | 0.57 | 0.84 | 1.13 | 1.52 |
| | 7 | Stock | 0.47 | 0.40 | 0.35 | 0.28 | 0.49 | 0.46 | 0.40 | 0.33 |
| | | Bond | 0.28 | 0.39 | 0.49 | 0.60 | 0.28 | 0.39 | 0.49 | 0.60 |
| | | Ratio | 0.59 | 0.97 | 1.42 | 2.16 | 0.57 | 0.86 | 1.22 | 1.82 |
| | 9 | Stock | 0.47 | 0.42 | 0.37 | 0.30 | 0.49 | 0.48 | 0.43 | 0.35 |
| | | Bond | 0.28 | 0.41 | 0.53 | 0.67 | 0.28 | 0.41 | 0.53 | 0.66 |
| | | Ratio | 0.59 | 0.98 | 1.43 | 2.23 | 0.57 | 0.86 | 1.22 | 1.87 |
| 6 | 6 | Stock | 0.24 | 0.24 | 0.24 | 0.24 | 0.25 | 0.27 | 0.28 | 0.29 |
| | | Bond | 0.16 | 0.28 | 0.38 | 0.52 | 0.16 | 0.28 | 0.38 | 0.52 |
| | | Ratio | 0.67 | 1.13 | 1.56 | 2.13 | 0.65 | 1.02 | 1.35 | 1.79 |
| | 8 | Stock | 0.23 | 0.20 | 0.19 | 0.19 | 0.24 | 0.23 | 0.22 | 0.22 |
| | | Bond | 0.16 | 0.28 | 0.38 | 0.52 | 0.16 | 0.28 | 0.38 | 0.52 |
| | | Ratio | 0.72 | 1.40 | 2.00 | 2.79 | 0.69 | 1.24 | 1.71 | 2.32 |
| | 10 | Stock | 0.23 | 0.19 | 0.17 | 0.16 | 0.24 | 0.22 | 0.20 | 0.19 |
| | | Bond | 0.17 | 0.30 | 0.41 | 0.53 | 0.17 | 0.30 | 0.40 | 0.53 |
| | | Ratio | 0.73 | 1.56 | 2.37 | 3.42 | 0.70 | 1.36 | 1.99 | 2.81 |
| | 12 | Stock | 0.23 | 0.20 | 0.17 | 0.14 | 0.24 | 0.23 | 0.20 | 0.17 |
| | | Bond | 0.17 | 0.32 | 0.44 | 0.56 | 0.17 | 0.32 | 0.43 | 0.55 |
| | | Ratio | 0.74 | 1.64 | 2.59 | 3.94 | 0.70 | 1.41 | 2.16 | 3.22 |

Finally, the results of the demand in case (a) and (b) are very similar. However, this fact is strongly link to the random market price of risk of case (b) being set equal to its long term mean. Indeed, illustrations available from the authors show that the speculative component of equity is strongly affected by the movements of the market price of risk.

While some of our results, not reported in the paper but available from the authors upon request, indicate a decreasing function of the bond-stock ratio with respect to wealth,[7] Figures (2a) and (2b) exhibit an increasing pattern of the bond-stock ratio with respect to the investor's wealth.

Indeed, from Proposition 2, mean-variance terms are decreasing function of relative risk aversion. Since the latter is increasing in function of wealth (Figure 1), therefore speculative demand increases with wealth. Therefore, the ratio's pattern is explained by M-B hedging motives. Moreover, because of some consumption smoothing effect, (see Wachter, 2002 and proposition 2), the consumption related M-B interest rate is smaller than the wealth related M-B component. This effect is amplified by the fact that our investor is more risk averse towards consumption than towards wealth. Besides, Proposition (1) shows that $\pi_C^{(i)}(t)$ is a decreasing function of wealth and proposition 2 demonstrates that the M-B hedging demand is a weighted average of the consumption and wealth related M-B hedging components with weights $\pi_C^{(i)}(t)$ and its complement to unity, respectively. As a consequence, the time preference hedging demand is increasing in investor's wealth and the pattern of the bond-stock ratio is explained.

**Figure 2. The bond-stock ratio.** The figure exhibits the bond stock ratio in function of wealth. The base case parameters are given in Table (1).




[7] This is the case, for instance, for high risk aversion towards wealth ( $\gamma = 6$ ) when risk aversion towards consumption is very high ( $\beta = 12$ ), or for low wealth related risk aversion ( $\gamma = 1$ ).

In addition, Figures (2a) and (2b) show a substantial decrease in the ratio when the maturity of the bond exceeds the investment horizon.

A detailed analysis of the stock and bond demands is provided in Tables (5a) and (5b), respectively, where we separate the total demand in the speculative and M-B hedging components. While for $\beta > \gamma$ the M-B component increases with the investment horizon, the speculative demand decreases. Indeed, horizon impacts the speculative components only through relative risk aversion. This result has a nice interpretation; a long-term investor is less concerned by short term asset prices movements when she has a consumption objective. We also highlight a well known result, that is, the M-B components increase along with the investor's horizon. However, these two M-B hedging components differ much in magnitude as well as in their relative size to the speculative terms, respectively. This feature explains a significant part of the various patterns underlined in the previous figures and tables.

**Table 5a: Decomposition of the total demand for stocks.** This table reports the mean-variance (M-V) and the Merton-Breeden (M-B) hedging demand for stocks. The base case parameters are given in Table (1).

| | | | Case A | | | | Case B | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | T_I | | | | T_I | | | |
| $\gamma$ | $\beta$ | | 1 | 5 | 10 | 20 | 1 | 5 | 10 | 20 |
| 1 | 1 | M-V | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 | 1.47 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 3 | M-V | 1.42 | 1.28 | 1.15 | 1.00 | 1.42 | 1.28 | 1.15 | 1.00 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.02 | 0.03 |
| | 5 | M-V | 1.44 | 1.34 | 1.24 | 1.13 | 1.44 | 1.34 | 1.24 | 1.13 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.01 |
| | 7 | M-V | 1.44 | 1.36 | 1.28 | 1.18 | 1.44 | 1.36 | 1.28 | 1.19 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 |
| 3 | 3 | M-V | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 | 0.49 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.04 | 0.06 | 0.07 |
| | 5 | M-V | 0.46 | 0.39 | 0.34 | 0.31 | 0.46 | 0.39 | 0.34 | 0.31 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.02 | 0.05 | 0.05 | 0.05 |
| | 7 | M-V | 0.47 | 0.40 | 0.35 | 0.28 | 0.47 | 0.40 | 0.35 | 0.28 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.02 | 0.05 | 0.06 | 0.05 |
| | 9 | M-V | 0.47 | 0.42 | 0.37 | 0.30 | 0.47 | 0.42 | 0.37 | 0.30 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.02 | 0.05 | 0.06 | 0.06 |
| 6 | 6 | M-V | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.03 | 0.04 | 0.04 |
| | 8 | M-V | 0.23 | 0.20 | 0.19 | 0.19 | 0.23 | 0.20 | 0.19 | 0.19 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.03 | 0.03 | 0.04 |
| | 10 | M-V | 0.23 | 0.19 | 0.17 | 0.16 | 0.23 | 0.19 | 0.17 | 0.16 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.03 | 0.03 | 0.03 |
| | 12 | M-V | 0.23 | 0.20 | 0.17 | 0.14 | 0.23 | 0.20 | 0.17 | 0.14 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.01 | 0.03 | 0.03 | 0.03 |

The impact of wealth related risk aversion on M-B hedging demand is positive. This behavior is well documented for the time preference M-B demand (Lioui and Poncet, 2001; Munk and Sørensen, 2004). Nevertheless, regarding the M-B market price of risk hedging demand, some illustrations show that it is not monotonic in risk aversion due to the imperfect framework of time-additive utility functions (Wachter, 2002; Munk and Sørensen, 2008). To consider an investor more risk averse towards consumption than towards wealth seems, at a first glance, to correct this imperfection.

**Table 5b: Decomposition of the total demand for bonds.** This table reports the mean-variance (M-V) and the Merton-Breeden (M-B) hedging demand for bonds. The base case parameters are given in Table (1).

| | | | Case A | | | | Case B | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\beta$ | | T_I | | | | T_I | | | |
| | | | 1 | 5 | 10 | 20 | 1 | 5 | 10 | 20 |
| 1 | 1 | M-V | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 | 0.72 |
| | | M-B | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | 3 | M-V | 0.70 | 0.63 | 0.57 | 0.49 | 0.70 | 0.63 | 0.57 | 0.49 |
| | | M-B | 0.00 | 0.02 | 0.06 | 0.15 | 0.00 | 0.02 | 0.06 | 0.15 |
| | 5 | M-V | 0.71 | 0.66 | 0.61 | 0.56 | 0.71 | 0.66 | 0.61 | 0.56 |
| | | M-B | 0.00 | 0.01 | 0.04 | 0.11 | 0.00 | 0.01 | 0.04 | 0.11 |
| | 7 | M-V | 0.71 | 0.67 | 0.63 | 0.58 | 0.71 | 0.67 | 0.63 | 0.58 |
| | | M-B | 0.00 | 0.01 | 0.04 | 0.09 | 0.00 | 0.01 | 0.04 | 0.09 |
| 3 | 3 | M-V | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 | 0.24 |
| | | M-B | 0.04 | 0.13 | 0.21 | 0.33 | 0.04 | 0.13 | 0.21 | 0.33 |
| | 5 | M-V | 0.23 | 0.19 | 0.17 | 0.15 | 0.23 | 0.19 | 0.17 | 0.15 |
| | | M-B | 0.04 | 0.17 | 0.28 | 0.41 | 0.04 | 0.17 | 0.28 | 0.40 |
| | 7 | M-V | 0.23 | 0.20 | 0.17 | 0.14 | 0.23 | 0.20 | 0.17 | 0.14 |
| | | M-B | 0.05 | 0.19 | 0.32 | 0.47 | 0.05 | 0.19 | 0.32 | 0.46 |
| | 9 | M-V | 0.23 | 0.21 | 0.18 | 0.15 | 0.23 | 0.21 | 0.18 | 0.15 |
| | | M-B | 0.05 | 0.20 | 0.35 | 0.52 | 0.05 | 0.20 | 0.35 | 0.52 |
| 6 | 6 | M-V | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 | 0.12 |
| | | M-B | 0.04 | 0.16 | 0.26 | 0.40 | 0.04 | 0.16 | 0.26 | 0.40 |
| | 8 | M-V | 0.11 | 0.10 | 0.09 | 0.09 | 0.11 | 0.10 | 0.09 | 0.09 |
| | | M-B | 0.05 | 0.18 | 0.29 | 0.43 | 0.05 | 0.18 | 0.29 | 0.42 |
| | 10 | M-V | 0.11 | 0.10 | 0.08 | 0.08 | 0.11 | 0.10 | 0.08 | 0.08 |
| | | M-B | 0.06 | 0.21 | 0.32 | 0.45 | 0.06 | 0.21 | 0.32 | 0.45 |
| | 12 | M-V | 0.11 | 0.10 | 0.08 | 0.07 | 0.11 | 0.10 | 0.08 | 0.07 |
| | | M-B | 0.06 | 0.22 | 0.35 | 0.49 | 0.06 | 0.22 | 0.35 | 0.48 |

Moreover, when the M-B market price of risk hedging demand is examined in light of the consumption related risk aversion the non monotonic pattern is preserved. It is even accentuated since, depending on the level of wealth risk aversion the hump can be reversed. As far as the speculative demands as a function of consumption risk aversion are concerned, their behavior simply follows from Table (3), so no further comments will be made. Finally, the M-B hedging demand is most of the time increasing in consumption risk aversion as documented in the literature (Munk and Sørensen, 2004). However, our framework in which the investor has a risk aversion higher towards consumption than towards wealth, show that for low level of wealth related risk aversion, some non-monotonic effects arises due to the consumption related risk aversion as documented in the literature on market price of risk hedging. This further proves that the time preference and market price of risk effects are hard to separate.

## 4. Concluding remarks

This paper studies the optimal bond-stock allocation with a focus on the impact of consumption, which is not studied in the previous literature. Indeed, Brennan et al. (1997) and Canner et al. (1997) demonstrate that investors should consider long investment horizon for such an allocation. The effect of consumption can not be neglected when such long horizons are advised. To fully elucidate the impact of consumption, we allow our investor to be more risk averse towards consumption than towards wealth. Furthermore, our study takes into account a dynamic opportunity set with two predictive variables, i.e. the instantaneous short rate and the equity market price of risk.

We show that the optimal bond-stock mix is significantly affected by the role of consumption. The speculative demand shortens with horizon because of longer consumption stream obligations that leads the investor to be less focus on short term price movements. Moreover, we underline the fact that the Merton-Breeden hedging demand, (see Merton, 1971, 1973; Breeden, 1979), stems mostly from time preference preoccupations rather than dividend yield hedging. As a consequence, we are able to show that the bond-stock ratio can be an increasing function of investor's wealth.

We also show that the relative risk aversion[8] as a function of consumption related risk aversion exhibits a hump. This hump is explained by the fact that it is in the objective, which movements the investor is the more afraid of, i.e. consumption, that she invests the less. This hump is transmitted to the speculative demand. Finally, our results also highlight a hump in the Merton-Breeden hedging demand behavior as a function of consumption risk aversion, even for the time preference component. This feature

[8] Relative risk aversion is defined using the indirect utility function, Merton (1971), in our framework where consumption related risk aversion differs from wealth related risk aversion.

differs from previous settings in which risk aversions equal or consumption is not acknowledged, Sørensen, (1999), Lioui and Poncet (2001), Munk and Sørensen, (2004).

This article can be extended in several directions. First, the analysis can be carried out for an investor who faces various constraints such as borrowing limitation and no short sales of the risky assets. Finally, we can examine the impact of consumption on the bond-stock mix in light of a richer dynamics governing innovations in the state variables.

**Appendix.**

*A.Case (a)*

The function $q_j^{(a)}(t,T_I), j\in\{\beta,\gamma\}$ is given by the following equation:

$$q_j^{(a)}(t,T_I)=\exp\left(H(j,t,T_I)-\left(1-\frac{1}{j}\right)D_\alpha(t,T_I)r(t)\right), j\in\{\beta,\gamma\} \tag{A.1}$$

where the deterministic function $H(j,t,T_I)$, $j\in\{\beta,\gamma\}$, is given by:

$$\begin{aligned}H(j,\gamma,T_I)&=-\frac{1}{2}\frac{1}{j}\left[1-\frac{1}{j}\right]\left[\lambda_P^2+\lambda_S^{(a)2}+2\rho_{SP}\lambda_P\lambda_S^{(a)}\right][T_I-t]\\&+\left[\frac{\vartheta_j}{\alpha}-\frac{\left(\sigma_r^{(j)}\right)^2}{2\alpha^2}\right].[T_I-t-D_\alpha(t,T_I)]+\frac{\left(\sigma_r^{(j)}\right)^2}{4\alpha}D_\alpha(t,T_I)^2\end{aligned} \tag{A.2}$$

Where the parameters $\vartheta_j,\sigma_r^{(j)}$, $j\in\{\beta,\gamma\}$ are given by $\theta_j\equiv\left[1-\frac{1}{j}\right]\theta+\frac{\sigma_r\lambda_P}{\alpha}\left(1-\frac{1}{j}\right)^2$ and $\sigma_r^{(j)}\equiv\sigma_r\frac{j-1}{j}$.

The temporal horizon $T_{IP}^{(a)}$ is implicitly defined as follows:

$$\sigma_P\left(t,T_{IP}^{(a)}\right)\equiv\int_t^{T_I}\frac{q_\beta^{(a)}(t,v)}{Q_\beta^{(a)}(t,T_I)}\sigma_P(t,v)dv \tag{A.3}$$

As a consequence, $T_{IP}^{(a)}$ is function of investors' relative risk aversion towards consumption as well as market (a)'s dynamic. Moreover, because it is obvious that (A.3) represents a weighted average formula, we have that: $T_{IP}^{(a)}\le T_I$.

*B. Case (b)*

The function $q_j^{(b)}(t,T_I), j\in\{\beta,\gamma\}$ is given by the following equation:

$$q_j^{(b)}(t,T_I)=P(t,T_I)^{\left(1-\frac{1}{j}\right)}\exp\left(G_j(t,T_I)+E_j(t,T_I)\lambda_S^{(b)}(t)+\frac{1}{2}F_j(t,T_I)\lambda_S^{(b)}(t)^2\right) \tag{B.1}$$

$P(t,T_I)$ is the price of a Vasicek type zero coupon given of maturity $T_I$. The deterministic functions $G_j(t,T_I), E_j(t,T_I), F_j(t,T_I)$, are as follow:

$$\begin{aligned}G_j(t,T_I)&\equiv A(j,T_I-t)+\frac{-\rho_{SP}\lambda_P B(j,T_I-t)}{\sqrt{1-\rho_{SP}^2}}+\frac{C(j,T_I-t)}{2}\frac{(\rho_{SP}\lambda_P)^2}{1-\rho_{SP}^2}-\frac{j-1}{j^2}\lambda_P^2(T_I-t)\\&-\frac{j-1}{j^2}\left\{-2\lambda_P\frac{\sigma_r}{\alpha}[T_I-t-D_\alpha(t,T_I)]+\left[\frac{\sigma_r}{\alpha}\right]^2[T_I-t-2D_\alpha(t,T_I)+D_{2\alpha}(t,T_I)]\right\}\end{aligned} \tag{B.2}$$

$$E_j\left(t,T_I\right) \equiv \frac{B\left(j,T_I-t\right)}{\sqrt{1-\rho_{SP}^2}} - \frac{\rho_{SP}\lambda_P}{1-\rho_{SP}^2} C\left(j,T_I-t\right) \tag{B.3}$$

$$F_j\left(t,T_I\right) = \frac{C\left(j,T_I-t\right)}{1-\rho_{SP}^2}. \tag{B.4}$$

Functions $A(j,u), B(j,u), C(j,u)$ are solutions of the following ordinary differential system:

$$\partial_u C(j,u) = \frac{\sigma_\lambda^2}{1-\rho_{SP}^2} C(j,u)^2 - 2\left[\kappa - \sigma_\lambda\left(1-\frac{1}{\gamma}\right)\right] C(j,u) - \frac{j-1}{j^2}, C(j,0)=0 \tag{B.5}$$

$$\begin{aligned}\partial_u B\left(j,u\right) = &\left[\frac{\sigma_\lambda^2}{1-\rho_{SP}^2} C\left(j,u\right) - \left[\kappa - \sigma_\lambda\left(1-\frac{1}{j}\right)\right]\right] B\left(j,u\right) \\ &+ \left[\kappa\frac{\overline{\lambda}_S - \rho_{SP}\lambda_P}{\sqrt{1-\rho_{SP}^2}} + \left(1-\frac{1}{j}\right)\rho_{SP}\left[\lambda_P - \sigma_P\left(u\right)\right]\right] \frac{\sigma_\lambda}{\sqrt{1-\rho_{SP}^2}} C\left(j,u\right),\ B\left(j,0\right)=0\end{aligned} \tag{B.6}$$

$$\begin{aligned}\partial_u A\left(j,u\right) = &\left[\frac{1}{2}\frac{\sigma_\lambda^2}{1-\rho_{SP}^2} + \kappa\frac{\overline{\lambda}_S - \rho_{SP}\lambda_P}{\sqrt{1-\rho_{SP}^2}} + \left(1-\frac{1}{j}\right)\rho_{SP}\left[\lambda_P - \sigma_P\left(u\right)\right]\right] B\left(j,u\right) \\ &+ \frac{1}{2}\frac{\sigma_\lambda^2}{1-\rho_{SP}^2} C\left(j,u\right), A\left(j,0\right)=0\end{aligned} \tag{B.7}$$

$\partial_u$ stands for the partial derivatives with respect to *u*.

The temporal horizons $T_{IS}^{(b)}, T_{IP}^{(b)}$ are implicitly defined as follows:

$$\sigma_P\left(t,T_{IP}^{(b)}\right) \equiv \int_t^{T_I} \frac{q_\beta^{(b)}(t,v)}{Q_\beta^{(b)}(t,T_I)} \sigma_P\left(t,v\right) dv \tag{B.8}$$

$$E_\beta\left(t,T_{IS}^{(b)}\right) + F_\beta\left(t,T_{IS}^{(b)}\right)\lambda_S^{(b)}(t) \equiv \int_t^{T_I} \frac{q_\beta^{(b)}(t,v)}{Q_\beta^{(b)}(t,T_I)} \left[E_\beta\left(t,v\right) + F_\beta\left(t,v\right)\lambda_S^{(b)}(t)\right] dv \tag{B.9}$$

As a consequence, $T_{IS}^{(b)}, T_{IP}^{(b)}$ is function of the investor's relative risk aversion towards consumption as well as case (b)'s dynamics. Moreover, it is obvious that Eqs. (B.8) and (B.9) represent weighted average formulae, so we have that: $T_{IS}^{(b)}, T_{IP}^{(b)} \leq T_I$.

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