Pricing Executive Stock Options under Employment Shocks

Julio Carmona
University of Alicante

Angel León
University of Alicante

Antoni Vaello
University of Illes Balears

January 15, 2009

(Preliminary Version)

Abstract

We obtain explicit expressions for valuing perpetual ESO using a stochastic discount factor derived from a maximization program with constrained holdings of company's stock a la Ingersoll. In contrast with previous utility-based models, we take into account that ESOs have stochastic lives due to the risk of employment termination. By using a simple Poissson process for the modeling of employment shocks, we analyze the determinants of both, the subjective valuation by employees and the objective valuation by firms. We also perform a detailed analysis of employees incentives, with particular emphasis in the uncertainty introduced by employment shocks.

JEL Classification: D91, G11, G13

Keywords: Executive Stock Options, Stochastic Discount Factor, Risk Aversion.

1 Introduction

Employee stock options (ESOs) are American call options that provide pay for performance. Firms use ESOs to align employees' incentives with the shareholders. To create such incentives, ESOs cannot be sold or transferred nor hedge it by short selling the firm's stock, but partial hedge is possible by trading correlated assets. Additionally, they can only be exercised after a given period of time, called vesting period, has elapsed. As a result of all those constraints, standard methods for valuing American options do not apply.

Given the increasing relevance of ESOs as a component of corporate compensation, the Financial Accounting Standards Board (FASB) revised the Financial Accounting Standard 123 (FAS 123R) in 2004. The FAS 123R does not specify a preference for a particular valuation method, but it specifies that the valuation method should be based on established principles of financial economics like time value of money and risk-neutral valuation (FAS 123R, paragraph A8). Moreover, the FAS 123 enumerates the factors required in the valuation technique at a minimum. Besides common factors (expected volatility, exercise price, current stock price, expected dividends,...), the standard remarks that the valuation technique should consider the expected term of the ESO taking into account both the contractual term of the option and the effects of employees' expected exercise (FAS 123R, paragraph A18). In this line, the FASB has proposed to use the standard Black-Scholes model for pricing European call options but replacing the ESO's maturity date by its observed average life. This observed average life is clearly affected by events that terminates the employment relationship, such as employees' voluntary resignation or dismissal. Jennergren and Näslund (1993) were the first to propose a modified Black-Scholes valuation model by incorporating an exogenous exit rate at which employees abandon their jobs. This essentially implies the introduction of a discount factor reflecting the probability that the employee remains in the firm before ESO's expiration.

A common finding in empirical evidence, among which Huddart and Lang (1996) and Bettis et al. (2005) are the main references, is that the observed average life is lower than the maturity date. Typically, employees tend to exercise ESOs well before their maturity date and mostly right after the end of the vesting period. This behavior is inconsistent with the exercise policy that an outside investor, who takes decisions optimally in an unrestricted environment, will follow. This difference in the exercise policy confirms that the same ESO has a different value for the employee and for the firm. Thus, the literature on ESO valuation has distinguished between a subjective value and an objective value, where the latter is usually defined as the amount that an unrestricted outside investor would pay for the ESOs.

Lambert et al. (1991) focus on the subjective value of ESOs by using an expected utility framework. They establish the subjective value of one ESO as its certainty-equivalence. As it is well known by now, the subjective valuation is very relevant because the market value of ESOs depends on its payoff and this payoff is determined by the employee's exercise policy. Many papers use the certainty-equivalent approach improving the initial model of Lambert et al. (1991). Huddart (1994) use this framework to provide an estimate of this subjective value determining the executive's exercise rule on a binomial tree. Kulatilaka and Marcus (1994) follow the same approach but they focus on the objective valuation for accounting purposes. Thus, they propose to compute the firm cost of one ESO by using the employee's exercise policy in an otherwise non restricted environment in which the firm can hedge its risk. The value of an ESO computed this way is termed as the objective value or true cost of ESOs for firms. Of course, since the employee's threshold price is lower than the unrestricted outside investor's, the firm's true cost will be lower than the fair market value of ESOs. Hall and Murphy (2002) and Cai and Vijh (2005) also use this methodology to obtain the objective value of ESOs in a more general setting, allowing for restricted stocks of the firm in the executive's wealth and introducing the market portfolio, respectively.

Our approach to derive both, the subjective and objective ESO value, follows a simplified version of Ingersoll's (2006) model. We retain his asset menu, so that the employee allocates her wealth across the market portfolio and risk free bonds, but reduce the risk factors to just the market risk. Further, as in Ingersoll's model, the employee is constrained to hold more of the company's stock than its corresponding share in the market portfolio. This could be rationalized on the grounds of good corporative image or, more generally, by the fact that the employee is not holding a well diversified portfolio. As a result, there are two sources of risk, one coming from the systematic risk factor underlying the market portfolio (non diversifiable) and the other coming from the idiosyncratic component of the company's stock (not correlated with market risk). In a well diversified portfolio the only source of risk would come exclusively from the market portfolio and any other idiosyncratic component would have vanished.

By maximizing a lifetime utility function, Ingersoll (2006) finds an stochastic discount factor which includes those two sources of risk, and that can be used to price the ESO by pricing the risk free bond and the company's stock. The model is extended to include a job termination risk. By using the usual no arbitrage condition, this discount factor allows us to obtain an stochastic differential equation which provides a closed form solution for the subjective valuation of perpetual ESOs with positive vesting period. Furthermore, by simply assuming either, a well diversified portfolio or a risk neutral agent, we can also obtain the solution for the ESO market value. Finally, by using the risk neutral solution with the employee's exercise policy derived in the subjective solution we obtain the objective ESO value.

Summarizing, the main contributions of the paper are the following ones. We extend the model of Ingersoll (2006) allowing for job termination and suboptimal ESO early exercise. We obtain a closed formula for the subjective and objective value of a perpetual ESO, where the objective value is a particular case of the former. Moreover, we document the small bias size of the perpetual ESO value respect to the finite maturity one. We also conduct a deep analysis about ESO incentives studying the ESO greek *delta* and *vega*.

2 Perpetual ESO Valuation

Our benchmark model will be a perpetual ESO with a stochastic live arising from exogenous employment shocks which forces the termination of the employment relationship, as in the model of Jennergren and Näslund (1993). These shocks can arise from either the employee's side, due to voluntary resignation, or from the firm's side, due to firm's bankruptcy or firing of employees. In any case, the employee is forced to exercise the option if the event occurs after the vesting period, or to forfeit it, if the vesting period has not come to end. The time at which the employment relationship is terminated is simply modeled as the first event of a Poisson process with hazard rate of λ per unit time. This Poisson process is assumed to be independent of any other stochastic process underlying our menu of assets. The hazard rate leads to jumps in the ESO price as in Jennergren and Näslund (1993). We assume that the job termination risk is not priced, that is, it can be diversified away. This assumption is very common in the literature. See for instance Jennergren and Näslund (1993), Carpenter (1998), Hull and White (2004), Carr and Linetsky (2000), Sircar and Xiong (2007) and Leung and Sircar (2007).

Those employment shocks forces exogenous exercise of ESOs. Endogenous exercise results from the optimizing behavior of the employee. Now, we turn to describe the constrained maximization problem faced by the employee. First, she faces the following menu of assets: a risk free bond, the market portfolio and the company's stock. The equations describing the dynamics of the company stock and the market portfolio would be given by

$$\begin{aligned}\frac{dS}{S} &= (\mu_S - q_S)dt + \sigma_S dZ_S \\ \frac{dM}{M} &= (\mu_M - q_M)dt + \sigma_M dZ_M\end{aligned}$$

where q_S and q_M denote the continuous dividend yield on the stock and the market portfolio, respectively. The company stock and the market portfolio are assumed to be imperfectly correlated. Formally, the Wiener processes satisfy the following relationship:

$$\sigma_S dZ_S = \beta\sigma_M dZ_M + \sigma_I dZ_I \tag{1}$$

where the parameter β is the conventional 'beta' in the Capital Asset Price Model (CAPM), for $\mathbb{E}\left[(dZ_M)^2\right] = \mathbb{E}\left[(dZ_I)^2\right] = 1$ and $\mathbb{E}\left[dZ_M dZ_I\right] = 0$. Notice that σ_I is not an independent parameter, since it must satisfy the restriction $\sigma_I^2 = \sigma_S^2 - \beta^2\sigma_M^2$ as equation (1) makes clear. In short, we can write the equation for the stock price dynamics as:

$$\frac{dS}{S} = (\mu_S - q_S)\, dt + \beta\sigma_M dZ_M + \sigma_I dZ_I$$

As Ingersoll (2006), we assume that the employee is infinitely lived and maximizes an expected lifetime utility of the constant relative risk aversion class. To capture the degree of no diversification, we define the parameter θ as the excess of company's stock holdings over the optimal level already incorporated in the market portfolio [1]. Therefore, the executive's problem is

$$\max_{C,\omega} \mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \frac{C_t^{1-\gamma}}{1-\gamma} dt \right\} \tag{2}$$

subject to the budget constraint dynamics

$$dW = \left\{ \left[r + \omega\left(\mu_M - r\right) + \theta\left(\mu_S - r\right) \right] W - C \right\} dt + \omega\sigma_M W dZ_M + \theta\sigma_S W dZ_S \tag{3}$$

with initial condition $W(0) = W_0$. For simplicity no wage income is assumed. If CAPM holds, i.e.

[1] Let $\underline{\theta}$ denote the minimum amount of the company stock that the employee is constrained to hold. If $\xi^\star$ denotes the optimal share of the company stock in the market portfolio and $\omega^\star$ the optimal share of the market portfolio in the employee's total portfolio, then θ would satisfy the following condition $\theta = \underline{\theta} - \omega^\star\xi^\star \geq 0$.

$\mu_S = r + \beta\big(\mu_M - r\big)$, we can rewrite equation (3) as

$$dW = \{(r + (\omega + \theta\beta)(\mu_M - r))W - C\} dt + \omega\sigma_M W dZ_M + \theta\sigma_S W dZ_S. \tag{4}$$

Using the orthogonal decomposition described in equation (1), we can further rewrite the wealth dynamics equation as

$$dW = \big\{\big(r + \big(\omega + \theta\beta\big)\big(\mu_M - r\big)\big) W - C\big\} dt + \big(\omega + \theta\beta\big)\sigma_M W dZ_M + \theta\sigma_I W dZ_I \tag{5}$$

It is verified that, conditional on the initial value of wealth:

$$\begin{aligned} \mathbb{E}_0\big[dW\big] &= \Big\{\Big[r + \big(\omega + \theta\beta\big)\big(\mu_M - r\big)\Big]W - C\Big\} dt \\ \mathbb{E}_0\big[dW^2\big] &= \Big\{\big(\omega + \theta\beta\big)^2\sigma_M^2 + \theta^2\sigma_I^2\Big\}W^2 dt. \end{aligned}$$

Given the above conditions, we can obtain the stochastic discount factor in the following lemma.

Lemma 1 *The stochastic discount factor, Θ, that prices the derivative will obey the following ordinary differential equation (ODE):*

$$\frac{d\Theta}{\Theta} = -\hat{r}dt - \left(\frac{\mu_M - r}{\sigma_M}\right) dZ_M - \gamma\theta\sigma_I dZ_I \tag{6}$$

where $\hat{r} = r - \gamma\theta^2\sigma_I^2$.

Proof.- See Appendix A.

Notice that, when the employee is either risk neutral, $\gamma = 0$, or has a well diversified portfolio, $\theta = 0$, the stochastic discount factor does not include any term reflecting the (diversifiable) idiosyncratic risk of the stock. Hence, the resulting value coincides with the risk neutral price for marketable options. As in the Black-Scholes-Merton model, what matters for the valuation of European options is not the agents' degree of risk aversion but their possibilities of diversification.

For pricing the ESO, we use the no arbitrage condition $\mathbb{E}_0\big\{d(\Theta V)\big\} = 0$ where V denotes the money value of the ESO. Thus,

$$\begin{aligned} 0 &= \mathbb{E}_0\left[d(\Theta V)\right] = \mathbb{E}_0\left(d(\Theta V)\middle|\text{no employment shock}\right)\left(1-\lambda dt\right) + \\ &+ \mathbb{E}_0\left(d(\Theta V)\middle|\text{ employment shock}\right)\lambda dt \end{aligned} \tag{7}$$

where $\Psi(S)$ denotes the payoff of the ESO holder if there is an employment shock defined as $\Psi(S) = (S-K)\mathbb{1}_{\{S>K\}}$, where $\mathbb{1}_A$ is an indicator function verifying that $\mathbb{1}_A = 1$ if A is true and $\mathbb{1}_A = 0$ otherwise. Given equation (7), we get the following result:

Lemma 2 *Under the non arbitrage condition, it holds that* $\mathbb{E}_0\{d(\Theta V)\} = 0$ *where* V *denotes the ESO money value and* Θ *is the employee's stochastic discount factor for valuation of the derivative. From this condition and assuming that CAPM holds, we get the following fundamental ODE for the subjective valuation of ESOs:*

$$\left(\frac{\sigma_S^2}{2}\right)V_{SS}S^2 + \left(\hat{r}-\hat{q}_S\right)V_S S - \left(\hat{r}+\lambda\right)V + \lambda\Psi(S) = 0 \tag{8}$$

where $\hat{r} = r - \gamma\theta^2\sigma_I^2$ *and* $\hat{q}_S = q_S + \gamma\theta(1-\theta)\sigma_I^2$.

Proof.- See Appendix B.

Equation (8) is the ODE defining the employee's value of the ESO in the continuation or waiting region. The structure of the problem implies that there is a threshold price, S^*, such that the optimal policy is waiting while $S < S^*$ and exercising as soon as $S \geq S^*$. Hence, at the boundary with the exercise region, the following conditions must hold:

$$V(S^*) = S^* - K \tag{9}$$

$$V'(S^*) = 1 \tag{10}$$

where the threshold price S^* is determined endogenously as part of the complete solution. Considering equations (8), (9) and (10), we obtain the following two propositions:

Proposition 3 *Assume that there is no vesting period and the ESO is a perpetual American call option,* *then the value for the ESO holder is given by*

$$V(S) = \begin{cases} \hat{A}_1 S^{\hat{\alpha_1}} & \text{if } S \le K \\ \hat{B}_1 S^{\hat{\alpha_1}} + \hat{B}_2 S^{\hat{\alpha_2}} + \lambda \left(\dfrac{S}{\lambda + \hat{q}_S} - \dfrac{K}{\lambda + \hat{r}} \right) & \text{if } K < S \le \hat{S}^* \\ S - K & \text{if } S > \hat{S}^* \end{cases} \tag{11}$$

where the values of $\hat{A}_1$, $\hat{B}_1$ and $\hat{B}_2$ are defined in Appendix C by equations (20), (24) and (21) respectively. Finally, the threshold price $\hat{S}^*$ is uniquely defined by solving the following equation:

$$\lambda \left(\frac{\hat{S}^*}{K} \right)^{\hat{\alpha_2}} = -(1 - \hat{\alpha_2})\hat{r} - \hat{\alpha_2}\hat{q}_S \left(\frac{\hat{S}^*}{K} \right). \tag{12}$$

Proof.- See Appendix C.

The first two rows of equation (11) show the ESO value when the price is below the optimal subjective threshold. Both belong to a situation in which the employee is better-off waiting rather than exercising the option. For an intuitive explanation of these gains let us consider the second waiting region. In this region, the first component comes from the possible increase in the future price of the underlying stock. The second one concerns the possibility of exercising the ESO if an employment shock occurs at any future time with a probability of λ. Hence, the term $\lambda \left(S/(\lambda + \hat{q}_S) - K/(\lambda + \hat{r}) \right)$ denotes the expected ESO present value when it is in the money.[2] Of course, in the first waiting region this term does not appear since the option is out of the money.

Figure 1 shows the typical shape of $V(S)$ derived in proposition 3. This figure displays several values for the stock excess holdings. Note that the situation of $\theta = 0$ is equivalent to the ESO risk neutral valuation. This value acts as an upper boundary for other situations in which the employee is less diversified.

[Figure 1 is about here]

[2] This expected present value is obtained by solving the following integral:

$$\int_0^\infty \lambda e^{-\lambda t} \left[e^{-rt} \left(S e^{(r - q_S)t} - K \right) \right] dt$$

where the time for the first employment shock follows an exponential distribution with mean $1/\lambda$.

A simplified situation that improves our understanding is provided by the case $\lambda = 0$ in the following corollary. This case leads to the vanilla perpetual American call option already described in Dixit and Pindyck (1994).

Corollary 4 *Assuming $\lambda = 0$ in proposition 3, the subjective value of the ESO at any time is given by*

$$V(S) = \begin{cases} \hat{A}_1 S^{\hat{\alpha}_1} & \text{if } S \leq \hat{S}^* \\ S - K & \text{if } S > \hat{S}^* \end{cases} \tag{13}$$

where the values of $\hat{A}_1$ and the threshold price $\hat{S}^*$ are given by

$$\hat{A}_1 = \frac{\hat{S}^* - K}{\left(\hat{S}^*\right)^{\hat{\alpha}_1}} \; ; \quad \hat{S}^* = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 - 1} K.$$

where $\hat{\alpha}_1$ is a positive function of $\hat{r}$ and a negative function of $\hat{q}_S$.

Denote V^{SUB} as the subjective ESO value and V^{RN} as the market value of ESO achieved when either $\theta = 0$ or $\gamma = 0$ (risk neutral valuation). Then, in the waiting region, V^{SUB} can be written as

$$V^{SUB} = \left(\frac{S}{\hat{S}^*}\right)^{\hat{\alpha}_1} \left(\hat{S}^* - K\right) = \frac{\left(\hat{\alpha}_1 - 1\right)^{(\hat{\alpha}_1 - 1)}}{\hat{\alpha}_1^{\hat{\alpha}_1}} K^{1-\hat{\alpha}_1}. \tag{14}$$

Accordingly, the effect on V^{SUB} of changes in the underlying parameters will all come through their effects on the positive root $\hat{\alpha}_1$. In Appendix D we show that the relationship between V^{SUB} and $\hat{\alpha}_1$ is negative, thus anything that reduces $\hat{\alpha}_1$ will increase the ESO value. Notice also that the relationship between $\hat{S}^*$ and $\hat{\alpha}_1$ is also negative and hence, V^{SUB} and $\hat{S}^*$ will tend to move in the same direction.

The perpetual ESO described in corollary 4 will also help us to understand the impact of changes in the components of total variance, i.e. $\sigma_S^2 = \beta^2 \sigma_M^2 + \sigma_I^2$. Although a higher variance tends to increase V^{SUB}, when this increase is due to a higher idiosyncratic volatility, the adjusted interest rate falls and the adjusted dividend yield increases. These changes tend to decrease V^{SUB}. In Appendix D we derive the precise conditions under which V^{SUB} decreases when σ_I increases. As one would expect, either a higher degree of risk aversion or a higher excess of stock holding makes more likely that V^{SUB} decreases.

Next, we introduce our main result concerning the subjective valuation of perpetual ESOs with a positive vesting period.

Proposition 5 *Let ν denote the length of the vesting period. The ESO expected value at the granting date, $t = 0$, conditional on information available at this date is given by*

$$V_0^{SUB} = e^{-\lambda\nu}\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}V(S_\nu)\right] \tag{15}$$

where the exponential term is the probability that the employee will remain employed at the end of the vesting period and

$$\begin{aligned}
\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}V(S_\nu)\right] &= e^{-\hat{r}\nu}\mathbb{E}_0\left[(S_\nu - K)\mathbb{1}_{\{S_\nu > \hat{S}^*\}}\right] + \\
&+ e^{-\hat{r}\nu}\mathbb{E}_0\left[\left(\hat{B}_1 S^{\hat{\alpha_1}} + \hat{B}_2 S^{\hat{\alpha_2}} + \lambda\left(\frac{S_\nu}{\lambda + \hat{q}_S} - \frac{K}{\lambda + \hat{r}}\right)\right)\mathbb{1}_{\{K < S_\nu \leq \hat{S}^*\}}\right] \\
&+ e^{-\hat{r}\nu}\mathbb{E}_0\left[(\hat{A}_1 S_\nu^{\hat{\alpha_1}})\mathbb{1}_{\{S_\nu \leq K\}}\right]
\end{aligned} \tag{16}$$

where for any a and $b \in \mathbb{R}$, it holds that

$$\begin{aligned}
\mathbb{E}_0\left[S_\nu^{\hat{\alpha}}\mathbb{1}_{\{a \leq S_\nu \leq b\}}\right] &= \exp\left\{\hat{\alpha}\left(\ln S_0 + \frac{\sigma^2}{2}(\hat{b} - 1)\nu\right) + \hat{\alpha}\frac{\sigma^2}{2}\nu\right\} \times \\
&\left\{\Phi\left(\frac{\ln b - \mu - \hat{\alpha}\sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu - \hat{\alpha}\sigma^2}{\sigma}\right)\right\}
\end{aligned}$$

with $\mu = \ln S_0 - (\hat{r} - \hat{q}_S - \sigma_S^2/2)\nu$ and $\sigma = \sigma_S\sqrt{\nu}$.

Proof.- See Appendix E.

We can obtain a version of proposition 5 for well diversified agents (equivalently, risk neutral agents) by simply setting the parameter θ to zero. That is,

Corollary 6 *When $\theta = 0$, the adjusted risk free rate, $\hat{r}$, and the adjusted dividend yield, $\hat{q}_S$, become r and q_S respectively so that the risk neutral ESO valuation is obtained as*

$$\begin{aligned}
V_0^{RN} = e^{-\lambda\nu}\Big\{ & e^{-r\nu}\mathbb{E}_0\left[(S_\nu - K)\mathbb{1}_{\{S_\nu > S^*\}}\right] + \\
& + e^{-r\nu}\mathbb{E}_0\left[\left(B_1 S^{\alpha_1} + B_2 S^{\alpha_2} + \lambda\left(\frac{S_\nu}{\lambda + q_S} - \frac{K}{\lambda + r}\right)\right)\mathbb{1}_{\{K < S_\nu \leq S^*\}}\right] \\
& + e^{-r\nu}\mathbb{E}_0\left[(A_1 S_\nu^{\alpha_1})\mathbb{1}_{\{S_\nu \leq K\}}\right]\Big\}
\end{aligned} \tag{17}$$

Following Hall and Murphy (2002), we can compute the objective value defined as the true cost of granting one ESO. This true cost is the amount received by the firm if the ESO would be sold to a well diversified investor with the exercise policy of a risk averse under-diversified employee, facing exogenous employment shocks, to whom the ESO was granted. We show that the ESO cost experiments a substantial fall with respect to the ESO market value. Notice that this true cost can be simply calculated by substituting the employee's threshold $\hat{S}^{\star}$ obtained in proposition 3 into the market ESO value given in (17).

3 Discussion

Here, we begin studying the impact on the ESO valuation due to changes in several parameters of interest. These results can be found in the section of sensitivity analysis. Second, we discuss the robustness of our perpetual ESO valuation by comparing with American-style ESOs with finite maturities.

For all situations, we assume the risk free interest rate $r = 6\%$, the continuous dividend yield $q_S = 1.5\%$ and the market volatility $\sigma_M = 20\%$. All these parameter values are taken on a yearly basis. Both the strike price, K, and the stock price at the granting date, S_0, are equal to \$30. This suggests that ESOs are granted at the money.

3.1 Sensitivity Analysis

We illustrate the effect of varying each of the following parameters: (i) the market beta, β, which can be either 0 or 1; (ii) the total yearly volatility of the stock return, σ_S, which can be 30%, 40% or 60%; (iii) the vesting period, ν, which can be either 0 or 3 years and hence, using propositions 3 and 5 to obtain respectively; (iv) the employment shock captured through the yearly Poisson intensity parameter, λ, with values of either 10% or 20%; (v) the excess stock holding, θ, ranging from 0% to 40%; and finally, (vi) the risk aversion parameter, γ, with value of 2 or 4. In short, this analysis is displayed on Table 1 throughout four panels according to different values of λ and ν, such that in all we allow for different values of σ_S, γ, θ and β. That is, panel A ($\lambda = 10\%$, $\nu = 0$), panel B ($\lambda = 10\%$, $\nu = 3$), panel C ($\lambda = 20\%$, $\nu = 0$) and panel D ($\lambda = 20\%$, $\nu = 3$). Next, we show how sensitive are ESO prices according to the above parameters.

First, the ESO market price, V^{RN}, is obtained under the restriction of $\theta = 0\%$. Note that V^{RN} increases with the total volatility of the stock return. This feature is independent of the risk decomposition into common risk or beta and specific or idiosyncratic risk, σ_I. The reason is that a well diversified agent does not worry about the size of σ_I. As expected, it is shown that V^{RN} acts as an upper boundary for V^{SUB}.

[Table 1 is about here]

Second, the higher the value of β the higher V^{SUB}. This effect has already been addressed by Tian (2004). A value of $\beta = 0$ suggests that the market portfolio is useless for hedging the risk of large holdings of the company stock. This affects negatively the ESO price with respect to an initial positive beta with the same size for the total volatility. Observe that under $\beta = 0$, $\sigma_S = \sigma_I$. Note also that, for any level of both the excess stock holding and risk aversion, if we increase the size of beta in any row (i.e. σ_S does not change its size) whatever panel in Table 1, it leads to an increase of common risk in exchange for a decrease in the specific risk. This effect leads to an increase of V^{SUB}.

Third, moving down across the same column of any panel, so that σ_S increases while the level of beta, risk aversion and excess stock holding remain unchanged, does not have a very clear effect on V^{SUB}. In most situations, there is evidence that higher values of σ_I produce a pattern of either a decreasing ESO value (see, for instance, the column of panel A corresponding to the values of $\beta = 1$ $\gamma = 2$ and $\theta = 30\%$) or increasing ESO value (see, for instance, the column of panel A corresponding to the values of $\beta = 1$ $\gamma = 2$ and $\theta = 10\%$). For the remaining situations, there is an (inverted) U-shaped behavior. See, for instance, column of panel A for the values of $\beta = 1$ $\gamma = 2$ and $\theta = 20\%$ for the U-shaped case, while column of panel B for the values of $\beta = 0$ $\gamma = 2$ and $\theta = 20\%$ for the inverted U-shaped case. In order to understand better this result for the idiosyncratic risk, Appendix D shows the analytical conditions that increases of σ_I lead to increases of V^{SUB} under the restricted version of our model which is the case of imposing both no vesting period and no employment shocks obtained in Corollary 4. It is also shown in Appendix D that a higher beta (which increases the total volatility) leads to a higher ESO value for this restricted model.

Fourth, the higher the risk aversion or the excess stock holding, the lower V^{SUB}. Fifth, the higher the employment shock intensity the lower V^{SUB}. Compare panels A and C when there is no vesting period, while panels B and D for the positive one. Last but not least, the higher the vesting period the lower V^{SUB}. Compare panels A and B for $\lambda = 10\%$ or panels C and D for $\lambda = 20\%$.

Finally, it is exhibited in Figure 2 the main findings concerning the impact on V^{OBJ} for different values of γ, θ and β. We take the values of $\sigma_S = 30\%$, $\nu = 3$ and $\lambda = 20\%$. The straight line at the top of the figure represents $V^{RN} = 6.240$ which corresponds to the first row in Panel D. It is shown that V^{OBJ} increases with β and it becomes higher the lower the risk aversion is. Henceforth, the higher θ the lower V^{OBJ}. So, the discount obtained as $V^{RN} - V^{OBJ}$ increases in any of the following situations: when the value of θ or γ increases, or when the value of β decreases.

[Figure 2 is about here]

3.2 Perpetual versus Finite Maturities

It becomes interesting to analyze if ESOs having finite maturities are adequately approximated by perpetual ones. We calculate ESO prices with finite maturities using the least-squares Monte Carlo algorithm (LSMC henceforth) of Longstaff and Schwartz (2001). We simulate the risk neutral price process but replacing r and q_s by $\hat{r}$ and $\hat{q_s}$, defined in Lemma 2, respectively.

At maturity, the ESO is exercised if it is in the money, then the subjective value of the ESO is $V_T^{SUB} = (S_T - K)^+$. One period before, at $T - \delta t$ where δt is the length of one time step, on one hand there is a probability equal to $1 - e^{\lambda\delta t}$ to abandon the firm and the payoff of the ESO would be $(S_{T-\delta t} - K)^+$. On the other hand, with probability $e^{\lambda\delta t}$ the employee remains in the firm and thus, he must decide either to hold or to exercise voluntary the ESO. The employee will exercise the ESO if $S_{T-\delta t} - K > e^{-\hat{r}\delta t}\mathbb{E}_{T-\delta t}\left[V_T^{SUB}\right]$, in this case the ESO value will be $S_{T-\delta t} - K$. Otherwise, the payoff will be the discounted expected one period ahead ESO value. Thus, the ESO value at any time t verifying that $T > t > \nu$ is computed as

$$V_t^{SUB} = e^{-\lambda\delta t}\left[X_t \mathbb{1}_{\{S_t - K < X_t\}} + (S_t - K)^+ \mathbb{1}_{\{S_t - K \geq X_t\}}\right] + \left(1 - e^{-\lambda\delta t}\right)(S_t - K)^+$$

where $X_t = e^{-\hat{r}\delta t}\mathbb{E}_t\left[V_{t+1}^{SUB}\right]$ is the discounted expectation of the ESO value[3]. The conditional expected

[3]Hull and White (2004) and Ammann and Seiz (2004) also introduce in the same way the exit rate for the backwards induction in their binomial tree models.

ESO value is computed by least-squares such that for those paths in the money, the one period ahead ESO value is regressed over some basis functions of the current stock price. We work backwards until the vesting or grant date with this scheme. We use 20,000 paths simulated with monthly frequency[4] and we take the average of 50 previous estimations using 25 different seeds plus the 25 antithetics.

[Figure 3 is about here]

Figure 3 exhibits alternative subjective ESO values without vesting period for different values of β and λ from Table 1. The values of the risk aversion and the total volatility are $\gamma = 2$ and $\sigma_s = 30\%$. The remaining parameters are the same as in Table 1. The selected time to maturities, (T), go from 5 years until 25 years, denoted as $V^{SUB}_{(T)}$. It is observed that ESO prices do not change from about 15 years until the end considering alternative values of θ, see the graphics on the left. The graphics of relative biases, $(V^{SUB} - V^{SUB}_{(T)})/V^{SUB}_{(T)}$, where V^{SUB} are the ESO values from Table 1 displayed on the right lead to a decreasing pattern as T increases. Indeed, this positive bias tends to be lower than 10% for the benchmark American-style ESO with a maturity of ten years. In short, the benchmark ESO value is well approximated through the perpetual one[5]. For the shortest maturity, $T = 5$, the largest bias is around 25% for $\lambda = 10\%$, while it decreases significantly to approximately 10% for $\lambda = 20\%$. Therefore, the larger the probability of employment shock, the better the approximation through the perpetual ESO becomes.

4 Incentive Effects

To analyze the ESO incentive effects, we focus on how the ESO subjective value changes in response to changes of the stock price (delta measure) and the volatility (vega measure). For the last case, we study the vega through the decomposition between systematic and idiosyncratic volatility. Note that the manager's characteristics, such as the risk aversion degree and diversification restrictions, are considered here since ESO incentives depend on the manager's subjective valuation. This leads that the proper measures become the subjective delta and vega which are perceived by the employee (not

[4]Stentoft (2004) obtains that the LSMC method with 10 exercise points per year produce very accurate prices compared with the ones obtained using a binomial model with 50,000 time steps. He argues that more accurate prices are obtained when increasing the simulated paths or the number of basis functions used as regressors.

[5]It is available upon request, though not reported here, a table similar to Table 1 for the case of an ESO with $T = 10$ years containing the ESO values underlying the four graphics on the left-hand side of the above figure.

the employer). We will compare these indexes with the market delta and vega corresponding to the either the employer or the employee without stockholding restrictions. For all situations, we assume a vesting period, ν, of three years and an intensity rate, λ, equal to 20%. Other parameters such as K, r, q_S and σ_M keep the same value as in Section 3. For the remaining parameters, i.e. β, γ and θ, their alternative sets of values are exhibited later in figures 4 and 5.

The main purpose of ESO packages is providing managers' incentives in order to raise shareholders' wealth (agency problem). This idea of aligning manager and shareholder interests relies upon the presumption that employees will maximize the firm value since they receive payouts contingent upon firm value. As noticed above, the option delta is the change in the subjective value, V^{SUB}, relative to the change in the company stock price, S. Since we cannot obtain a closed-form expression for this partial derivative, we approximate the delta by its finite counterpart using increments of one per cent in the stock price. Pictures on the left-hand side of figure 4 show the corresponding subjective deltas for several values of β, γ and θ. It is shown that the delta is always positive but with a decreasing slope which tends to be rather constant when the option is deeply in the money. Note also that the delta for the ESO market value, V^{RN}, acts as an upper bound. We conclude, in line with Ingersoll (2006), that this market delta overstates the executive's incentives. We also observe a decrease in the subjective delta as the level for risk aversion increases. This effect becomes more significant when θ increases when comparing picture 'a' ('b') with 'c' ('d') for the same level of β. Note that changes in beta are not important if you compare picture 'a' ('c') with 'b' ('d') for the same level of θ. Hence, the incentives to raise shareholders' wealth tend to reduce with a lower portfolio diversification of the executive. This evidence can also be shown as the difference between the market delta and the subjective one increases with θ, see again pictures 'a' ('b') and 'c' ('d').

As expected, the case of $\lambda = 0$ only makes delta values higher (a drop in λ from 20% to 0% roughly doubles the delta values) but it does not change the qualitative behavior, so it is not reported here. In short, we find that a higher likelihood of employment termination may reduce substantially the executive's incentives to increase shareholders' wealth.

[Figure 4 is about here]

While options may provide incentives for executives to work harder, they can also induce suboptimal risk-taking behavior (moral hazard problem). Hence, it becomes interesting to analyze the subjective vegas which measure the incentives related with the volatility. A higher volatility of the underlying asset return implies a higher value of traditional call options. Thus, there is a presumption that ESOs rise the incentive of taking riskier projects with higher returns. However, taking a new investment project can change both the systematic and the idiosyncratic volatilities. Since the executive's subjective valuation is influenced negatively by the idiosyncratic volatility, see Lemma 2 or Corollary 4, the source of the change in total volatility is relevant to establish how the manager's incentives are affected. Looking at pictures on the right-hand side of figure 4, we can see that the systematic-risk vega is positive with a decreasing slope for all reasonable values of β. In most situations, except for picture 'h', the vega for the ESO risk neutral or market value acts as a lower boundary. The behavior for vega seems quite predictable. In all cases, a higher market beta creates incentives to take up investment projects with returns highly correlated with the market portfolio return since it increases the subjective ESO value. Looking at pictures 'e' ('f') and 'g' ('h'), the systematic-risk vega measure changes due to increases in systematic risk while keeping constant the idiosyncratic level at 20% (30 %). We observe that the effects for the executive's degree of risk aversion do not become so important as in the case for the subjective delta. It is also shown that the size for the difference between the market systematic-risk vega and the subjective one increases for lower values of portfolio diversification, compare pictures 'e' ('f') and 'g' ('h'), when β increases. The effect of changing the idiosyncratic volatility level is less important.

Finally, we make the same analysis for the vega but changing now the composition of total volatility and keeping its size fixed, see figure 5. This new analysis is different to the one carried out before in the sense that total volatility, σ_S, raised because β increased. As we move through the x-axis to the right, σ_I increases while β decreases at the same time so as to hold a fixed value for σ_S across the different graphics in figure 5. It is shown that vega becomes negative being its size higher (lower) for higher (lower) values of the risk aversion degree (total volatility). The size of vega increases with the size of the idiosyncratic volatility which is in line with Ingersoll (2006).

[Figure 5 is about here]

5 Conclusions

In this paper we have examined how the valuation of perpetual ESOs can be achieved by using a stochastic discount factor derived from the constrained intertemporal optimization problem faced by the employee. Following Ingersoll (2006), the constraint comes from the obligation to hold a proportion of the company's stock higher than the optimal one. We obtain a stochastic discount factor that, by pricing the risk free asset, the market portfolio and the company's stock, can also price the option. In a first step we obtain a closed formula for the subjective value of ESOs with and without a vesting period. The subjective valuation of ESOs allows us to obtain the firm's cost by obtaining the subjective threshold price that triggers immediate exercise by the employee.

Despite considering perpetual ESOs, our valuation method approximates reasonably well the more realistic finite maturity case. In our sensitive analysis we have found that higher values for the degree of risk aversion, the excess stock holding or the risk of employment termination, reduces substantially the ESO subjective value. We must remark an interesting result of our sensitive analysis. That is, in the presence of employment shocks, there is no need for the subjective value of ESO to be positively related with the threshold that define the executive's optimal exercise policy. In particular, we have found that a higher idiosyncratic volatility may increase the subjective ESO value even though the threshold clearly falls. Thus, a higher value of the subjective ESO might not be positively correlated with a lower expected time to exercise.

References

[1] Ammann, M. and R. Seiz 2004. Valuing employee stock options: Does the model matter? *Financial Analysts Journal* 60(5):21–37.

[2] Bettis, J. C., J. M. Bizjak, and M. L. Lemmon 2005. Exercise behavior, valuation, and the incentive effects of employee stock options. *Journal of Financial Economics* 76(2):445–470.

[3] Black, F. and M. Scholes 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* 81(3):637–659.

[4] Cai, J. and A. M. Vijh 2005. Executive stock and option valuation in a two state-variable framework. *Journal of Derivatives* 12(3):9–27.

[5] Calvet, A. L. and A. H. Rahman 2006. The subjective valuation of indexed stock options and their incentive effects. *The Financial Review* 41:205–227.

[6] Carpenter, J. N. 1998. The exercise and valuation of executive stock options. *Journal of Financial Economics* 48:127–158.

[7] Cochrane, J. H. 2005. *Asset Pricing.* : Princeton University Press.

[8] Cochrane, J. H. and J. Saa-Requejo 2001. Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *Journal of Political Economy* 108(1):79–119.

[9] Cvitanic, J., Z. Wiener, and F. Zapatero 2008. Analytic pricing of employee stock options. *Review of Financial Studies* 21(2):683–724.

[10] Dixit, A. and R. Pyndick 1994. *Investment under Uncertainty.* : Princeton University Press.

[11] Financial Accounting Standard Board 1995. Statement of financial accounting standards n° 123: Accounting for stock-based compensation. *FASB.*

[12] Financial Accounting Standard Board. 2004. Statement of financial accounting standards n° 123 (revised 2004): Share-based payments. *FASB.*

[13] Hall, B. J. and K. J. Murphy 2000. Optimal exercise prices for executive stock options. *American Economic Review* 90:209–214.

[14] Hall, B. J. and K. J. Murphy 2002. Stock options for undiversified executives. *Journal of Accounting and Economics* 33(1):3–42.

[15] Henderson, V. 2005. The impact of the market portfolio on the valuation, incentives and optimality of executive stock options. *Quantitative Finance* 5(1):35–47.

[16] Huddart, S. 1994. Employee stock options. *Journal of Accounting and Economics* 18(2):207–231.

[17] Huddart, S. and M. Lang 1996. Employee stock option exercises. An empirical analysis. *Journal of Accounting and Economics* 21(1):5–43.

[18] Hull, J. C. and A. White 2004. How to value employee stock options. *Financial Analysts Journal* 60(1):114–19.

[19] Ingersoll, J. E. 2006. The subjective and objective evaluation of incentive stock options. *Journal of Business* 79(2):453–487.

[20] International Accounting Standard Board 2004. International financial reporting standard n^{o} 2: Share-based payment: 2004. *IASB* (London).

[21] Jennergren, L. P. and B. Näslund 1993. A comment on "valuation of executive stock options and the FASB proposal". *Accounting Review* 68(1):179–183.

[22] Johnson, S. and Y. Tian 2000. Indexed executive stock options. *Journal of Financial Economics* 57(1):35–64.

[23] Jorgensen, P. L. 2002. American-style indexed executive stock options. *European Finance Review* 6:321–358.

[24] Kulatilaka, N. and A. J. Marcus 1994. Valuing employee stock options. *Financial Analysts Journal* 50(6):46–56.

[25] Lambert, R. A., D. F. Larcker, and R. E. Verrecchia 1991. Portfolio considerations in valuing executive compensation. *Journal of Accounting Research* 29(1):129–149.

[26] Sircar, R. and W. Xiong 2007. A general framework for evaluating executive stock options. *Journal of Economic Dynamics and Control* 31(7):2317–2349.

[27] Tian, Y. S. 2004. Too much of a good incentive? the case of executive stock options. *Journal of Banking and Finance* 28:1225–1245.

[28] Yermack, D. 1998. Companies' modest claims about the value of CEO stock options awards. *Review of Quantitative Finance and Accounting* 10:207–226.

Appendix A

The Bellman's equation corresponding to the problem of maximizing (2) subject to the budget constraint (3) is:

$$0 = \max_{C,\omega} \left\{ \frac{C^{1-\gamma}}{1-\gamma} - \rho J + \left[\left(r + (\omega + \theta\beta)(\mu_M - r) \right) W - C \right] J_W + \frac{1}{2} \left[(\omega + \theta\beta)^2 \sigma_M^2 + \theta^2 \sigma_I^2 \right] W^2 J_{WW} \right\}$$

The first order conditions are:

$$\begin{aligned} C &= J_W^{-1/\gamma} \\ (\omega + \theta\beta) &= \frac{(\mu_M - r)}{\sigma_M^2} \left(\frac{-J_W}{W J_{WW}} \right) \end{aligned}$$

By substituting into the Bellman's equation we get after simplifying:

$$0 = \frac{\gamma}{1-\gamma} J_W^{1-1/\gamma} - \rho J + rWJ_W - \frac{1}{2} \left(\frac{\mu_M - r}{\sigma_M} \right)^2 \left(\frac{J_W^2}{J_{WW}} \right) + \frac{1}{2} \theta^2 \sigma_I^2 W^2 J_{WW}$$

Using the trial solution $J = \dfrac{b}{1-\gamma} W^{1-\gamma}$ we end up with the following solution for the optimal consumption and the market portfolio share in wealth:

$$C = aW \ ; \quad \omega = \frac{1}{\gamma} \frac{\mu_M - r}{\sigma_M^2} - \theta\beta$$

where

$$a \equiv b^{-1/\gamma} = \frac{\rho}{\gamma} - \frac{1-\gamma}{\gamma} r - \frac{1-\gamma}{2\gamma^2} \left(\frac{\mu_M - r}{\sigma_M} \right)^2 + \frac{1-\gamma}{2} \theta^2 \sigma_I^2$$

Substituting the optimal values of C and ω in the equation for the dynamics of wealth, we get:

$$\frac{dW}{W} = \left[r - a + \frac{1}{\gamma} \left(\frac{\mu_M - r}{\sigma_M} \right)^2 \right] dt + \frac{1}{\gamma} \left(\frac{\mu_M - r}{\sigma_M} \right) dZ_M + \theta \sigma_I dZ_I$$

Now, since the stochastic discount factor, Θ is defined as the discounted value of the marginal utility of wealth:

$$\Theta = e^{-\rho t} J_W = e^{-\rho t} b W^{-\gamma}$$

we apply Ito's lemma to get:

$$d\Theta = \frac{\partial \Theta}{\partial t} dt + \frac{\partial \Theta}{\partial W} dW + \frac{1}{2} \frac{\partial^2 \Theta}{\partial W^2} (dW)^2 = -\rho \Theta dt - \gamma \Theta \left(\frac{dW}{W} \right) + \frac{\gamma(\gamma+1)}{2} \Theta \left(\frac{dW}{W} \right)^2$$

By straightforward substitution and after some simplifications we get equation (6).

Appendix B

Equation (6) together with the money value of the ESO, V :

$$dV = \left[\left(\mu_S - q_S\right)V_S S + \frac{\sigma_S^2}{2}V_{SS}S^2\right]dt + \beta\sigma_M V_S S dZ_M + \sigma_I V_S S dZ_I$$

allow us to obtain equation our fundamental equation (8). Indeed, by omitting terms of order higher than dt:

$$\begin{aligned}
\mathbb{E}_0\left[Vd\Theta\right] &= -\left(r - \gamma\theta^2\sigma_I^2\right)V\Theta dt \\
\mathbb{E}_0\left[\Theta dV\right] &= \left(\mu_S - q_S\right)V_S S\Theta dt + \frac{\sigma_S^2}{2}V_{SS}S^2\Theta dt \\
\mathbb{E}_0\left[d\Theta dV\right] &= -\left(\mu_S - r\right)V_S S\Theta dt - \gamma\theta\sigma_I^2 V_S S\Theta dt
\end{aligned}$$

where$\mathbb{E}_0$ denotes the conditional expectation operator under the real measure and the CAPM condition, $\mu_S = r + \beta\left(\mu_M - r\right)$, has been used to obtain the third equation. Now, by straightforward substitution, we get:

$$0 = \mathbb{E}_0\left(Vd\Theta + \Theta dV + d\Theta dV\right)\left(1 - \lambda dt\right) + \mathbb{E}_0\left(Vd\Theta + \Theta\left(\Psi\left(S\right) - V\right) + d\Theta\left(\Psi\left(S\right) - V\right)\right)\lambda dt.$$

Hence,

$$-\left(r - \gamma\theta^2\sigma_I^2\right)V + \left(r - q_S - \gamma\theta\sigma_I^2\right)V_S S + \frac{\sigma_S^2}{2}V_{SS}S^2 + \lambda\left[\Psi\left(S\right) - V\right] = 0.$$

Finally, by defining $\hat{r} = \left(r - \gamma\theta^2\sigma_I^2\right)$ and $\hat{q}_S = \left(q_S + \gamma\theta\left(1-\theta\right)\sigma_I^2\right)$ we obtain equation (8).

Appendix C

The solution to equation (8) can be easily shown to be

$$V\left(S\right) = \begin{cases} \hat{A}_1 S^{\hat{\alpha_1}} & \text{if } S < K \\ \hat{B}_1 S^{\hat{\alpha_1}} + \hat{B}_2 S^{\hat{\alpha_2}} + \left(\dfrac{\lambda S}{\lambda + \hat{q}_S} - \dfrac{\lambda K}{\lambda + \hat{r}}\right) & \text{if } S \geq K \end{cases} \tag{18}$$

where $\hat{\alpha_1}$ and $\hat{\alpha_2}$ are, respectively, the positive and negative root of the quadratic equation $\hat{\alpha}^2 + \left(\hat{b} - 1\right)\hat{\alpha} - \hat{c}$, where:

$$\hat{c} \equiv \frac{2}{\sigma_S^2}\left(\hat{r} + \lambda\right) = -\hat{\alpha_1}\hat{\alpha_2}\ ;\quad \hat{b} \equiv \frac{2}{\sigma_S^2}\left(\hat{r} - \hat{q}_S\right) = 1 - \hat{\alpha_1} - \hat{\alpha_2} \tag{19}$$

In (8) the negative root has been eliminated in the region $S < K$ by imposing the boundary condition $\lim_{S\to 0} V\left(S\right) = 0$.

The constants $\hat{A}_1$ and $\hat{B}_2$ can be solved in terms of $\hat{B}_1$ and K by using the usual conditions of value matching and smooth pasting for $S = K$:

$$\begin{aligned}
\hat{A}_1 K^{\hat{\alpha_1}} &= \hat{B}_1 K^{\hat{\alpha_1}} + \hat{B}_2 K^{\hat{\alpha_2}} + \left(\frac{\lambda}{\lambda + \hat{q}_S}\right)\left(\frac{\hat{r} - \hat{q}_S}{\lambda + \hat{r}}\right) K \\
\hat{\alpha_1} \hat{A}_1 K^{\hat{\alpha_1}} &= \hat{\alpha_1} \hat{B}_1 K^{\hat{\alpha_1}} + \hat{\alpha_2} \hat{B}_2 K^{\hat{\alpha_2}} + \left(\frac{\lambda}{\lambda + \hat{q}_S}\right) K
\end{aligned}$$

whose solution is:

$$\hat{A}_1 = \hat{B}_1 + \frac{2\lambda}{\sigma_S^2}\left(\frac{1}{\hat{\alpha_1}(\hat{\alpha_1} - 1)}\right)\frac{K^{1-\hat{\alpha_1}}}{(\hat{\alpha_1} - \hat{\alpha_2})} \tag{20}$$

$$\hat{B}_2 = \frac{2\lambda}{\sigma_S^2}\left(\frac{1}{\hat{\alpha_2}(\hat{\alpha_2} - 1)}\right)\frac{K^{1-\hat{\alpha_2}}}{(\hat{\alpha_1} - \hat{\alpha_2})} \tag{21}$$

To determine the remaining constant, $\hat{B}_1$, and the threshold price, $S^{\star}$, we use equations (9) and (10) to get:

$$\hat{B}_1 \hat{S^*}^{\hat{\alpha_1}} + \hat{B}_2 \hat{S^*}^{\hat{\alpha_2}} + \left(\frac{\lambda \hat{S}^*}{\lambda + \hat{q}_S} - \frac{\lambda K}{\lambda + \hat{r}}\right) = \hat{S}^* - K \tag{22}$$

$$\hat{\alpha_1} \hat{B}_1 \hat{S^*}^{\hat{\alpha_1}} + \hat{\alpha_2} \hat{B}_2 \hat{S^*}^{\hat{\alpha_2}} + \frac{\lambda \hat{S}^*}{\lambda + \hat{q}_S} = \hat{S}^* \tag{23}$$

Solving first for $\hat{B}_1$ in equation (23) one gets:

$$\hat{B}_1 = \frac{-\hat{\alpha_2}}{\hat{\alpha_1}} \hat{B}_2 (\hat{S}^*)^{\hat{\alpha_2} - \hat{\alpha_1}} + \frac{1}{\hat{\alpha_1}}\left(1 - \frac{\lambda}{\lambda + \hat{q}_S}\right)(\hat{S}^*)^{1-\hat{\alpha_1}} \tag{24}$$

Finally, by combining equations (23) and (23) we get the implicit equation for solving for $\hat{S}^*$ which, by using the relations given in equation (19), can be written as equation (12) in the main text.

Appendix D

For a perpetual ESO with no vesting and no employment shocks, its value in the waiting region is given by

$$V^{SUB} = \left(\frac{S}{\hat{S}^{\star}}\right)^{\hat{\alpha}_1} (\hat{S}^{\star} - K) = \frac{(\hat{\alpha}_1 - 1)^{(\hat{\alpha_1} - 1)}}{\hat{\alpha}_1^{\hat{\alpha}_1}} K^{1-\hat{\alpha}_1}$$

using the notation of the main text.

Clearly, the impact of changes in the underlying parameters comes from its effect on $\hat{\alpha}_1$. In particular, a change in any of these parameters that increases $\hat{\alpha}_1$ has a negative impact on the subjective valuation of ESO. This is because

$$\left(\frac{\partial V^{SUB}}{\partial \hat{\alpha}_1}\right) = \frac{(\hat{\alpha}_1 - 1)^{\left(\hat{\alpha}_1 - 1\right)}}{\hat{\alpha}_1^{\hat{\alpha}_1}} K^{1-\hat{\alpha}_1}\left\{\ln\left(\frac{\hat{\alpha}_1 - 1}{\hat{\alpha}_1}\right) - \ln K\right\}$$

which is negative by taking $K > \left(\frac{\hat{\alpha}_1 - 1}{\hat{\alpha}_1}\right)$. Therefore, by computing the effect on the positive root we have also obtained the effect on the subjective value of ESO. To this end the following lemma, which is stated without proof, will be used

Lemma 7 *The partial derivatives $\partial\hat{\alpha}_1/\partial\hat{b}$ and $\partial\hat{\alpha}_1/\partial\hat{c}$ are given by*

$$\begin{aligned}
\frac{\partial\hat{\alpha}_1}{\partial\hat{b}} &= -\frac{1}{2}\left\{1+\frac{\left(\frac{1-\hat{b}}{2}\right)^2}{\sqrt{\left(\frac{1-\hat{b}}{2}\right)^2+\hat{c}}}\right\} = \frac{-\hat{\alpha}_1}{(\hat{\alpha}_1-\hat{\alpha}_2)} < 0 \\
\frac{\partial\hat{\alpha}_1}{\partial\hat{c}} &= \frac{1}{2}\left\{\frac{1}{\sqrt{\left(\frac{1-\hat{b}}{2}\right)^2+\hat{c}}}\right\} = \frac{1}{(\hat{\alpha}_1-\hat{\alpha}_2)} > 0
\end{aligned}$$

Using this lemma we obtain the following result:

Proposition 8 *For nonnegative values of market beta, an increase in market beta always increases the ESO subjective value. However, an increase in idiosyncratic volatility will increase ESO subjective value iff*

$$\hat{\alpha}_1(\hat{\alpha}_1-1) > 2\gamma\theta(\hat{\alpha}_1-\theta)$$

Proof.- From the definitions of $\hat{b}$ and $\hat{c}$ we compute the impact of changes in β and σ_I on $\hat{\alpha}_1$. Using the risk decomposition $\sigma_S^2 = \beta^2\sigma_M^2 + \sigma_I^2$, it is easy to find that

$$\frac{\partial\hat{b}}{\partial\beta^2} = -\frac{\sigma_M^2}{\sigma_S^2}\,\hat{b} \quad ; \quad \frac{\partial\hat{c}}{\partial\beta^2} = -\frac{\sigma_M^2}{\sigma_S^2}\,\hat{c}$$

and

$$\frac{\partial\hat{b}}{\partial\sigma_I^2} = -\left(\frac{\hat{b}}{\sigma_S^2}+\frac{2\gamma\theta}{\sigma_S^2}\right) \quad ; \quad \frac{\partial\hat{c}}{\partial\sigma_I^2} = -\left(\frac{\hat{c}}{\sigma_S^2}+\frac{2\gamma\theta^2}{\sigma_S^2}\right)$$

Thus

$$\frac{\partial\hat{\alpha}_1}{\partial\beta^2} = \frac{\partial\hat{\alpha}_1}{\partial\hat{b}}\frac{\partial\hat{b}}{\partial\beta^2} + \frac{\partial\hat{\alpha}_1}{\partial\hat{c}}\frac{\partial\hat{c}}{\partial\beta^2} = \left(\frac{\sigma_M^2}{\sigma_S^2}\right)\frac{\hat{\alpha}_1(1-\hat{\alpha}_1)}{(\hat{\alpha}_1-\hat{\alpha}_2)}$$

which is clearly negative as long as $\hat{\alpha}_1 > 1$. Similarly

$$\frac{\partial\hat{\alpha}_1}{\partial\sigma_I^2} = \frac{\partial\hat{\alpha}_1}{\partial\hat{b}}\frac{\partial\hat{b}}{\partial\sigma_I^2} + \frac{\partial\hat{\alpha}_1}{\partial\hat{c}}\frac{\partial\hat{c}}{\partial\sigma_I^2} = \frac{\hat{\alpha}_1(1-\hat{\alpha}_1)}{\sigma_S^2(\hat{\alpha}_1-\hat{\alpha}_2)} + \frac{2\gamma\theta(\hat{\alpha}_1-\theta)}{\sigma_S^2(\hat{\alpha}_1-\hat{\alpha}_2)}$$

Clearly the first term is negative (since $\hat{\alpha}_1 > 1$) whereas the second one is positive (since $\theta < 1$). From here we obtain the sign condition stated in the proposition.

□

Appendix E

We want to solve the following conditional expectation:

$$
\begin{aligned}
&\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}V(S_\nu)\right] = \hat{A}_1 E_0\left[\frac{\Theta_\nu}{\Theta_0}S_\nu^{\hat{\alpha_1}}\mathbb{1}_{\{S_\nu\le K\}}\right] + \hat{B}_1\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}S_\nu^{\hat{\alpha_1}}\mathbb{1}_{\{K\le S_\nu\le \hat{S^*}\}}\right] + \\
&+\hat{B}_2\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}S_\nu^{\hat{\alpha_2}}\mathbb{1}_{\{K\le S_\nu\le \hat{S^*}\}}\right] + \frac{\lambda}{\lambda+\hat{q}_S}\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}S_\nu\mathbb{1}_{\{K\le S_\nu\le \hat{S^*}\}}\right] - \frac{\lambda}{\lambda+\hat{r}}K\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}\mathbb{1}_{\{K\le S_\nu\le \hat{S^*}\}}\right] + \\
&+\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}S_\nu\mathbb{1}_{\{S_\nu\ge \hat{S^*}\}}\right] - K\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}\mathbb{1}_{\{S_\nu\ge \hat{S^*}\}}\right]
\end{aligned}
$$

Thus, all expectations take the general form $\mathbb{E}_0\left[\frac{\Theta_\nu}{\Theta_0}S_\nu^\alpha\mathbb{1}_{\{a\le S_\nu\le b\}}\right]$ for α any given real number. Given the stochastic dynamics driving $S_\nu^{\hat{\alpha}}$ and (Θ_ν/Θ_0) we have explicit expressions for each one of them, namely:

$$
\begin{aligned}
S_\nu^\alpha &= S_0^\alpha\cdot\exp\left\{\alpha\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)\nu + \alpha\sigma_S\sqrt{\nu}\varepsilon\right\}\\
\frac{\Theta_\nu}{\Theta_0} &= \exp\left\{-\left(\hat{r} + \frac{1}{2}\left(\frac{\mu_M - r}{\sigma_M}\right)^2 + \frac{\gamma^2\theta^2\sigma_I^2}{2}\right)\nu - \left(\frac{\mu_M - r}{\sigma_M}\right)\sqrt{\nu}\varepsilon_M - \gamma\theta\sigma_I\sqrt{\nu}\varepsilon_I\right\}
\end{aligned}
$$

where ε, ε_M and ε_I are *independent* standard normal variables satisfying $\sigma_S\varepsilon = \beta\sigma_M\varepsilon_M + \sigma_I\varepsilon_I$. Then, the expectation we seek to solve is given by a double integral of the form:

$$
\begin{aligned}
&\int_{\varepsilon_M}\int_{\varepsilon_I} S_0^\alpha\exp\left\{\alpha\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)\nu + \alpha\left(\beta\sigma_M\varepsilon_M + \sigma_I\varepsilon_I\right)\sqrt{\nu}\right\}\times\\
&\exp\left\{-\left(\hat{r} + \frac{1}{2}\left(\frac{\mu_M - r}{\sigma_M}\right)^2 + \frac{\gamma^2\theta^2\sigma_I^2}{2}\right)\nu - \left(\frac{\mu_M - r}{\sigma_M}\right)\sqrt{\nu}\varepsilon_M - \gamma\theta\sigma_I\sqrt{\nu}\varepsilon_I\right\}\times\\
&\phi(\varepsilon_M)\phi(\varepsilon_I)d\varepsilon_M d\varepsilon_I
\end{aligned}
\tag{25}
$$

where $\phi(\,\cdot\,)$ denotes the density function of a standard normal variable. Notice that the range of integration for ε_M and ε_I must be such that $a \le S_\nu \le b$.

Following Cochrane and Saa-Requejo (1999), we define the new variables:

$$
\delta_1 = \frac{\beta\sigma_M\varepsilon_M + \sigma_I\varepsilon_I}{\sigma_S} \quad ; \quad \delta_2 = \frac{\sigma_I\varepsilon_M - \beta\sigma_M\varepsilon_I}{\sigma_S}
$$

Notice that δ_1 and δ_2 are too *independent standard normal variables.* By reversing the change we have the following expression in terms of ε_M and ε_I:

$$
\varepsilon_M = \frac{\beta\sigma_M\delta_1 + \sigma_I\delta_2}{\sigma_S} \quad ; \quad \varepsilon_I = \frac{\sigma_I\delta_1 - \beta\sigma_M\delta_2}{\sigma_S}
$$

After substitution into equation (25) we get the following expression:

$$
\begin{aligned}
&S_0^{\alpha}\exp\left\{\alpha\left(\mu_S - q_S - \frac{\sigma_S^2}{2}\right)\nu - \left(\hat{r} + \frac{1}{2}\left(\frac{\mu_M - r}{\sigma_M}\right)^2 + \frac{\gamma^2\theta^2\sigma_I^2}{2}\right)\nu\right\}\times \\
&\int_{\delta_1}\int_{\delta_2}\exp\left\{\left[\alpha\sigma_S - \left(\frac{\beta(\mu_M - r) + \gamma\theta\sigma_I^2}{\sigma_S}\right)\right]\sqrt{\nu}\delta_1\right\}\times \\
&\exp\left\{-\left[\left(\frac{(\mu_M - r)}{\sigma_M}\right) - \beta\sigma_M\gamma\theta\right]\frac{\sigma_I}{\sigma_S}\sqrt{\nu}\delta_2\right\}\phi(\delta_1)\phi(\delta_2)d\delta_1 d\delta_2
\end{aligned}
\tag{26}
$$

We turn next to the specification of the integration region for each of the new variables. Clearly, for $a \leq S_\nu \leq b$ we have the following boundaries in terms of the δ_1 variable:

$$
\frac{\ln(a/S_0) - (\mu_S - q_S - \sigma_S^2/2)\nu}{\sigma_S\sqrt{\nu}} \leq \delta_1 \leq \frac{\ln(b/S_0) - (\mu_S - q_S - \sigma_S^2/2)\nu}{\sigma_S\sqrt{\nu}}
$$

or more compactly $A \leq \delta_1 \leq B$. By the other hand the range of integration for δ_2 is unrestricted. Hence, by omitting the exponential term that appears outside the double integral (26), we are left with:

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{\delta_2}\exp\left\{-\left[\left(\frac{\mu_M - r}{\sigma_S}\right) - \beta\sigma_M\gamma\theta\right]\left(\frac{\sigma_I}{\sigma_S}\right)\sqrt{\nu}\delta_2 - \frac{1}{2}\delta_2^2\right\}d\delta_2\times \\
&\frac{1}{\sqrt{2\pi}}\int_A^B\exp\left\{\left[\alpha\sigma_S - \left(\frac{\beta(\mu_M - r) + \gamma\theta\sigma_I^2}{\sigma_S}\right)\right]\sqrt{\nu}\delta_1 - \frac{1}{2}\delta_1^2\right\}d\delta_1
\end{aligned}
$$

Each integral is solved by completing the square. Thus, for the first integral, we get:

$$
\exp\left\{\frac{1}{2}\left[\left(\frac{\mu_M - r}{\sigma_M}\right) - \beta\sigma_M\gamma\theta\right]^2\left(\frac{\sigma_I}{\sigma_S}\right)^2\nu\right\}
$$

whereas for the second integral we obtain:

$$
\exp\left\{\frac{1}{2}\left[\alpha\sigma_S - \left(\frac{\beta(\mu_M - r) + \gamma\theta\sigma_I^2}{\sigma_S}\right)\right]^2\nu\right\}\times\left\{\Phi\left[\frac{\ln(b) - \mu - \alpha\sigma^2}{\sigma}\right] - \Phi\left[\frac{\ln(a) - \mu - \alpha\sigma^2}{\sigma}\right]\right\}
$$

for $\mu = \ln(S_0) + \left(\hat{r} - \hat{q}_S - (\sigma_S^2/2)\right)\nu$ and $\sigma = \sigma_s\sqrt{\nu}$. In the computation of this integral we have made use of the relationship $\hat{r} - \hat{q}_S = r - q_S - \gamma\theta\sigma_I^2$ and the CAPM condition $(\mu_s - r) = \beta(\mu_M - r)$.

Finally and after some algebra, the product of the three remaining exponentials can be greatly simplified to:

$$
\exp\{-\hat{r}\nu\}\exp\left\{\alpha\mu + \alpha^2\frac{\sigma^2}{2}\right\}
$$

Summing up, we have obtained that:

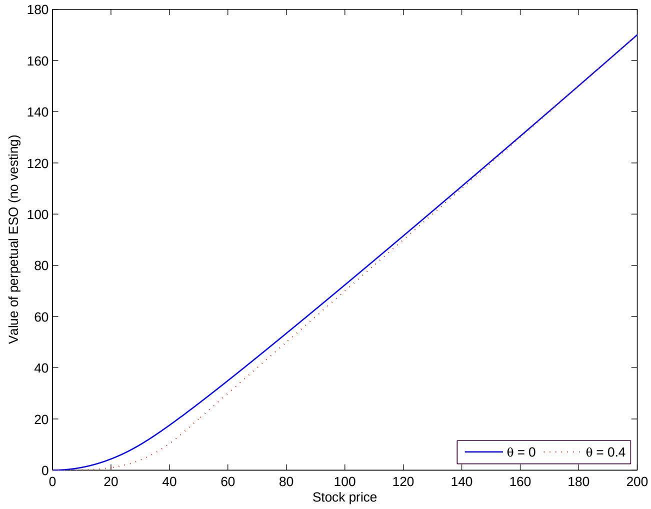
$$
\mathbb{E}_0\left[\left(\frac{\Theta_\nu}{\Theta_0}S_\nu^{\alpha}\right)\mathbb{1}_{\{a\leq S_\nu\leq b\}}\right] = \exp\{-\hat{r}\nu\}\exp\{\alpha\mu + \alpha^2\sigma^2\}\times\left\{\Phi\left(\frac{\ln b - \mu - \alpha\sigma^2}{\sigma}\right) - \Phi\left(\frac{\ln a - \mu - \alpha\sigma^2}{\sigma}\right)\right\}
$$

Table 1: Perpetual subjective Valuation.

		$\beta = 0$					$\beta = 1$				
	θ	0.00	0.10	0.20	0.30	0.40	0.00	0.10	0.20	0.30	0.40
γ	σ_s	Panel A: $\lambda = 0.1.\ \nu = 0$									
2	0.30	11.000	8.802	7.404	6.420	5.683	11.000	9.673	8.714	7.987	7.416
	0.40	12.859	9.617	7.813	6.624	5.768	12.859	10.266	8.694	7.611	6.810
	0.60	16.248	10.866	8.410	6.923	5.911	16.248	11.273	8.899	7.432	6.419
4	0.30	11.000	7.273	5.415	4.281	3.516	11.000	8.605	7.127	6.105	5.348
	0.40	12.859	7.644	5.456	4.215	3.416	12.859	8.527	6.453	5.190	4.332
	0.60	16.248	8.174	5.538	4.185	3.345	16.248	8.656	6.007	4.596	3.719
		Panel B: $\lambda = 0.1.\ \nu = 3$									
2	0.30	9.778	7.692	6.365	5.415	4.689	9.778	8.516	7.613	6.928	6.392
	0.40	11.324	8.232	6.450	5.221	4.310	11.324	8.858	7.337	6.259	5.446
	0.60	14.095	8.756	6.054	4.356	3.216	14.095	9.187	6.616	4.961	3.823
4	0.30	9.778	6.233	4.382	3.169	2.311	9.778	7.505	6.098	5.101	4.343
	0.40	11.324	6.259	3.892	2.466	1.560	11.324	7.159	5.012	3.617	2.645
	0.60	14.095	5.733	2.636	1.205	0.533	14.095	6.296	3.193	1.635	0.828
		Panel C: $\lambda = 0.2.\ \nu = 0$									
2	0.30	8.296	6.930	5.994	5.306	4.774	8.296	7.485	6.863	6.370	5.969
	0.40	9.951	7.863	6.597	5.720	5.066	9.951	8.297	7.216	6.434	5.837
	0.60	13.080	9.380	7.507	6.311	5.468	13.080	9.677	7.884	6.718	5.885
4	0.30	8.296	5.920	4.612	3.766	3.170	8.296	6.803	5.807	5.086	4.536
	0.40	9.951	6.492	4.856	3.865	3.197	9.951	7.116	5.610	4.639	3.952
	0.60	13.080	7.340	5.179	3.990	3.238	13.080	7.716	5.573	4.360	3.574
		Panel D: $\lambda = 0.2.\ \nu = 3$									
2	0.30	6.240	5.062	4.263	3.668	3.199	6.240	5.538	5.009	4.593	4.258
	0.40	7.365	5.560	4.433	3.625	3.011	7.365	5.939	4.996	4.300	3.761
	0.60	9.450	6.120	4.290	3.108	2.304	9.450	6.404	4.674	3.528	2.729
4	0.30	6.240	4.192	3.019	2.215	1.630	6.240	4.954	4.100	3.470	2.978
	0.40	7.365	4.320	2.743	1.758	1.120	7.365	4.894	3.495	2.550	1.878
	0.60	9.450	4.077	1.901	0.874	0.389	9.450	4.464	2.296	1.184	0.602

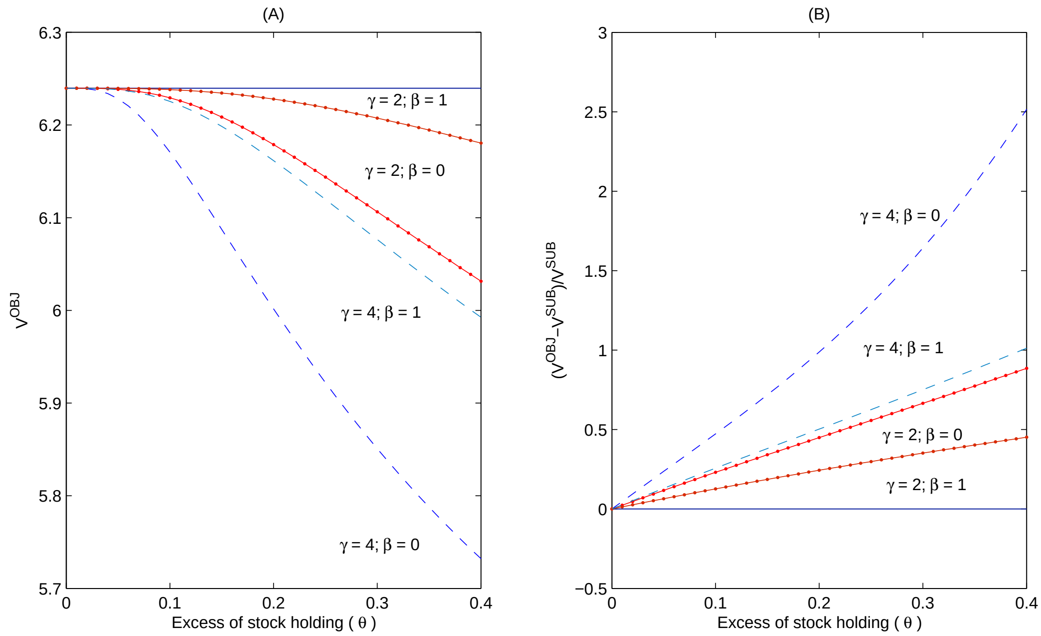
This table shows the subjective ESO value obtained simulating the ESO value for the first waiting region described in equation (11) when there is no vesting period and equation (16) otherwise. The first column, γ, denotes the risk-aversion coefficients considered. The second column, σ_s, contains the different levels of firm stock volatility. The following 5 columns are obtained with $\beta = 0$ and the last 5 with $\beta = 1$. Each column hold a different value of θ ranging from 0 to 0.4. Moreover, the table is divided in four panels for different combinations of the employee shock rate, λ, and the ESO vesting period ν.

Figure 1: Subjective value of ESO as a function of the stock price for several values of θ. No vesting



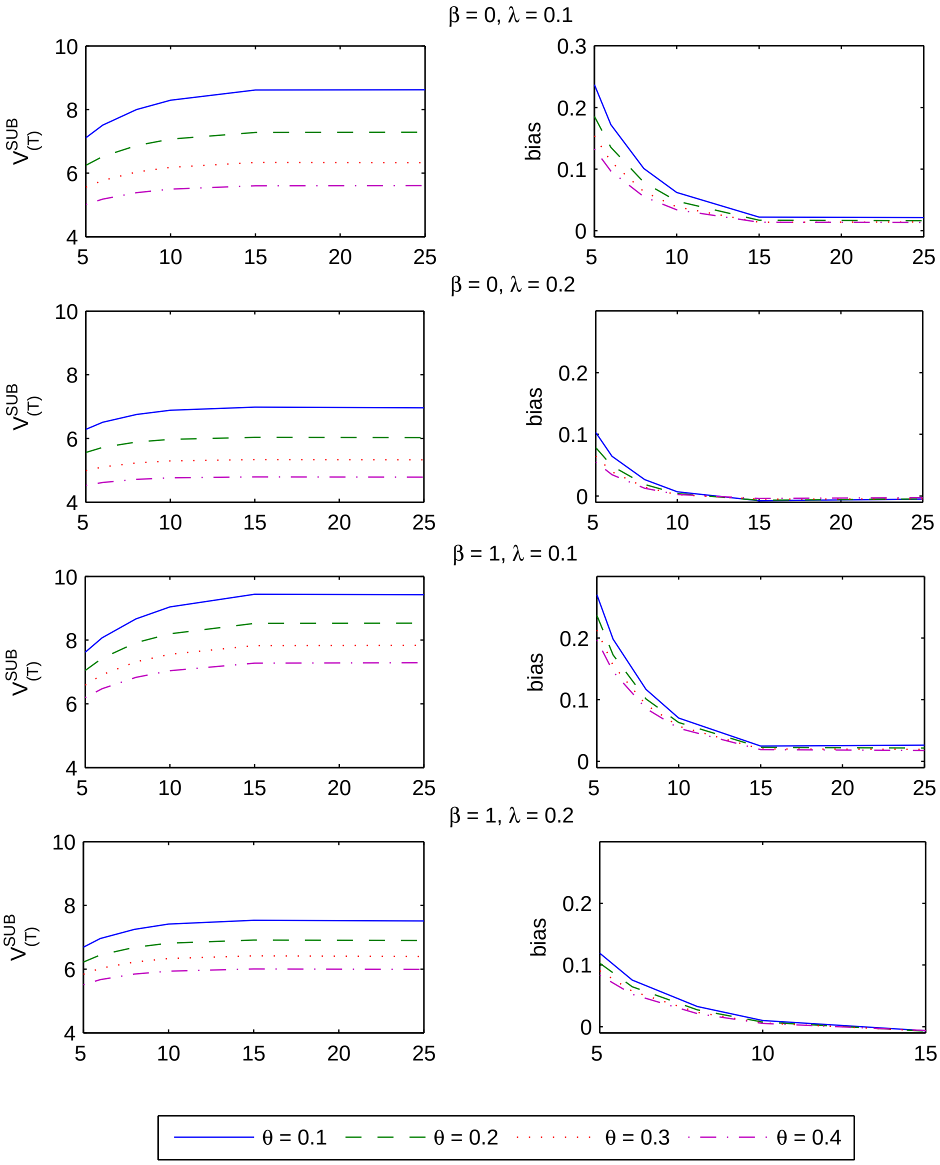
The figure plots the subjective value of ESO as a function of the stock price for different values of θ. The value of the remaining parameters are $r = 6\%$, $q_S = 1.5\%$, $\lambda = 20\%$, $\sigma_M = 20\%$, $\sigma_S = 40\%$ and $K = \$30$. The thick line ($\theta = 0$), which represents the market value V^{RN}, acts as an upper bound for the subjective valuation of ESO

Figure 2: Firm cost of perpetual ESO. Vesting = 3 years



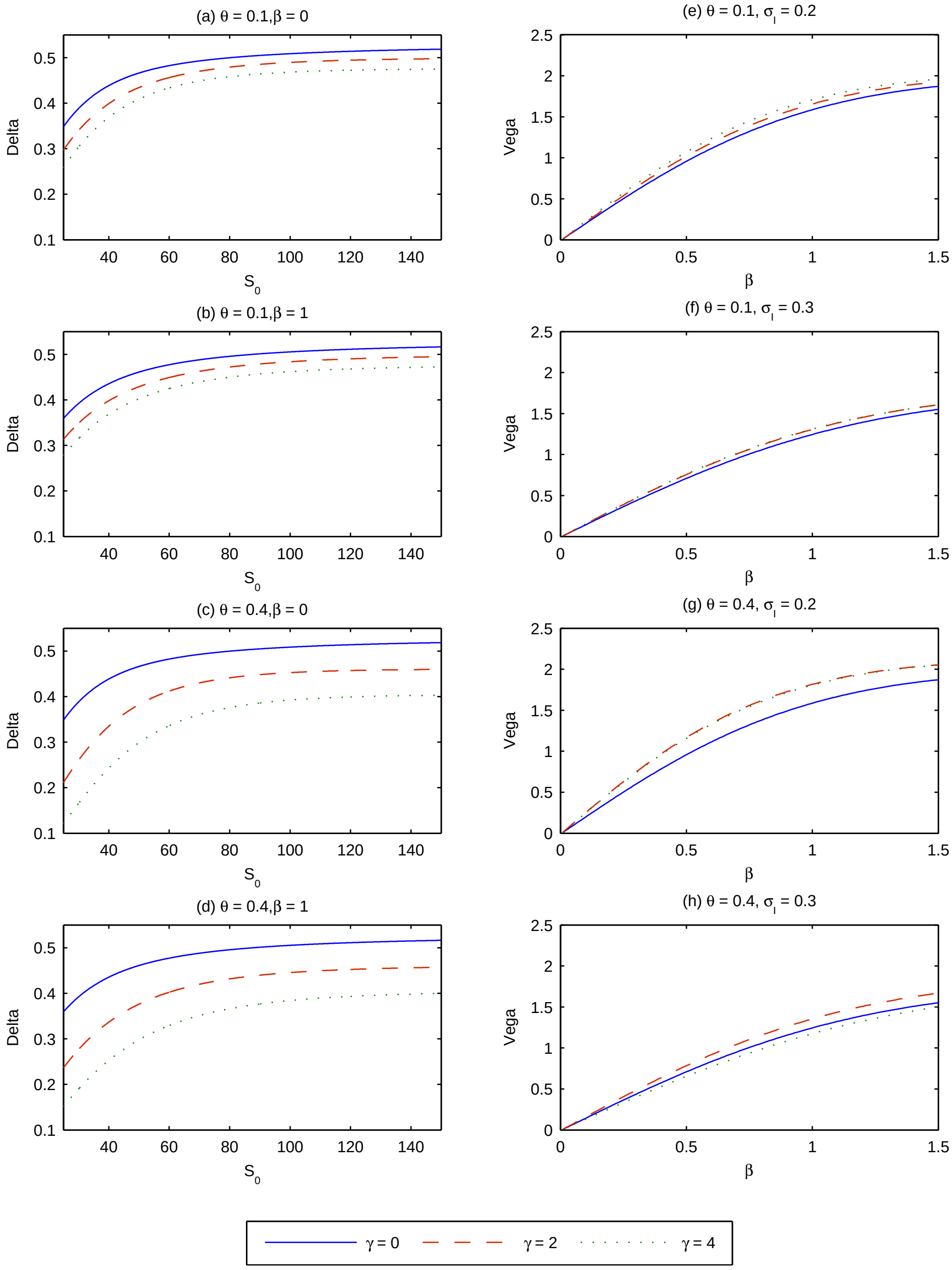
Panel A represents the firm cost, V^{OBJ}, of a perpetual ESO for several values of γ and β. Other values of the parameters are $\lambda = 20\%$ and $\nu = 3$ years. Panel B represents the deviations of the objective value from the subjective value in terms of the latter. In both panels, the thick line represents the risk neutral value, V^{RN} of the perpetual ESO.

Figure 3: Maturity Plot



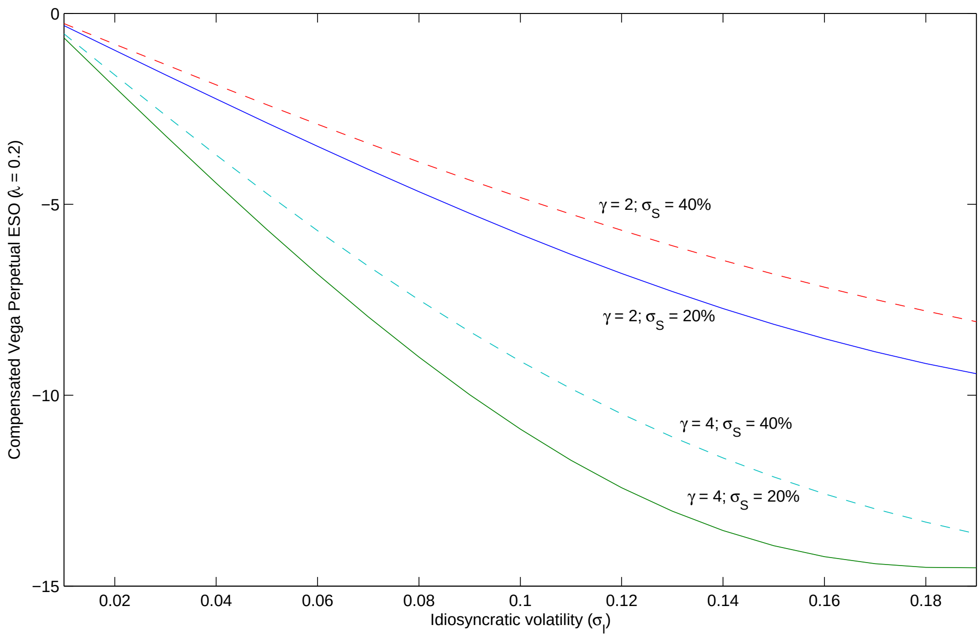
This figure shows the subjective value of the ESO (left hand graphics) for different maturities (in x-axis) ranging from 5 to 25 years. Each graphic hold a different pair of values for (β, λ), and each line corresponds to a different value of θ. The right hand graphics display the relative bias of the perpetual ESO computed as $(V^{SUB} - V^{SUB}_{(T)})/V^{SUB}_{(T)}$, where V^{SUB} are the ESO values from Table 1. $V^{SUB}_{(T)}$ has been obtained through simulations. We use 20,000 paths and we take the average over 50 previous estimates obtained with 25 seeds and the 25 antithetics.

Figure 4: Option Greeks



Panels (a) to (d) plots the Delta of the subjective ESO. The parameters not shown takes the following values: $r = 6\%$, $q_S = 1.5\%$, $\sigma_M = 20\%$, $\sigma_I = 30\%$, $\lambda = 20\%$ and $K = 30$. In the numerical computation of Delta we have used V^{SUB} as given in proposition 5 for successive values of S is steps of 1 per cent. Panels (e) to (h) plots the systematic Vega of the subjective ESO. The parameters not shown takes the following values: $r = 6\%$, $q_S = 1.5\%$, $\sigma_M = 20\%$, $\lambda = 20\%$ and $K = 30$. For the numerical computation of the systematic Vega we have also used V^{SUB} as given in proposition 5, for successive values of β is steps of 1 percentage point.

Figure 5: Compensated Option Vega



The figure plots the numerical partial derivative of V^{SUB} with respect to changes in σ_I so that the total volatility, σ_S, does not change. The parameters not shown takes the following values: $r = 6\%$, $q_S = 1.5\%$, $\sigma_M = 20\%$ and $K = 30$.