The Value of Internal Funds

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Abstract

The paper analyzes the value of internal funds and thereby derives optimal saving policy. Using a real options approach, the paper focuses on a capacity expansion problem in which a firm can hedge its future dependence on external capital markets by retaining cash. Firms can increase their value if they optimally trade off costs of external finance against agency costs of free cash flow. Departing from standard textbook approaches, the paper shows that one dollar of cash can be valued at a premium to its notional amount. It turns out that the relative gain from saving is most significant for firms with low levels of expected profitability. Finally, the value of internal funds is negatively related to volatility which marks a difference to existing literature.

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1 Introduction

It is known since Modigliani and Miller [13, 14] that in a frictionless world financing and payout policy does not affect firm value. Academic research over the past 50 years has shown that if one or several of these assumptions are relaxed, the predictions regarding the effect on firm value change dramatically.

Using an asymmetric information framework, Myers and Majluf [15] argue that a firm's financing decision follows a pecking order where corporations prefer internally generated cash over external funding and debt to equity. Their argument relies on asymmetric information between the firm and its investors and on the fact that this information can not be transferred costlessly. Specifically, they argue that concerns over asymmetric information favor aggressive cash retention. However, as Jensen [11] recognizes, there are also agency problems due to asymmetric information between management and shareholders which makes it costly to hoard cash within the corporate shell.

This paper focuses on the trade-off between costs of external and internal finance and thereby analyzes the value of internal funds and optimal retention policy. Within a real options framework, the paper answers the question whether a firm can increase its value by retaining cash rather than paying out dividends. Specifically, I model an all-equity financed firm which has the option to expand capacity. The justification for not including debt in the model is given by the desire to avoid capturing effects that cash serves as some form of negative debt. In other words, the focus on an all-equity financed firm directly implies that the value of cash or internally generated funds is not related to the option to avoid bankruptcy when the firm is doing badly.

The firm can exercise its growth option by paying the corresponding investment costs. If the firm has no cash at hand, the necessary amount has to be raised externally which unfortunately comes at a cost. On the other hand, holding cash within the firm also decreases shareholder value as management might waste resources which in turn induces costly monitoring activities by shareholders. The firm therefore has to trade off costs of external finance against agency costs of free cash flow to optimally exercise its option and maximize firm value.

The paper quantifies the gain from saving cash to which I will simply refer to as the value of internal funds. For what follows, the terms cash and internal funds are used interchangeably. Clearly, the firm will only want to retain earnings when the potential gain by doing so is not offset by the corresponding agency costs of free cash flow. The paper further analyzes the marginal value of cash to the firm and shows that even in absence of bankruptcy a firm might value one dollar of additional funds at more than its notional amount. The paper goes on to show that the effect of volatility on the value of internal funds is ambiguous and mostly negative. This is because cash derives its value by possibly avoiding costs of external finance when exercising the option. However, in the majority of cases costs of external finance lose their relative value when volatility is increased. Finally, the relative gain from saving is most significant for firms with low levels of profitability.

Optimal retention policy depends on the marginal value of cash, the severity of agency costs and the level of current cash flow. The model solves for optimal saving policy and also describes the marginal value of internal funds depending on different allocations of cash and cash flow. The marginal value of cash is influenced by costs of external finance, agency costs of free cash flow, the level of the firm's cash account and current cash flow. Firms value cash most when financing costs are high, agency costs small and the probability of exercising the option is relatively high.

Finally, the paper also provides a simplified version of the model which permits for a closed-form solution for the upper bound of the value of internal funds. This is useful as a first approximation to determine whether saving cash makes sense at all. Besides, it helps to gain basic insights of why volatility mostly has a negative effect on the value of internal funds.

In a related paper, Gamba and Triantis [8] determine optimal capital structure of a firm which can invest in profitable growth opportunities. They use a neoclassical model in which the firm is partly financed with equity and debt and can decide whether it retains earnings, pays a dividend or pays down debt. Saving cash serves two functions. First, it allows the firm to avoid defaulting in low profitability states as the cash on hand decreases its net debt exposure. Second, by making external financing costly it allows the firm to prevent additional financing costs when growth opportunities are exercised in high profitability states. They find that the value of financial flexibility can be quite large in the presence of profitable growth opportunities or when the firm is exposed to negative income shocks. In a similar fashion, Asvanunt et al. [1] derive optimal capital structure for a firm in the presence of growth opportunities when the firm is allowed to save cash to avoid costly external financing.

This paper differs from Gamba and Triantis and Asvanunt et al. on several aspects. First, by focusing on an all-equity financed firm it derives a pure value of cash which is not influenced by the fact that cash serves as some form of negative debt. Gamba and Triantis acknowledge that the value of financial flexibility is driven by two factors, namely the firms ability to optimally exercise its growth opportunities and the reduction in its net debt exposure. Second, contrary to both papers I assume quadratic agency costs of free cash flow to account for the fact that managers might engage in empire building when cash reserves are abundant. The assumption of convex agency costs considers the fact that retaining low levels of cash is cheaper than building up huge cash reserves. Third, I am able to derive a closed-form solution for the upper bound of the value of internal funds. This is again due to the fact that the focus on an all-equity financed firm allows me to separate the value of cash from debt related effects. Using this upper bound, I derive comparative statics which are then compared to the full model.

Other related work includes Boyle and Guthrie [2] who analyze a firm's dynamic investment decision where the firm is allowed to save cash to relax an exogenously given financing constraint resulting from asymmetric information. They show that due to the possibility of future earnings shocks, a firm may be willing to exercise its growth option prior to the benchmark case established by an otherwise unconstrained firm. This setup is different from my model, where the firm starts as a constrained firm and can reduce its dependence on external capital markets by engaging in precautionary saving.[1]

[1]Hirth and Uhrig-Homburg [10] extend their work and introduce financing costs into the original model. However, similar to the underlying paper by Boyle and Guthrie the focus is not on the value of cash but on optimal investment timing. Another difference concerns the general setup of the model which will be discussed in section [2].

Finally, using a representative agent's framework, Eisfeldt and Rampini [6] study level and dynamics of the value of aggregate liquidity when external shocks occur. They are the first to explicitly combine two previously mentioned frictions, namely costs of external finance and agency costs of free cash-flow in order to investigate the relationship between the value of liquidity and financing shortfalls. They find that aggregate value is highest when investment opportunities are abundant but levels of current cash flow are low.

Empirical evidence regarding payout policy by Fama and French [7] shows that the proportion of dividend payers in the US has fallen from 66.5% percent in 1978 to 20.8% in 1999. Their studies reveal that small firms with abundant growth opportunities and low profit levels generally don't pay dividends. This is confirmed by DeAngelo, DeAngelo and Skinner [3] although they observe that aggregate real dividends increased over the two decades mentioned above. DeAngelo, DeAngelo and Stulz [4] formulate a life-cycle theory of payout decisions and show that dividends are most likely to be paid out when firms have a high ratio of retained earnings to total equity.

The paper proceeds as follows. In section 2, I first introduce the model and the corresponding valuation equations. Besides, I derive a closed-form solution for a simplified version of the model and perform comparative statics. Section 3 finally computes the value of internal funds for different parameter values and tests whether the comparative statics of the simplified model also hold for the general case. Section 4 concludes.

2 The Model

This paper derives the value of internal funds. As Modigliani and Miller [13, 14] have shown, in a frictionless world financing, payout and investment policy are independent of each other. It does not matter whether a firm raises cash externally or uses internally generated funds to pay for an investment. To make internal financing matter, I therefore introduce two frictions, namely costs of external finance and agency costs of free cash flow. The value of internal funds is derived by calculating the value to internally finance an investment. For reasons of brevity, I may also refer to this value as simply the financing option.

Similar to Dixit and Pindyck [5] and McDonald and Siegel [12], I focus on the case when a firm faces a growth option whose value has to be determined. More specifically, the firm has the option to expand capacity by paying some investment costs IC. Departing from traditional real option models, the paper focuses on the question of how these investment costs are paid. The intuition is as follows. If at the time of capacity expansion the firm has sufficient cash available, it will be able to internally finance the investment at its true investment costs IC. However, if it turns out that liquid funds are insufficient - which might be the case if the company had paid out all its cash as dividends to its shareholders - the firm has to pay IC plus the corresponding costs of external finance for raising the entire amount. Alternatively, it is also possible that the firm has some cash at hand and only needs to raise the remaining part it lacks. In this case, total investment costs would be higher than when the firm is able to internally finance the investment but they still would be lower than when all funds have to be raised externally. Denoting IC_E and IC_S as total investment costs when the firm has to raise all or some amount externally, it follows that $IC \leq IC_S \leq IC_E$. However, their exists a trade-off as holding cash is costly because managers might just simply waste it.

To test whether there is a value of internal funds, I proceed as follows. I first establish a benchmark case by determining firm and option value for an all-equity firm which pays out all earnings as a dividend to its shareholders. I then derive the value of the financing option by allowing the firm to save for future investment and comparing corresponding option and firm values to the benchmark case. I conclude the section by deriving a closed-form solution for the upper bound of the financing option.

2.1 Basic Setup and Benchmark Case

As mentioned before, the value of internal funds is determined in the context of a capacity expansion problem. Thereby, I make use of the basic idea developed by McDonald and Siegel [12] and Dixit and Pindyck [5].

Consider a firm which produces a single product and operates at some initial capacity level K_0.[2] The cash flow produced by the firm is risky and follows a Geometric Brownian Motion

$$dx = \mu x dt + \sigma x dW^Q \tag{1}$$

where dW^Q is a standard Brownian motion under the riskneutral measure Q and μ and σ are mean and volatility of the growth rate of x. I further assume that there exists a traded asset being perfectly correlated with the

[2] For presentational purposes, the initial capacity level is normalized to 1.

firms cash flow which has the following dynamics $dX = rXdt + \sigma X dW^Q$ where $r > \mu$ and $\delta \equiv r - \mu$.

The firm is all-equity financed such that all earnings accrue to shareholders either via dividend payments or via capital gains. If the firm retains its earnings, it can put the money on the cash account where it earns the riskless return r.

However, following Jensen [11] saving cash is costly as management might be more likely to engage in empire building when cash reserves are abundant. Shareholders therefore would want to monitor the firm which unfortunately comes at a cost. I follow Eisfeldt and Rampini [6] in assuming that only the fraction of the operating cash flow which is retained within the firm is subject to quadratic agency costs. Their main argument is that liquid funds can be allocated to a financial intermediary such that each period only the retained fraction of earnings has to be monitored. Letting C denote the cash account and combining above, we get that

$$dC = \left\{ \alpha x - \frac{\phi}{2}(\alpha x)^2 + rC \right\} dt \tag{2}$$

which also implies an instantaneous dividend payment equal to $(1 - \alpha)x$. To make the saving decision potentially matter, I further assume that the firm starts with no initial cash at hand, i.e. $C_0 = 0$.

The explicit treatment of agency costs of free cash flow marks a sharp distinction to the models of Asvanunt et al. [1] and Gamba and Triantis [8] as saving becomes increasingly expensive the higher the fraction of retained earnings.[3] This setup is also different from Boyle and Guthrie [2] who assume distinct dynamics for operating profits and cash account. This is due to the fact that they investigate the possibility of future financing shortfalls and its implications for optimal exercise policy compared to an otherwise unconstrained firm.[4] In this paper, the focus is on another aspect. Starting with a firm which has to finance the whole project externally, I analyze how much value the firm would add by not paying out dividends and instead saving the cash to reduce future financing needs.

The firm has the option to increase capacity to a higher level K_1 by paying the necessary investment costs IC. However, if it lacks internal funds it has to raise all or part of the missing amount externally. External financing unfortunately comes at a cost. Specifically, I will consider the following general cost function $e(c)$, where

$$e(c) = \gamma_0 + \gamma_1 c_t + \gamma_2 c_t^2 \tag{3}$$

and $c_t = IC - C_t$ for the case when $C_t \leq IC$ and zero else. The specification of this function has been taken and adapted from Hennessy and Whited [9] who structurally estimate external financing costs. Total costs of capacity expansion will therefore be given by the sum of investment costs and costs of external finance, i.e. $IC_{E,S} = IC + e(c)$.

[3] Specifically, Gamba and Triantis assume that there is a tax disadvantage of keeping the cash within the firm resulting in a linear treatment of agency costs. Asvanunt et al. assume that the return on the cash account is lower than the risk-free rate r, i.e. $r_x < r$.

[4] Boyle and Guthrie assume that prior to exercising the growth option the firm consists of assets in place G and the cash account X. Assets in place generate an income stream equal to $\nu G dt + \phi G dZ$ which directly affects the cash account whose dynamics are given by $dX = rXdt + \nu G dt + \phi G dZ$.

Total firm value is finally given by the sum of expected dividend payments and expected capital gains which include the cash retained within the firm and the capital gain due to potential capacity expansion.

Proposition 1 *Total firm value, denoted by $F(x, C)$ is a function of both state variables x and C and has to satisfy the following Hamilton-Jacobi-Bellman equation under the risk-neutral measure Q*

$$rF = \max_{\alpha} \left\{ (1-\alpha)x + (r-\delta)xF_x + (\alpha x - \frac{\phi}{2}(\alpha x)^2 + rC)F_C + 1/2\sigma^2 x^2 F_{xx} \right\} \tag{4}$$

with the additional requirement that $\alpha \in [0, 1]$.

Proof: See Appendix.

In order to determine total firm value and the value of internal funds, one has to solve the HJB-equation with respect to the following boundary conditions.

$$\begin{gathered} F(0, C_t) = C_t \\ F(x^*, C_\tau) = \tfrac{K_1 x^*}{\delta} + C_\tau - IC_S \\ F_x(x^*, C_\tau) = \tfrac{K_1}{\delta} \end{gathered} \tag{5}$$

The first condition says that if the value of the cash flow hits zero, the firm is liquidated and is only worth the value of of the cash account, C_t. The second condition implies that at the time of exercising the option the firm receives the payoff of the capacity expansion, pays the corresponding costs IC_S and is worth the amount of cash it has on hand. The last condition is the traditional smooth-pasting condition ensuring optimal exercise policy. The value-matching condition reflects the fact that after exercising the option all future earnings are paid out as dividends such that $F(x_\tau, C_\tau) = E_\tau^Q \left[\int_\tau^\infty e^{-r(t-\tau)} K_1 x_t dt\right] + C_\tau$.

2.2 The Value of Internal Funds

The value of internal funds is derived by comparing total firm value under optimal saving policy to the case when all earnings are paid out as dividends. The value of this benchmark case is derived by solving the following PDE.

Proposition 2 *The benchmark case is assumed to be an all-equity firm which pays out all earnings as a dividend to its shareholders. Total firm value, denoted as $F^B(x, C)$ satisfies the following PDE*

$$rF = x + (r - \delta)xF_x + rCF_C + 1/2\sigma^2 x^2 F_{xx} \tag{6}$$

For the case when $C_0 = 0$ the PDE reduces to an ODE with the following analytic solution

$$F(x)^B = \frac{x}{\delta} + A_2 x^{\beta_1} \tag{7}$$

where $A_2 = \left(\frac{\Delta K x_E^*}{\delta} - IC_E\right)\left(\frac{1}{x_E^*}\right)^{\beta_1}$ and τ equals the exercise time of the option which is formally defined as $\{\tau := \inf\{u \geqslant 0 : x_u = x_E^*\}\}$.

Proof: See Appendix.

For most cases there exists no analytical solution satisfying both PDE and corresponding boundary conditions. Even if one abstracted for the moment from agency costs of free cash flow, the problem is due to IC_S in equation [5] which is defined as the sum of IC and $e(C, IC)$. Assuming for example that $F(x, C) = \nu C + Ax^{\beta} + \gamma x$ we can observe that the first boundary condition would imply that ν equals one which contradicts the value-matching condition. I therefore choose to solve the partial differential equation numerically by resorting to finite difference methods, i.e. Crank Nicholson Scheme. The PDE is solved on a grid with nodes $(x_j, C_i) : j = 1, ..., M, i = 1, ..., N$ where $x_j = jdx$ and $dC = C_i - C_{i-1}$. Partial derivatives are approximated by

$$\begin{aligned} F_x &= \tfrac{1}{2}\left(\tfrac{F_{i-1,j+1}-F_{i-1,j-1}}{2dx} + \tfrac{F_{i,j+1}-F_{i,j-1}}{2dx}\right) \\ F_{xx} &= \tfrac{1}{2}\left(\tfrac{F_{i-1,j+1}-2F_{i-1,j}+F_{i-1,j-1}}{(dx)^2} + \tfrac{F_{i,j+1}-2F_{i,j}+F_{i,j-1}}{(dx)^2}\right) \\ F_C &= \tfrac{F_{i,j}-F_{i-1,j}}{dC} \end{aligned} \tag{8}$$

which implies that the resulting difference equation at node (x_j, C_i) can be formulated as

$$-a_j F_{i-1,j-1} - (b_j - d_{i,j})F_{i-1,j} - c_j F_{i-1,j+1} = a_j F_{i,j-1} + (b_j + d_{i,j})F_{i,j} + c_j F_{i,j+1} + e_j \tag{9}$$

where

$$\begin{aligned} a_j &= \tfrac{\sigma^2 j^2 - \mu j}{4} \\ d_{i,j} &= \tfrac{jdx - \phi/2(\alpha j dx)^2 + ridc}{dc} \\ b_j &= -\tfrac{\sigma^2 j^2 + r}{2} \\ c_j &= \tfrac{\sigma^2 j^2 + \mu j}{4} \\ e_j &= (1-\alpha) jdx \end{aligned} \tag{10}$$

This equation is defined for $2 \leq j \leq M$ and $2 \leq i \leq N$. One alternative would be to solve the PDE by using the boundary conditions defined in [5]. However, this is computationally demanding as one has to consider the entire grid. To overcome this drawback, I will divide the state space of C into two different regimes. We know that if the firm has sufficient cash available, i.e. $C \geq IC$, it will not retain its earnings as it only incurs costs of holding cash within the firm. Therefore, for $C \geq IC$ we will have that $\alpha = 0$. It is also known that because $C \geq IC$ the firm will not need to access costly external capital markets and as a consequence it will be able to finance the investment at its true costs IC. However, when $\alpha = 0$ and $C \geq IC$ the value of an option to invest can be derived analytically as there is no contradiction between the boundary conditions anymore. Thus, as long as $x < x^*$ we know that for $C \geq IC$ the boundary conditions are given by

$$\begin{aligned} F(0, C_t) &= C_t \\ F(x^*, C_\tau) &= A(x^*)^{\beta_1} + \tfrac{K_0 x^*}{\delta} + C_\tau - IC \\ F_x(x^*, C_\tau) &= \beta_1 A(x^*)^{\beta_1 - 1} + \tfrac{K_0}{\delta} \end{aligned} \tag{11}$$

When $C < IC$, the boundary conditions still read as before and are given by

$$\begin{aligned} F(0, C_t) &= C_t \\ F(x^*, C_\tau) &= \tfrac{K_1 x^*}{\delta} + C_\tau - IC_S \\ F_x(x^*, C_\tau) &= \tfrac{K_1}{\delta} \end{aligned} \tag{12}$$

This has the advantage that the grid for C has an upper limit equal to the value of IC which drastically decreases computational requirements.

While for the benchmark case under $C_0 = 0$, optimal trigger level was independent of the two state variables, exercise policy when allowing the firm to retain cash depends on the level of C which in turn is affected by the retention rate α and operating profit x. Using the subscript S to denote the case when the firm is allowed to save the cash, we have that $x_S^* = f(C(x, \alpha))$. The optimal exercise point will depend on the level of cash the firm has available which in turn will be affected by the firms operating profit and its retention rate.

Definition 1 *The value of internal funds is defined as the change in total firm value due to the ability of a firm to internally finance an investment. Specifically, it is given by*

$$R(x, C) \equiv F(x, C) - F^B(x, C) \tag{13}$$

where C is the shortcut for $C(x, \alpha)$.

While the measure above gives an absolute answer to the value of internal funds, it can't be used to judge whether the amount gained or lost due to not paying out dividends is economically significant. I therefore compare the corresponding value to the initial value of the capacity expansion option for the benchmark firm. One can then judge by how much the firm can relatively increase its initial option value if it chooses to (partly) internally finance the investment.

Definition 2 *The economic significance of the value of internal funds is defined by comparing the corresponding option value to the capacity expansion option of the benchmark case. Specifically, it is defined as*

$$S(x, C) \equiv \frac{F(x, C) - F^B(x, C)}{A(x, C)} \tag{14}$$

where $A(x, C) \equiv F^B(x, C) - \frac{x}{\delta} - C$.

Before I proceed to the numerical implementation, I present a simplified version of the model which allows for a closed form solution.

2.3 A Simplified Model

This section introduces a simplified model to determine the maximum attainable value of internal funds. It should be noted that this is a hypothetical value which must not be confused with the true value derived in the previous section. It is hypothetical in the sense that it yields the maximum value without considering the attainability of the solution and agency costs of free cash flow. Nevertheless, it is very useful as its closed-form solution allows us to perform comparative statics which then can be compared against the dynamics of the true value.

I therefore consider the following fictitious example. Consider the case of a firm which has an initial cash balance C_0 greater than the necessary investment costs. Let's further assume that the firm is not subject to agency costs of free cash flow. The investment environment described in the previous section is still valid and the firm therefore plans to increase its capacity level. However, the firm needs to decide whether it wants to pay out all its cash holdings and future earnings as dividends or not. By doing so it would have to finance the project completely externally and investment costs would increase to IC_E. The firm therefore decides to calculate the value of internal funds and to assess its economic significance.

The firm evaluates two scenarios. Under the first one, it keeps the cash within the firm and retains all future earnings. Due to the dynamics of the cash account we know that $C_t > IC \; \forall \, t$ such that investment costs when exercising the option will equal IC. Following traditional real option models, we can apply value matching and smooth pasting condition plus imply the usual condition that $F^H(0, C_t) = C_t$ where the superscript H indicates that this is a hypothetical example. It follows that the PDE [4] has to be solved with respect to the following boundary conditions

$$\begin{aligned} F^H(0, C_t) &= C_t \\ F^H(x^*, C_\tau) &= \tfrac{K_1 x^*}{\delta} + C_\tau - IC \\ F^H_x(x^*, C_\tau) &= \tfrac{K_1}{\delta} \end{aligned} \tag{15}$$

Assuming that the solution is given by $F^H(x, C) = \nu C + A x^\beta + \gamma x K_{0,1}$, the PDE has an explicit solution as its dependence on C is linear in all boundary conditions. Following Definition [1], the upper bound for the value of internal funds is given by the difference in firm values for the hypothetical scenario and the benchmark case.

Proposition 3 *The upper bound for the value of internal funds is only a function of x and is given by*

$$R(x) = x^{\beta_1} \left(A_1 - A_2 \right) \tag{16}$$

where $A_1 = \left(\frac{\Delta K x^*}{\delta} - IC\right)\left(\frac{1}{x^*}\right)^{\beta_1}$, $A_2 = \left(\frac{\Delta K x_E^*}{\delta} - IC_E\right)\left(\frac{1}{x_E^*}\right)^{\beta_1}$ *and the variables* x^* *and* x_E^* *indicate optimal trigger levels under the different cost structures.*

Proof: See Appendix.

Depending on individual firm characteristics, such as assumed factor of capacity expansion, costs of external finance, drift rate, volatility and risk-free rate, the upper bound $R(x)$ will take on different values. The advantage of calculating this upper bound is that the closed form solution helps us to easily get implications under which scenarios there might be an option value at all.

As a first step, we can confirm our intuition that the existence of costs of external finance makes internal financing potentially interesting.

Proposition 4 *The marginal effect of proportional and convex issuance costs on the upper bound for the value of internal funds is positive and given by*

$$\frac{\partial R(x)}{\partial \gamma} = \left(\frac{\Delta K x^*}{\delta} - IC\right)\left(\frac{x}{x^*}\right)^{\beta_1}(\beta_1 - 1) \tag{17}$$

Proof: See Appendix.

When analyzing the value of internal funds or its upper bound, one also has to assess its economic significance which has been introduced in Definition [2]. Depending on individual parameter values such as assumed drift rates, risk-free rate and volatility the economic significance will differ. However, it can be shown that it is unrelated to the assumed factor of capacity expansion.

Proposition 5 *The economic significance of the upper bound for the value of internal funds is unrelated to the assumed capacity expansion factor. That is,* $\frac{\partial S(x)}{\partial \Delta K} = 0$.

Proof: See Appendix.

A final interesting feature to notice is that the upper bound of the value of internal cash is ambiguously related to volatility. This is due to its definition as it is derived as the difference between the value of a growth option for an unconstrained and a constrained firm. In other words, cash derives its value by possibly avoiding costs of external finance when exercising the option. However, for most cases costs of external finance lose their relative importance when volatility is increased which induces a negative relation between value of internal funds and volatility.

Proposition 6 *The effect of volatility on the upper bound for the value internal funds is ambiguous. Specifically, it depends on the costs of external finance and the parameter* β_1 *which in turn is influenced by the difference in drift rates, the risk-free rate and volatility. The concrete expression for the partial derivative of the value function with respect to volatility is given by*

$$\frac{\partial R(x)}{\partial \sigma} = A_1 x^{\beta_1} \frac{\partial \beta_1}{\partial \sigma} \left\{ \log\left(\frac{x}{x^*}\right) - (1+\gamma)^{1-\beta_1} \log\left(\frac{x}{x^*(1+\gamma)}\right) \right\} \qquad (18)$$

Proof: See Appendix.

In the following section I will numerically derive the value of internal funds. I will also assess its economic significance and compare it to its upper bound to determine whether precautionary saving makes sense in an uncertain environment under the availability of growth options. As a last step, I will compare the predictions gained from the simplified model to the true dynamics of the full model.

3 Numerical Analysis

3.1 Input Parameters

I will quantify the value of internal funds for two different firm types. A small firm with high growth opportunities and relatively low level of profitability, and a large firm with limited expansion capabilities, low risk and relatively high level of profitability. This will allow me to characterize the circumstances under which there exists an economic significant option value.

The difficulty in describing different firm types concerns the underlying assumptions. It is known from Dixit and Pindyck [5] that the factor of capacity expansion has implications for the value of an option to invest. However, I have shown in the previous section that the economic significance of the financing option is irrelevant of the factor of capacity expansion which fortunately eliminates the sensitivity of the results toward this input parameter. Without any further implications I can therefore assume that small firms have the option to increase their capacity levels by a factor of one half while a large firm can do so by only 5%. Volatility assumptions are made with respect to empirical stock market data. For the base case, I assume that firms face 19% and 14% of risk in their cash flow which equals annualized standard deviations for firms listed on the Russel 2000 and the S&P 500.[5] The costs

[5]Volatilities are estimated using monthly stock market data for the Russel 2000 and the S&P 500 from Datastream. Both return series are calculated using all historical price data for the respective index.

of capacity expansion are assumed to equal 10 units. The risk-free rate is matched to empirical Fed Funds data and is assumed to equal 6% whereas I assume that δ equals 5% and 1% for the base case.[6] The main intuition for the positive relationship between size and drift rates is that opportunity costs of not having a project should be higher for small firms compared to large firms. A complementary explanation has been given by Fama and French [7] who observe that small and relatively unprofitable firms generally do not pay dividends. The following table summarizes the assumptions regarding firm invididual parameter values.

	Small	Large
ΔK	50%	5%
σ	19%	14%
δ	5%	1%
r_f	6%	6%
IC	10	10

Table 1: Assumptions regarding firm individual parameter values

The estimates for the costs of external finance are taken from Hennessy and Whited [9] who perform a structural estimation and distinguish between small and large firms. Assuming a linear-quadratic weakly convex function given by

$$e = \gamma_0 + \gamma_1 c_t + \gamma_2 c_t^2 \tag{19}$$

it turns out that $\gamma_{1,2}$ are significant at the 5% level whereas the fixed cost

[6]The average risk-free rate is estimated by using the mean of historical monthly Fed Fund Rates available since 1955.

parameter γ_0 is significant at the 10% level. While in general the model is able to easily account for the fixed cost parameter, I choose not to include it in the numerical implementation of the model to avoid biasing results. Hennessy and Whited further report different parameter estimates for small and large firms and thereby confirm the fact that financing is more costly for small firms. The following table summarizes different cost structures for small and large firms.[7]

	γ_0	γ_1	γ_2
Small	0	0.120	0.0004
Large	0	0.053	0.0002

Table 2: Different scenarios of financing costs

Concerning agency costs of free cash flow, I follow Eisfeldt and Rampini [6] in assuming that the parameter ϕ equals 0.05 for the base case.

3.2 The Value of Internal Funds

Focusing on the two types of firms, I will derive the upper bound for the financing option for each firm type and compare it to the numerically derived value when saving is endogenous.[8] For what follows, I will analyze option values for each type separately.

To assure accurateness of the numerical results, the Crank Nicholson scheme is first solved for the case when an analytical solution is available. Assump-

[7]Specifically, these are estimates of external costs of equity. However, they also perform an estimation of external costs of debt which are even higher.

[8]For the calculation of the upper bound, I assume that $x_0 = 1$ and $C_0 = 0$.

tions concerning grid size and maximum values of the state variables are then made to replicate the result.[9]

3.2.1 The Case of a Small Firm

A small firm is traditionally characterized by the availability of large growth opportunities. Having shown in the previous section that the economic significance of the financing option is unrelated to the potential factor of capacity expansion, the robustness of the results increases as the guess about the size of expansion factor becomes irrelevant.

The base case of the small firm is characterized by assuming a standard deviation of its cash flow of 19% and a value of δ equal to 5%. The assumption regarding cash flow risk is made with respect to the long-run risk of monthly stock returns of the Russel 2000 while the assumptions concerning δ reflects the intuition that opportunity costs of not taking a project should be relatively high for a small firm. Besides, empirical studies show that it is small and less profitable firms which mostly choose not to pay dividends.[10]

The calculation of the upper bound for the financing option is straightforward and follows equation [16]. For the case of convex issuance costs, the value of the option to invest if the firm has to finance the project externally is given by 2.121 units. The upper bound for the value of internal funds is

[9] Specifically, I choose dx, dC and X_{max} such that the numerical solution is correct up to some error level ϵ with the requirement that $\left|R(x) - R(x)^{NUM}\right| \leq \epsilon$. The model is finally solved by setting the maximum value of the cash flow X_{max} equal to 10, its step size dx to 0.05 and $dc = 0.0025$. The error level ϵ is set equal to $5 \cdot 10^{-4}$. Comparative statics in the following section are computed with step sizes equal to $dx = 0.1$ and $dc = 0.01$ respectively.

[10] See Table [1] for a summary of all firm-individual input parameters

given by 0.280 which implies that the firm could increase option value by 13.19% if it is able to finance the project internally.

The interesting question is whether when accounting for endogenized saving, the increase in option value will be close to this hypothetical upper bound. I start by analyzing the pure attainability of this maximum value and therefore first exclude agency costs of free cash flow. Applying Crank Nicholson method, the model reveals that firms can indeed increase option value by 0.243 or 11.45% if they retain earnings within the firm rather than paying them out. In other words, when making saving truly endogeneous the firm is able to achieve 87% of its upper bound for the value of internal funds.

Accounting for agency costs of free cash flow and assuming that the firm retains all cash within the firm, the relative gain from not paying out dividends decreases to 3.73%. However, this comparison is somehow not fair as it exogeneously imposes that all earnings are kept within the firm. The value maximizing policy is given by choosing an optimal dynamic retention policy which leads to an increase of $S(x_0, C_0)$ to 6.6 %. It is interesting to see that the economic significance of the value of internal funds ranges from zero to 13.18 % depending on the chosen allocation of x and C. This is visualized in Figure [1].

Concerning optimal saving policy, one can see that for values of x close to zero the firm chooses to pay out most funds as dividends. The reason is that agency costs of free cash flow dominate as the probability of exercising the option is low. However, for slightly higher values of x it is optimal to retain part of the cash flow in order to reduce future financing costs. Moreover,

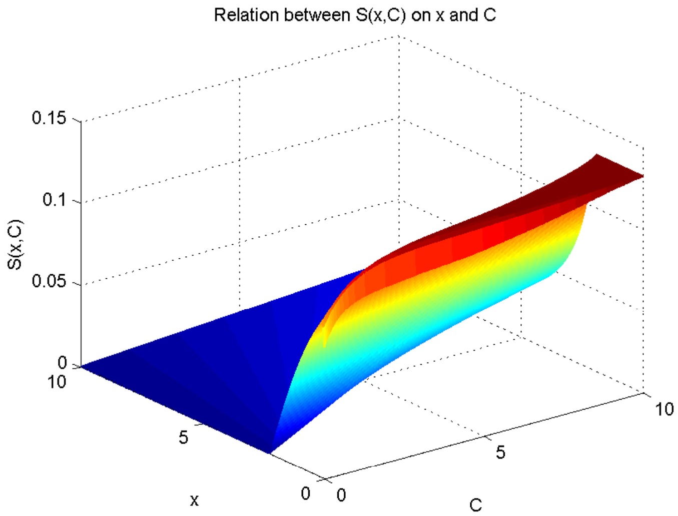

Figure 1: The influence of x and C on the value of S(x,C).

by simultaneously increasing C we can see that the optimal retention ratio increases to as much as 100%.

The results of this stylized example show that even if the firm retained all earnings within the firm and followed a myopic saving policy, it would still generate value compared to the benchmark case in the presence of growth options. However, the example also implies that because of agency costs of free cash flow optimal saving policy varies across the state space. Clearly, the implications regarding payout policy depend crucially on the functional form of agency costs and on the assumptions regarding the cost parameter ϕ. Changing the value of ϕ to 0.025 the relative gain from saving evaluated at x_0 and C_0 increases to 8.55%.

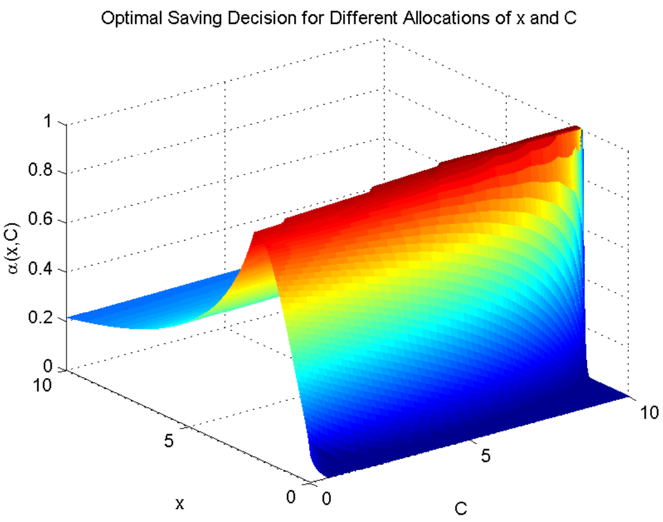

Figure 2: Optimal saving policy for different allocations of x and C.

3.2.2 The Case of a Large Firm

Large firms are generally characterized through a lower level of risk and little growth opportunities. Similar to the preceding case, I assume that annual standard deviation of cash flow can be approximated by the long run volatility of monthly stock returns of the S&P 500. The level of expected profitability is assumed to equal 5%.

The upper bound for the value of internal funds is given by 0.024. Its economic significance is small and equals 0.88% of the option for the case of complete external financing. Due to the higher trigger level for exercising the option, the firm is able to nearly achieve maximum option value when saving is endogeneous, i.e. $R(x, C) = 0.0235$. By introducing agency costs of free cash flow the value of internal funds drops to 0.2084 when allowing the firm to choose an optimal saving policy. Given that the economic significance of the value of internal funds is less than 1%, it seems save to conclude that the option to retain cash is less valuable than for the example in the preceeding subsection.

3.3 Comparative Statics

This section seeks to answer three specific questions.[11] First, what is the marginal value of internal funds across different allocations of cash and cash flow? In other words, how does the marginal value of cash change across the state space? Second, how does the value of internal funds depend on volatility? And finally, what is the influence of expected profitability on the value of internal funds?

What is the reason for analyzing the marginal value of internal funds? It is known from the previous section that the marginal value of cash is a main determinant of optimal payout policy. In absence of agency costs, the firm would always fully retain its earnings as long as the marginal value of cash exceeds one. Otherwise it would be indifferent between retaining cash and paying dividends. Including agency costs of free cash flow, the optimal retention policy is given by the following rule.

$$\alpha^* = \frac{F_C - 1}{\phi x F_C} \tag{20}$$

with the additional requirement that $\alpha^* \in [0, 1]$. We can therefore analyze the marginal value of cash implied by optimal firm behavior across different allocations of cash and cash flow. The following figure visualizes the result.

[11] All calculations in this section are made with respect to the stylized example of a small firm.

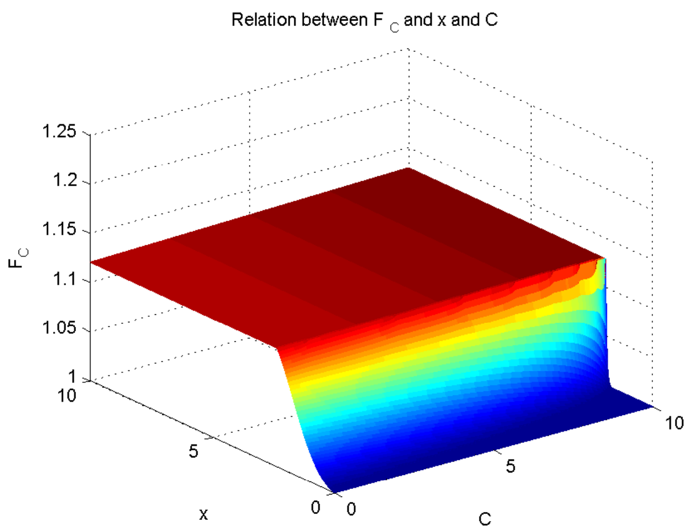

Figure 3: The marginal value of cash across different allocations of x and C.

One can see that for low levels of cash and cash flow the marginal value of internal funds equals one. This can be explained by the fact that the probability of exercising the option is very low and thus agency costs of free cash flow play a dominant role. Increasing the cash balance even leaves the marginal value of internal funds at one for a wider range of values of x. In other words, if the firm already has a lot of cash and current cash flow is low then it does not value an extra dollar at a premium to its notional amount. On the other hand, increasing x quickly leads to an increase in F_C above one. Thus, the more likely it is that the option gets exercised the more value the firm places on internal funds. Depending on the allocation of x and C in the state space, the value premium of cash can be as high as 12.8%.

The second question concerns the effect of volatility on the value of internal funds. I have shown in Proposition [6] that the effect of volatility on the upper bound of the financing option is ambiguous which can be seen in Figure [4].

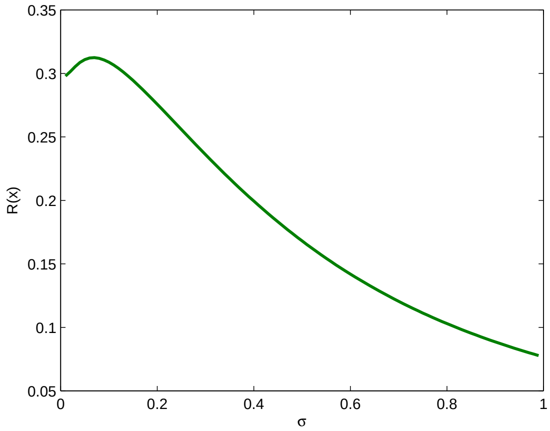

Figure 4: Ambiguous relationship between volatility and the upper bound of the financing option.

The main issue again concerns the full model. It turns out that the behavior of the financing option with respect to volatility is similar to the behavior of its upper bound. Due to the presence of agency costs of free cash flow, it makes a difference whether the firm saves all earnings or whether it chooses an optimal payout policy for each volatility level. If all funds are kept within the firm the value of internal funds even gets negative for some volatility levels whereas if payout policy is chosen optimally the value of internal funds stays positive. Nevertheless, in both cases the value of internal funds is decreasing in volatility. The intuition is that cash derives its value by possibly avoiding costs of external finance. However, these costs lose their relative importance for moderate to high levels of volatility thereby implying the negative relation between the value of internal funds and uncertainty.

A final and interesting aspect which has not been explored analytically concerns the impact of expected profitability on the value of internal funds.

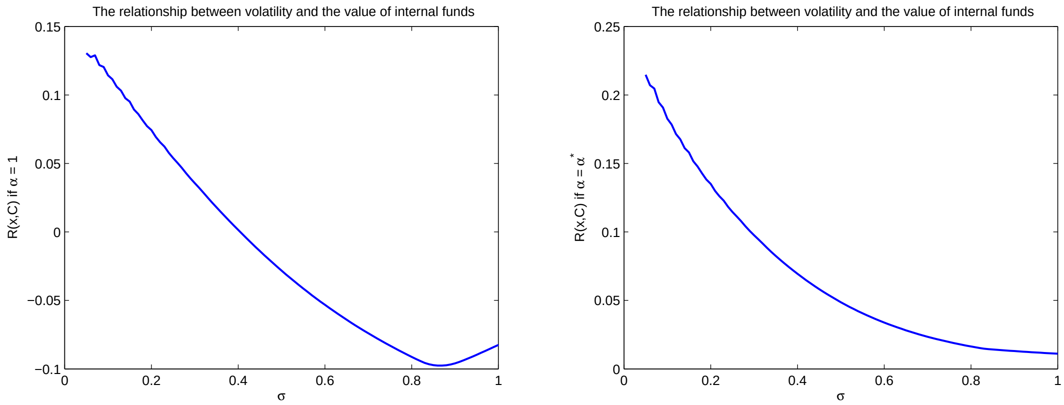

Figure 5: The impact of volatility on the value of internal funds for different payout policies

Specifically, the question is how profitability affects the economic significance of the value of cash. I will therefore compare comparative statics of the simplified model to those of the full model including agency costs of free cash flow. Without agency costs, one can observe an inverse relationship between the economic significance of the financing option and profitability. Thus, the lower expected profitability the higher is the potential gain from saving. On the other hand, by including agency costs the relationship is more ambiguous. While it still holds true that high levels of profitability lower the relative gain from saving it is also the case that negative expected profitability decreases option value. This can be explained by the fact that if the firm is really doing badly, the probability of exercising the option is low such that agency costs of free cash flow dominate the value of internal funds. This can effect be seen in Figure [6].

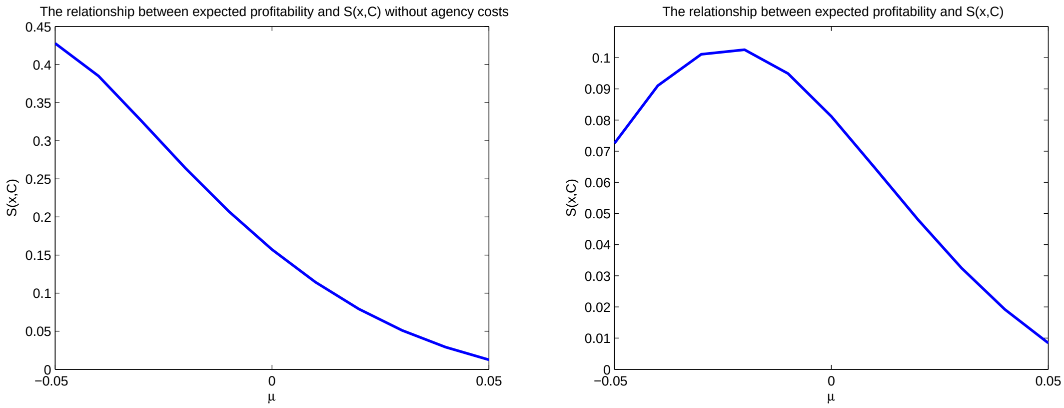

Figure 6: The relationship between expected profitability and the economic significance of the financing option

4 Conclusion

The paper analyzes the value of internal funds and thereby also characterizes optimal saving policy. Specifically, the value of internal funds is derived within a real options framework in which a firm has the option to expand capacity by paying the corresponding investment costs. Departing from traditional real options models, the paper focuses on the question of how these costs are actually paid. To make this question potentially matter, I introduce two frictions, namely costs of external finance and agency costs of free cash flow.

In general, one can observe that the value of internal funds depends on firm individual parameter values. For firms with low to moderate levels of profitability or high opportunity costs of not owning the complete project the relative gain from saving can be as high as 13.18%. Departing from standard textbook approaches, the paper shows that one dollar of cash can be valued at a substantial premium to its notional value.

Concerning comparative statics, I have further shown that the effect of volatility on the value of internal funds is ambiguous and mostly negative. Using the results derived from the upper bound of the financing option, we can see that it is affected by costs of external finance and the positive root of the fundamental quadratic which in turn is influenced by assumed drift rates, risk-free rate and again volatility. The intuitive reason for the negative relationship is that cash derives its value by possibly avoiding costs of external finance. For higher levels of volatility costs of external finance lose their relative importance thereby implying the negative relation between value of internal funds and volatility. For the full model including agency costs of free cash flow, the dynamics are similar if the payout decision is made optimally. Otherwise for full payout strategies, the value of internal funds can also get negative.

The paper contributes on different aspects to the literature. First, by focusing on an all-equity financed firm the paper is able to provide a pure value of cash which is not distorted by the possibility of avoiding bankruptcy costs. In other words, cash does not serve to reduce the net debt exposure of the firm and thus can't be viewed as some form of negative debt. Second, by choosing this approach I am able to derive a closed form solution to the upper bound of the financing option. This allows for a quick assessment of whether there exists a value of internal funds at all and it also permits the computation of comparative statics to obtain implications regarding payout policy. The negative effect of volatility on option value also holds true for the full model when saving is completely endogenous. Third, the paper uses quadratic agency costs of free cash flow to model the costs of retaining cash within the firm. This allows to consider the fact that the costs of holding cash are convex in the amount of funds retained.

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A Appendix

A.1 Proof of Proposition [1]

Proof. Using the fact that $\mu = r - \delta$ we can write that $dx = (r-\delta)xdt + \sigma x dW^Q$. Let's suppose we construct a risk-free portfolio by holding θ_1 units of the firm and shorting θ_2 units of the traded asset. The long position of the portfolio entitles us to an instantaneous dividend payment $\theta_1(1-\alpha)x$.

The value of the portfolio P is given by $(\theta_1 F - \theta_2 X)$ and it follows that the total return from holding the portfolio over a short time interval dt equals

$$dP = \theta_1 \left((1-\alpha)xdt + dF\right) - \theta_2 dX \tag{21}$$

Applying Ito's Lemma leaves us with

$$dP = \theta_1 \left((1-\alpha)xdt + F_x dx + F_C dC + \frac{1}{2}\sigma^2 x^2 F_{xx} dt \right) - \theta_2 dX \tag{22}$$

For $\theta_1 = 1$, it immediately follows that θ_2 equals $\left(\frac{F_x x}{X}\right)$ which then implies that $dP = rPdt$. Combining above and using the fact that $P = (F - \frac{xF_x}{X}X)$ we obtain that

$$rF = (1-\alpha)x + (r-\delta)xF_x + (\alpha x - \frac{\phi}{2}(\alpha x)^2 + rC)F_C + 1/2\sigma^2 x^2 F_{xx} \tag{23}$$

The only step missing is to treat α as a stochastic optimal control by imposing that

$$rF = \max_{\alpha} \left\{ (1-\alpha)x + (r-\delta)xF_x + (\alpha x - \frac{\phi}{2}(\alpha x)^2 + rC)F_C + 1/2\sigma^2 x^2 F_{xx} \right\} \tag{24}$$

with the additional requirement that $\alpha \in [0,1]$.

■

A.2 Proof of Proposition [2]

Proof. By assumption α is set to 0 such that the PDE in equation [4] simplifies to

$$rF = x + (r-\delta)xF_x + rCF_C + 1/2\sigma^2 x^2 F_{xx} \tag{25}$$

The additional assumption that $C_0 = 0$ reduces the PDE to an ODE. For this case, we also know that total investment costs are independent of C as the whole amount has to be raised externally, i.e. $IE_E = IC + e(IC)$. Then the problem reduces to the case of Dixit and Pindyck [5] with the well-known solution of the value of the capacity expansion option given by $\left(\frac{\Delta K x_E^*}{\delta} - IC_E\right)\left(\frac{x}{x_E^*}\right)^{\beta_1}$. The corresponding optimal trigger level is given by

$$x_E^* = \frac{\beta_1}{(\beta_1 - 1)\Delta K}\delta IC_E \tag{26}$$

where β_1 is the positive root of the fundamental quadratic

$$\frac{1}{2}\beta(\beta-1) + \mu\beta - r = 0 \tag{27}$$

■

A.3 Proof of Proposition [3]

Proof. Let's start by assuming that the solution to the underlying PDE [4] and the boundary conditions for the unconstrained case [15] is given by $F(x,C) = \alpha C + A_1 x^{\beta_1} + \gamma x$ where the subscript (1) denotes option value for the unconstrained firm.

It immediately follows that option value and trigger level are given by

$$x^* = \frac{\beta_1}{(\beta_1 - 1)\Delta K}\delta IC \tag{28}$$

$$A_1 x^{\beta_1} = \left(\frac{\Delta K x^*}{\delta} - IC\right)\left(\frac{x}{x^*}\right)^{\beta_1} \tag{29}$$

where β_1 is the positive root of the fundamental quadratic

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0 \tag{30}$$

Similarly we know from proposition [2] that option value and trigger level for the constrained case equal to

$$x_E^* = \frac{\beta_1}{(\beta_1 - 1)\Delta K}\delta IC_E \tag{31}$$

and

$$A_2 x^{\beta_1} = \left(\frac{\Delta K x_E^*}{\delta} - IC_E\right)\left(\frac{x}{x_E^*}\right)^{\beta_1} \tag{32}$$

Using the definition of $R(x)$ we obtain that

$$R(x) = x^{\beta_1}(A_1 - A_2) \tag{33}$$

which proves the assertion. ■

A.4 Proof of Proposition [4]

Proof. Notice that costs of external finance are defined as $e = \gamma_0 + \gamma_1 c_t + \gamma_2 c_t^2$. For $c_t = IC$ and without the fixed cost parameter, costs of external finance are $e(IC) = \gamma_1 IC + \gamma_2 IC^2$. Rearranging gives that $IC_E = IC(1 + \gamma)$ where $\gamma = \gamma_1 + \gamma_2 IC$. It follows that $x_E^* = x^*(1 + \gamma)$. Due to the definition of the financing option we also have that $R(x) = x^{\beta_1}(A_1 - A_2)$. Substitution into the expression for the growth option under full payout we obtain that

$$A_2 x^{\beta_1} = \left(\frac{\Delta K x^*(1+\gamma)}{\delta} - IC(1+\gamma)\right)\left(\frac{x}{x^*(1+\gamma)}\right)^{\beta_1} \tag{34}$$

Taking the partial derivate of $R(x)$ with respect to γ yields

$$\frac{\partial R(x)}{\partial \gamma} = 0 - \underbrace{\left(\frac{\Delta K x_E^*}{\delta} - IC_E\right)\left(\frac{x}{x_E^*}\right)^{\beta_1} \underbrace{(1-\beta_1)}_{<0 \text{ as } \beta_1 > 1}}_{>0} \tag{35}$$

which proves the proposition above. ■

A.5 Proof of Proposition [5]

Proof. Notice that by definition we have that $S(x) = \frac{A_1 x^{\beta_1}}{A_2 x^{\beta_1}} - 1$. Noting that $x^* = \frac{\beta_1}{(\beta_1 - 1)\Delta K} IC\delta$ and taking the derivative of the whole expression with respect to ΔK we obtain the desired result. ■

A.6 Proof of Proposition [6]

Proof. Concerning the partial derivative of any growth option with respect to volatility, it is sufficient to observe that

$$\frac{\partial A x^{\beta_1}}{\partial \sigma} = A x^{\beta_1} \log\left(\frac{x}{x^*}\right)\frac{\partial \beta_1}{\partial \sigma} \tag{36}$$

as $\frac{\partial A x^{\beta_1}}{\partial x^*}\frac{\partial x^*}{\partial \beta_1}$ equals zero. Given that the positive solution to the fundamental quadratic is characterized by the same parameters for both the constrained and unconstrained firm, we only need to know that $\frac{\partial \beta_1}{\partial \sigma} < 0$. Further details can be found in Dixit & Pindyck [5].

Applying above to $\frac{\partial R(x)}{\partial \sigma}$ we get that

$$\frac{\partial R(x)}{\partial \sigma} = \frac{\partial \beta_1}{\partial \sigma} \left\{ A_1 x^{\beta_1} \log \left(\frac{x}{x^*} \right) - A_2 x^{\beta_1} \log \left(\frac{x}{x_E^*} \right) \right\} \tag{37}$$

Using the fact that $x_E^* = x^*(1+\gamma)$ where $\gamma = (\gamma_1 + \gamma_2 IC)$ and that $A_2 x^{\beta_1} = A_1 x^{\beta_1} (1+\gamma)^{1-\beta_1}$, we can rewrite the equation as

$$\frac{\partial R(x)}{\partial \sigma} = \frac{\partial \beta_1}{\partial \sigma} \left\{ A_1 x^{\beta_1} \log \left(\frac{x}{x^*} \right) - A_1 x^{\beta_1} (1+\gamma)^{1-\beta_1} \log \left(\frac{x}{x^*(1+\gamma)} \right) \right\} \tag{38}$$

which again can be rewritten as

$$\frac{\partial R(x)}{\partial \sigma} = A_1 x^{\beta_1} \frac{\partial \beta_1}{\partial \sigma} \left\{ \log \left(\frac{x}{x^*} \right) - (1+\gamma)^{1-\beta_1} \log \left(\frac{x}{x^*(1+\gamma)} \right) \right\} \tag{39}$$

Due to the fact that $x < x^* < x^*(1+\gamma)$ we know that $\log \left(\frac{x}{x^*} \right) > \log \left(\frac{x}{x^*(1+\gamma)} \right)$. The question whether the expression in the bracket is positive or negative will depend on $(1+\gamma)^{1-\beta_1}$ which will lie between 0 and 1 for different values of γ and β_1.

■