Is the Skew Priced in Structured Retail Products?

Evidence From the German Secondary Market

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Abstract

We investigate the pricing of bonus certificates, a popular type of structured retail products which features an embedded down-and-out put option. Due to the volatility skew, such a product cannot be valued in straightforward manner using classical Black-Scholes analysis. Therefore, we consider the skew-consistent stochastic volatility model of Heston (1993) and analyze its pricing differentials in comparison to several variants of the Black-Scholes model. Evaluating a data sample of 808 bonus certificates over a period of 14 months, covering 107,711 quotes, we examine whether the skew is actually priced in the products. Indeed, by comparing model values with actual market prices, we find evidence that issuers take the volatility skew into account. Moreover, when investigating the issuers' margins with the Heston model, we find that margins are of similar size among different issuers, but are larger compared to other structured retail products with embedded plain-vanilla options.

Keywords: option pricing, stochastic volatility, implied volatility, volatility skew, volatility smile, structured financial product

JEL Classification: G13, G21

1 Introduction

Modern valuation of derivative products is based on a stochastic process of the underlying security. The classical approach of Black and Scholes (1973) and Merton (1973) assumes geometric Brownian motion, which implies a lognormal distribution of future underlying prices. Despite the widespread use of this model in theory and practice, there is strong evidence that the assumption of a constant volatility does not hold in reality (e.g., Rubinstein (1994)). The prices of traded options imply different volatilities, depending on the strike price and the time to maturity. As a consequence, the true distribution of future underlying prices deviates from lognormality in some way. However, despite its inability to account for this deviation, the Black-Scholes model is commonly applied to price plain-vanilla options by using an implied volatility which corresponds to the particular strike price and maturity date of the considered option.

For a traded option, calculating an implied volatility means replacing the true risk-neutral future price distribution of the underlying security at maturity—which may be of arbitrary shape—by a lognormal distribution with the same expected underlying price and a volatility that leads to the same expected option payoff. It is easy to imagine that such an approach may involve some kind of "violence", which makes the implied volatility suitable only for pricing similar options in terms of strike price and time to maturity. In particular, it is highly questionable whether the implied volatility surface obtained from traded plain-vanilla options is directly transferable to the pricing of non-vanilla (exotic) instruments. Instead, a closer look at the future underlying price distribution, or, even one step back, at the stochastic process, is necessary.

The implied volatility of many stocks and stock indices as a function of the strike price for a fixed maturity exhibits a concavely decreasing shape which is known as the volatility skew. Such a skew is consistent with the option pricing model of Heston (1993), who assumes a volatility which is itself stochastic. If the volatility is negatively correlated with the underlying price, volatility tends to increase for decreasing underlying prices and vice versa. This effect leads to a more leptokurtic distribution of stock returns, which is in line with the empirical distribution (causing the volatility skew within the Black-Scholes model).

In this paper, prices of retail derivative products involving non-vanilla options are investigated with regard to the reflection of the skew. Particularly, we analyze whether the price setting for these products takes the volatility skew into account. Retail derivatives are offered by financial institutions in various forms. Usually, they are exchange-traded, where the issuer also functions as the market maker. Observed prices of retail derivatives thus do not necessarily reflect their fair values, but rather the price-setting policy of the issuers.

This price-setting behavior has been the subject of several empirical studies in the past years. Most of them focus on products with embedded plain-vanilla options, in particular for Switzerland, Wasserfallen and Schenk (1996) (capital-guaranteed products), Burth et al. (2001) (reverse convertibles and discount certificates), and Grünbichler and Wohlwend (2005) (non-capital-guaranteed products), for Germany, Wilkens et al. (2003) (reverse convertibles and discount certificates), and Baule et al. (2008) (discount certificates), and for the U.S., Benet et al. (2006) (reverse convertibles). Muck (2006) and Wilkens and Stoimenov (2007) investigate speculative turbo certificates which feature knock-out options, but which are very insensitive to volatility due to their similarity to common forward contracts. Stoimenov and Wilkens (2005) analyze the pricing of various types of retail derivatives including products with embedded exotic options. However, the authors focus solely on classical Black-Scholes analysis.

The paper is also related to the literature branch dealing with implied volatility models. Bates (2003) provides an overview of empirical option pricing. Dumas et al. (1998) calibrate deterministic implied volatility functions to S&P 500 index options and test the predictive and hedging performance for out-of-sample data. The alternative models are found to be inferior with respect to the constant volatility (Black-Scholes) model. The same result is reported by Brandt and Wu (2002), who fit an implied binomial tree to European-style FTSE 100 index options and examine the prices for corresponding American-style options. Buraschi and Jackwerth (2001) develop a statistical test and, using S&P 500 index options data, find evidence that options are non-redundant securities, which underlines the need for additional risk factors such as stochastic volatility and jumps. Also based on S&P 500 data, Bakshi et al. (1997) evaluate the performance of several smile-consistent option pricing models. While for pricing purposes, introducing both stochastic volatility and jumps yields the best results, for hedging, stochastic volatility alone is superior.

This paper is the first to investigate the pricing of retail derivatives featuring exotic options with a model that takes the volatility skew into account. Besides a general analysis of the margins and their influencing parameters, we take a look in particular at the price differences between a Black-Scholes approach and the Heston model. The analysis is based on bonus certificates, a very popular type of retail derivative which features an embedded down-and-out put option.[1] The paper contributes to the field in several ways. First, we analyze the effect of the volatility skew on theoretical values of bonus certificates. Second, we investigate whether the price setting of issuers (as market makers) indeed takes the skew effect into account. Third, we analyze the margins of bonus certificates and their influencing parameters with a skew-consistent model.

The paper proceeds as follows. Section 2 describes the valuation of bonus certificates; particularly, we review the Black-Scholes option-pricing framework and the Heston model to value the embedded barrier option. Section 3 presents our empirical methodology and the data. In Section 4 we report the results of our empirical analyses. Section 5 concludes the paper.

[1] These products are also known as Participation Securities with Contingent Protection.

2 Valuation of Bonus Certificates

2.1 Valuation by Duplication

At maturity T, a bonus certificate promises the holder an amount equal to the price of the underlying security S_T. Furthermore, if the underlying price has not reached (or fallen below) a pre-determined barrier H at any point of time between issuance (T^*) and maturity of the certificate, a bonus payment applies in a way that the minimum repayment equals a bonus level K. Thus, with the first-passage time $\tau = \inf\{t > T^* : S_t \leq H\}$, the payoff BC_T of the bonus certificate at maturity is given by

$$BC_T = S_T + \max\{K - S_T; 0\}1_{\tau > T}, \tag{1}$$

where $1_{\{.\}}$ is the indicator function. The payoff scheme (1) can be duplicated by a long position in a zero-strike call on the underlying security and a long position in a down-and-out barrier put option. If the underlying security pays no dividends within the lifetime of the certificate, the value of the zero strike call equals the underlying price. Provided there exists no risk of default, the value BC'_t of the bonus certificate at any time $t \leq T$ is thus given by

$$BC'_t = S_t + pdo_t, \tag{2}$$

where pdo_t denotes the value of the down-and-out put option with barrier H and strike price K. However, as bonus certificates are unsecured notes which may fail to pay in the event of an issuer's default, the value must be adjusted for this credit risk. Therefore, we apply the model of Hull and White (1995), which assumes independence between market and credit risk.[2] Accordingly, the value is given by

$$BC_t = BC'_t(1 - e^{-c(T-t)}) = (S_t + pdo_t)(1 - e^{-c(T-t)}), \tag{3}$$

[2]See Baule et al. (2008) for a discussion of approaches to value structured products subject to credit risk.

where c denotes the credit spread of the issuer.

Whereas the underlying price can be readily observed for traded assets, there are no exchange-traded barrier options available.[3] We consider two different valuation approaches for the barrier option. First, we employ the Black-Scholes framework, which provides a closed-form solution for the barrier option value. However, this approach assumes that underlying returns are normally distributed. As such, the Black-Scholes model is not consistent with the volatility skew. Second, we consider the influence of stochastic volatility within the model proposed by Heston (1993). In contrast to the Black-Scholes approach, the Heston model is able to account for the volatility skew.

2.2 Black-Scholes Framework

Within the Black-Scholes option pricing framework, the underlying price S_t is assumed to follow a geometric Brownian motion under the risk-neutral measure,

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dW_t, \tag{4}$$

where r is the risk-free interest rate, σ is the constant volatility, and W_t is a standard Wiener process. The specification of the price process (4) implies that underlying returns are normally distributed.

Using no-arbitrage arguments, the value of a down-and-out put pdo_t can be derived by the following analytic formula, first provided by Merton (1973):

$$pdo_t = p_t - pdi_t, \tag{5}$$

where p_t denotes the value of a plain-vanilla put option with

$$p_t = Ke^{-r(T-t)}\Phi(-d_1 + \sigma\sqrt{T-t}) - S_t\Phi(-d_1), \tag{6}$$

[3] An exception is the Australian market, where some barrier warrants are traded; see Easton et al. (2004).

where

$$d_1 = \frac{\ln(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \tag{7}$$

and pdi_t is the value of a down-and-in put option with

$$\begin{aligned} pdi_t &= -S_t\Phi(-x_1) + Ke^{-r(T-t)}\Phi(-x_1 + \sigma\sqrt{T-t}) + S_t(H/S_t)^{2\lambda}[\Phi(y) - \Phi(y_1)] \\ &\quad - Ke^{-r(T-t)}(H/S_t)^{2\lambda-2}[\Phi(y - \sigma\sqrt{T-t}) - \Phi(y_1 - \sigma\sqrt{T-t})], \end{aligned} \tag{8}$$

where

$$\lambda = \frac{r + \sigma^2/2}{\sigma^2}, \tag{9}$$

$$y = \frac{\ln[H^2/(S_t K)]}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \tag{10}$$

$$x_1 = \frac{\ln(S_t/H)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \tag{11}$$

$$y_1 = \frac{\ln(H/S_t)}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}. \tag{12}$$

As defined above, H denotes the barrier level, K is the strike price, and $T-t$ is the time to maturity. $\Phi(\cdot)$ represents the cumulative normal distribution function. Except for the volatility, σ, all parameters required for the valuation are readily observable. Besides applying historical underlying price variations for the estimation, the volatility can be derived from quoted option prices. This implied volatility represents a market appraisal of the future volatility of the underlying asset over the lifetime of the option and is therefore considered to be a forward-looking estimate.

However, if the market expects the actual distribution of underlying returns to be not normal, implied volatilities vary with the strike price and the time to maturity of the option. Particularly, for stock index options the implied volatility usually exhibits a skew, i.e., it decreases with increasing strike prices.[4] The skew can be attributed to the leptokurtic distribution of underlying

[4]There is evidence that the skew is more pronounced for index options compared to individual stock options; see e.g., Bakshi et al. (2003), Bollen and Whaley (2003), and Branger and Schlag (2004).

returns with a fat left tail compared to the normal distribution assumed by Black-Scholes. Practitioners often circumvent the problem of varying implied volatilities by employing different volatilities for different strikes and times to maturity in the Black-Scholes model. These volatilities are implied by the prices of quoted options that most closely match the strike and the time to maturity. However, for barrier options this procedure cannot be applied in a straightforward way since they are not actively traded and, hence, market prices are not observable. Therefore, the implied volatility of plain-vanilla options has to be utilized as a proxy. As barrier options are not only defined by the strike price, but also by a barrier level, it is yet unclear for which strike price this implied volatility shall be calculated. In the following, we examine three different plain-vanilla options to estimate the required volatility. First, we use implied volatilities of at-the-money plain-vanilla options ("at-the-money volatility"). Second, we employ implied volatilities of options where the strike price of the plain-vanilla option matches the strike price of the barrier option ("strike volatility"). Third, we consider plain-vanilla options with a strike price corresponding to the barrier level ("barrier volatility").

2.3 Heston Model

The Heston model assumes that under the risk-neutral measure, the underlying price S_t follows the Itô process

$$\frac{dS_t}{S_t} = r\,dt + \sqrt{v_t}\,dW_t^S. \tag{13}$$

In contrast to the geometric Brownian motion assumed by Black-Scholes, the variance v_t evolves stochastically following a mean-reversion process

$$dv_t = \kappa(\theta - v_t)\,dt + \eta\sqrt{v_t}\,dW_t^v, \tag{14}$$

where κ denotes the mean-reversion speed, θ the long-run mean, and η the volatility of the volatility. The changes in the underlying price and the volatility are correlated,

$$E\left[dW_t^S dW_t^v\right] = \rho\, dt, \tag{15}$$

with ρ being the correlation coefficient.

If the correlation is negative, the model is consistent with the observed skew; as for decreasing underlying prices, volatility tends to increase and vice versa.

The solution for the price of a plain-vanilla call option can be found using a Fourier transform technique. This yields a semi-analytic formula which involves the evaluation of a complex integral. Within the integrand, a complex logarithm is to be calculated, which must be conducted with special care according to the non-uniqueness of the complex logarithm. We follow the approach of Kahl and Jäckel (2005) for the numerical integration, using an adaptive quadrature method.

However, to value a barrier option within the Heston model, no such semi-analytic procedure is feasible. Therefore, Monte Carlo simulation is applied. We prefer a simple Euler discretization over more complex schemes because of the regularity of its convergence, which allows us to apply Richardson extrapolation.[5] We use 100,000 simulation runs with a time step of 0.01 years. To enhance the accuracy, several techniques proposed by Glassermann (2004) are applied. In particular, besides Richardson extrapolation, we use antithetic variables, control variates with the corresponding Black-Scholes value, and an adjustment for the probability of intrastep barrier hitting.

[5]See Glassermann (2004), pp. 356–358.

2.4 Model Impact on Barrier Option Values

Before conducting our empirical investigation, we first generically analyze the impact of the different models, namely Black-Scholes and Heston, on barrier option values. Particularly, we calculate the differences between the Black-Scholes and the Heston model values for different underlying prices.[6]

We consider a generic down-and-out put with a period of one year to maturity. The option has a strike price of $K = 1$ and a barrier at $H = 0.6$. To value the barrier option within the Heston model, we employ a representative set of parameters with initial variance $v_t = 0.02$, long-run mean $\theta = 0.05$, mean-reversion speed $\kappa = 1.5$, volatility of volatility $\eta = 0.6$, and correlation coefficient $\rho = -0.6$. The risk-free interest rate is set at $r = 0.04$. The corresponding Black-Scholes model values are obtained by calculating plain-vanilla option values with the Heston model and employing the respective implied volatilities in (5). As described above, we consider three different volatility estimates for the Black-Scholes model (strike volatility, barrier volatility, and at-the-money volatility of plain-vanilla options).

[INSERT FIGURE 1 HERE]

Figure 1 displays the deviations of the three Black-Scholes values from the Heston values for different underlying prices. For the Black-Scholes model with strike volatility, the barrier option value is overestimated compared to the Heston model, except for underlying prices near the barrier level. The model with at-the-money volatility yields similar results. Only for in-the-money underlying prices are the Black-Scholes values slightly lower than those for the Heston model. For the model with barrier volatility, values are lower than in the Heston model for

[6] For a similar analysis see Hull and Suo (2000). The authors report considerable deviations of the Heston and Black-Scholes model values for different barrier levels.

underlying prices close to the barrier, whereas they are larger close to and above the strike price of the barrier option.

Especially for underlying prices at and slightly below the strike price, where most certificates are issued, any variant of the Black-Scholes model tends to overestimate the value of the down-and-out put and thus to underestimate the value of the bonus certificate. The reason for this is that volatility influences the value of the down-and-out put in two ways. First, a higher volatility increases the option value because of the asymmetric payoff profile. Second, a higher volatility decreases the option value according to the increased knock-out probability. If the volatility structure exhibits a skew, the second (negative) effect is more pronounced compared to a flat volatility. As a consequence, in such a situation, any choice of flat volatility can be inappropriate in explaining the barrier option value. To sum up, it is questionable whether a Black-Scholes-type model is sufficiently suitable to value bonus certificates and unclear, which choice of volatility to use, if any, to lead to the comparatively best results.

3 Methodology and Data

3.1 Certificates Data

Our empirical investigation is based on bonus certificates on the German blue-chip stock index DAX. The data set comprises bid and ask quotes for these products traded in the German market between November 1, 2006 and December 28, 2007. For each certificate we obtain the last bid and ask quotes on the respective day from the European Warrant Exchange (EUWAX). The sample comprises 845 different certificates with a total of 116,204 pairs of bid-ask quotes. In order to avoid biases due to small option time values, we restrict our analysis to certificates with at least 30 calendar days to maturity. We further eliminate quotes from the sample for which an implied volatility for the corresponding certificate could not be adequately estimated from the options market. Furthermore, we focus on issuers with at least 50 certificates outstanding. The remaining set includes 808 certificates and 107,711 bid-ask quotes. The composition of the sample is given in Table 1.

[INSERT TABLE 1 HERE]

The average time to maturity at issuance amounts to 2.73 years. Since the issuance of bonus certificates increased rapidly in 2007, leading to a significant share of recently issued certificates, the effective average time to maturity of the sample is still comparatively high with a value of 2.16 years. Likewise, the average daily number of products with at least one bid-ask quote per day increased considerably from 180 in November 2006 to 620 in December 2007. The reported relative bid-ask spread is calculated by relating the absolute spread to the midpoint of the bid and ask quotes. The average spread of the total sample is fairly small with a value of 0.10 %. However, we find considerable differences in the spreads across the issuers.

To value bonus certificates, we calculate the values of both components of the replicating portfolio, the long zero-strike call and the long down-and-out put option on the DAX. Since the DAX is defined as a performance index, where dividends are considered within its calculation, the value of the zero-strike call equals the index level. We obtain the level of the DAX to the tick at the time of each certificate quote from Frankfurt electronic trading (XETRA).

3.2 Market Data and Model Calibration

The value of the down-and-out put within the Black-Scholes framework is calculated by formula (5). As mentioned above, the required volatility σ is deducted from the volatility surface. We obtain daily settlement prices of call options on the German DAX traded at the EUREX exchange. For each option, we further obtain the corresponding level of the DAX at the time the settlement price is recorded. To determine the volatility σ for a barrier option, we apply a weighting scheme where we interpolate in the two-dimensional space formed by the parameters time to maturity and strike price. Each interpolation involves implied volatilities of four traded options of the two option series with time to maturity closest above and closest below the time to maturity of the respective certificate. Since we consider three versions of the Black-Scholes model in terms of the applied implied volatility, we employ three strike prices: For each series, we select those two options which are closest above and closest below (i) the DAX at-the-money level, (ii) the strike, and (iii) the barrier level of the certificate. Eventually, the interpolated implied volatility is used in formula (5) to calculate the barrier option model value. Within this calculation, we use the level of the DAX at the time of the corresponding bid-ask quotes for the certificate.

Risk-free interest rates are extracted from German government bonds. To estimate interest rates that match the option's time to maturity, we apply Svensson's (1994) extension of the approach proposed by Nelson and Siegel (1987) in modeling the zero-coupon yield curve. Parameter estimates for the parametric Svensson function are provided by the Deutsche Bundesbank on a daily basis. For the risk of issuer default, we deduct credit spreads from the iBoxx index for corporate financials with maturity 1–3 years on a daily basis.

For the calibration of the Heston model, we apply the same information as for the Black-Scholes model, i.e., daily settlement prices from call prices at the EUREX. There is no standard calibration procedure for the Heston model. We follow Gatheral (2006), who suggests a stepwise estimation of the parameters to obtain reasonable values as a starting point for an optimization. First, the parameters κ, η, and ρ are fitted to the skew, according to the following formula:

$$\frac{\partial}{\partial x}\sigma_{BS}^2 = \frac{\rho\eta}{\kappa'(T-t)}\left(1-\frac{1-e^{-\kappa'(T-t)}}{\kappa'(T-t)}\right), \tag{16}$$

where σ_{BS} is the Black-Scholes implied volatility,

$$x = r(T-t) + \log\frac{S_t}{K}, \tag{17}$$

and

$$\kappa' = \kappa - \frac{\rho\eta}{2}. \tag{18}$$

Then, the initial and long-term variance are fitted to the at-the-money volatility term structure:

$$\sigma^2_{BS}\Big|_{K=e^{r(T-t)}S_t} = (v_t - \theta')\left(\frac{1-e^{-\kappa'(T-t)}}{\kappa'(T-t)}\right) + \theta', \tag{19}$$

where

$$\theta' = \theta\frac{\kappa}{\kappa'}. \tag{20}$$

Based on this initial parameter set, we fit the observed implied volatility surface. To obtain stable results, we apply a two-step procedure. In the first step, we deduct the volatility level and skew for each EUREX option maturity and strike levels at-the-money, 20% in-the-money and 20% out-of-the-money. In the second step, the Heston parameters are fitted to this volatility information by minimizing the sum of weighted squared errors between model and market values. To quantify the over- or underpricing of a bonus certificate i, the calculated Black-Scholes (BS) and Heston (He) model values $BC^{model}_{i,t}$ are compared to the observed market prices $BC^{obs}_{i,t}$. We consider the relative price deviations,

$$\Delta V^{model}_{i,t} = \frac{BC^{obs}_{i,t} - BC^{model}_{i,t}}{BC^{model}_{i,t}}, \quad model = BS, He, \tag{21}$$

and employ the mid point of the bid-ask quotes as observed price $BC^{obs}_{i,t}$.

4 Results

4.1 Evolution of the Volatility Skew

Prior to the pricing analysis, we examine the magnitude of the volatility skew during the sample period. Particularly, we compare the implied volatilities of (synthetic) out-of-the-money, at-the-money, and in-the-money DAX options with a period of one year to maturity. The strike level of the out-of-the-money (in-the-money) option is set at 80% (120%) of the at-the-money level. As described in Section 3, the implied volatilities of the synthetic options are obtained by interpolating in the two-dimensional space formed by the parameters time to maturity and strike price. Figure 2 displays the evolvement of the resulting implied volatilities over the course of the period of examination.

[INSERT FIGURE 2 HERE]

Obviously, the implied volatility exhibits a significant skew; the mean difference between the out-of-the-money and the in-the-money volatility is 9.37%. Whereas the three displayed implied volatilities increase slightly on average over the course of the considered period, the difference remains fairly stable (standard deviation of 0.75% for the difference of the out-of-the-money and in-the-money volatility).

4.2 Average Price Differences

Having shown the size and importance of the volatility skew, we now turn to the investigation of the issuers' price setting. Table 2 reports the average differences ΔV_t as defined by (21) between the market prices and the model values of the bonus certificates for each issuer.

[INSERT TABLE 2 HERE]

When looking at the average price differences of the total sample, we find that the difference for the Heston model is higher than that for each of the three considered versions of the Black-Scholes model. Thus, the Heston model yields lower model values on average than the Black-Scholes model. Comparing the three Black-Scholes variants, we find a clear ranking order, where the barrier volatility version leads to the highest, and the strike volatility version to the lowest average differences. Although the differences vary across the issuers, these findings still hold when considering the issuers separately. Furthermore, we find negative differences between market and model values for the Black-Scholes variants with strike volatility, as well as with at-the-money volatility. This result provides a first indication of the inadequacy of these models since negative differences would imply (on average) negative issuer margins.

Comparing the price differences of bonus certificates based on the Heston model with other types of structured retail products, margins for bonus certificates are comparatively high. According to Table 2, average Heston model price differences range from 1.66% (Commerzbank) to 4.61% (Goldman Sachs). In contrast to these finding, a recent study of discount certificates has found average margins below 1%.[7] This is in line with Stoimenov and Wilkens (2005), who report higher margins for more exotic instruments compared to plain-vanilla structures. However, it should be noted that the average remaining time to maturity of bonus certificates in our sample is also larger than it is for typical discount certificates. The last column of Table 2 reports average annualized margins, defined as the relative price difference divided by the remaining time to maturity. Based on this measure, deviations between the issuers are still existent, but with a range of 1.13% to 1.83%, are considerably smaller. The average value of 1.53% per annum is still large compared to plain-vanilla instruments.

4.3 Explanation of Price Differences

In this section, we analyze the price differences based on the four models in more detail. In particular, we apply a statistical model for the composition of price differences and examine to which extent they can be explained for the four pricing models. Since the skew is well documented within the considered sample (see Section 4.1), the price differences implied by the (skew-consistent) Heston model should yield the best result in this regard if the issuer actually

[7]See Baule et al. (2008).

accounts for the skew.

Besides the pricing model applied by the issuer, the price difference is subject to further potential influencing factors. First, according to the life cycle hypothesis stated by Wilkens et al. (2003), it is known that price differences at the secondary market are a decreasing function of the remaining time to maturity of the certificate. It can be argued that at maturity, price differences must be zero because of transparency reasons. Close to maturity, the value of the certificate nearly equals its intrinsic value, which leaves no space for any surcharges. The life cycle hypothesis has been confirmed by several further empirical studies, e.g., Stoimenov and Wilkens (2005) and Baule et al. (2008). Therefore, we incorporate the relative remaining lifetime to maturity

$$L_{i,t} = \frac{T_i - t}{T_i - T_i^*} \tag{22}$$

as an explanatory variable, with T_i^* being the time of issuance.

Second, the moneyness

$$M_{i,t} = \frac{S_t - K_i}{K_i} \tag{23}$$

is a further possible explanatory variable for price differences. For structured products with embedded plain-vanilla options, the moneyness has been found to have an impact on the price difference in some cases.[8] The reason for this is that moneyness can be seen as a measure of the issuer's leeway in incorporating a profit margin since it determines how obvious the fair ("true") value of the derivative is to investors. In the case of bonus certificates, the leeway increases close to the barrier, i.e., for larger negative values of the moneyness. Hence, we expect the moneyness to have a negative impact on the price difference.

In addition to these two variables, we include the calendar time $Y_t = t - t^*$, with t^* being the beginning of the examination period (November 1, 2006), to control for potential overall changes

[8]See e.g., Wilkens et al. (2003).

in the issuer's price setting over the course of time. In summation, we consider the following model for the relative price difference:

$$\Delta V_{i,t} = \beta_0 + \beta_1 L_{i,t} + \beta_2 Y_t + \beta_3 M_{i,t} + \epsilon_{i,t}. \tag{24}$$

The statistical model is applied to the four pricing models (three variants of Black-Scholes and Heston). The assessment of which pricing model is closest to actual price setting in reality is done via the standard deviation s_{Res} of the regression residuals. The lower this standard deviation, the better the linear model's fit to price differences and hence the "closer" the respective model to the issuer's actual price-setting model.

Within our data sample we deal with the problem that the observations are not independent. Instead, for each certificate, the time series of price differences exhibits a high degree of autocorrelation. To overcome this problem, for each certificate, we randomly select one observation out of the respective time series. In doing this, we are able to consider the cross section of certificates while comprising the total sample period.

[INSERT TABLE 3 HERE]

Regression results are provided in Table 3. The standard deviations s_{Res} of the regression residuals indicate a clear ranking order of the four models. Among the Black-Scholes models, barrier volatility is comparatively the best variant for all issuers with s_{Res} ranging from 0.0085 to 0.0204. However, for all issuers, the barrier volatility variant is outperformed by the Heston model with s_{Res} ranging from 0.0064 to 0.0175, indicating a significant improvement compared to the Black-Scholes regressions. The results are supported by the regression R^2s, which reach the highest values for all issuers except Goldman Sachs in case of the Heston model. These findings provide indications that the skew is priced by all issuers.

With regard to the influencing factors discussed above, we find a negative relationship between moneyness, $M_{i,t}$, and price differences for the Heston model, as well as for the barrier Black-Scholes variant, for all issuers. In most cases the relationship is highly significant at the 0.1% level. Hence, as expected, price differences increase with decreasing moneyness. Results for the other Black-Scholes variants are ambiguous regarding direction and significance of the moneyness. We attribute this effect to the inadequacy of these variants in explaining the actual price-setting model.

According to the life cycle hypothesis and the definition of the relative lifetime, $L_{i,t}$, we expect the variable β_1 for the relative lifetime to be positive. Indeed, for the Heston model we obtain positive coefficients, which are, however, only significant (at the 0.1% level) for Goldman Sachs and Sal. Oppenheim. Regarding the Black-Scholes variants, results are again ambiguous. Thus, if we conclude that the Heston model is most suitable, the findings are consistent with our expectations and support the results of prior studies with respect to the life cycle hypothesis, although with decreased significance (significance in only two of five cases).[9]

Furthermore, Heston model price differences increase with the calendar time, Y_t. Except for Société Générale the relationship is significant at least at the 1% level. Again, when considering the Black-Scholes variants, results are indistinct with the exception of Commerzbank, where we find a significant positive relationship in all cases.

To sum up, while results for the Black-Scholes variants are ambiguous, based on the Heston model, we find a statistically positive relationship between relative lifetime and price difference for two issuers and a significant negative relationship between moneyness and price difference for the Heston model for all issuers.

[9] In interpreting the results regarding the relationship of price differences and the relative lifetime, one has to consider the specific sample composition, which does not cover a representative sample of the complete life cycle. Since bonus certificates are innovative products, with the majority of the considered products issued in 2007 (average relative lifetime of the sample is 0.87), results of the linear slope estimation may be indistinct.

4.4 Consideration of the Skew

The results presented so far provide support for the hypothesis that the issuers indeed consider the skew in their pricing of structured retail products. However, it has not yet been statistically confirmed, and the comparably good explanatory power of the Black-Scholes variant with barrier volatility casts some doubt as to whether the results can withstand rigorous testing. Theoretically, to identify the pricing model closest to the issuer's, as with a factor analysis one could try to simultaneously regress observed market prices onto all considered model values while controlling for the identified influencing parameters as follows:[10]

$$\begin{aligned} BC_{i,t}^{obs} &= \alpha_1 BC_{i,t}^{BS-atm} + \alpha_2 BC_{i,t}^{BS-strike} + \alpha_3 BC_{i,t}^{BS-barrier} + \alpha_4 BC_{i,t}^{Heston} \\ &\quad + \beta_0 + \beta_1 L'_{i,t} + \beta_2 Y'_t + \beta_3 M'_{i,t} + \epsilon_{i,t}. \end{aligned} \tag{25}$$

One could then identify the actually applied pricing model by the significance of the regression coefficients—similar to factor loadings.

But since the absolute model values are highly correlated, such an approach would be useless due to multicollinearity. We therefore modify the approach in such a way that we consider the model variants separately. Furthermore, as in the preceding analysis, we rely on relative price differences rather than on absolute prices. Specifically, we test if the issuers apply one of the Black-Scholes variants without considering the skew using a regression of the form

$$\Delta V_{i,t}^{BS} = \alpha SKEW_{i,t} + \beta_0 + \beta_1 L_{i,t} + \beta_2 Y_t + \beta_3 M_{i,t} + \epsilon_{i,t}. \tag{26}$$

The null hypothesis is $\alpha = 0$.

To operationalize the variable $SKEW_{i,t}$, we utilize the Heston model. As the Heston model

[10]In contrast to the regression (24), the independent variable is not standardized here. Therefore, the control variables must also be multiplied with the absolute value of of the certificate, i.e., $L'_{i,t} = L_{i,t} \cdot BC_{i,t}$, $Y'_t = Y_t \cdot BC_{i,t}$, and $M'_{i,t} = M_{i,t} \cdot BC_{i,t}$.

accounts for the skew, the relative difference between the Heston and the respective Black-Scholes value is a suitable measure for the influence of the skew on the certificate value. We therefore define

$$D_{i,t} = \frac{BC_{i,t}^{He} - BC_{i,t}^{BS}}{BC_{i,t}^{BS}} \tag{27}$$

and run the regression

$$\Delta V_{i,t}^{BS} = \alpha D_{i,t} + \beta_0 + \beta_1 L_{i,t} + \beta_2 Y_t + \beta_3 M_{i,t} + \epsilon_{i,t} \tag{28}$$

for each of the three Black-Scholes variants.

If issuers account for the skew, the coefficient α for the variable $D_{i,t}$—representing the skew-consistent Heston model—is significantly positive. Furthermore, the coefficient should be exactly 1 if the issuer relied on the Heston model (as specifically applied in this study). However, as there are various ways to calibrate the Heston model and moreover different models which also account for the skew, we expect α to be at least approximately 1.

Again, we employ the random selection procedure described above to obtain an adequate sample for the regressions.

[INSERT TABLE 4 HERE]

Regression results are reported in Table 4. First and most striking, the coefficient α for the model difference is significantly positive at the 0.1% level for all issuers and models. Thus, the null hypothesis—that issuers apply Black-Scholes without considering the skew—can be rejected for each of the three model variants.

Furthermore, the regressions yield values for α which are close to 1. These findings provide evidence that issuers apply a model which leads to similar results as the Heston model with the described calibration.

To check the robustness of the results, we conducted the same analysis with different random samples, all of which lead to fairly similar results. So the title question can be answered with a definite "yes"—all issuers of bonus certificates incorporate the volatility skew into their retail prices.

5 Conclusion

This paper is the first to investigate the price-setting for bonus certificates, a popular type of retail derivative which features an embedded barrier option. We examine in particular whether issuers take the volatility skew into account. Besides the widely used Black-Scholes model, we employ the Heston stochastic volatility model, which is able to account for the well-documented skew.

A theoretical analysis of a generic barrier option reveals that Heston and Black-Scholes model values differ considerably depending on the underlying price and the employed estimate for the Black-Scholes stock price volatility. These findings cast doubt on the adequacy of a Black-Scholes-type model in valuing bonus certificates. Indeed, when assessing the considered models by the explanatory power for observed market prices, measured by the standard deviations of regression residuals for the relative price differences, the Heston model outperforms all three Black-Scholes variants. By applying a rigorous statistical test we can confirm these indications that issuers incorporate the volatility skew in their price setting, i.e., they deploy a valuation approach beyond the plain Black-Scholes model.

Our findings suggest that an analysis of structured financial products with embedded barrier options should not be based solely on Black-Scholes. In this sense, the paper joins the work of Muck (2006), who argues the necessity of applying more advanced models for the valuation of turbo certificates.

Although we can conclude that issuers account for the skew, we have no evidence that they actually rely on the Heston model. Since the calibration of the Heston model involves five parameters for which several calibration approaches are applicable, the calibration introduces some kind of noise. Hence, even if an issuer relies on the Heston model, calibration issues may lead to biases in model values. Moreover, there are other smile-consistent option-pricing models, such as extensions of the Heston model (e.g., allowing for discontinuous jumps in stock prices or stochastic interest rates) or local volatility models. Nonetheless, our results show that the actual pricing model provides results similar to those for the Heston model.

Based on the Heston model, we further analyze the structure of issuer margins. Besides supporting the life cycle hypothesis for bonus certificates, we find evidence that margins are a decreasing function of the certificate's moneyness. Concerning the average size of the margins, they are relatively high (amounting to 4.7% for Goldman Sachs and 2.3%–2.9% for other issuers), compared to recent studies of retail derivatives with embedded plain-vanilla options. These figures become less pronounced in the light of the rather lengthy lifetime of bonus certificates. However, average annualized margins are still relatively large at 1.5% on average.

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Issuer	# Certificates	# Quotes	Avg. Maturity	Avg. Spread
BNP Paribas	260	23,022	2.08	0.14 %
Commerzbank	129	22,479	1.60	0.07 %
Goldman Sachs	266	45,969	2.69	0.08 %
Sal. Oppenheim	89	10,569	1.64	0.12 %
Société Générale	64	5,672	1.41	0.25 %
Total	808	107,711	2.16	0.10 %

Table 1. *Composition of the certificates data sample*

	Black-Scholes			Heston	Heston ann.
Issuer	atm	strike	barrier		
BNP Paribas	0.85%	−0.30%	1.83%	2.33%	1.30%
	(1.48%)	(1.89%)	(2.07%)	(1.75%)	(1.07%)
Commerzbank	0.56%	0.31%	1.01%	1.66%	1.13%
	(1.31%)	(1.22%)	(1.42%)	(1.23%)	(0.91%)
Goldman Sachs	2.65%	1.81%	3.44%	4.61%	1.77%
	(2.57%)	(2.51%)	(2.85%)	(2.20%)	(0.54%)
Sal. Oppenheim	−0.39%	−1.76%	1.51%	2.65%	1.71%
	(1.51%)	(2.01%)	(1.64%)	(1.17%)	(0.59%)
Société Générale	−0.44%	−1.75%	1.24%	2.45%	1.83%
	(1.29%)	(1.58%)	(1.70%)	(1.22%)	(1.03%)
Total	1.37%	0.45%	2.28%	3.20%	1.53%
	(2.31%)	(2.43%)	(2.51%)	(2.19%)	(0.84%)

Table 2. *Average relative differences between market prices and model values by issuer. Black-Scholes model values are calculated with at-the-money volatility ("atm"), strike volatility ("strike") and barrier volatility ("barrier"). For the Heston model, the last column also in addition displays the annualized margin, i.e., the average relative difference divided by the remaining time to maturity. Standard deviations are reported in parentheses.*

	BS (atm)	BS (strike)	BS (barrier)	Heston
BNP Paribas ($N = 259$)				
Constant	+0.0256*	−0.0210*	+0.0291***	+0.0052
	(0.0119)	(0.0123)	(0.0082)	(0.0064)
Relative Lifetime	−0.0216	+0.0268*	−0.0252**	+0.0041
	(0.0132)	(0.0137)	(0.0091)	(0.0071)
Time	−0.0028	−0.0061	+0.0098*	+0.0197***
	(0.0066)	(0.0069)	(0.0046)	(0.0035)
Moneyness	−0.0564***	+0.0630***	−0.1098***	−0.0686***
	(0.0131)	(0.0136)	(0.0090)	(0.0070)
R^2	0.1132	0.1596	0.6286	0.6895
s_{Res}	0.0205	0.0213	0.0141	0.0110
Commerzbank ($N = 128$)				
Constant	+0.0038	−0.0011	−0.0001	+0.0048
	(0.0054)	(0.0058)	(0.0040)	(0.0033)
Relative Lifetime	−0.0062	−0.0014	+0.0037	+0.0050
	(0.0071)	(0.0077)	(0.0053)	(0.0044)
Time	+0.0126***	+0.0105**	+0.0139***	+0.0141***
	(0.0031)	(0.0034)	(0.0023)	(0.0019)
Moneyness	−0.0858***	−0.0643***	−0.0967***	−0.0777***
	(0.0128)	(0.0138)	(0.0094)	(0.0079)
R^2	0.4910	0.3510	0.7448	0.7563
s_{Res}	0.0115	0.0124	0.0085	0.0071
Goldman Sachs ($N = 265$)				
Constant	−0.0318*	−0.0481***	−0.0246*	−0.0159*
	(0.0123)	(0.0134)	(0.0098)	(0.0085)
Relative Lifetime	+0.0548***	+0.0804***	+0.0469***	+0.0565***
	(0.0144)	(0.0156)	(0.0115)	(0.0099)
Time	+0.0051	−0.0019	+0.0086*	+0.0098**
	(0.0046)	(0.0050)	(0.0036)	(0.0031)
Moneyness	−0.1009***	+0.0104	−0.1564***	−0.0864***
	(0.0148)	(0.0160)	(0.0118)	(0.0101)
R^2	0.3925	0.1335	0.6440	0.5416
s_{Res}	0.0256	0.0278	0.0204	0.0175

Table 3. *Results of regressions of price differences for the four models with respect to the relative remaining time to maturity (relative lifetime), the (absolute) time since issuance, and the moneyness. N denotes the number of observations, and s_{Res} is the standard deviation of the residuals. Significance at the 10% level is indicated with *, at the 1% level with **, and at the 0.1% level with ***. Standard errors are reported in parentheses.*

	BS (atm)	BS (strike)	BS (barrier)	Heston
Sal. Oppenheim ($N = 88$)				
Constant	$+0.0083$	-0.0154	$+0.0117$	-0.0188*
	(0.0159)	(0.0136)	(0.0114)	(0.0081)
Relative Lifetime	-0.0063	$+0.0274$*	-0.0062	$+0.0409$***
	(0.0188)	(0.0160)	(0.0135)	(0.0096)
Time	-0.0152*	-0.0134*	-0.0108*	$+0.0135$**
	(0.0088)	(0.0075)	(0.0063)	(0.0045)
Moneyness	-0.0405	$+0.1367$***	-0.1478***	-0.0297*
	(0.0284)	(0.0243)	(0.0204)	(0.0145)
R^2	0.0461	0.4756	0.5715	0.5778
s_{Res}	0.0195	0.0166	0.0140	0.0099
Société Générale ($N = 63$)				
Constant	$+0.0507$**	-0.0175	$+0.0317$*	$+0.0101$
	(0.0185)	(0.0166)	(0.0148)	(0.0088)
Relative Lifetime	-0.0525**	$+0.0193$	-0.0264*	$+0.0084$
	(0.0196)	(0.0176)	(0.0157)	(0.0094)
Time	-0.0236*	-0.0095	-0.0112	$+0.0042$
	(0.0098)	(0.0088)	(0.0078)	(0.0047)
Moneyness	-0.0704*	$+0.1143$	-0.1501***	-0.0622***
	(0.0300)	(0.0269)	(0.0240)	(0.0143)
R^2	0.1450	0.5479	0.6312	0.6881
s_{Res}	0.0134	0.0120	0.0107	0.0064

Table 3. *(continued) Results of regressions of price differences for the four models with respect to the relative remaining time to maturity (relative lifetime), the (absolute) time since issuance, and the moneyness. N denotes the number of observations, and s_{Res} is the standard deviation of the residuals. Significance at the 10% level is indicated with *, at the 1% level with **, and at the 0.1% level with ***. Standard errors are reported in parentheses.*

	BS (atm)	BS (strike)	BS (barrier)
BNP Paribas ($N = 259$)			
Constant	+0.0030	+0.0031	+0.0083
	(0.0062)	(0.0060)	(0.0065)
Model Difference	+1.1506***	+0.9866***	+0.9276***
	(0.0433)	(0.0341)	(0.0700)
Relative Lifetime	+0.0073	+0.0064	−0.0007
	(0.0069)	(0.0067)	(0.0073)
Time	+0.0218***	+0.0182***	+0.0184***
	(0.0035)	(0.0034)	(0.0036)
Moneyness	−0.0686***	−0.0587***	−0.0729***
	(0.0068)	(0.0078)	(0.0075)
R^2	0.7648	0.8034	0.7799
Commerzbank ($N = 128$)			
Constant	+0.0052	+0.0056*	+0.0068*
	(0.0032)	(0.0033)	(0.0034)
Model Difference	+1.2578***	+1.1279***	+1.3801***
	(0.0826)	(0.0677)	(0.1762)
Relative Lifetime	+0.0076*	+0.0057	+0.0053
	(0.0043)	(0.0043)	(0.0043)
Time	+0.0143***	+0.0142***	+0.0141***
	(0.0019)	(0.0019)	(0.0019)
Moneyness	−0.0752***	−0.0781***	−0.0707***
	(0.0076)	(0.0078)	(0.0084)
R^2	0.8227	0.7995	0.8298
Goldman Sachs ($N = 265$)			
Constant	−0.0118	−0.0093	−0.0159*
	(0.0083)	(0.0082)	(0.0084)
Model Difference	+1.2035***	+1.2001***	+0.9324***
	(0.0666)	(0.0557)	(0.0941)
Relative Lifetime	+0.0549***	+0.0517***	+0.0545***
	(0.0096)	(0.0095)	(0.0099)
Time	+0.0103***	+0.0109***	+0.0096**
	(0.0031)	(0.0030)	(0.0031)
Moneyness	−0.0842***	−0.0962***	−0.0942***
	(0.0099)	(0.0108)	(0.0119)
R^2	0.7303	0.6884	0.7414

Table 4. *Results of regressions of price differences for the three Black-Scholes variants with respect to the relative difference of the Heston model value and the respective Black-Scholes model variant, the relative lifetime, the (absolute) time since issuance, and the moneyness. N denotes the number of observations. Significance at the 10% level is indicated with *, at the 1% level with **, and at the 0.1% level with ***. Standard errors are reported in parentheses.*

	BS (atm)	BS (strike)	BS (barrier)
Sal. Oppenheim ($N = 88$)			
Constant	−0.0228**	−0.0167*	−0.0188*
	(0.0077)	(0.0076)	(0.0087)
Model Difference	+1.2495***	+1.0661***	+1.0533***
	(0.0718)	(0.0774)	(0.1133)
Relative Lifetime	+0.0494***	+0.0396***	+0.0413***
	(0.0094)	(0.0090)	(0.0108)
Time	+0.0189***	+0.0134**	+0.0139**
	(0.0046)	(0.0046)	(0.0052)
Moneyness	−0.0273*	−0.0340*	−0.0270
	(0.0134)	(0.0184)	(0.0194)
R^2	0.7930	0.8392	0.7888
Société Générale ($N = 63$)			
Constant	+0.0151*	+0.0062	+0.0142
	(0.0084)	(0.0082)	(0.0086)
Model Difference	+0.8958***	+0.9156***	+0.8550***
	(0.0562)	(0.0650)	(0.0764)
Relative Lifetime	+0.0010	+0.0108	+0.0021
	(0.0092)	(0.0085)	(0.0093)
Time	+0.0005	+0.0027	+0.0014
	(0.0045)	(0.0043)	(0.0046)
Moneyness	−0.0614***	−0.0388*	−0.0769***
	(0.0132)	(0.0169)	(0.0152)
R^2	0.8387	0.8963	0.8819

Table 4. *(continued) Results of regressions of price differences for the three Black-Scholes variants with respect to the relative difference of the Heston model value and the respective Black-Scholes model variant, the relative lifetime, the (absolute) time since issuance, and the moneyness. N denotes the number of observations. Significance at the 10% level is indicated with *, at the 1% level with **, and at the 0.1% level with ***. Standard errors are reported in parentheses.*

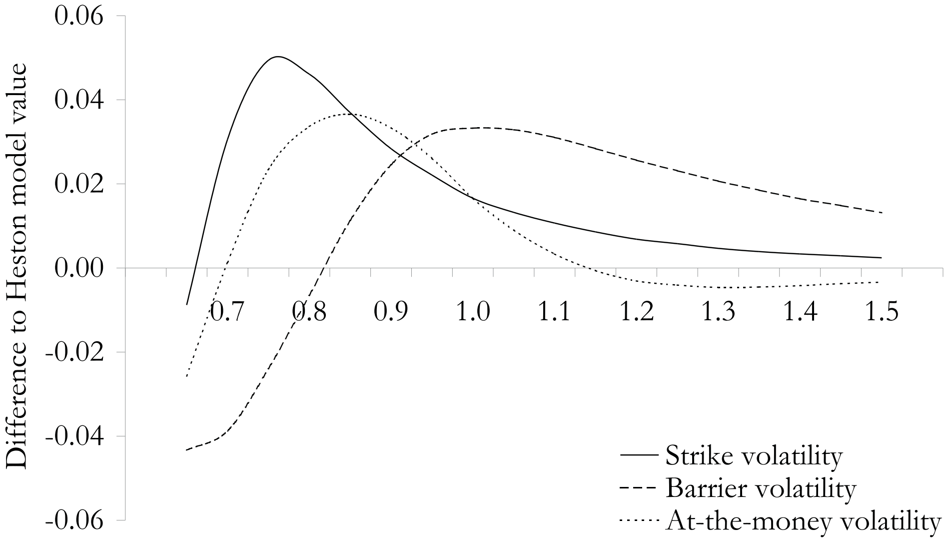

Figure 1. *Differences between the Black-Scholes and Heston model values of a down-and-out put option with strike* $K = 1$ *and barrier* $H = 0.6$ *for underlying prices between 0.65 and 1.5. Implied volatilities for the Black-Scholes model are obtained as strike volatility (solid line), barrier volatility (dashed line), and at-the-money volatility (dotted line) of plain-vanilla options whose values are consistent with the Heston model.*

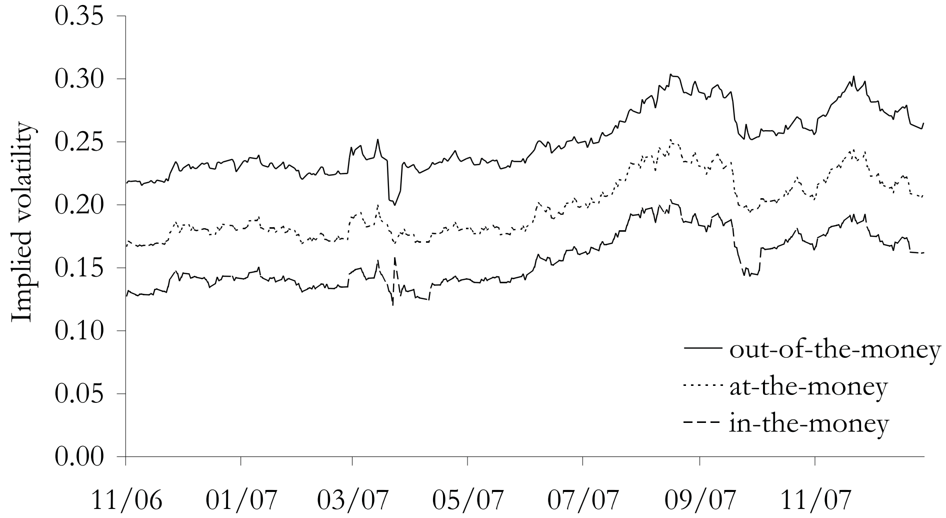

Figure 2. *Implied volatility for out-of-the-money (80% of the at-the-money level—solid line), at-the-money (dotted line) and in-the-money (120% of the at-the-money level—dashed line) DAX options during the period 01/11/2006 through 28/12/2007.*