

Incomplete information, idiosyncratic volatility and stock returns¹

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Abstract

We develop a model of firm investment under incomplete information that explains why idiosyncratic volatility and stock returns are related. When the unobserved state variable proxies for business cycles, we show that a properly calibrated version of the model generates a negative relation due to the natural asymmetry in the length of expansions and recessions. We further show that, conditional on earning surprises, the relation between idiosyncratic volatility and stock returns is positive after good news and negative after bad news. This result provides new insights on the nature of stock return predictability.

Keywords: Idiosyncratic volatility, incomplete information, cross-section of returns, q -theory of investment.

JEL Classification. G12, D83, D92.

1 Introduction

According to textbook asset pricing theory, investors are only compensated for bearing aggregate risk and, as a result, idiosyncratic volatility should not be priced. However, numerous recent empirical studies have documented a relation between stock returns and idiosyncratic volatility. In particular, [Ang, Hodrick, Xing, and Zhang \(2006\)](#), [Jiang, Xu, and Yao \(2007\)](#) and [Brockman and Yan \(2008\)](#) provide strong evidence of a negative relation for the US stock market, and [Ang, Hodrick, Xing, and Zhang \(2008\)](#) confirm in a recent study that a similar relation also holds in other markets. There is however no consensus as to the direction of this effect. Indeed, [Malkiel and Xu \(2001\)](#), [Spiegel and Wang \(2005\)](#) and [Fu \(2005\)](#) find positive relations between idiosyncratic volatility and expected returns, while [Longstaff \(1989\)](#) finds a weakly negative relation.

In this paper we propose a model of firm valuation under incomplete information that is able to explain the ambiguous link between idiosyncratic volatility and stock returns. Firms in our model are unable to perfectly anticipate the growth rate of their cash flows, but learn about it by observing a firm specific signal. As a result, the shocks perceived by a firm, the so-called innovation process, differ from those which are measured by the econometrician who conducts unconditional tests based on the whole history of stock returns. Indeed, the innovation process is the sum of two terms: the underlying true idiosyncratic shock and the error that the firm makes in estimating the growth rate of its cash flows. In contrast, the econometrician uses the actual underlying distribution to construct his tests and, hence, is able to measure the true idiosyncratic shocks of the firms.

Conditional on the information available to the firm the estimation error is equal to zero on average, and it follows that the expected returns satisfy a

conditional version of the CAPM where idiosyncratic volatility plays no role. In contrast, relative to the true underlying distribution, the firm's estimation error is different from zero and thus appears multiplied by idiosyncratic volatility in the expected stock returns as measured by an econometrician performing unconditional tests similar to those of [Ang et al. \(2006; 2008\)](#), [Jiang et al. \(2007\)](#) and [Brockman and Yan \(2008\)](#) among others. This is the mechanism which generates a relation between idiosyncratic volatility and stock returns. It is important to observe that the deviation from the CAPM which is implied by our model is not due to a missing factor. Indeed, the additional term in the expression of a firm's expected stock return is generated by the firm's own estimation errors and, hence, does not represent remuneration for exposure to a systematic risk factor. The presence of such a component in expected returns is entirely due to learning and could not be generated by introducing additional state variables into an otherwise standard model.

Since our aim is to explain properties of the cross-section of stock returns we need a valuation model in which heterogeneity among firms arises endogenously through time. Furthermore, we want to be able to calibrate the model to observable firm and industry characteristics. In order to achieve these goals, we focus on a simple version of the q -theoretic model of investment with adjustment costs which has been successful in describing many properties of the cross section of stock returns (see [Liu, Whited, and Zhang \(2007\)](#), [Li, Livdan, and Zhang \(2007\)](#) and the references therein). Specifically, we assume that each firm is endowed with a constant returns to scale production function and faces quadratic capital adjustment costs. The growth rate of the firm specific output price follows a two state Markov

chain which is common to all firms, and thus proxies for business cycles.¹ We assume that firms only observe the output prices and must therefore estimate the current value of the growth rate.

This simple specification delivers a closed form expression for the value of the firm which allows for a transparent analysis of the relation between idiosyncratic volatility and stock returns. In particular, the model shows that, relative to the true distribution from which the shocks are drawn, the expected excess stock return is the sum of two terms. The first term is the usual remuneration for exposure to aggregate risk, namely the product of the firm's beta and the market price of risk. The second term is the product of the firm's idiosyncratic volatility and a normalized forecast error. This term is unique to our incomplete information setting and is the channel through which idiosyncratic volatility and stock returns are related.

The empirical evidence documenting the idiosyncratic volatility anomaly relies on sorting stocks into portfolios on the basis of past idiosyncratic volatilities. In our model, the sign and magnitude of the relation that would be obtained using a similar construction depends on the distribution of the forecast errors among the sorted portfolios. Since the unobserved growth rate is common to all firms, the forecast errors at a given point in time all have the same sign. Firms underestimate the growth rate and, thus, make positive forecast errors, during expansion phases and overestimate the growth rate during recessions. The magnitude of these errors and their relative frequency depend on the distribution of the business cycles. Since the latter are asymmetric with long expansions and short recessions, the unconditional mean of the firms' forecasts is close to the value of the growth rate

¹Similar specifications have been used in the asset pricing literature by [Veronesi \(1999; 2000\)](#) and [David \(1997\)](#), in the corporate finance literature by [Hackbarth, Miao, and Morellec \(2006\)](#) and in the investment literature by [Guo, Miao, and Morellec \(2005\)](#) and [Eberly, Rebello, and Vincent \(2006\)](#).

that prevails during expansion phases. As a result, positive forecast errors are frequent and small, while negative forecast errors are infrequent but of larger magnitude. The relation between idiosyncratic volatility and stock returns is negative if the contribution of the negative forecast errors induced by recessions dominates, and positive otherwise. To determine which of the two cases prevails, we first calibrate the model to match US business cycles as well as important firm and industry characteristics. We then replicate the portfolio construction of [Ang et al. \(2006\)](#) on panels of data simulated from the calibrated model and obtain similar results. On average, portfolios of firms with high idiosyncratic volatility generate lower returns and have lower alphas, indicating that negative forecast errors dominate. We further confirm that the asymmetry in the distribution of the growth rate is the key element needed to generate a negative relation by simulating the model with a symmetric distribution of business cycles. In that case, there is no significant relation between idiosyncratic volatility and stock returns.

While the mechanism which links idiosyncratic volatility to stock returns is entirely due to incomplete information, the sign of the relation that we obtain in our model comes from the specific choice of business cycles as the underlying unobserved state variable. A different choice of the unobserved state variable could induce a positive relation. For example, a model with time dependent transitions between expansions and recessions implies a time varying distribution of forecast errors and, hence, could produce alternating episodes of positive and negative relation between idiosyncratic volatility and stock returns. Such a setting could help understand the diverging results in the empirical literature documenting the idiosyncratic volatility anomaly.

In addition to providing an explanation of the idiosyncratic volatility anomaly, our model also has some novel empirically testable implications for

the cross section of stock returns. First, we show that the response of returns to earning surprises depends on idiosyncratic volatility in an asymmetric way. Specifically, the model implies that, following good news, firms with larger idiosyncratic volatility should produce larger returns. Following bad news, the relation is reversed and firms with larger idiosyncratic volatility should produce lower returns. This implication relates to the vast literature on stock return predictability, see [Ball and Brown \(1968\)](#), [Watts \(1978\)](#), [Foster, Olsen, and Shevlin \(1984\)](#) and [Bernard and Thomas \(1990\)](#) among others. Our contribution is to show, theoretically, why the reaction to news should be stronger among high idiosyncratic volatility firms. Recent results in the empirical literature support this prediction. In particular, [Zhang \(2006\)](#) studies the link between information uncertainty and stock returns. Sorting firms on past return volatility, he documents that firms with high volatility perform relatively better following good news and relatively worse following bad news. Our prediction is consistent with this finding because in our model firms with large total volatility are also those with large idiosyncratic volatility.

The deviation from the traditional CAPM implied by our setting is driven by earning forecast errors. Therefore the second implication of our model is that, controlling for earning surprises, idiosyncratic volatility should not have explanatory power for the cross section of stock returns. In other words, introducing this control variable should reduce the volatility anomaly. Here again, there is recent empirical evidence in the literature to support the prediction of our model. [Jiang et al. \(2007\)](#) replicate the findings in [Ang et al. \(2006\)](#) by showing that idiosyncratic volatility has a negative and significant coefficient in the standard Fama-MacBeth regressions. However, when controlling for analyst forecast errors, they obtain a non-significant

coefficient for idiosyncratic volatility. Since such forecasting errors are a reasonable proxy for the additional term implied by our model, the results of [Jiang et al. \(2007\)](#) provide strong support for our prediction.

The remainder of the paper is organized as follows. In [Section 2](#), we formulate our incomplete information model and derive an analytical solution for the value of an individual firm. In [Section 3](#), we provide a theoretical analysis of the relation between idiosyncratic volatility and stock returns and we derive testable implications. In [Section 4](#), we detail the simulation methodology and the calibration of the model. We also present the results of regressions performed on artificial panel data and study the determinants of the relation by varying key parameters. [Section 5](#) concludes.

2 The model

In this section we construct a model of capital investment with adjustment costs under incomplete information. Heterogeneity among firms arises endogenously as each firm faces a specific output price that comprises both an idiosyncratic and an aggregate shock.

As in [Berk, Green, and Naik \(1999\)](#), we focus on a partial equilibrium model in the sense that we take the pricing kernel as given. This gives us the tractability we need in order to focus on the relation between idiosyncratic volatility and stock returns.

2.1 Information structure

We consider a continuous time model of an economy in which firms sell their output at a firm specific price X_i . This firm specific price process has a stochastic growth rate which is common to all firms and is affected by both idiosyncratic and aggregate shocks.

Assumption 1: For each i , the firm specific price X_i evolves according to

$$dX_{it} = X_{it}\theta_t dt + X_{it}\sigma \left(\rho dB_{at} + \sqrt{1 - \rho^2} dW_{it} \right) \quad (1)$$

where B_a , W_i are independent Wiener processes and (ρ, σ) are constants. The process for the growth rate θ is described below.

The constant $\rho\sigma$ measures the exposure of the firm specific price process to the aggregate shock B_a which is common to all the firms. The constant $\sigma\sqrt{1 - \rho^2}$ measures the exposure of the price process to the firm specific Wiener process W_i . The instantaneous volatility of the price process is identical across firms and equal to σ . Similarly, the instantaneous correlation between the prices faced by firms i and $j \neq i$ is identical across firms and equal to ρ . The firm specific prices grow at the rate θ which is common to all firms and satisfies the following.

Assumption 2: The growth rate of the price processes follows a two-state, continuous time Markov chain with generator matrix ²

$$\Gamma = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}. \quad (2)$$

The two states of the Markov chain are denoted by $\theta_h > 0$ and $\theta_l < 0$ and are referred to as the low and the high state of the economy.

The growth rate of the price processes can jump from one state to another, simultaneously for all firms. Furthermore, the above assumption implies that the transition times between the high and the low state on one hand, and between the low and high state on the other hand, are exponentially

²See [Karlin and Taylor \(1975, Chapter 4\)](#) for a precise definition of the generator matrix. The transition matrix of the chain over a period of length t can be obtained from the generator matrix simply as $\exp(\Gamma t)$.

distributed with parameters λ and μ . This is a simple way of introducing business cycles into the model.³ In the high state, the economy is expanding and all firms benefit from the positive trend in prices. In the low state, the economy is contracting and all firms suffer from the negative trend in prices. Nevertheless, in both states firms may be subject to specific adverse or beneficial market conditions which are modeled through their exposure to the idiosyncratic shocks.

A key feature of our model is that agents have incomplete information about the growth rate of the output price processes. Firms are run by managers who act in the best interest of shareholders. These managers play no role other than processing information and implementing the corporate strategy that maximizes shareholder value. In the model, the managers base their anticipations on the observation of different signals and hence have different perceptions about the current state of the economy. In particular, we make the following assumption.

Assumption 3: *The manager of firm i only observes the realizations of the aggregate shock B_a and the price process X_i faced by his firm.*

Since the manager only observes the aggregate shock and the price faced by his firm, his estimation of the current state depends only the realizations of his firm's price process but not on the price processes of the other firms in the economy. An identical information processing behavior can be obtained by assuming that the managers observe all the price processes but do not recognize the fact that the growth rate is common to all firms. The above assumption is thus behavioral as it implies that the managers are biased

³The interpretation of θ as a model of the business cycle indicates precisely how to calibrate the generator matrix. In the simulations of the model we use the frequencies of the business cycles as reported by the NBER to calibrate the transitions between the two states of the growth rate and investment rates to calibrate its values.

in the way they learn about the state of the economy. The [Brown and Rozeff \(1979\)](#) model, which is one of the most popular earnings forecast model, relies exclusively on past earnings and does not use any economy wide variables. Since the firm's output is locally deterministic, forecasting prices or earnings is equivalent in our setting, and it follows that Assumption 3 is in line with the standard forecasting practice.

To gauge the robustness of our results to Assumption 3, we consider in Section 4.4 an alternative specification of the model where the growth rates are identically and independently distributed across firms. In such a setting, the information processing behavior implied by Assumption 3 is optimal, and we show that the implications of this alternative specification for the relation between idiosyncratic volatility and stock returns are qualitatively similar to those of our base case model. While the latter requires Assumption 3, it has a clear advantage compared to a specification with iid growth rates since the assumption of a common growth rate makes it possible to calibrate the model to the business cycle.

Since the manager of firm i has complete information about the aggregate shocks, he bases his estimation of the growth rate on the observation of a firm specific signal s_i which evolves according to

$$ds_{it} = dX_{it}/X_{it} - \rho\sigma dB_{at} \equiv \theta_t dt + (1/\epsilon)dW_{it}. \quad (3)$$

The constant $\epsilon = 1/\sigma\sqrt{1-\rho^2}$ measures the precision of the signal. When ϵ is high, the signal has very little dispersion around the true value of the growth rate and firms are able to estimate θ accurately. On the contrary, when ϵ is low, the variance of the signal is very large and firms are thus unable to estimate the growth rate accurately.

Let \mathcal{F}_{it} denote the information set which is available at time t to the manager of firm i . This information set contains all past realizations of the aggregate shock and the firm specific price process. Let E_{it} denote the expectation conditional on \mathcal{F}_{it} and the manager's prior; and

$$m_{it} = E_{it}[\theta_t] \equiv E[\theta_t | \mathcal{F}_{it}] \quad (4)$$

denote the manager's estimation of the current growth rate of the price process. The following well-known lemma (see for example [Liptser and Shiryaev \(2001, p.372\)](#) or [David \(1997\)](#)) shows that the evolution of the estimated growth rate can be described by a diffusion process.

Lemma 1: *Assume that the initial prior of manager i is represented by $p_i \in [\theta_l, \theta_h]$. Then his estimation evolves according to*

$$dm_{it} = (\lambda + \mu)(\bar{m} - m_{it})dt + \epsilon(m_{it} - \theta_l)(\theta_h - m_{it})dB_{it} \quad (5)$$

subject to the initial condition $m_{i0} = p_i$. In this equation, the constant \bar{m} is the unconditional mean of the growth rate, and

$$B_{it} = \int_0^t \epsilon(ds_{iu} - m_{iu}du). \quad (6)$$

is a standard Wiener process with respect to the information set \mathcal{F}_{it} which is available to the manager of firm i .

The dynamics of the estimated growth rate given in the above lemma are quite intuitive. The stochastic shock dB_i is the normalized innovation in the firm's signal, that is the difference between the observed signal and its expected value divided by the volatility of the signal observed by the firm. The specific form of the volatility of m_i guarantees that the firm's estimation

takes values in the interval $[\theta_l, \theta_h]$ induced by the support of the growth rate. Finally, the drift is a mean reverting component which pushes back the firm's estimation towards the unconditional mean

$$\bar{m} = \theta_l + \frac{\mu}{\mu + \lambda}(\theta_h - \theta_l) \quad (7)$$

of the growth rate process. The fact that the coefficient of mean reversion increases with $\lambda + \mu$ is due to the property that the speed of convergence of the Markov chain towards its stationary distribution increases with the frequency of the shifts, see [Karlin and Taylor \(1975, Chap. 4\)](#).

Equipped with the definition of the innovation process, we can rewrite the dynamics of the firm specific price process as

$$dX_{it} = X_{it}m_{it}dt + X_{it}(\rho\sigma dB_{at} + (1/\epsilon)dB_{it}). \quad (8)$$

In conjunction with equation (5), this equation shows that the information set which is available to firm i coincides with the information set generated by the pair (m_i, X_i) . It follows that the relevant state variables for the firm valuation problem are the firm's price process and its estimation of the current growth rate.

In order to complete the description of the information structure in the model, we need to specify what information is available to the investors in the market. This is the purpose of the following assumption.

Assumption 4: *The manager of firm i publicly releases the values of m_i and X_i . Investors take these values and the dynamics (5), (8) as given for all firms in the economy.*

The first part of the above assumption insures that, for each firm i , the

manager's forecast of the firm's growth rate is readily available to investors. Since investors take the dynamics of m_i and X_i as given, this further implies that all agents in the model agree on the state variables that are relevant to the firm and, hence, also on the value of the firm.

There is empirical evidence, that managers do provide such information to investors, either directly or indirectly through analysts covering the firm. In particular, [Ajinkya and Gift \(1984\)](#) find that managers release earnings forecast in order to move the investors' earnings expectations towards the management forecast. Similarly, [Graham, Harvey, and Rajgopal \(2005\)](#) find that CFOs provide earnings guidance to analysts if there is a significant gap between analysts' forecasts and internal projections.

The second part of Assumption 4 is quite natural in the context of our model as it implies that the information available to investors is simply the aggregation of the information available to the managers. However, it is important to note that investors are not trying to estimate the true state from their observation of the processes m_i and X_i . They take the dynamics in equations (5) and (8) as given and do not internalize the fact these arise from the filtering of the growth rate by the managers.⁴

2.2 Firm valuation

Each firm uses capital K and labor L to produce output according to the isoelastic Cobb-Douglas production function

$$Y(K, L) = AK^{1-\zeta}L^\zeta$$

⁴An identical information structure can be obtained by assuming that investors do not realize that the growth rate is common to all firms and, hence, estimate the growth rate of firm i by considering X_i and B_a only.

where A is a nonnegative constant and $\zeta \in (0, 1)$ represents the constant share of labor. The firm pays a fixed wage w and can costlessly adjust its labor input. As a result, the firm's operating profit is given by

$$\pi(X_{it}, K_{it}) = \max_{L \geq 0} X_{it}Y(K_{it}, L) - wL = DX_{it}^{\Phi}K_{it}, \quad (9)$$

where $\Phi = 1/(1 - \zeta)$ and $D \equiv D(A)$ is a nonnegative constant which we normalize to one by choosing the value of the constant A .

The firm undertakes gross investment I_i and incurs depreciation at a constant rate $\delta \geq 0$. Consequently, the dynamics of its capital stock are

$$dK_{it} = (I_{it} - \delta K_{it})dt. \quad (10)$$

Investment is reversible but capital cannot be adjusted costlessly. Following [Abel and Eberly \(1997\)](#) we assume that the instantaneous investment cost function is given by

$$\phi(I_{it}) = bI_{it} + \frac{1}{2\gamma}I_{it}^2 \quad (11)$$

where $b \geq 0$ represents the purchase price of one unit of capital and $\gamma > 0$ is a constant that measures the severity of the adjustment costs. The fact that ϕ is convex reflect the fact that the more units of additional capital the firm tries to incorporate into the existing one, the less effective those units are at expanding firm capacity on the margin.⁵

Following [Hall \(2001\)](#) and [Zhang \(2005\)](#) we assume that the firm can costlessly issue new equity if its operating cash flows are not sufficient to

⁵The specification of the capital adjustment technology can be generalized to include numerous features such as asymmetric adjustment costs ([Zhang \(2005\)](#), [Hall \(2001\)](#)), fixed costs ([Abel and Eberly \(1994; 1997\)](#), [Cooper \(2006\)](#)) or irreversibility ([Abel and Eberly \(1994\)](#)). We choose to focus on the simple specification in equation (11) because it allows for an explicit solution to the firm valuation problem.

finance new investments. On the other hand, when operating cash flows are larger than investment expenses the firm pays dividend to its shareholders. Accordingly, the total cash flow paid to the shareholders of firm i at time t is given by

$$C_{it} = \pi(X_{it}, K_{it}) - \phi(I_{it}) = X_{it}^\Phi K_{it} - bI_{it} - \frac{1}{2\gamma} I_{it}^2. \quad (12)$$

Abstracting from agency issues, we assume that the manager of the firm acts in the best interest of shareholders and hence chooses an investment strategy that maximizes the market value of the firm. To identify the latter, we now define a stochastic discount factor.

Assumption 5: *Financial markets are complete. Assets can be valued by discounting future cash flows using the stochastic discount factor*

$$\xi_t = \exp\left(-rt - \kappa B_{at} - \frac{\kappa^2}{2} t\right). \quad (13)$$

In this equation, the constants r and κ represent, respectively, the risk free rate and the market price of aggregate risk.

The specification of the stochastic discount factor is quite natural in the context of our model. Indeed, only the aggregate shocks which are common to all firms carries a risk premium. Furthermore, the fact that all investors observe the aggregate shock B_a implies that they have the same perception of the stochastic discount factor and this property is crucial to guarantee that they agree on the prices of all traded assets.⁶

⁶The specification of the stochastic discount factor could be generalized to allow for a stochastic risk free rate and a stochastic risk premium as in [Berk et al. \(1999\)](#) or [Zhang \(2005\)](#). We focus on a specification where these components are constant for tractability. However, the mechanism which drives our result does not rely on this assumption and hence would still prevail under more general specifications.

We further assume that over a small time interval of length dt the firm stops its activity with probability Λdt so that the average lifetime of a firm is $1/\Lambda$. After it ceases operations, the firm has no more value. As is well known, the possibility of liquidation can be accounted for in the valuation by increasing the risk free rate from r to $r + \Lambda$ in the computation of the discounted value of the firm's future cash flows.⁷

Putting together the various pieces of the model, we can now formally define the value of firm i , conditional on being active, as

$$V_{it} = \max_{I_s} E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} (\pi(X_{is}, K_{is}) - \phi(I_s)) ds, \quad (14)$$

where $\xi_{t,s} = \xi_t/\xi_s$ is the stochastic discount factor at time t for cash flows which are paid at time $s \geq t$. The following proposition derives an analytical solution for the value of the firm.

Proposition 1: *Assume that the parameters of the model satisfy*

$$\frac{r + \Lambda}{\Phi} > \max \left[\theta_h - \rho\sigma\kappa - \frac{1}{2}\sigma^2(1 - \Phi); 2\theta_h - 2\rho\sigma\kappa - \sigma^2(1 - 2\Phi) \right]. \quad (15)$$

Then, conditional on being active, the value of a firm i and its optimal investment policy are given by

$$V_{it} = Q(m_{it}, X_{it})K_{it} + G(m_{it}, X_{it}), \quad (16)$$

$$I_{it} = \gamma Q(m_{it}, X_{it}) - \gamma b. \quad (17)$$

In the above equations, the marginal value of the firm's capital, Q , and the

⁷See for example Duffie, Schroder, and Skiadas (1996), Duffie and Singleton (1999) and Collin-Dufresne, Goldstein, and Hugonnier (2004).

market value of the firm's growth options, G , are defined by

$$Q(m_{it}, X_{it}) = (q_0 + q_1 m_{it}) X_{it}^\Phi, \quad (18)$$

$$G(m_{it}, X_{it}) = g_0 + g_1(m_{it}) X_{it}^\Phi + g_2(m_{it}) X_{it}^{2\Phi}, \quad (19)$$

where the constants q_0 , q_1 and g_0 and the functions $g_1(m)$ and $g_2(m)$ are defined in the appendix.

The above proposition is in line with the neoclassical q -theory of investment according to which a firm invests when the value of an additional unit of capital exceeds its purchase price and disinvests otherwise. As in [Abel and Eberly \(1997\)](#), the combination of constant returns to scale and capital independent adjustment costs implies that both the marginal value of capital and the value of the firm's growth options are independent of firm size as measured by its capital stock.

Specific to our analysis is the fact that there are two state variables influencing the firm's investment behavior: the specific price X_i and the expected growth rate m_i . Both of these variables affect the marginal value of capital and hence condition the firm's investment policy and the evolution of its capital stock. Since the marginal value of capital is a linear function of the estimated growth rate it follows that

$$Q(m_{it}, X_{it}) = E_{it}[Q(\theta_t, X_{it})]$$

where $Q(\theta_t, X_{it})$ is the marginal value of capital that would prevail in a full information context.⁸ Since investment is linear in the marginal value of capital, this further implies that investment under incomplete information is

⁸When the growth rate is constant ($\theta_h = \theta_l = \theta$) and investors have full information, the expectation becomes irrelevant. In that case, the marginal value of capital is given by $Q_i = X_i^\Phi/B$ for some nonnegative constant B as in [Abel and Eberly \(1997\)](#).

the expectation of its full information counterpart. Incomplete information reduces the optimal investment level in the high state and increases it in the low state. This effect is illustrated by Figure 1 which plots a simulated path of the firm's optimal investment policy under both complete and incomplete information.

Insert Figure 1 about here

In the simulated path of Figure 1, the high state is more likely than the low state ($\mu \gg \lambda$) and therefore the unconditional mean \bar{m} of the growth rate is closer to the high value of the growth rate θ_h . Since the estimated growth rate reverts to its mean \bar{m} , this implies that the firm's investment adjusts slowly in the low state and rather fast in the high state. In the postwar US economy, business cycles present a similar asymmetry with short recessions and long expansions. We discuss in Section 4.3 the crucial role of this asymmetry in our explanation of the negative relation between idiosyncratic volatility and stock returns.

3 Idiosyncratic volatility and stock returns

This section derives the relation between idiosyncratic volatility and stock returns implied by the model. We first discuss our theoretical findings in Section 3.1 and then discuss testable implications in Section 3.2.

3.1 Theoretical findings

When performing unconditional tests on data generated from the model, we do not capture the distributional properties of the returns as perceived by investors. Instead, we measure a combination of perceived returns, which are

solely due to exposure to aggregate risk, and forecast errors which are due to incomplete information. In other words, basic regression results provide coefficient estimates which are drawn from the true underlying distribution, and not from the perceived conditional distribution of stock returns.

To clearly identify the respective contributions of exposure to aggregate risk and forecast errors to stock returns, we start by analyzing the dynamics of the firm value. Applying Itô's lemma to the expression of the firm value given in Proposition 1 we obtain

$$\begin{aligned} \frac{dV_{it} + C_{it}dt}{V_{it-}} &= (r + a_{it}\kappa + \Lambda(1 - N_{it}))dt \\ &+ a_{it}dB_{at} + \iota_{it}dB_{it} - dN_{it} \end{aligned} \quad (20)$$

where $1 - N_i$ is the indicator function that the firm is active,

$$a_{it} = \rho\sigma X_{it} \frac{V_x(m_{it}, X_{it})}{V(m_{it}, X_{it})} \quad (21)$$

denotes the firm's aggregate, or systematic, volatility and

$$\iota_{it} = (1/\epsilon)X_{it} \frac{V_x(m_{it}, X_{it})}{V(m_{it}, X_{it})} + \epsilon(m_{it} - \theta_1)(\theta_h - m_{it}) \frac{V_m(m_{it}, X_{it})}{V(m_{it}, X_{it})} \quad (22)$$

denotes the firm's idiosyncratic volatility. Both of these volatilities contain a term which comes from the sensitivity of the firm value to variations in the output price. However, idiosyncratic volatility is also driven by a specific component which comes from incomplete information.

Since $E_{it}[dN_{it}] = \Lambda(1 - N_{it})dt$ by definition, equation (20) shows that, conditional on the information available to investors, the expected stock return depends only on the firm's exposure to aggregate risk as measured by the aggregate volatility a_i . Therefore, from the point of view of investors,

a version of the intertemporal CAPM holds in the sense that

$$E_{it} \left[\frac{dV_{it} + C_{it}dt}{V_{it-}dt} \right] = r + a_{it}\kappa.$$

The econometrician's perspective is different. Using the link between the innovation process and the original Wiener process, we can write the dynamics of the firm value as

$$\begin{aligned} \frac{dV_{it} + C_{it}dt}{V_{it-}} &= (r + a_{it}\kappa + \iota_{it}\eta_{it} + \Lambda(1 - N_{it}))dt \\ &\quad + a_{it}dB_{at} + \iota_{it}dW_{it} - dN_{it}, \end{aligned}$$

where $\eta_{it} = \epsilon(\theta_t - m_{it})$. Relative to equation (20), the drift now contains an additional component which depends on the idiosyncratic volatility of the firm ι_i and the manager's forecast error $\theta - m_i$. Conditional on the information available to investors this component is null on average since

$$E_{it}[\iota_{it}\eta_{it}] = E_{it}[\iota_{it}\epsilon(\theta_t - m_{it})] = \iota_{it}\epsilon(E_{it}[\theta_t] - m_{it}) \equiv 0,$$

by definition of the manager's forecast m_i . However, conditional on the whole information set (i.e. knowing the true state of the economy) this term becomes observable and hence satisfies

$$E_t[\iota_{it}\eta_{it}] = \iota_{it}\eta_{it} \neq 0. \tag{23}$$

This difference in the measurements of the average returns by investors on the one hand and the econometrician on the other is the key mechanism that allows us to obtain a link between idiosyncratic volatility and stock returns. When $\theta_t > m_{it}$ ($\theta_t < m_{it}$) there is a positive (negative) shock which is

interpreted as being part of the innovation and therefore does not contribute to the investors' perception of the expected return. This shock will however affect a time series estimation of the mean stock returns because these are drawn from the true distribution which includes the additional term $\iota_i \eta_i$ in the drift of the firm value process.

We summarize the previous discussion and present our main result on the expected excess return equation in the following proposition.

Proposition 2: *The instantaneous expected excess return conditional on the whole information set is given by*

$$E_t \left[\frac{dV_{it} + C_{it}dt}{V_{it}-dt} \right] - r = a_{it}\kappa + \iota_{it}\eta_{it}. \quad (24)$$

where a_i and ι_i denote the firm's aggregate and idiosyncratic volatility and η_i is the normalized forecast error.

Equation (24) is not a multi-factor specification in the tradition of [Merton \(1973\)](#) intertemporal CAPM. The first term on the right hand side is a remuneration for the exposure to aggregate risk. The second term, however, comes from the forecast error and is not a remuneration for risk. This term depends on the level of idiosyncratic volatility and on the manager's forecast, which are both firm specific, but it also depends on the current state of the economy θ . Since the latter is common to all firms, and can only take two values, all forecast errors have the same sign.⁹ They do however differ in their magnitude, since a firm's assessment of its current growth rate depends on the trajectory of its specific price process.

⁹The two state process underlying this result allows for a simple interpretation of the common growth rate as a proxy for the evolution of the business cycle. It is however not necessary for the validity of our analysis since Proposition 2 holds for any specification of the common growth rate process.

In the model, the cash flow dynamics follow from the firms' investment decisions but this property is not necessary for the validity of Proposition 2. In particular, the return decomposition given in equation (24) holds for any specification of the cash flows as long as the information structure and the state variables are kept the same. The endogenous cash flow specification on which we focus allows for a straightforward calibration of the model to observed firms and investment characteristics. Furthermore, our model implies that the heterogeneity among firms, and hence among idiosyncratic volatilities, arises endogenously as firms react optimally to changing market conditions. This makes the model more realistic, and the level of heterogeneity more plausible, than if we had exogenously postulated a cash flow process for each firm.

Proposition 2 describes the risk return relation conditional on the whole information set. In practice, an econometrician performing unconditional tests on data generated from the model would rely on a much smaller information set. In particular, portfolio regressions similar to the one we conduct in Section 4.3 are constructed from realized stock returns which are averaged across time and stocks. Even in such a case, our analysis remains valid. To see this, consider the sample average excess return on an equally weighted portfolio of n stocks over a period of length Δ starting at time t , that is

$$A_t = \int_t^{t+\Delta} \frac{1}{n\Delta} \sum_{i=1}^n \left(\frac{dV_{is} + C_{is}ds}{V_{is-}} - rds \right).$$

To infer the mean excess return on the portfolio, an econometrician computes the time series average of successive realizations of A . Using our previous

results we may decompose each realization into the sum of three terms

$$A_t = \int_t^{t+\Delta} \frac{1}{n\Delta} \sum_{i=1}^n (a_{is} dB_{as} + \iota_{is} dW_{it} - dN_{is} + \Lambda(1 - N_{is}) ds) \\ + \int_t^{t+\Delta} \frac{1}{n\Delta} \sum_{i=1}^n (a_{is} \kappa) ds + \int_t^{t+\Delta} \frac{1}{n\Delta} \sum_{i=1}^n (\iota_{is} \eta_{is}) ds.$$

When averaged across time, the three terms in the above decomposition behave differently. The first term averages to zero as it represents the average shock incurred by the portfolio. The second term is the standard reward for the exposure of the portfolio to aggregate risk. Even if the firms' forecasts are conditionally unbiased, the last term is non zero on average because the estimation error are multiplied by the firms' idiosyncratic volatilities.

The sign and magnitude of the effect depend on the joint distribution of forecast errors and volatilities. Due to the complex path dependence of these variables, this distribution cannot be computed in closed form. However, the assumption that the unobservable state variable proxies for the business cycle gears the results toward a negative relation. Since business cycles are asymmetric with long expansions and short recessions, the unconditional mean of the forecast, \bar{m} , is close to the high level of the state variable, θ_h . This induces estimation errors that are on average large when negative and small when positive. We thus expect to observe a negative relation between idiosyncratic volatility and stock returns. We confirm this in Section III.C where we show that when calibrated to match moments of firms and industry characteristics, the model generates the negative average relation documented by [Ang et al. \(2006; 2008\)](#) and [Jiang et al. \(2007\)](#).

It is important to note that a different choice of the underlying state variable could induce a positive relation between idiosyncratic volatility and stock returns. For example, a specification where the unconditional mean

of the forecast would be closer to the low level of the state variable would yield a positive relation. One could also introduce a richer dynamic where the long term mean would itself be time varying implying periods of positive relation alternating with periods of negative relation. Such a setting could help understand the diverging results in the literature concerning the sign of the idiosyncratic volatility effect.

3.2 Testable implications and empirical support

The previous results can be used to derive two novel implications which are related to earning forecasts and idiosyncratic volatility. While we do not perform a formal test of these predictions, we show that they are strongly supported by recent findings in the empirical asset pricing literature.

Proposition 2 shows that the loading of stock returns on forecast errors is given by the firm's idiosyncratic volatility. This suggests that stocks with larger idiosyncratic volatility should be more responsive to forecast errors. In particular when the realized growth rate is higher than anticipated, i.e. when $\theta > m_i$, firms with larger idiosyncratic volatility should have higher returns than firms with lower idiosyncratic volatility. On the contrary, these firms should have lower returns when $\theta < m_i$. This leads to the following testable implication.

Implication 1: *Following good news, firms with larger idiosyncratic volatility should produce relatively larger returns, and following bad news they should produce relatively lower returns.*

There is a vast literature documenting the predictability of stock returns following earning announcements, see [Ball and Brown \(1968\)](#), [Watts \(1978\)](#), [Foster et al. \(1984\)](#) and [Bernard and Thomas \(1990\)](#) among others. The fact that good (bad) news are followed by positive (negative) returns is

usually referred to in that literature as the post-earning announcement drift. Most of the theories proposed to explain this anomaly are behavioral. In particular, [Bernard and Thomas \(1990\)](#) suggest that investors underreact to news while [Barberis, Shleifer, and Vishny \(1998\)](#) rely on the representative heuristic and conservatism bias. Our model proposes an explanation based on incomplete information and, in addition, predicts that the effect should be stronger among high idiosyncratic volatility firms.¹⁰

Recent results in the empirical literature provide support to the above implication. In his study of the link between information uncertainty and stock returns, [Zhang \(2006\)](#) provides a detailed analysis of the properties of portfolios sorted on the basis of different proxies for information uncertainty. One of the six proposed proxies is the stock volatility, which is measured by the standard deviation of weekly excess returns.¹¹ Defining good and bad news according to the direction of the forecast revisions made by analysts, [Zhang \(2006\)](#) finds that firms with high volatility produce relatively lower returns following bad news and relatively higher returns following good news. More precisely, a portfolio long in the high volatility stocks and short in the low volatility stock has a monthly average return of -1.47% after bad news and 0.44% after good news over the sample period from January 1983 to December 2001.¹² These results provide strong support for Implication 1 as long as revisions of analysts' forecasts qualify as a good proxy for the forecast errors which appear in equation (24).

According to Proposition 2, the expected excess return of a firm is the

¹⁰Since information is revealed continuously through time there is no formal earnings announcement in our model. However, Implication 1 deals with instantaneous returns and thus describes the local relation between news and stock returns.

¹¹In general, firms with high total volatility do not necessarily have high idiosyncratic volatility. In our model the two quantities are linked through equations (21)–(22) and we show in Section 4.3 that sorting stocks on the basis of total or idiosyncratic volatility results in the same portfolios.

¹²see [Zhang \(2006, Table III\)](#).

sum of two components. The first one is generated by exposure to aggregate risk and can be measured by the firm’s market beta. The second one is an idiosyncratic component, which is the product of the firm’s idiosyncratic volatility and a forecast error. This suggests that if one could control for these firm specific forecast errors, then idiosyncratic volatility should not play any role in explaining the cross section of stock returns. This instantaneous forecast error is difficult to measure empirically but earnings forecast errors should provide a reasonable proxy because, in our model, earnings are linear in the firm specific price process. This naturally leads to the following implication.

Implication 2: *Controlling for earning forecast errors, idiosyncratic volatility should not have explanatory power for the cross-section of returns.*

There is recent evidence in the literature that supports this prediction of our model. As part of their study of the information content of idiosyncratic volatility, [Jiang et al. \(2007\)](#) estimate the following linear model

$$\begin{aligned} \text{Return}_{t+1} = & b_0 + b_1\text{IVOL} + b_2 \ln(\text{SIZE}) + b_3 \ln(\text{B/M}) + b_4\text{PrRet} \\ & + b_5\text{LEV} + b_6\text{LIQ} + b_7\text{SHOCK} + \varepsilon \end{aligned} \quad (25)$$

where IVOL is a measure of idiosyncratic volatility and the variable SHOCK stands for different unexpected earning measures. In particular, two of the measures they use are analyst forecast errors which defined as follows: “realized quarterly earning per share in excess of the mean of analysts forecasts at the last month of the portfolio formation quarter, divided by the previous year’s book value of equity per share”. As explained above, this variable provides a reasonable proxy for the forecast error which appears in the instantaneous return equation (24).

Jiang et al. (2007) estimate the Fama–MacBeth regression in equation (25). Omitting the SHOCK variable they obtain a negative and significant relation between idiosyncratic volatility and stock returns. When introducing current (FERQ0) and one step ahead (FERQ1) earning forecast errors, idiosyncratic volatility becomes insignificant as predicted by our model. The authors provide an explanation which is related to selective corporate information disclosure. Our model does not validate or invalidate their explanation as it addresses the problem in a very different way. However, it is noteworthy that our model is able to explain the direction of the effect following good or bad news, while theirs does not.

The two implications discussed here have only been addressed separately in the literature. The strength of the incomplete information framework that we propose is that it can simultaneously explain the significant role played by the forecast revisions in the relation between idiosyncratic volatility and stock returns, and the asymmetric nature of the return reaction of high volatility stocks following good and bad news.

4 Implementation of the model

In this section we use simulated data to show that our model is able to qualitatively replicate the idiosyncratic volatility anomaly. We describe the simulation methodology in Section 4.1 and discuss the choice of parameters in Section 4.2. The results of the tests performed on the simulated data are presented in Section 4.3.

4.1 Simulation methodology

The model developed in the previous sections is set in continuous time and, hence, needs to be discretized before it can be simulated. To this end, we ap-

ply the standard Euler scheme which allows for a transparent discretization of the model's dynamics.

Let Δ be a fixed time step, e.g. one day. In accordance with the convention taken in Proposition 2, the return of firm i is computed as

$$\text{Return}_{it} = \frac{V_{it} + C_{it}\Delta}{V_{it-\Delta}} - 1,$$

where V_i is the firm's value process as defined in Proposition 1, and

$$C_{it} = X_{it}^\Phi K_{it} - \phi(I_{it})$$

is the cash flow of the firm. To compute beta coefficients, we exogenously define a return process for the market portfolio by letting

$$dM_t = (r + \bar{\sigma}\kappa)M_t dt + \bar{\sigma}M_t dB_{at}.$$

Given this process, the return on the market is computed as

$$\text{Market Return}_t = \frac{M_t}{M_{t-\Delta}} - 1,$$

and the beta of firm i is obtained by regressing the return of the firm on the return of the market. In the above specification, the constant volatility parameter $\bar{\sigma}$ has no effect on the estimation other than scaling the values of the beta coefficients. We choose its value in such a way that the average beta of the firms across all the simulations is close to one.

In order to replicate the tests of [Ang et al. \(2006; 2008\)](#) we simulate 10,000 artificial panels of data each with 500 firms sampled at a daily frequency for a period of 10 years. All the statistics which are discussed below are obtained by averaging across simulations.

4.2 Calibration

The model has 13 parameters, which can be separated in three groups. The composition of these groups as well as the chosen parameter values are reported in Table 1.

Insert Table 1 about here

The first group contains the parameters whose values are directly measurable from the data or can be obtained from previous studies. The depreciation rate is set equal to $\delta = 12\%$ following [Cooper and Haltiwanger \(2006\)](#). Following [Kydland and Prescott \(1982\)](#) we set the share of labor in the production function to $\zeta = 0.7$ so that the price elasticity of operating profits is given by $\Phi = 10/3$. The market price of risk κ and the interest rate are set equal to 0.3 and 4.8%, respectively, in order to match the average Sharpe ratio and nominal interest rate in the US over the past 100 years as reported by [Shiller \(2005\)](#). Finally, we set the liquidation intensity to $\Lambda = 9.2\%$ so that firms operate for approximately 11 years on average.¹³

The parameters in the second group define the transition matrix of the growth rate process. To match the duration and frequency of business cycles in the postwar US as measured by the [NBER \(2008\)](#), we set the transition intensities to $\lambda = 0.233$ and $\mu = 1.154$. This parametrization implies that, on average, the length of a complete cycle is 61.9 months, with expansion phases of 51.5 months and contractions of 10.4 months.

The parameters of the third group are chosen in such a way that the moments obtained from the simulated data match a number of empirical moments of investment dynamics, stock returns, and book to market ratios.

¹³While this figure might seem small it is comparable to the average lifetime implied by the default rates on speculative grade bond, see [Duffie and Singleton \(2003\)](#).

The parameters of the adjustment cost function play an important role in the determination of the size of growth options relative to the total firm value. We choose γ and b to match the mean and standard deviation of book-to-market ratios. Following [Pontiff and Schall \(1998\)](#), we construct a book-to-market ratio index over the period 1920–2008 by using the Dow Jones Industrial Average Index at a monthly frequency along with the previous year book value obtained from [ValueLine \(2006\)](#). For the entire sample the average is 69% and the standard deviation is 27%. In contrast, over the period 1981–2003, for which we also have investment data, the mean and standard deviation of the book-to-market ratio are 42% and 27% respectively. The simulated data generates an average book-to-market ratio of 50% with a monthly standard deviation of 32%.

The states of the growth rate process are set to $\theta_h = 0.707\%$ and $\theta_l = -5.863\%$ to match the mean investment and disinvestment rate reported in [Abel and Eberly \(1999a\)](#) and [Eberly et al. \(2006\)](#). The model produces a mean investment rate of 10% and a mean disinvestment rate of 3%, while empirical values are 15% and 2% respectively. Finally we set the total volatility of the price process to $\sigma = 21\%$ and the correlation between firm specific prices and aggregate shocks to $\rho = 0.1$. This produces an average standard deviation of stock returns of 31% which lies well within the range of empirically reported values (see [Campbell, Lettau, Malkiel, and Xu \(2001\)](#) and [Vualteenaho \(2001\)](#)).

Insert Table 2 about here

Table 2 summarizes the moments obtained in the simulated data along with their empirical counterparts. It shows that our model generates values which are in line with key moments of industry and firms characteristics

and provides reasonable level of stock return volatility. The model performs however very poorly regarding the share of adjustment costs as it generates a value twice as high as its empirical counterpart. This is a well-known shortcoming of adjustment cost models, see [Chirinko \(1993\)](#). We could improve the performance of the model along this dimension by modifying the cost function to allow for features such as fixed costs or partial irreversibility. We choose to focus on the simple specification in equation (11) because it allows for an explicit solution to the firm valuation problem.

4.3 Results

To test the ability of our model to generate the idiosyncratic volatility anomaly we now use the simulated panels of data to construct portfolios sorted on different measures of idiosyncratic volatility. We replicate the portfolio construction of [Ang et al. \(2006\)](#). To this end, we use one month of daily observations to construct the measures of idiosyncratic volatility and consider a one month portfolio holding period. [Ang et al. \(2006\)](#) use two different measures of idiosyncratic volatility: (i) standard deviation of past returns and (ii) standard deviation of the residuals of a three factor Fama–French model. We focus here on the first measure and replace the second one by the standard deviation of the residuals of a market model since the construction of the Fama–French factors is problematic in our partial equilibrium framework.

Insert Table 3 about here

Table 3 displays summary statistic from our simulation for five quintile portfolios ranked by increasing measure of idiosyncratic volatility along with the benchmark results of [Ang et al. \(2006\)](#). As shown by the first and

the fourth column of Panel A, the model is able to generate the volatility anomaly. Indeed, the portfolios' alphas are both decreasing in the measure of idiosyncratic volatility and highly statistically significant. This shows that portfolios which contain high idiosyncratic volatility firms have lower risk adjusted performance. Furthermore, the average portfolio returns are also decreasing in idiosyncratic volatility. The magnitude of the effect generated by our model is however much lower than the empirical results. [Ang et al. \(2006\)](#) obtain an alpha of -16.20% per year for a portfolio long in the fifth quintile and short in the first quintile, while our model yields an alpha of -1.99% per year. Similarly, [Ang et al. \(2006\)](#) report a negative average return of -11.64% per year for this long-short portfolio while our model generates a negative return of -1.69% per year.¹⁴ It is important to note that the empirical results reported in [Ang et al. \(2006\)](#) are mainly driven by extremely low returns in the fifth quintile.¹⁵ Indeed, the difference in average return between the fourth and fifth quintile is ten times larger than between the first on fourth quintile. If we exclude the fifth quintile, our results are much more in line with theirs. In particular, for a portfolio long in the fourth quintile and short in the first, [Ang et al. \(2006\)](#) obtain an alpha of -5.04% and an average return of -0.84% while our model yields an alpha of -1.63% and an average return of -1.37% .

The last three columns of Table 3 report the average idiosyncratic volatility effect, idiosyncratic volatility and aggregate volatility of the quintiles portfolios. Comparing the two panels of the table shows that sorting on either total volatility or residual volatility correctly identifies the firms with

¹⁴In their paper [Ang et al. \(2006\)](#) report a monthly alpha of -1.35% and an average monthly return of -0.97% for the long-short portfolio. The numbers reported here are obtained from theirs by multiplying by twelve.

¹⁵In order to obtain such large effects, we would need our model to generate more heterogeneity among firms. This could potentially be achieved by initially endowing firms with heterogenous cash flow dynamics and production technologies.

largest idiosyncratic volatility. As expected from the fact that returns are generated according to equation (24), the measured alphas are almost identical to the true idiosyncratic volatility effect reported in the fifth column. In accordance with the prediction of our model the magnitude of the effect is larger for high idiosyncratic volatility firms.

Insert Figure 2 about here

As explained in Section 3.1 the sign of the idiosyncratic volatility effect depends on the joint distribution of future forecast errors and idiosyncratic volatilities. Figure 2 displays the idiosyncratic volatility effect as a function of the forecast error and the empirical distribution of the normalized forecast error η conditional on the true value of the growth rate. As can be seen from the figure, the distribution is negatively skewed in both states. This is due to the fact that, since expansions are more frequent than recessions for the chosen parameter values, the unconditional mean \bar{m} of the forecast is close to the high value of the growth rate. In the high state, when the forecast error is positive, the effect is small as m_i tends to be close to θ_h due to mean reversion. However in the low state, the effect is much larger since the distribution of the forecast errors is concentrated on the left.

Insert Figure 3 about here

The idiosyncratic volatility effect which is reported in Table 3 results from the combination of the effects which occur in the low and high states. While the high state is more frequent, the overall effect is dominated by the negative contribution of the low state. In a model with symmetric regimes and transition probabilities, the idiosyncratic volatility effect should be absent since the contribution of the two states should cancel out. Figure

3 confirms this intuition by showing that in such a model the idiosyncratic volatility effect and the distribution of the forecast errors are both perfectly symmetric. Panel A of Table 4 reports the summary statistics obtained from this model and shows that the alphas of the quintile portfolios are not significantly different from zero. However, the long–short portfolio generates a positive average return due to its net exposure to aggregate risk.

Insert Table 4 about here

The average return generated by the long–short portfolio is not only due to the idiosyncratic volatility effect as it also reflects the different exposure of the quintile portfolios to aggregate risk. In Table 3, the firms with highest idiosyncratic volatility also have the highest aggregate volatility. The long–short portfolio therefore exhibits a positive exposure to aggregate risk which contributes positively to its average return. This exposure could completely offset the idiosyncratic volatility effect. However the alphas of the portfolios should still decrease with idiosyncratic volatility, since larger exposure to aggregate risk increases not only the portfolio returns but also their betas. This intuition is confirmed by Panel B of Table 4 which reports the summary statistics for a model where the correlation coefficient ρ is set equal to 0.5. With this parametrization, aggregate volatilities are much larger and the long–short portfolio generates a positive return even though the idiosyncratic volatility effect is still present. Indeed, the alphas of the portfolios are significantly negative and decrease with idiosyncratic volatility.

4.4 An alternative specification

The assumption of a common growth rate is convenient since it allows for a simple calibration of the transition matrix. However, Assumption 3 can be

seen as a shortcoming of the model as it implies that managers do not learn from observing other firms price processes. We show in this section that the link between idiosyncratic volatility and stock returns that we identify does not require such a behavioral assumption.

Assume that the firm specific price process evolves according to

$$dX_{it} = X_{it}\theta_{it}dt + X_{it}\sigma \left(\rho dB_{at} + \sqrt{1 - \rho^2} dW_{it} \right), \quad (26)$$

where the only modification, compared to our base case model, is that the growth rate θ_i is now firm specific rather than common to all firms. We maintain the assumption that θ_i follows a two state Markov chain and further assume that it is identically and independently distributed across firms.

Contrary to our base case model, we now assume that each manager observes the aggregate shock and the price processes of all the firms in the economy and must estimate the current value of his firm's growth rate from his observations. Since the growth rates are iid across firms, the observation of another firms' price process does not carry any relevant information for the estimation of a firm's growth rate. As a result, the only signal which is relevant to the manager of firm i is given by

$$ds_{it} = dX_{it}/X_{it} - \rho\sigma dB_{at} \equiv \theta_{it}dt + (1/\epsilon)dW_{it}, \quad (27)$$

and it follows that the manager's estimation reported in equation (5) is now optimal without any behavioral assumption.

The equations which describe the dynamics of the firm value process remain valid under this alternative specification provided that one replaces each instance of θ by θ_i . In particular, the relevant idiosyncratic volatility

component in stock returns is now entirely firm specific, and given by

$$\eta_{it} = \epsilon(\theta_{it} - m_{it}).$$

Since the growth rate is no longer common to all firms, this idiosyncratic volatility component changes sign across firms. However, the law of large numbers implies that, at any point in time, a fraction $\mu/(\lambda + \mu)$ of firms are in the up state while the remaining fraction is in the low state. This suggests that the idiosyncratic volatility effect documented in the previous section should still prevail in this alternative model provided that (i) the high state is more probable than the low state and (ii) firms estimation errors are larger in the low state than in the high state.

Insert Table 5 about here

In order to confirm this intuition, we repeat the simulation and portfolio construction procedure described in the previous section and present the results for two sets of parameters in Table 5. The top panel of the table shows that, when using the base case parameters of Table 1, we obtain an idiosyncratic volatility effect $\nu_i \eta_i$ which is monotonically decreasing with idiosyncratic volatility as in the original specification of the model. However, the magnitude of this effect is much smaller than in the original model and does not compensate the effect of systematic volatility. As a result, the average return of the long short portfolio is very slightly positive but its alpha is negative and statistically significant.

The bottom panel of Table 5 reports the results for a parametrization of the model with symmetric regimes and transition probabilities similar to used in Figure 3. As shown by columns $\nu \eta$ and α , the idiosyncratic volatility effect is entirely absent in this model. In particular, none of the portfolios

contain a significant idiosyncratic volatility component and the increase in return apparent in column R can be entirely attributed to the portfolios' exposure to systematic risk. This confirms the fact that, in the context of our model, the driving force behind the idiosyncratic volatility effect is the asymmetric nature of business cycles.

5 Conclusion

In this paper, we propose a model of firm valuation under incomplete information that is able to explain the ambiguous link between idiosyncratic volatility and stock returns. While the relation between idiosyncratic volatility and stock returns generated by our model is entirely due to our incomplete information assumption, we show that its sign depends on the choice of the underlying unobserved state variable. When this state variable proxies for the business cycle we show that, due to the natural asymmetry between expansions and recessions, a properly calibrated version of the model is able to replicate the negative relation documented by [Ang et al. \(2006; 2008\)](#), [Brockman and Yan \(2008\)](#) and [Jiang et al. \(2007\)](#) among others.

In addition to explaining the idiosyncratic volatility anomaly, our model also has novel implications for the cross-section of stock returns. First, our model predicts that firms with larger idiosyncratic volatility should display lower returns following bad news and higher returns following good news. We present results in the recent literature (see [Zhang \(2006\)](#)) which document such an effect. It is important to note that our model generates these properties endogenously through a rational mechanism. Second, our model predicts that controlling for earning forecast errors should mitigate the idiosyncratic volatility anomaly. This implication is supported by the recent findings of [Jiang et al. \(2007\)](#) who show that introducing contem-

poraneous earnings forecasts errors renders the coefficient of idiosyncratic volatility insignificant in standard Fama–MacBeth regressions.

Appendix

Proof of Lemma 1. The proof of this well-known filtering result can be found in numerous places among which [Liptser and Shiryaev \(2001, p.372\)](#). QED.

In order to facilitate the proof of Proposition 1, we start by presenting a useful technical result.

Lemma 2: *Let a, b, c and δ be arbitrary constants and consider the, possibly infinite, process defined by*

$$S_t = E_{it} \int_t^\infty e^{-a(s-t)} \xi_{t,s} X_{is}^b (c + \delta m_{is}) ds.$$

If the parameters of the model are such that

$$\min_{m \in [\theta_l, \theta_h]} \left[r + a + b\rho\sigma\kappa + \frac{1}{2}b(1-b)\sigma^2 - bm \right] > 0, \quad (28)$$

then the process S is well defined and given by $S_t = (C + Dm_{it})X_{it}^b$ where the constants C and D are the unique solutions to

$$\begin{aligned} bC + bD(\theta_l + \theta_h) &= D(r + a + b\rho\sigma\kappa + \lambda + \mu + \frac{1}{2}b(1-b)\sigma^2) - \delta, \\ C(r + a + \frac{1}{2}b(1-b)\sigma^2) &= D((\lambda + \mu)\bar{m} - b\theta_l\theta_h) + c. \end{aligned}$$

Proof. The fact that (28) is sufficient for the finiteness of S follows from the boundedness of m_i and the fact that ξX_t^b is an exponential process with constant volatility, we omit the details.

In order to establish the second part, let the constants C and D be as in the statement and consider the process defined by

$$M_t = e^{-at} \xi_t X_{it}^b (C + Dm_{it}) + \int_0^t e^{-as} \xi_s X_{is}^b (c + \delta m_{is}) ds.$$

Using the dynamics of the pair (m_i, X_i) in conjunction with the definition of the constants (C, D) and applying Itô's lemma we deduce that M is a local martingale. Using the boundedness of m_i in conjunction with (28) and well-know results on geometric Brownian motion we can show that

$$E_{i0} \sup_{t \in [0, T]} |M_t|^2 < \infty,$$

for any finite time T . This implies that the local martingale M is a true martingale at least up to time T and it follows that

$$\begin{aligned} e^{-at} X_{it}^b (C + Dm_{it}) \\ = E_{it} \left[e^{-aT} \xi_{t,T} X_{iT}^b (C + Dm_{iT}) + \int_t^T e^{-as} \xi_{t,s} X_{is}^b (c + \delta m_{is}) ds \right]. \end{aligned}$$

Taking the limit as $T \rightarrow \infty$ on both sides of the previous expression and using the dominated convergence theorem we obtain

$$X_{it}^b (C + Dm_{it}) = S_t + \lim_{T \rightarrow \infty} E_{it} \left[e^{-a(T-t)} \xi_{t,T} X_{iT}^b (C + Dm_{iT}) \right]$$

and the proof will be complete once we show that the second term on the right is equal to zero. This follows from the boundedness of m_i and the assumed validity of (28), we omit the details. QED.

Proof of Proposition 1. Let I be an arbitrary investment policy for firm i , denote by K_i the associated capital stock and let

$$S_t = E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} (\pi(X_{is}, K_{is}) - \phi(I_s)) ds$$

denote the corresponding value process, conditional on the firm being active.

Using the dynamics of the capital stock process we obtain

$$K_{is} = e^{-\delta(s-t)} K_{it} + \int_t^s e^{-\delta(\tau-t)} I_\tau d\tau.$$

Substituting this back into the expression of the firm value process and using the definition of the firm's production function in conjunction with the law of iterated expectations we obtain

$$S_t = Q_{it} K_{it} + E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} (Q_{is} I_s - \phi(I_s)) ds$$

where the process

$$Q_{it} = E_{it} \int_t^\infty e^{-(\delta+\Lambda)(s-t)} \xi_{t,s} X_{is}^\Phi ds$$

gives the marginal valuation of the firm's capital. Assuming that the above expectations are well defined at the optimum (this will be verified below), this shows that the optimal investment policy is given by

$$I_{it}^* = \operatorname{argmax}_{x \in \mathbb{R}} (x Q_{it} - \phi(x)) = \gamma Q_{it} - \gamma b$$

and all that is left to do is to compute the marginal value of capital and the value of the firm's growth options:

$$G_t = V_{it} - Q_{it} K_{it} = \frac{\gamma}{2} E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} (Q_{is} - b)^2 ds. \quad (29)$$

Let us start by computing the firm's marginal value of capital. Using the result of Lemma 2 we have that if

$$r + \Lambda + \delta + \Phi (\rho \sigma \kappa - \theta_h + \frac{1}{2} \Phi (1 - \Phi) \sigma^2) > 0,$$

then Q_i is well defined and given by equation (18) where q_0 and q_1 solve the linear system of two equations with two unknowns given by

$$\begin{aligned} q_0\Phi &= -q_1\Phi(\theta_1 + \theta_h) \\ &+ q_1\left(r + \Lambda + \delta + \Phi\rho\sigma\kappa + \lambda + \mu + \frac{1}{2}\Phi(1 - \Phi)\sigma^2\right), \\ q_1\Phi\theta_1\theta_h &= 1 + q_1\bar{m}(\lambda + \mu) \\ &- q_0\left(r + \Lambda + \delta + \Phi\rho\sigma\kappa + \frac{1}{2}\Phi(1 - \Phi)\sigma^2\right) \end{aligned}$$

Substituting equation (18) into equation (29) and simplifying the terms we obtain that the value of the firm growth options is given by

$$\begin{aligned} G_t &= g_0 + q_1^2 E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} m_{is}^2 X_{is}^{2\Phi} ds \\ &+ E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} \left((q_0^2 + 2q_0q_1m_{is}) X_{is}^{2\Phi} - 2b(q_0 + q_1m_{it}) X_{it}^\Phi \right) ds \end{aligned}$$

where $g_0 \equiv \gamma b^2 / (2(r + \Lambda))$. Using Lemma 2 we have that the last expectation on the right hand side is well defined provided that

$$\frac{r + \Lambda}{\Phi} > \max \left[\theta_h - \rho\sigma\kappa - \frac{1}{2}(1 - \Phi)\sigma^2; 2\theta_h - 2\rho\sigma\kappa - (1 - 2\Phi)\sigma^2 \right], \quad (30)$$

and that in this case it is given by

$$(g_{10} + g_{11}m_{it})X_{it}^\Phi + (g_{20} + g_{21}m_{it})X_{it}^{2\Phi}$$

where the constants g_{10} , g_{11} , g_{20} and g_{21} solve the linear system of four

equations with four unknowns given by

$$\begin{aligned}
g_{10}\Phi &= 2bq_1 - g_{11}\Phi(\theta_1 + \theta_h), \\
&+ g_{11}(r + \Lambda + \Phi\rho\sigma\kappa + \lambda + \mu + \frac{1}{2}\Phi(1 - \Phi)\sigma^2), \\
g_{11}\Phi\theta_1\theta_h &= g_{11}\bar{m}(\lambda + \mu) - 2bq_0 \\
&- g_{10}(r + \Lambda + \Phi\rho\sigma\kappa\frac{1}{2}\Phi(1 - \Phi)\sigma^2) \\
2g_{20}\Phi &= -2g_{21}\Phi(\theta_1 + \theta_h) - 2q_0q_1 \\
&+ g_{21}(r + \Lambda + 2\Phi\rho\sigma\kappa + \Phi(1 - 2\Phi)\sigma^2 + \lambda + \mu), \\
2g_{21}\Phi\theta_1\theta_h &= g_{21}\bar{m}(\lambda + \mu) + q_0^2 \\
&- g_{20}(r + \Lambda + 2\Phi\rho\sigma\kappa + \Phi(1 - 2\Phi)\sigma^2).
\end{aligned}$$

In order to obtain the value of the firm's growth options, it now only remains to compute the first expectation

$$\Psi(m_{it}, X_{it}) = E_{it} \int_t^\infty e^{-\Lambda(s-t)} \xi_{t,s} m_{is}^2 X_{is}^{2\Phi} ds.$$

Since m_i is bounded it follows from (30) that the function Ψ is well defined. On the other hand, using the fact that the dynamics of the price process are linear, it can be shown that the function Ψ is homogenous of degree 2Φ with respect to x and it follows that

$$\Psi(m, x) = x^{2\Phi} H(m)$$

for some bounded function $H : [\theta_1, \theta_h] \rightarrow \mathbb{R}$. Assuming that this function is

smooth and applying Itô's lemma we deduce that

$$\begin{aligned} m^2 &= (r + \Lambda + 2\Phi\rho\sigma\kappa + \Phi(1 - 2\Phi)\sigma^2 - 2\Phi m)H(m) \\ &\quad - ((\lambda + \mu)(\bar{m} - m) + 2\Phi(m - \theta_l)(\theta_h - m))H'(m) \\ &\quad - (1/2)(\epsilon(m - \theta_l)(\theta_h - m))^2 H''(m). \end{aligned}$$

Unfortunately, this ordinary differential equation does not admit a closed form solution because of the quadratic term on the right hand side. As a result we will need to solve it numerically. To this end we have to specify boundary conditions. Using the fact that θ_l and θ_h are entrance boundaries for the process m_i (see [David \(1997\)](#)) it can be shown that these boundary conditions take the form

$$\lim_{m \rightarrow \theta_l, \theta_h} |H''(m)| < \infty.$$

In practice we impose some finite values for the second derivative of the unknown function on the boundaries of the domain and verify that our numerical solution is insensitive to the choice of these values.

Putting everything together we have that the value of an arbitrary firm is given by equations (16), (18) and (19) with

$$\begin{aligned} g_1(m) &= (\gamma/2)(g_{10} + g_{11}m), \\ g_2(m) &= (\gamma/2)(g_{20} + g_{21}m + H(m)), \end{aligned}$$

provided that the parameters of the model are such that inequality (30) holds true. QED.

References

- A. Abel and J. Eberly. A unified model of investment under uncertainty. *American Economic Review*, pages 1369–1384, 1994.
- A. Abel and J. Eberly. An exact solution for the investment and market value of a firm facing uncertainty, adjustment costs, and irreversibility. *Journal of Economic Dynamics and Control*, 21:831–852, 1997.
- A. Abel and J. Eberly. The effects of irreversibility and uncertainty on capital accumulation. *Journal of Monetary Economics*, 44:339–377, 1999a.
- A. Abel and J. Eberly. Investment and q with fixed costs: An empirical analysis. *Working paper, University of Pennsylvania*, 1999b.
- B. Ajinkya and M. Gift. Corporate managers earnings forecasts and symmetrical adjustments of market expectations. *Journal of Accounting Research*, 22:425–444, 1984.
- A. Ang, R. Hodrick, Y. Xing, and X. Zhang. The cross-section of volatility and expected returns. *Journal of Finance*, 51:259–299, 2006.
- A. Ang, R. Hodrick, Y. Xing, and X. Zhang. High idiosyncratic volatility and low returns: International and further u.s. evidence. *Forthcoming in the Journal of Financial Economics*, 2008.
- R. Ball and P. Brown. An empirical evaluation of accounting income numbers. *Journal of Accounting Research*, 6:159–178, 1968.
- N. Barberis, A. Shleifer, and R. Vishny. A model of investor sentiment. *Journal of Financial Economics*, 49:307–343, 1998.
- S. Barnett and P. Sakellaris. A new look at firm market value, investment and adjustment cost. *The Review of Economics and Statistics*, 81:250–260, 1999.
- J. Berk, R. Green, and V. Naik. Optimal investment, growth options and security returns. *Journal of Finance*, 54:1153–1607, 1999.
- V. Bernard and J. Thomas. Evidence that stock prices do not fully reflect the implications of current earnings for future earnings. *Journal of Accounting and Economics*, 13:305–340, 1990.
- P. Brockman and X.S. Yan. The time-series behavior and pricing of idiosyncratic

- volatility: Evidence from 1926 to 1962, 2008. Working paper University of Missouri at Columbia. Available online at ssrn.com/abstract=1117284.
- L. D. Brown and M. S. Rozeff. Univariate time-series models of quarterly accounting earnings per share: a proposed model. *Journal of Accounting Research*, 17:179–189, 1979.
- J. Campbell, M. Lettau, B. Malkiel, and Y. Xu. Have individual stocks become more volatile? an empirical exploration of idiosyncratic risk. *Journal of Finance*, 56:1–43, 2001.
- R. Chirinko. Business fixed investment spending: Modeling strategies, empirical results, and policy implications. *Journal of Economic Literature*, 31:1875–1911, 1993.
- P. Collin-Dufresne, R. Goldstein, and J. Hugonnier. A general formula for valuing defaultable securities. *Econometrica*, 72(5):1377–1407, 2004.
- I. Cooper. Asset pricing implications of nonconvex adjustment costs and irreversibility of investment. *Journal of Finance*, 61:139–170, 2006.
- R. Cooper and J. Haltiwanger. On the nature of capital adjustment costs. *Review of Economic Studies*, 73:611–633, 2006.
- A. David. Fluctuating confidence in stock markets: Implications for returns and volatility. *Journal of Financial and quantitative Analysis*, 32:427–462, 1997.
- D. Duffie and K. Singleton. Modelling term structures of defaultable bonds. *Review of Financial Studies*, 12:687–720, 1999.
- D. Duffie and K. Singleton. *Credit Risk*. Princeton University Press, 2003.
- D. Duffie, M. Schroder, and C. Skiadas. Recursive valuation of defaultable securities and the timing of resolution of uncertainty. *Annals of Applied Probability*, 6:1075–1090, 1996.
- J. Eberly, S. Rebello, and N. Vincent. Investment and value: A neoclassical benchmark. *Working paper, Northwestern University*, 2006.
- G. Foster, C. Olsen, and T. Shevlin. Earnings releases, anomalies, and the behavior of security returns. *The Accounting Review*, 59:574–603, 1984.
- F. Fu. Idiosyncratic risk and the cross-section of expected returns. *Working paper, University of Rochester*, 2005.

- J. Graham, C. Harvey, and S. Rajgopal. The economic implications of corporate financial reporting. *Journal of Accounting and Economics*, 40:3–73, 2005.
- X. Guo, J. Miao, and E. Morellec. Irreversible investment with regime shifts. *Journal of Economic Theory*, 122:37–59, 2005.
- D. Hackbarth, J. Miao, and E. Morellec. Capital structure, credit risk, and macroeconomic conditions. *Journal of Financial Economics*, 82:519–550, 2006.
- R. Hall. The stock market and capital accumulation. *American Economic Review*, 91:1185–1202, 2001.
- G. Jiang, D. Xu, and T. Yao. The information content of idiosyncratic volatility. *Forthcoming in the Journal of Financial and Quantitative Analysis*, 2007.
- S. Karlin and H. M. Taylor. *A first course in stochastic processes*. Academic Press, 2nd edition, 1975.
- F. Kydland and E. Prescott. Time to build and aggregate fluctuations. *Econometrica*, 50:1345–1370, 1982.
- E. Li, D. Livdan, and L. Zhang. Anomalies. *Working paper, University of Michigan*, 2007.
- R. Liptser and A. Shiryaev. *Statistics of Random Processes, Volume I: General Theory*. Springer Verlag, 2nd edition, 2001.
- L. Liu, T. Whited, and L. Zhang. Investment-based expected stock returns. *Working paper, University of Michigan*, 2007.
- F. Longstaff. Temporal aggregation and the continuous-time capital asset pricing model. *Journal of Finance*, 44:871–887, 1989.
- B. Malkiel and Y. Xu. Idiosyncratic risk and security returns. *Working paper, University of Texas at Dallas*, 2001.
- R. Merton. An intertemporal capital asset pricing model. *Econometrica*, 41:867–887, 1973.
- NBER. Business cycles expansions and contractions, 2008. Data available online at www.nber.org/cycles.html.
- J. Pontiff and L. Schall. Book-to-market ratios as predictors of market returns. *Journal of Financial Economics*, 49:141–160, 1998.
- R. Shiller. *Irrational Exuberance*. Princeton University Press, 2nd edition, 2005.

Data available online at www.econ.yale.edu/~shiller.

- M. Spiegel and X. Wang. Cross-sectional variation in stock returns: Liquidity and idiosyncratic risk. *Working paper, Yale*, 2005.
- ValueLine. A long term perspective, 2006. Document available online at www.valueline.com.
- P. Veronesi. Stock market overreactions to bad news in good times: a rational expectations equilibrium model. *Review of Financial Studies*, 12:975–1007, 1999.
- P. Veronesi. How does information quality affect stock returns? *Journal of Finance*, 55:807–837, 2000.
- T. Vualteenaho. What drives firm level stock returns? *Journal of Finance*, 57:233–264, 2001.
- R. Watts. Systematic abnormal returns after quarterly earnings announcements. *Journal of Financial Economics*, 6:127–150, 1978.
- F. Zhang. Information uncertainty and stock returns. *Journal of Finance*, 61:105–137, 2006.
- L. Zhang. The value premium. *Journal of Finance*, 60:67–103, 2005.

Table 1
Benchmark parameter values

	Group I		Group II		Group III							
ζ	δ	κ	Λ	r	λ	μ	γ	b	σ	ρ	θ_l	θ_h
0.7	0.12	0.30	0.092	0.048	0.233	1.154	15350	4000	6.30%	10%	-5.863%	0.707%

Notes: This table lists the parameter values used to simulate the model. We break the parameters into three groups. Group I includes the depreciation rate of capital δ , the market price of risk and the riskfree rate r . While the two former are chosen to match prior empirical studies, the latter has to be set high enough to satisfy the condition of Proposition 1. Group II includes the intensity parameters λ and μ which are chosen to match the frequency of the business cycles as reported by the NBER. The parameters in the final group are calibrated in such a way that the model matches key moments of firm and industry characteristics, see Table 2 for a list of these moments.

Table 2
Key moments under the benchmark parameters

Moments	Data	Model
Average rate of investment	0.15	0.10
Average rate of disinvestment	0.02	0.03
Average share of adjustment costs	0.13	0.32
Average book-to-market ratio	0.42	0.50
Standard deviation of book-to-market ratio	0.27	0.31
Average volatility of stock returns	0.25–0.32	0.31

Notes: This table reports a set of moments generated by the model under the parameter values of Table 1. The investment and disinvestment moments are from [Abel and Eberly \(1999b\)](#) and [Eberly et al. \(2006\)](#). The average share of adjustment costs is from [Barnett and Sakellaris \(1999\)](#). The empirical average and standard deviation of book-to-market are computed using the reconstructed times series of book-to-market of the Dow Jones index over the period 1981–2003. The data sources for the range of volatilities of stock returns are [Campbell et al. \(2001\)](#) and [Vualteenaho \(2001\)](#). All moments are expressed on an annual basis.

Table 3
Properties of portfolios sorted on volatility

	α		β	R	$\iota\eta$	ι	a
Benchmark: Ang et al. (2006)							
1	1.68			12.72			
2	1.56			13.80			
3	0.84			14.64			
4	-3.36			11.88			
5	-14.52			1.08			
5-1	-16.20			-11.64			
A. Sorted on total volatility							
1	-0.841	(-21.64)	0.804	4.748	-0.839	27.18	2.44
2	-1.595	(-41.01)	0.999	4.157	-1.685	33.58	3.03
3	-2.092	(-53.81)	1.075	3.722	-2.199	35.97	3.25
4	-2.470	(-63.52)	1.120	3.381	-2.556	37.25	3.36
5	-2.834	(-72.88)	1.165	3.055	-2.906	38.29	3.46
5-1	-1.992	(-51.24)		-1.693			
B. Sorted on CAPM residual volatility							
1	-0.862	(-22.18)	0.819	4.740	-0.861	27.26	2.45
2	-1.634	(-42.20)	1.012	4.129	-1.701	33.64	3.03
3	-2.108	(-54.22)	1.082	3.712	-2.205	35.98	3.25
4	-2.464	(-63.38)	1.122	3.388	-2.555	37.25	3.36
5	-2.816	(-72.42)	1.153	3.062	-2.899	38.28	3.45
5-1	-1.953	(-50.24)		-1.678			

Notes: The first panel of this table reports the estimation results of [Ang et al. \(2006\)](#) for portfolios sorted on total volatility. Panels A and B report summary statistics obtained in our simulations for portfolios sorted on total return volatility and on idiosyncratic volatility relative to the CAPM. Portfolios are formed every months, based on volatility computed using daily observations over the previous month. Portfolio 1 (5) is the portfolio with the lowest (highest) volatilities. The statistics in the columns labeled α , R , $\iota\eta$, ι and a are measured in yearly percentage terms. The α and β columns report Jensen's alpha and the portfolio beta with respect to the CAPM. The R column reports the average gross return. The $\iota\eta$ column reports the idiosyncratic volatility effect. Finally, the columns labeled ι and a report the idiosyncratic and aggregate volatilities as defined by equations (22) and (21). All values are computed by taking averages across 10,000 simulations of a panel of 500 firms for 10 years with a daily time step. The parameters used in the simulations are reported in Table 1.

Table 4
Properties of portfolios sorted on total volatility under
alternative parameter values

	α	β	R	$\iota\eta$	ι	a	
A. Symmetric shocks							
1	-0.1041	(-0.51)	0.783	5.522	0.001	23.57	2.37
2	-0.0069	(-0.30)	0.969	5.791	-0.004	29.46	2.96
3	-0.0601	(-0.44)	1.038	5.988	-0.007	31.31	3.14
4	-0.0891	(-0.27)	1.083	5.805	-0.009	32.41	3.25
5	-0.0550	(-0.24)	1.133	5.887	0.013	33.45	3.59
5-1	0.049	(0.23)		0.510			
B. High aggregate volatility							
1	-1.272	(-32.71)	3.235	6.915	-0.878	20.46	9.79
2	-1.769	(-45.51)	4.108	7.095	-1.537	26.09	12.41
3	-2.179	(-56.05)	4.584	7.005	-2.058	29.13	13.81
4	-2.517	(-64.74)	4.913	6.875	-2.529	31.17	14.75
5	-2.936	(-75.53)	5.226	6.658	-3.063	32.94	15.57
5-1	-1.664	(-42.81)		-0.003			

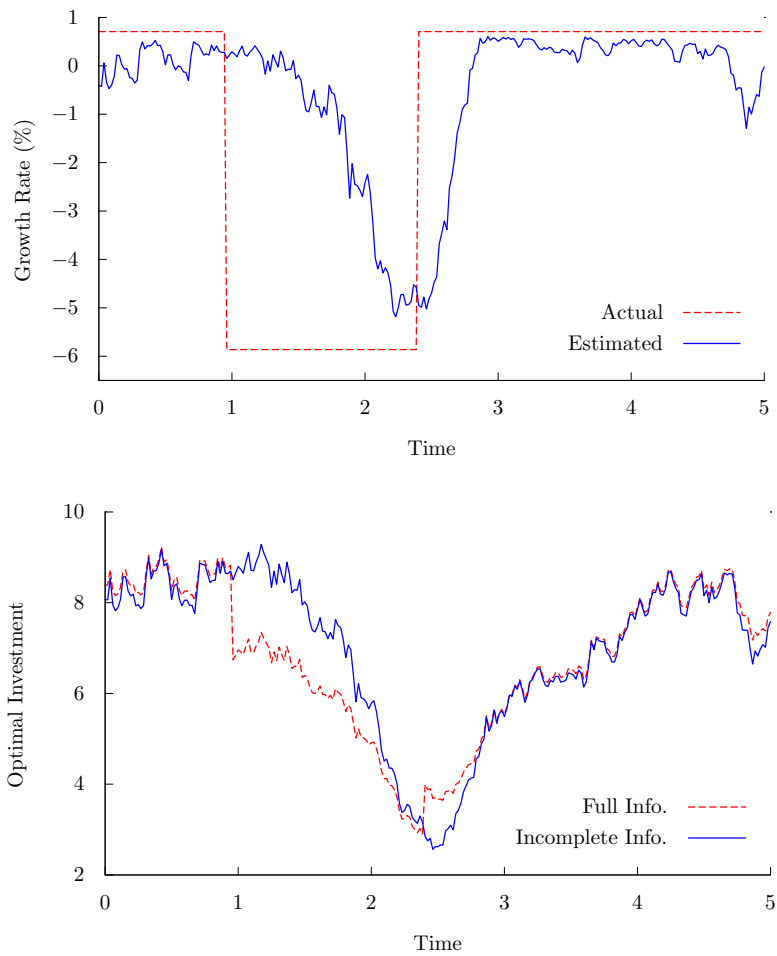
Notes: This table reports summary statistics for portfolios sorted on total volatility. The parameters used in the simulations are those of Table 1 except for $\mu = \lambda = 1$, $-\theta_l = \theta_h = 0.707\%$ in Panel A and $\rho = 0.5$ in Panel B. The statistics in the columns labeled α , R , $\iota\eta$, ι and a are measured in yearly percentage terms. The α and β columns report Jensen's alpha and the portfolio beta with respect to the CAPM. The R column reports the average gross return. The $\iota\eta$ column reports the idiosyncratic volatility effect. Finally, the columns labeled ι and a report the idiosyncratic and aggregate volatilities as defined by equations (22) and (21). All values are computed by taking averages across 10,000 simulations of a panel of 100 firms for 10 years with a daily time step.

Table 5
Properties of portfolios sorted on total volatility under
alternative model specification

	α		β	R	$\nu\eta$	ι	a
A. Base case parameters							
1	-0.016	(-0.42)	0.864	5.646	0.028	29.15	2.63
2	0.042	(1.08)	1.051	5.851	0.021	35.25	3.19
3	0.012	(0.31)	1.103	5.864	0.007	36.80	3.33
4	-0.046	(-1.17)	1.136	5.836	-0.011	37.69	3.41
5	-0.076	(-1.95)	1.174	5.839	-0.051	38.50	3.49
5-1	-0.060	(-1.93)		0.002			
B. Symmetric shocks							
1	-0.044	(-1.12)	0.785	5.546	-0.003	24.11	2.38
2	0.007	(0.19)	0.987	5.764	0.000	30.23	2.99
3	0.038	(0.98)	1.050	5.846	0.002	32.08	3.17
4	0.012	(0.31)	1.093	5.858	0.000	33.19	3.28
5	-0.030	(-0.77)	1.140	5.858	0.002	34.28	3.39
5-1	0.014	(0.35)		0.312			

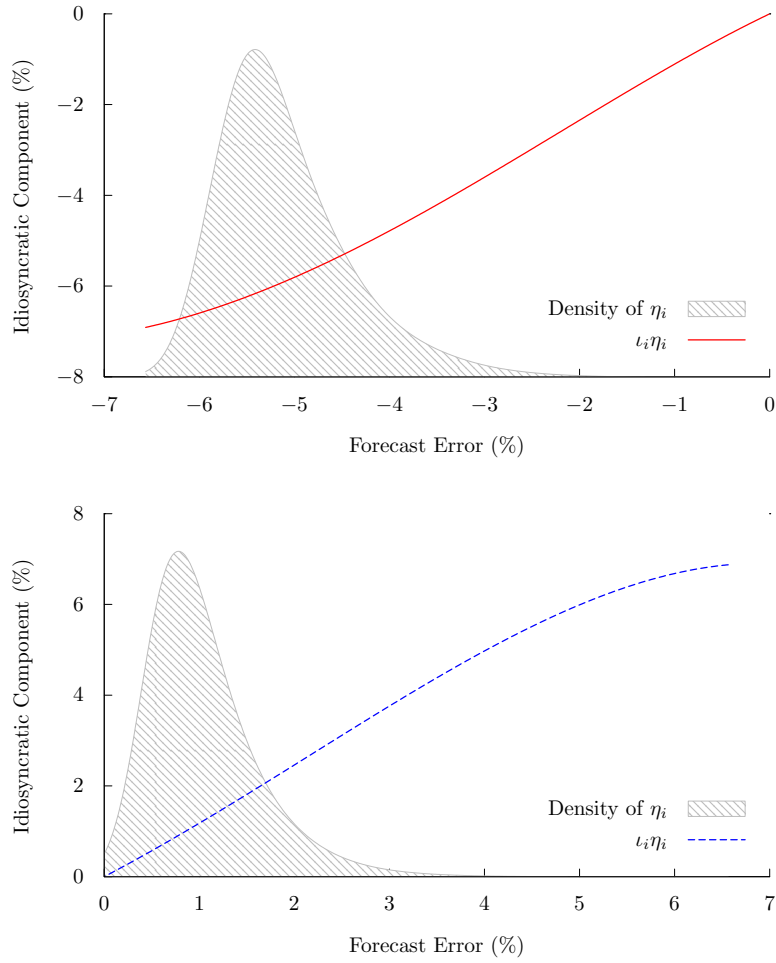
Notes: This table reports summary statistics for portfolios sorted on total volatility. The parameters used in the simulations are those of Table 1 in Panel A and $\mu = \lambda = 1$, $-\theta_1 = \theta_h = 0.707\%$ in Panel B. The statistics in the columns labeled α , R , $\nu\eta$, ι and a are measured in yearly percentage terms. The α and β columns report Jensen's alpha and the portfolio beta with respect to the CAPM. The R column reports the average gross return. The $\nu\eta$ column reports the idiosyncratic volatility effect. Finally, the columns labeled ι and a report the idiosyncratic and aggregate volatilities as defined by equations (22) and (21). All values are computed by taking averages across 10,000 simulations of a panel of 500 firms for 10 years with a daily time step.

Figure 1: Estimation and optimal investment policy



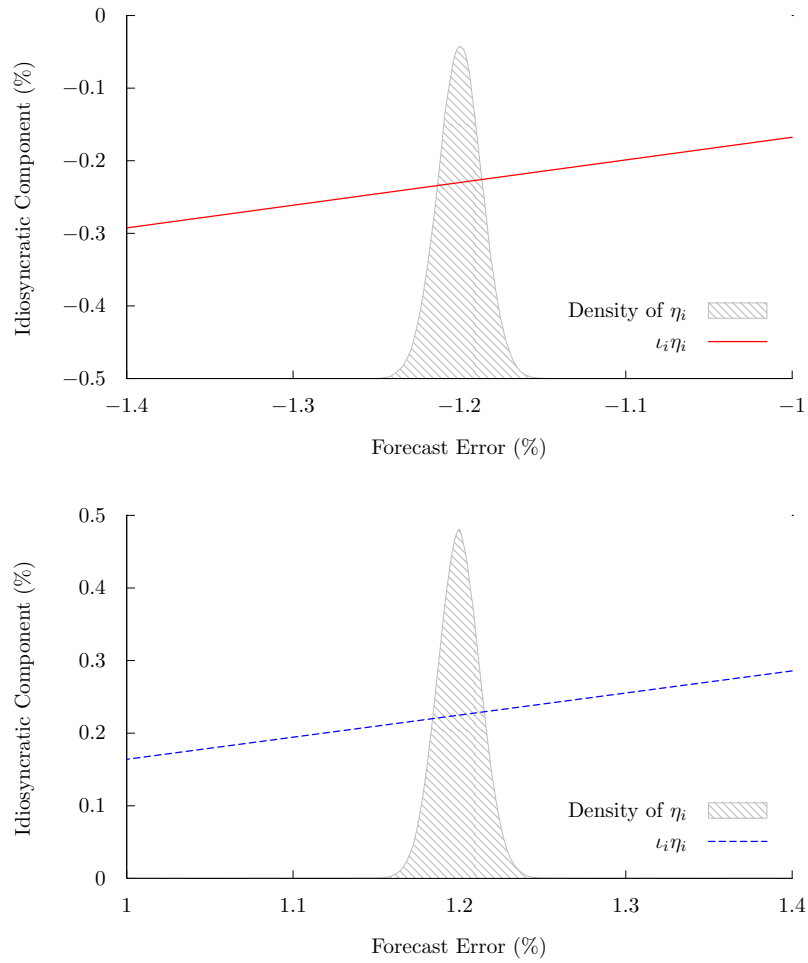
Notes: The top panel displays a sample path of θ_t and the corresponding path of the estimation m_{it} . The bottom panel displays the associated investment policy under full information (dashed line) and incomplete information (solid line). The parameters used for this figure are given in Table 1.

Figure 2: Idiosyncratic component in return and forecast error



Notes: The top panel displays the idiosyncratic component in returns $\iota_i \eta_i$ and the empirical distribution of the forecast error η_i conditional on being in the low state ($\theta_t = \theta_l$). The bottom panel displays the same quantities conditional on being in the high state ($\theta_t = \theta_h$). In both panels the state variable is set to its average value. The data used for this figure is from a simulated panel of a thousand firms with the parameters of Table 1.

Figure 3: Idiosyncratic component in return and forecast error with symmetric shocks



Notes: The top panel displays the idiosyncratic component in returns $\iota_i \eta_i$ and the empirical distribution of the forecast error η_i conditional on being in the low state ($\theta_t = \theta_l$). The bottom panel displays the same quantities conditional on being in the high state ($\theta_t = \theta_h$). In both panels the state variable is set to its average value. The data used for this figure is from a simulated panel of a thousand firms with the parameters of Table 1 except for $-\theta_l = \theta_h = 1.2\%$ and $\mu = \lambda = 1$.