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The *Option*-iPoD. The Probability of Default Implied by Option Prices Based on Entropy

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The *option*-iPoD. The Probability of Default Implied by Option Prices based on Entropy

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Abstract

We present a framework to derive the probability of default implied by the price of equity options. The framework does not require any strong statistical assumption, and provides results that are informative on the expected developments of balance sheet variables, such as assets, equity and leverage, and on the Greek letters (delta, gamma and vega). We show how to extend the framework by using information from the price of a zero-coupon bond and CDS-spreads. In the episode of the collapse of Bear Stearns, *option*-iPoD was able to early signal market sentiment.

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I. Introduction

This paper develops a framework to derive a market-based measure of probability of default. The probability of default is inferred from equity options, by applying the principle of minimum cross-entropy, Cover and Thomas (2006). Contrary to other approaches based on option prices, such as Hull et al. (2004), we do not impose any distributional assumption on the asset whose probability of default we are interested in. Furthermore, there is no need to specify any assumption on recovery rates, as typical when obtaining probability of default from credit derivatives, such as CDS-spreads, Duffie and Singleton (2003).

The probability of default is defined as the probability that the value of the underlying asset will go below a threshold level, the default barrier. While based on the balance sheet structure suggested by Merton (1974), *option*-iPoD significantly departs from existing methodologies, as it does not assume any ad-hoc default barrier. To the contrary, the default barrier is endogenously determined.

The framework exploits the entire information set available from option prices, so as to capture the well documented volatility smile and skew, and results extremely informative about the expected value of balance sheet variables. Since the probability distribution is recovered, one can also obtain the implied expected value of equity and leverage. For risk-management purposes, the implied asset volatility, and the Greeks (delta, gamma, and vega) are also determined. We discuss how to derive a term structure of probability of default, based on the maturity of option contracts, and how to extend the framework to incorporate information from the price of zero-coupon bonds to obtain a credit-default spread.

The paper is organized as follows. Section II describes the problem. Section III proposes the solution method. Section IV discusses the limitations of using equity options in the current framework. Section V presents the empirical implementation. Section VI illustrates results from a sample of major U.S. banks. Section VII illustrates the developments of *option*-iPoD and Moody's KMV Expected Default Frequency during the period leading to the collapse of Bear Stearns. Section VIII indicates some caveats, Section IX extends the framework and incorporates zero-coupon bonds into the analysis, and Section X concludes.

II. The Problem

The problem is to determine a market-based measure of probability of default (PoD). If one indicates by V the value of an asset, PoD is generally defined as:

$$PoD(X) = \int_0^X f_v dv$$

Where f_v is the probability density function of the value of the asset, and X is the default threshold, i.e., the value that triggers the default. When the value of the asset goes below X, the company defaults.

In order to estimate $PoD(X)$, we will use equity options. A call-option is a contract that gives the holder the right to buy the underlying asset (equity), at a predetermined price, or the strike price. Therefore, at expiration, the payoff of a call-option written on a stock is given by

$$C_T^K = \max(E_T - K;0)$$

where K is the strike price and E_T is the price of the stock at expiration (T).
The *option*-iPoD is determined by assuming the balance sheet structure suggested by Merton (1974). To finance its assets V, the company has only two sources of financing: debt (D), and equity (E). Equity is a junior claim on the value of the assets, i.e., when the company defaults debt is repaid first and equity holders receive the remainder. As such, at each point in time, the payoff to an equity-holder is:

$$E = \max(V - D;0)$$

As a result, an option written on a stock can be regarded as an option on an option, Hull et. al (2004).

$$C_T^K = \max(E_T - K;0) = \max(\max(V_T - D;0) - K;0) = \max(V_T - D - K;0)$$

To obtain *option*-iPoD, we need to determine D, and the probability that V_T will end-up below D.

Different approaches would be able to analyze this problem. For example, one could specify an ad-hoc probability density function for V_T - such as a normal density or a mixture of normals – or fit a polynomial function or splines to meet some pre-specified constraints. In this paper, we will solve for $PoD(D)$ by focusing on the cross-entropy functional introduced by Kullback and Leibler (1951).

The principle of maximum entropy and the related minimum cross-entropy make it possible to recover the probability distribution of a random variable. The recovered distribution is solely driven by what the researcher can observe. There is no additional assumption. In this sense, the maximum entropy distribution is the closest to the true distribution, as long as the true distribution is reflected into observable data, Jaynes (1957). When the researcher has additional information concerning the distribution she would like to recover, the principle can also incorporate this prior-information. In this case, the principle of maximum entropy is extended into the principle of minimum cross-entropy (or relative entropy), Cover and Thomas (2006). Buchen and Kelly (1996) show how to recover the probability distribution of an underlying asset by applying maximum entropy and minimum cross-entropy when option-prices are available.

Here, we show how to extend this approach to obtain the probability of default implied by option-prices, the *option*-iPoD.

Our problem is the following:

$$\min_{D}\left\{\min_{f(V_T)}\int_{V_T=0}^{\infty} f(V_T)\log\left[\frac{f(V_T)}{f^0(V_T)}\right]dV_T\right\} \tag{1}$$

Where $f^0(V_T)$ is the prior probability density function of the value of the asset, representing the researcher's prior-knowledge on $f(V_T)$, the posterior density. $f(V)\log\left[\frac{f(V)}{f^0(V)}\right]$ is the cross-entropy between $f(V)$ and $f^0(V)$ as defined by Kullback and Leibler (1951). The cross-entropy represents the degree of uncertainty around $f(V)$.[2] The set-up is extremely flexible. For instance, $f^0(V_T)$ can be easily included in the problem, but is not required. The constraints that drive the probability density function are only observable information. In our problem, we will impose the balance sheet structure suggested by Merton, i.e. the value of equity corresponds to the value of a call option written on the value of assets:

$$E_0 = e^{-rT}\int_{V_T=0}^{\infty}\max(V_T - D;0)f(V_T)dV_T = e^{-rT}\int_{V_T=D}^{\infty}(V_T - D)f(V_T)dV_T \tag{2}$$

Equation (2) says that the present value of the stock price at expiration of the option contract must correspond to the stock price observed today, E_0. In addition, we constraint $f(V)$ by asking the posterior density to be able to price observable option prices:

$$C_0^i = e^{-rT}\int_{V_T=0}^{\infty}\max(V_T - D - K_i;0)f(V_T)dV_T = e^{-rT}\int_{V_T=D+K_i}^{\infty}(V_T - D - K_i)f(V_T)dV_T \tag{3}$$

[2] The cross-entropy can be interpreted as a measure of relative distance between the prior and the posterior density function. Cover and Thomas (2006) discuss the statistical properties of entropy. We posit the problem in terms of minimum cross-entropy. There is no loss of generality, since the maximum entropy distribution corresponds to the minimum cross-entropy distribution when the prior is the uniform distribution, Buchen and Kelly (1996).

Equation (3) says that the present value of the call-option payoff at expiration must correspond to the observed call-option price today, C_0^i, where $i = 1,2....n$ indicates the number of available option contracts. Finally, we require an additivity constraint on the posterior density function:

$$1 = \int_{V_T=0}^{\infty} f(V_T) dV_T \tag{4}$$

III. SOLUTION

The problem is solved sequentially. First, we solve the optimization problem for $f(V_T)$. $f(V_T)$ will be in function of the free parameter D. Second, and given the optimal $f(V_T)$, we solve for D. The Lagrangian is:

$$L = \int_{V_T=0}^{\infty} f(V_T)\log\left[\frac{f(V_T)}{f^0(V_T)}\right]dV_T + \lambda_0\left[1 - \int_{V_T=0}^{\infty} f(V_T)dV_T\right] + \lambda_1\left[E_0 - e^{-rT}\int_{V_T=D}^{\infty}(V_T - D)f(V_T)dV_T\right]$$
$$+\sum_{i=1}^{n}\lambda_{2,i}\left[C_0^i - e^{-rT}\int_{V_T=D+K_i}^{\infty}(V_T - D - K_i)f(V_T)dV_T\right]$$

the FOC for $f(V_T)$ requires the Fréchet derivative of L to be equal to zero:

$$\left.\frac{\partial L(f+\varepsilon g)}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$$

for some other density function g, which immediately implies, Cover and Thomas (2006):

$$f(V_T) = f^0(V_T)\cdot \exp\left[\lambda_0 - 1 + \lambda_1 e^{-rT}\mathbf{1}_{V_T>D}(V_T - D) + \sum_{i=1}^{n}\lambda_{2,i}e^{-rT}\mathbf{1}_{V_T>D+K_i}(V_T - D - K_i)\right]$$

where $\mathbf{1}_{V_T>D}$ corresponds to the indicator function that takes the value of one whenever $V_T > D$, and zero otherwise, and $\mathbf{1}_{V_T>D+K_i}$ is defined whenever $V_T > D + K_i$. This expression reduces to:

$$f(V_T,\lambda) = \frac{1}{\mu(\lambda)} f^0(V_T)\cdot \exp\left[\lambda_1 e^{-rT}\mathbf{1}_{V_T>D}(V_T - D) + \sum_{i=1}^{n}\lambda_{2,i}e^{-rT}\mathbf{1}_{V_T>D+K_i}(V_T - D - K_i)\right] \tag{5}$$

with $\mu(\lambda) = \int_{V_T=0}^{\infty} f^0(V_T) \cdot \exp\left[\lambda_1 e^{-rT} \mathbf{1}_{V_T > D}(V_T - D) + \sum_{i=1}^{n} \lambda_{2,i} e^{-rT} \mathbf{1}_{V_T > D + K_i}(V_T - D - K_i)\right] dV_T$,
since

$$\exp[\lambda_0 - 1] = \frac{1}{\mu(\lambda)}$$

in order for equation (4) to be satisfied. Equation (5) can be substituted back into the Lagrangian to find the FOC for the lambdas, requiring:

$$\frac{\partial L(f(V_T, \lambda), \lambda)}{\partial \lambda} = 0$$

These are given by the following system:

$$\frac{1}{\mu(\lambda)} \frac{\partial \mu(\lambda)}{\partial \lambda_1} = E_0$$

$$\frac{1}{\mu(\lambda)} \frac{\partial \mu(\lambda)}{\partial \lambda_{2,i}} = C_0^i \quad \text{for} \quad i = 1,2....n \tag{6}$$

The system of equations is nonlinear, and will be solved numerically.[3] Once a solution to equation (6) is obtained, we can substitute it back into (5), to obtain:

$$f^*(V_T, D) \tag{7}$$

$f^*(V_T, D)$ depends on the free parameter D, the default barrier. To solve for D, one can substitute equation (7) into the original Lagrangian $L(f^*(V_T, D))$. The optimal D will be determined by:

$$\lim_{\Delta \to 0} \frac{L(f^*(V_T, D + \Delta)) - L(f^*(V_T, D))}{D + \Delta} = 0 \tag{8}$$

Empirically, equation (8) will also be solved numerically.

The economic interpretation of the solution is as follows. First, we started by specifying an initial probability density function, the prior $f^0(V_T)$, defined on the random variable of

[3] We used a standard Newton method, as described, for example, in Djafari (2000). Buchen and Kelly (1996) and Avellaneda (1998) show that the objective function is convex, and that the solution is unique.

interest (V_T). The FOC describes how to optimally modify the prior and construct a posterior density $f(V_T)$ that is able to satisfy the price constraints observed in the market. In equation (5), we can note that the Lagrange multipliers are the optimal weights given to each constraint when moving from $f^0(V_T)$ to $f(V_T)$. These multipliers have a direct economic interpretation: they represent the shadow cost of each constraint and indicate how difficult (costly) it is for the prior probability density function to meet these constraints.

IV. What can Equity Options Say About Default?

The answer is: not everything that we would like, but enough to be able to confidently implement the solution proposed in Section III.

Equity options are not well suited to describe the default state, the state in which the asset value (V) is lower than the default barrier (D). This is because the entire information set we are using comprises stock and option prices, which refer to the non-default state. Since $E = \max(V - D;0)$, we do not have information on what happens when $(V - D) < 0$. In fact, there is no trading of stocks, no trading of options in that state! Formally, we can see this by looking at the FOC for $f(V_T)$. When $(V - D) < 0$, $f(V_T)$ is driven by the prior $f^0(V_T)$, and the Lagrange multipliers associated with the market-price constraints in equation (2)-(3) do not help shaping $f(V_T)$. In fact, there is no market (no price) for equity and options. Put it differently, when $(V - D) \le 0$, $E = 0$ and $C = 0$, and from the perspective of an option (or equity) investor there is no difference whether $V = D$ or $V < D$ since her payoff does not change.[4]

While equity options do not contain information on the shape of the probability density function in the default state, they do contain information on the cumulative distribution function, the probability of default. As the payoff to an option investor is unaffected when $V \le D$, the option prices will reflect cumulative information, without distinguishing between the probability of *entering* into the default state ($V = D$), from the probability of *being* in the default state ($V < D$). When $V = D$, the density obtained from option prices will therefore need to have sufficient "mass" to take into account values for which $V < D$ as well.
In the default state, the constraints that contain information on the range of values V_T can take are no more active. As a consequence, the FOC can only ask λ_0, the Lagrange multiplier associated with the additivity constraint in equation (4), to provide the cumulative and residual information on $f(V_T)$.

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[4] A similar argument would show that the payoff to a put investor would be $P = K$ (or $P = -K$ for a short position) when $V \le D$.

V. Empirical Implementation

We need at least two option contracts to solve the problem. Intuitively, one contract is used to shape the density $f^*(V_T, D)$, while the other contract is needed to pin down D. More contracts are clearly useful to obtain a better representation of the data, but two options are sufficient to implement, in practice, this framework.

Suppose that two option contracts are available. The contracts are written on the same stock, and expire the same date. The empirical strategy consists of using one option contract, (C_1, K_1), to solve the first optimization problem in (1), and obtain $f^*(V_T, D)$ in (7). A second step, outside of the first optimization problem, uses the second option contract, (C_2, K_2), to search for the D that: (i) is consistent with (7)-(8); and (ii) satisfy the price constraint on the second option contract, i.e., is able to price the second option contract.

To implement the numerical procedure, we first need to discretize the domain of V. The following step procedure is implemented taking into account the observations in Section IV:

(i) we calibrate the maximum value that V can take, Vmax. Vmax is based on the book value of assets, the average growth rate of the last four quarters of the book value of assets and its standard deviation;

(ii) starting from a suitable initial guess of D, call it D_0, we divide the domain of V in two sub-intervals, $DS^0 = [0; D_0]$ describing the default state, and $NDS^0 = [D_0 + \varepsilon; V\max]$ indicating the non-default state.

(iii) when $V_T \in DS^0$, we discretize the domain by allowing V_T to take only two values: 0 or D_0, and we start by setting $f^0(V_T = 0) = 0$ and $f^0(V_T = D_0) = 0$, requiring $PoD(D_0) = 0$.

(iv) when $V_T \in NDS^0$, we discretize the domain by constructing 100 equally spaced values V_T can take. We assign the same likelihood to each of these values (the prior is a uniform distribution) so that under $f^0(V_T)$, $\Pr(V_T \in NDS^0) = 1 - PoD(D_0) = 1$.

(v) we solve numerically for a new D, D' and $f(V_T, D')$ for which $f(V_T = 0) = 0$, $PoD(D') = f(V_T = D')$ and $\Pr(V_T \in NDS') = 1 - PoD(D') = 1 - f(V_T = D')$.

(vi) once a solution is obtained, we repeat steps (iii)-(v). However, we now *fix* the range of values V can take between $DS' = [0; D']$ and $NDS' = [D' + \varepsilon; V\max]$, and we determine new values for $f(V_T, D'')$ and D'';

(vii) the procedure stops when $D'' = D' = D^*$, so that $f^*(V_T, D') = f^*(V_T, D'') = f^*(V_T, D^*)$, and $\Pr(V_T \in NDS^*) = 1 - PoD(D^*) = 1 - f^*(V_T = D^*)$.

(viii) *option*-iPoD corresponds to $PoD(D^*)$.

VI. Results

We apply the above framework to the ten largest U.S. banking groups. We include five bank holding companies: Bank of America, Citigroup, JPMorgan, Wachovia, Wells Fargo, and five investment banks: Bear Stearns, Goldman Sachs, Lehman Brothers, Merrill Lynch, and Morgan Stanley.

As the financial crisis erupted in the summer of 2007, it seems interesting to see how the framework is able to capture the developments in market sentiment. Option contracts have different expirations: January, March, April, May, June, July, August, and September comprise the entire set of expirations for the chosen sample. However, banks do not have option contracts expiring at all available dates. Typically, a standard option cycle is followed over the calendar year. For example, Citigroup's cycle comprises January, March, June and September expirations. During the year, further expirations generally become available. Option contracts expiring in the month of January of the following two years are also available. As an example, table 1 summarizes all option contracts on February 12, 2008.

Table 1. Option contracts cycles. Available expiration dates on February 12, 2008

Bank of America	Citigroup	JP Morgan	Wachovia	Wells Fargo	Bear Stearns	Goldman Sachs	Lehman Brothers	Merrill Lynch	Morgan Stanley
3/22/2008	3/22/2008	3/22/2008	3/22/2008	3/22/2008	3/22/2008	4/19/2008	4/19/2008	4/19/2008	4/19/2008
4/19/2008	6/21/2008	6/21/2008	4/19/2008	4/19/2008	4/19/2008	7/19/2008	7/19/2008	7/19/2008	7/19/2008
5/17/2008	9/20/2008	9/20/2008	7/19/2008	7/19/2008	7/19/2008	1/17/2009	1/17/2009	1/17/2009	1/17/2009
8/16/2008	1/17/2009	1/17/2009	1/17/2009	1/17/2009	1/17/2009	1/16/2010	1/16/2010	1/16/2010	1/16/2010
1/17/2009	1/16/2010	1/16/2010	1/16/2010	1/16/2010	1/16/2010				
1/16/2010									

Source: Bloomberg

For each contract, there is a range of available strike prices. This range is not constant throughout the life of the contract, i.e., generally more strikes become available as the expiration approaches. Furthermore, volumes differ substantially within the same contract; higher volumes tend to be registered close to the value of the underlying asset (at-the-money using Wall Street jargon) than further away from that value (out-of-the-money or in-the-money depending whether one is interested in a call or a put contract). The strike prices apply to both call and put options expiring on the same day.

Table 2 presents all strikes available for the put/call contracts on Citigroup expiring on June 21, 2008, as of February 12, 2008. The trading volumes on February 12, 2008 are also reported.

Table 2. Citigroup. Strikes, Volume and Weights

Today's Date	Expiration Date	Stock Price Today	Strike Price	Volume 1/ Call	Put	Weights 2/ Call	Put
2/12/2008	6/21/2008	26.21	15.0	0	26	0.00000	0.00118
2/12/2008	6/21/2008	26.21	17.5	0	120	0.00000	0.00544
2/12/2008	6/21/2008	26.21	20.0	184	6153	0.00891	0.27883
2/12/2008	6/21/2008	26.21	22.5	35	7254	0.00169	0.32873
2/12/2008	6/21/2008	26.21	25.0	197	7316	0.00953	0.33154
2/12/2008	6/21/2008	26.21	27.5	7196	275	0.34827	0.01246
2/12/2008	6/21/2008	26.21	30.0	5288	675	0.25593	0.03059
2/12/2008	6/21/2008	26.21	32.5	644	175	0.03117	0.00793
2/12/2008	6/21/2008	26.21	35.0	6437	60	0.31154	0.00272
2/12/2008	6/21/2008	26.21	37.5	621	10	0.03006	0.00045
2/12/2008	6/21/2008	26.21	40.0	20	3	0.00097	0.00014
2/12/2008	6/21/2008	26.21	42.5	25	0	0.00121	0.00000
2/12/2008	6/21/2008	26.21	45.0	15	0	0.00073	0.00000
2/12/2008	6/21/2008	26.21	47.5	0	0	0.00000	0.00000
2/12/2008	6/21/2008	26.21	50.0	0	0	0.00000	0.00000
2/12/2008	6/21/2008	26.21	55.0	0	0	0.00000	0.00000

Source: Bloomberg and author's calculations
1/ one contract corresponds to 100 stocks
2/ call (or put) contract volume divided by total call (or put) volume

To estimate *option*-iPoD for Citigroup on February 12, 2008, we use the entire information set of table 2. As such, the framework naturally handles the well documented deviations from the lognormal distribution, observed as volatility smile and skew. These are optimally captured by the Lagrange multipliers associated to each constraint imposed during optimization, equation (5). Since call and put options are related by the put-call parity, we can restrict our attention to call contracts.[5]

Within these contracts, it seems reasonable to assign different weights to different contracts. Otherwise, each contract would have the same relevance during optimization.[6] We weight the contracts by their volume, so that the call option with K = 27.5 will have the largest weight, and K = 45 will have the smallest, Table 2. Furthermore, we assume a uniform distribution as a prior distribution for $f^0(V_T)$.[7]

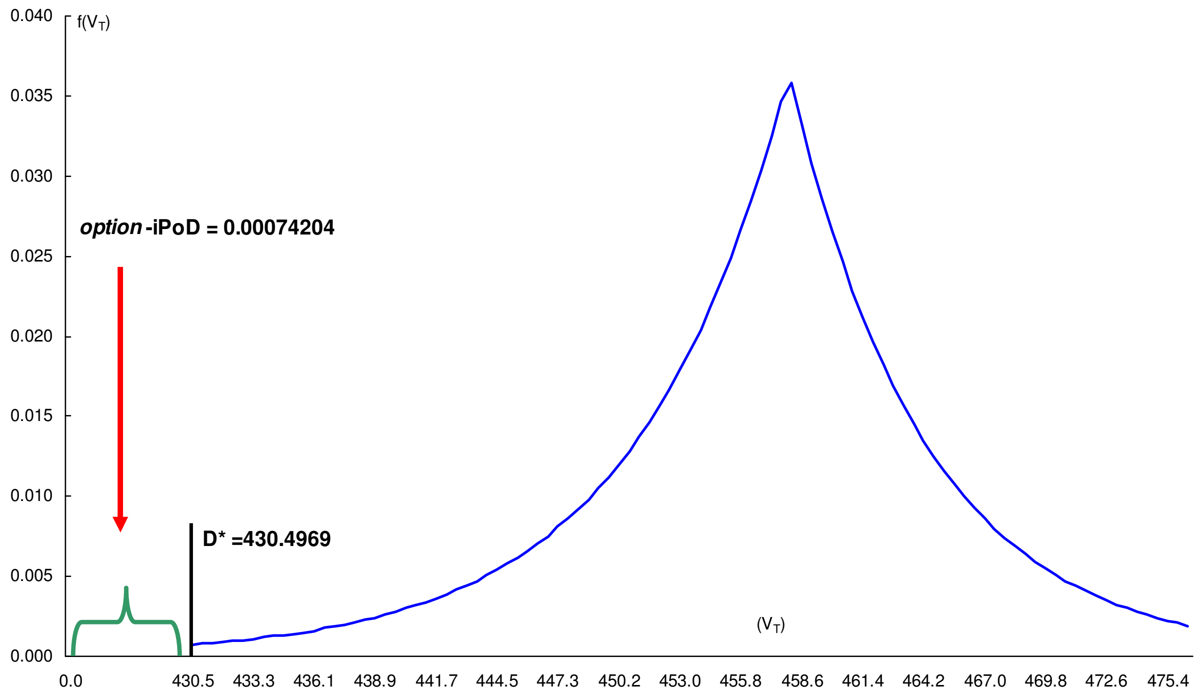
Figure 1 presents *option*-iPoD for Citigroup, and the entire estimated probability density function $f(V_T)$. As discussed in Section IV, the framework is not able to describe the probability density function in the default state, but is informative on the cumulative distribution function, the probability of default.

[5] Results are not significantly affected when put contracts are used.

[6] These weights are clearly different from the Lagrange multipliers, which are the optimal weights that are chosen endogenously during optimization.

[7] This is a conservative assumption. We have experimented with several alternatives, including calibrating a lognormal distribution based on quarterly balance sheet information. Results are not significantly affected.

Figure 1. Citigroup, February 12, 2008: *option*-iPoD and the Probability Density Function 1/



1/ the expiration of the option contracts is June 21, 2008. The value of assets V_T is $ per share outstanding.

Table 3 presents other relevant output. Since the probability density function is recovered, the model delivers useful output for risk-management purposes. In particular, the implied asset volatility and the Greek letters – delta, gamma, and vega – describing the different dimension of risk-exposure in an option position are obtained.

Table 3. Citigroup: Summary of Results

Input	
Today's date	2/12/2008
Expiration date of the contracts	6/21/2008
Stock Price today	26.21
Output	
Balance Sheet developments	
Expected value of the Stock Price at expiration	26.41
Default Barrier	430.50
***Option-* iPOD**	**0.0007420400**
Expected value of the Assets at expiration 1/	456.91
Max Value of Assets at expiration 1/	476.37
Book Value of Assets totay 2/	437.01
Expected Value of Leverage at expiration	17.30
Book Value of Leverage 3/	19.21
Market Value of Leverage 4/	16.20
Risk-Management output	
Implied Asset Volatility 5/	22.27
Skewness	-1.83
Kurtosis	27.53
Delta 6/	0.15
Gamma 6/	0.02
Vega 6/	0.12

Source: author's calculations

1/ per share outstanding

2/ Book value of Assets divided by number of shares outstanding

3/ Book value of Assets divided by book value of equity

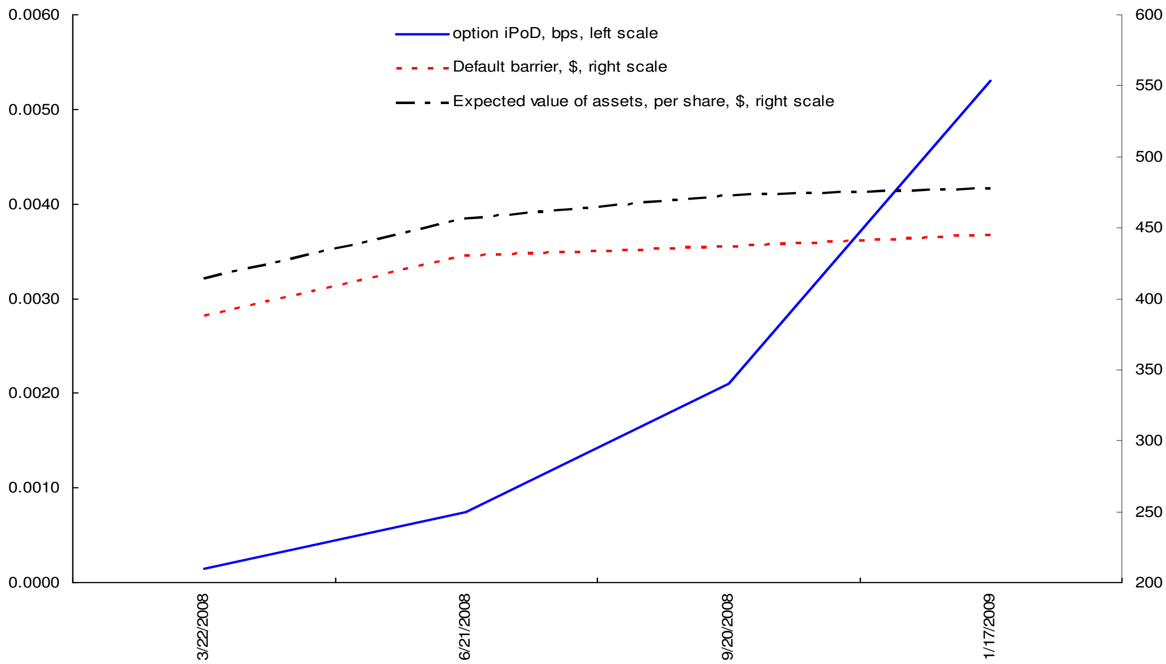
4/ Book value of Assets divided by market value of equity

5/ annualized

6/ for an underlying stock price (or strike price) of 27.5

Each day, markets trade more than one expiration date on the same stock. As such, the framework is able to deliver an entire term structure of *option*-iPoDs, Figure 2.

Figure 2. Citigroup: Term-structure of *option*-iPoD on Feb 12, 2008



The model provides the market-expected values of several balance sheet variables. It is interesting to note that leverage, defined as the ratio of book value of assets to book value of equity, has recently attracted considerable attention, since its procyclicality has been indicated as one possible source of concern for financial stability, Shin (2008). Interestingly, our framework indicates that the market expects some de-leveraging for Citigroup in the next months, Figure 3.[8]

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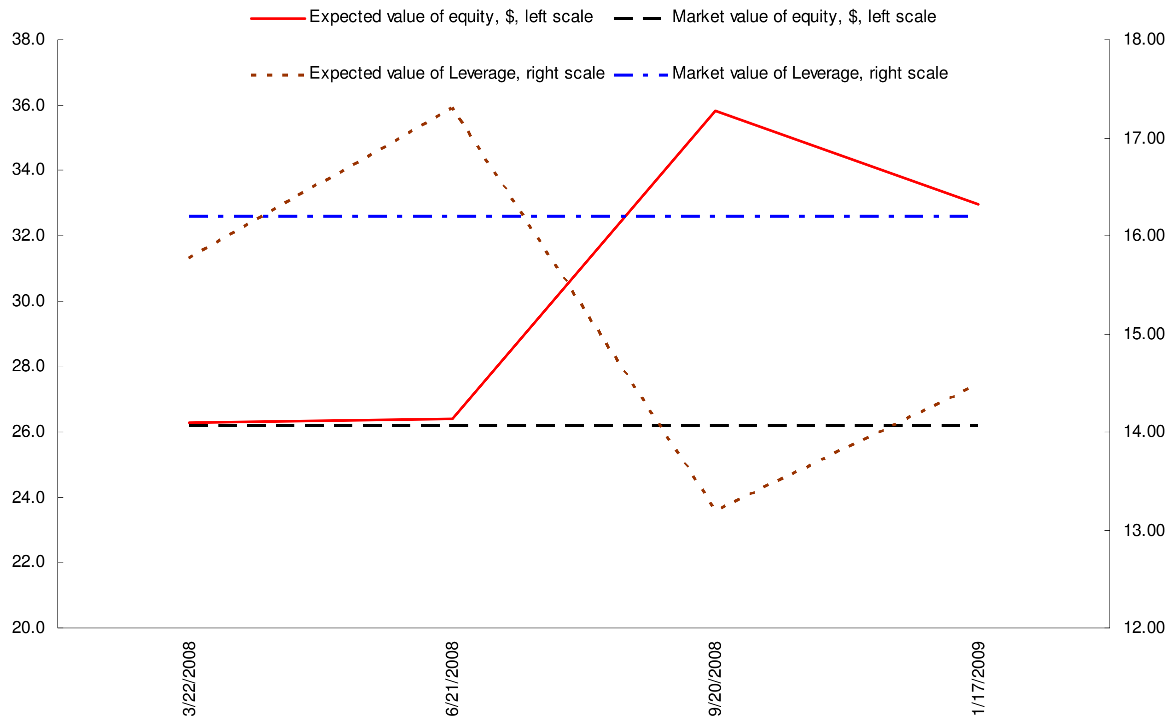
[8] Appendix I presents results for the other banks in Table 1.

Figure 3. Citigroup: Expected Balance Sheet Developments on Feb 12, 2008



We can say even more. We can compute the likelihood that a certain balance sheet ratio will end up above (or below) a pre-specified threshold. Table 4 computes the probability that the leverage ratio will be below some illustrative values. Due to the risk-neutrality of the framework, caution should be used when interpreting Tables 3–4. Nevertheless, this type of output might be particularly appealing for bank regulators.

Table 4. Citigroup: Leverage-at-Risk

Leverage 1/	Equity 2/	Probability
< 20	> 5	0.73
< 25	> 4	0.87
< 30	> 3.3	0.92

Source: author's calculations

1/ ratio of Assets to Equity

2/ ratio of Equity to Assets, in percent

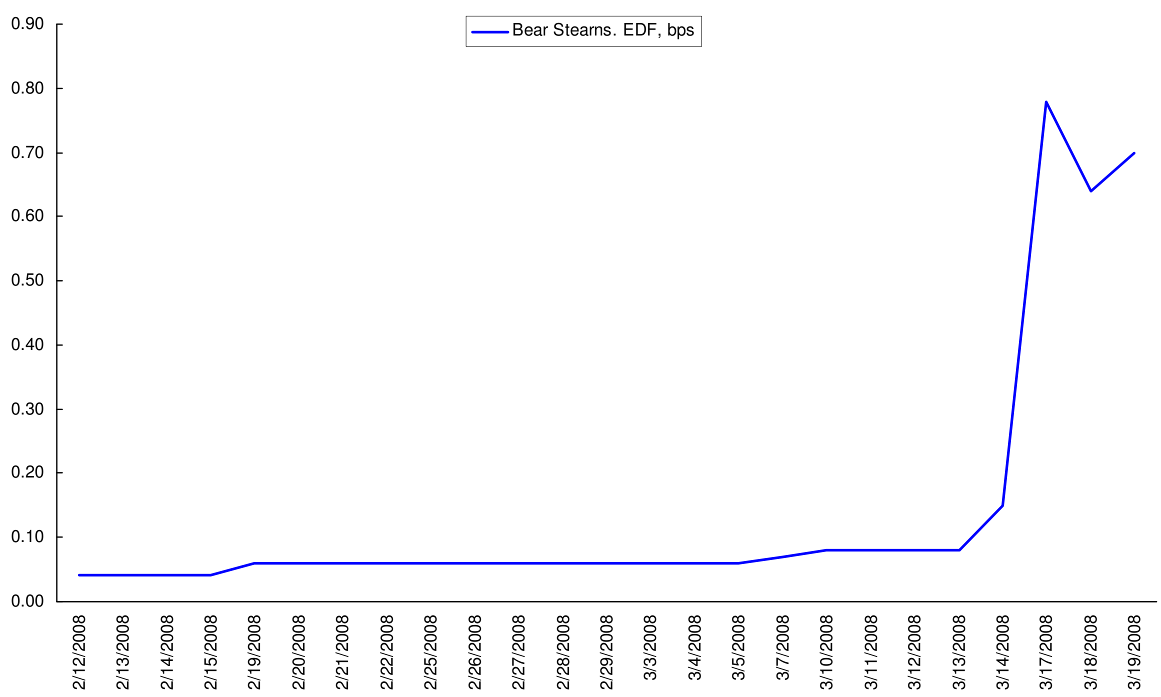
VII. LISTEN TO *OPTION* -IPOD. THE COLLAPSE OF BEAR STEARNS

Bear Stearns collapsed on March 14, 2008. During the following weekend, a rescue plan was put in place by the Federal Reserve Bank (FED). The plan involved the acquisition of Bear Stearns by JP Morgan Chase. It seems interesting to see how and if *option*-iPoD was able to capture market sentiment around this event. In addition, we present the developments of Moody's KMV Expected Default Frequency (EDF^{TM}) during the same time period.

We have to say upfront that *option*-iPoD is not directly comparable with EDF^{TM}. EDF^{TM} represents the real-world probability of default in one year time, as estimated by Moody's. On the other hand, *option*-iPoD represents the risk-neutral probability of default at the expiration of the chosen contract. As such, in the absence of a correction for risk-aversion and a time-transformation, the level of *option*-iPoD cannot be directly compared with the level of EDF^{TM}. However, both indicators are forward-looking and market-based. Therefore, the percentage change in the indicator should provide a measure of the change in market perception of the probability of default.

Figure 4 presents Moody's KMV EDF^{TM} for Bear Stearns, during February 12–March 19, 2008.

Figure 4. Moody's KMV Expected Default Frequency in one year

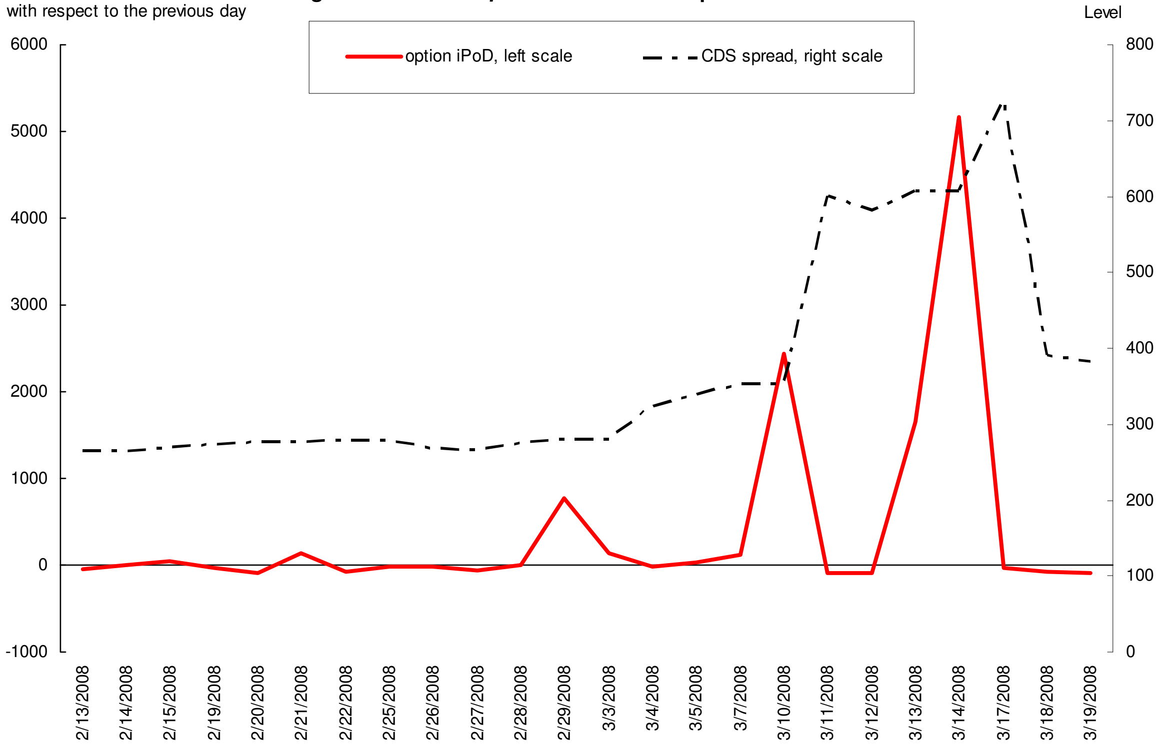


Source: Moody's Creditedge

EDF™ remained almost constant until March 13, and presented a small increase on March 14. Bear Stearns collapsed on March 14. EDF™ reached its peak on March 17, when the FED rescue plan had already been announced and remained more or less at that level on March 18–19.

We focus on the put/call option contract on Bear Stearns expiring on March 22, 2008, the closest to the March 14 collapse, and we compute *option*-iPoD during February 12–March 19, 2008. Figure 5 illustrates the developments of *option*-iPoD and the five-years senior CDS spread on Bear Stearns.

Figure 5. Listen to *option*-iPoD. The collapse of Bear Stearns

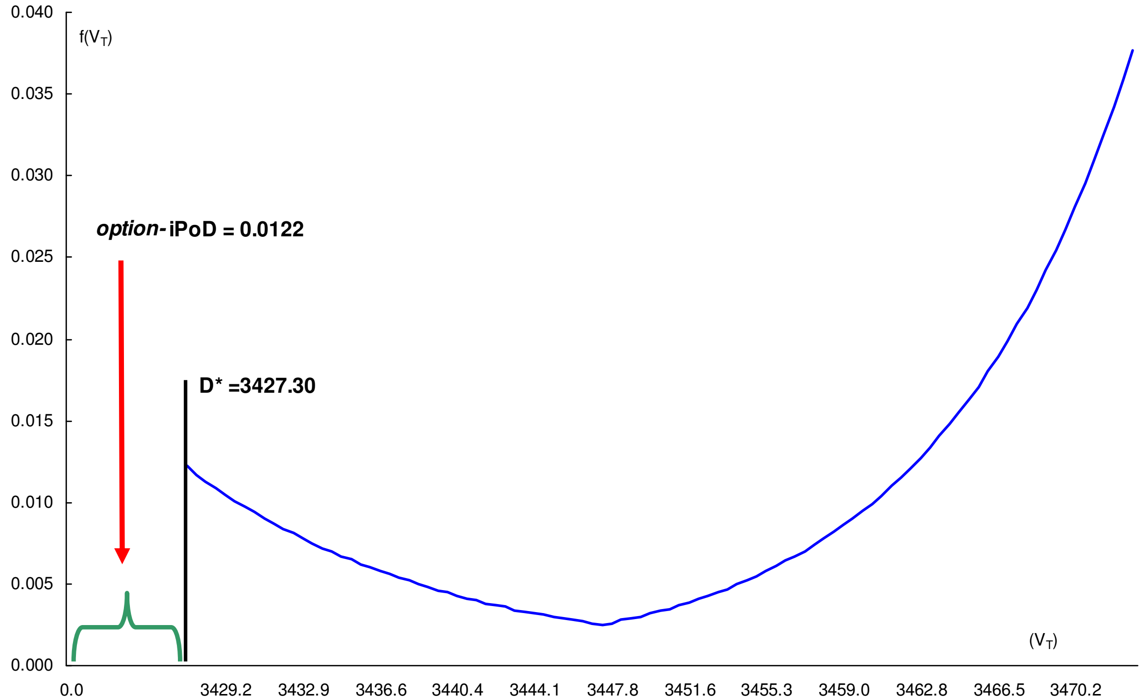


Option-iPoD started to indicate some market nervousness on February 21. On February 29, *option*-iPoD jumped by a factor of 766 with respect to the previous day. The following week a relative calm seemed to return, but on March 10 *option*-iPoD jumped again, a jump 4 times bigger than the previous one, i.e., *option*-iPoD was signaling Bear Stearns was under considerable stress. *option*-iPoD reached its peak on March 14, but quickly dropped on March 17, following the rescue plan announced by the FED. In addition, during this episode, changes in *option*-iPoD appear to be a leading indicator for changes in the level of CDS-spread.

We do not attempt to explain the differences between *option*-iPoD and EDF™. Contrary to existing methodologies, a key feature of *option*-iPoD is that our default barrier is endogenously determined and time-varying, thus contributing to incorporate market sentiment more efficiently. This feature appears particularly relevant when analyzing the default probability of corporations, such as banks, which perform extensive off-balance sheet activities.[9]

It is particularly interesting to look at the probability density function estimated on March 14, Figure 6. The bell shape is reversed, the distribution is bi-modal, indicating considerable market uncertainty about the future of Bear Stearns.

Figure 6. Bear Stearns, March 14, 2008: *option*-iPoD and the Probability Density Function 1/



1/ the expiration of the option contracts is March 22, 2008. The value of assets V_T is \$ per share putstanding.

VIII. Caveats

The framework we are proposing is not definitive. In particular, we have not shown that the endogenous default barrier is uniquely defined. Empirically, this has not been a problem, since we have always found a unique solution. Nevertheless, we acknowledge this issue to require further analysis.

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[9] Moody's KMV's methodology adopts a default barrier based on on-balance sheet liabilities, and generally corresponding to short-term liabilities plus half long-term liabilities, Creditedge User Guide (2007), Dwyer, Guo and Hood (2006), Crosbie and Bohn (2003).

As the discussion in Section IV highlights, it appears difficult to derive a credit-default swap spread from this framework. In order to model the default state, and the price to pay to insure against default, the framework could be usefully extended by using information from the price of bonds. Since bonds are senior claims with respect to equity (stock), prices of bonds provide information on the default state, and on CDS-spreads.

IX. Zero-Coupon *Option*-iPoD

We can extend the framework, as indicated in Section VIII, by incorporating information from the price of bonds. We will argue that while theoretically appealing, the extension may not be the best empirical solution to estimate the probability of default.[10]

Each day, markets trade a certain number of bonds on the same company. In fact, bonds differ by type (zero-coupon, coupon, and convertible bonds are the most common) and maturity. Different bonds represent different senior claims on the value of the assets. As such, it is empirically cumbersome to obtain a seniority tree describing the entire structure of claims traded in the markets.

For the sake of exposition, let us suppose that the company of interest finances its assets (V) with equity (E), and a zero-coupon bond, whose face-value is (D), and whose maturity is assumed to be T, the maturity of the option contracts. A zero-coupon bond is a bond that costs PB_0 today and repays \$1 of face-value at time T, but does not pay any coupon before maturity. Since $PB_0 < 1$, PB_0 provides information on the probability of default, i.e., the probability that the company will default and the bond holder will not receive \$1 at time T.

It is relatively straightforward to incorporate this extra-structure into the framework. We simply need to ask our new posterior density $f^{ZC}(V_T)$, where ZC stands for zero-coupon, to be able to price the zero-coupon bond, on top of the earlier option price constraints. Since the bond is paid also when $V \le D$, this extra constraint will drive the shape of $f^{ZC}(V_T)$ in the default state.

The new problem is:

$$\min_{D}\left\{\min_{f^{ZC}(V_T)} \int_{V_T=0}^{\infty} f^{ZC}(V_T)\log\left[\frac{f^{ZC}(V_T)}{f^{0}(V_T)}\right]dV_T\right\} \qquad (1)'$$

---

[10] Chan-Lau (2006) reviews a number of techniques to determine a market-based probability of default. Zou (2003) develops a similar framework to analyze default probability.

subject to the following constraints:

$$E_0 = e^{-rT} \int_{V_T=D}^{\infty} (V_T - D) f^{ZC}(V_T) dV_T \tag{2'}$$

$$C_0^i = e^{-rT} \int_{V_T=D+K_i}^{\infty} (V_T - D - K_i) f^{ZC}(V_T) dV_T \tag{3'}$$

$$PB_0 = e^{-rT} \left\{ \int_{V_T=0}^{D} \left(\frac{V_T}{D} \right) f^{ZC}(V_T) dV_T + \int_{V_T=D}^{\infty} f^{ZC}(V_T) dV_T \right\} \tag{4'}$$

$$1 = \int_{V_T=0}^{\infty} f^{ZC}(V_T) dV_T \tag{5'}$$

$i = 1,....n$ indicates the available option contracts. (4)' says that today, the price of a zero-coupon bond must correspond to the present value of its expected payoff. Similarly to Section III., the solution to (1)' is:

$$f^{ZC}(V_T, \lambda) = f^{0}(V_T) \cdot \gamma \tag{6'}$$

where

$$\gamma = \exp\left[\lambda_0 - 1 + \lambda_1 e^{-rT} \mathbf{1}_{V_T > D}(V_T - D) + \sum_{i=1}^{n} \lambda_{2,i} e^{-rT} \mathbf{1}_{V_T > D+K_i}(V_T - D - K_i) + \lambda_3 e^{-rT} \left(\mathbf{1}_{V_T > D} + \mathbf{1}_{V_T \le D} \left(\frac{V_T}{D} \right) \right) \right]$$

and $i = 1,2....,n$.

The zero-coupon *option*-iPoD is:

$$\int_{V_T=0}^{D} f^{ZC}(V_T) dV_T$$

In the CDS market, a recovery rate of 40 percent is typically assumed in order to back-up the probability of default, Duffie and Singleton (2003). [11] Interestingly, our framework delivers an endogenous expected recovery rate:

$$\int_{V_T=0}^{D} \left(\frac{V_T}{D} \right) f^{ZC}(V_T) dV_T$$

Since the yield to maturity is given by:

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[11] Altman (2006) reviews the recent literature on recovery rate estimation and its relationship with default.

$$PB_0 = e^{-yT}$$

(4)' implies that the credit-default spread is:

$$s = y - r = -\frac{1}{T}\ln\left(\int_{V_T=0}^{D}\left(\frac{V_T}{D}\right)f^{ZC}(V_T)dV_T + \int_{V_T=D}^{\infty}f^{ZC}(V_T)dV_T\right) \qquad (6)'$$

From the empirical point of view, zero-coupon *option*-iPoD turned out to be impracticable. When brought to the data, the numerical algorithm described in Section V has rarely converged. In particular, constraint (5)' is seldomly satisfied, so that the resulting zero-coupon *option*-iPoD remains vague. This result makes a lot of sense. We have been really stretching the Merton (1974) framework, and it is not surprising to us that most of the times data reject this extension.[12]

As mentioned, markets trade several types of bonds on the same company with a different seniority structure. Furthermore, the maturity of the zero-coupon bond would very rarely correspond to the maturity of the option contracts. In our opinion, these empirical limitations appear difficult to solve, and lead to the advice to listen, in practice, to *option*-iPoD of Section V.[13]

X. CONCLUSIONS

We have presented a framework that uses the principle of minimum cross-entropy to derive the probability of default implied by the prices of equity options. The framework is flexible, does not require any strong statistical assumption, and provides results that are extremely informative on the expected developments of balance sheet variables, such as the value of assets, equity and leverage. In addition, the framework delivers useful output variables for risk-management purposes. In the episode of the collapse of Bear Stearns, *option*-iPoD was able to early signal market sentiment. These results should not be considered as an attempt to assess the stability of the U.S. banking sector at this difficult juncture, see Capuano and Segoviano (2008) for an in-depth stability analysis of the U.S. banking sector.

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[12] We have been somewhat more successful by letting the posterior density price directly the CDS-spread that comes from the market, Appendix II presents results for Citigroup. This means to replace (4)' with (6)' in the optimization algorithm, where s is the CDS-spread quoted by the market. Results for other U.S. banks are available, upon request.

[13] An alternative way to obtain zero-coupon *option*-iPoD would be to assume a recovery rate, say 40 percent, to back-up from (4)' both D and $f^{ZC}(V_T)$. Empirically, this strategy would suffer from the limitations discussed above, since the seniority structure of different bonds would remain unknown. Furthermore, there is debate on the empirical validity of a fixed recovery rate assumption in the pricing of credit derivatives, see Andritzky and Singh (2006) for sovereign defaults and Altman (2006) for corporate defaults.

We have shown how to extend the framework by using information from the price of a zero-coupon bond. However, we have encountered serious data limitations that suggest not to use, in practice, the extension that accounts for a zero-coupon bond. Following Miller and Liu (2002), we intend to extend *option*-iPoD in a multivariate framework.

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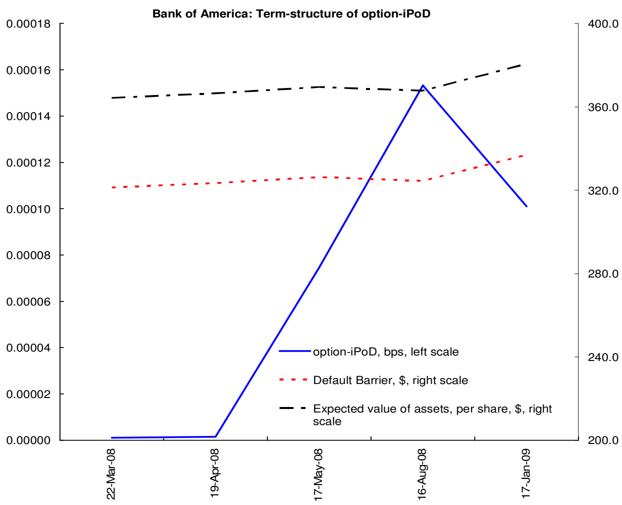
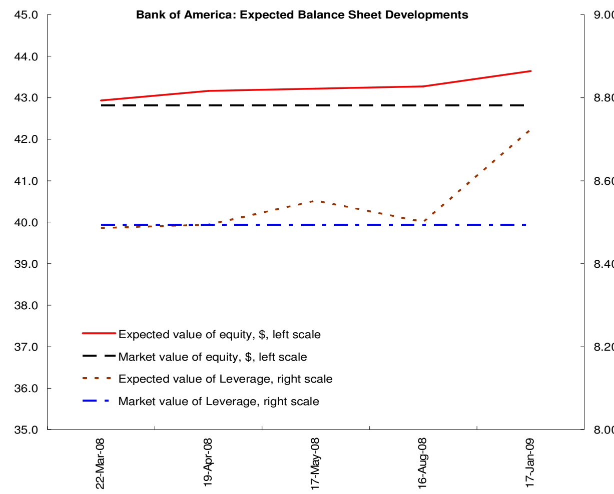
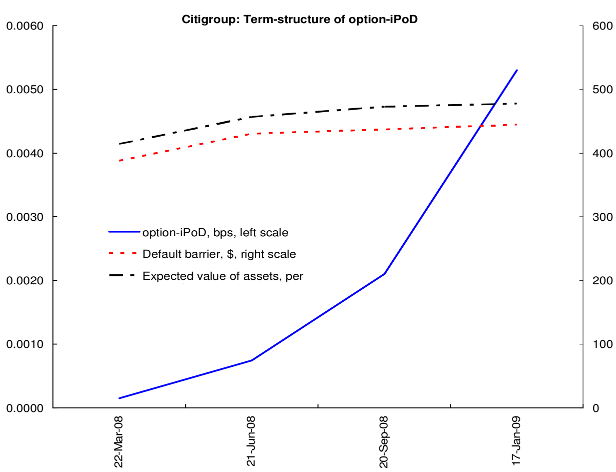
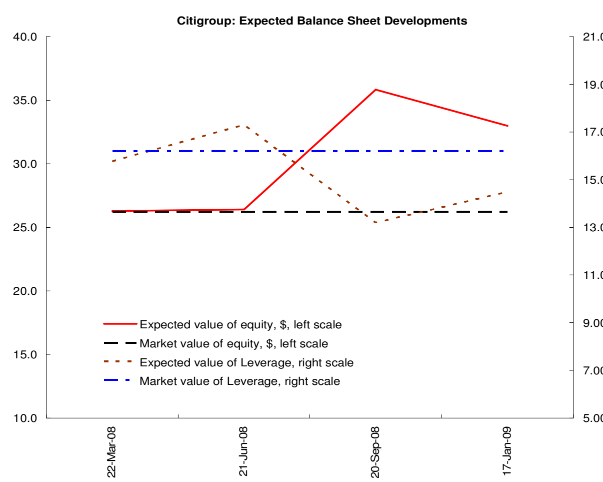
Appendix I. Results From The Ten Largest U.S. Financial Institutions

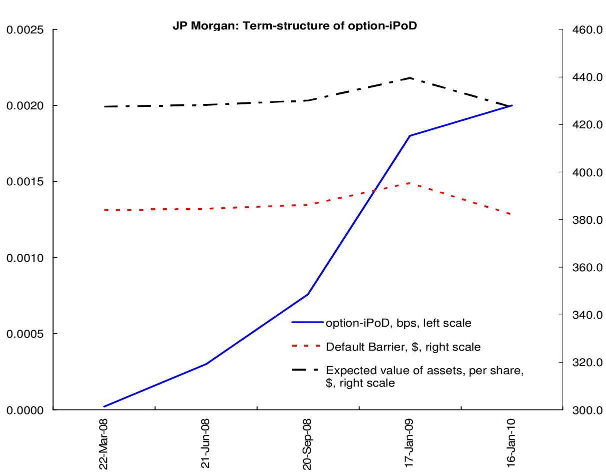
Figure 1.A. Bank Holding Companies: *option*-iPoDs and Balance Sheet Developments on February 12, 2008

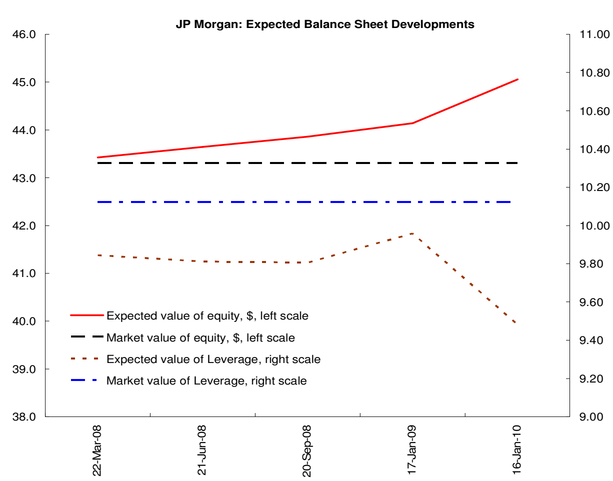






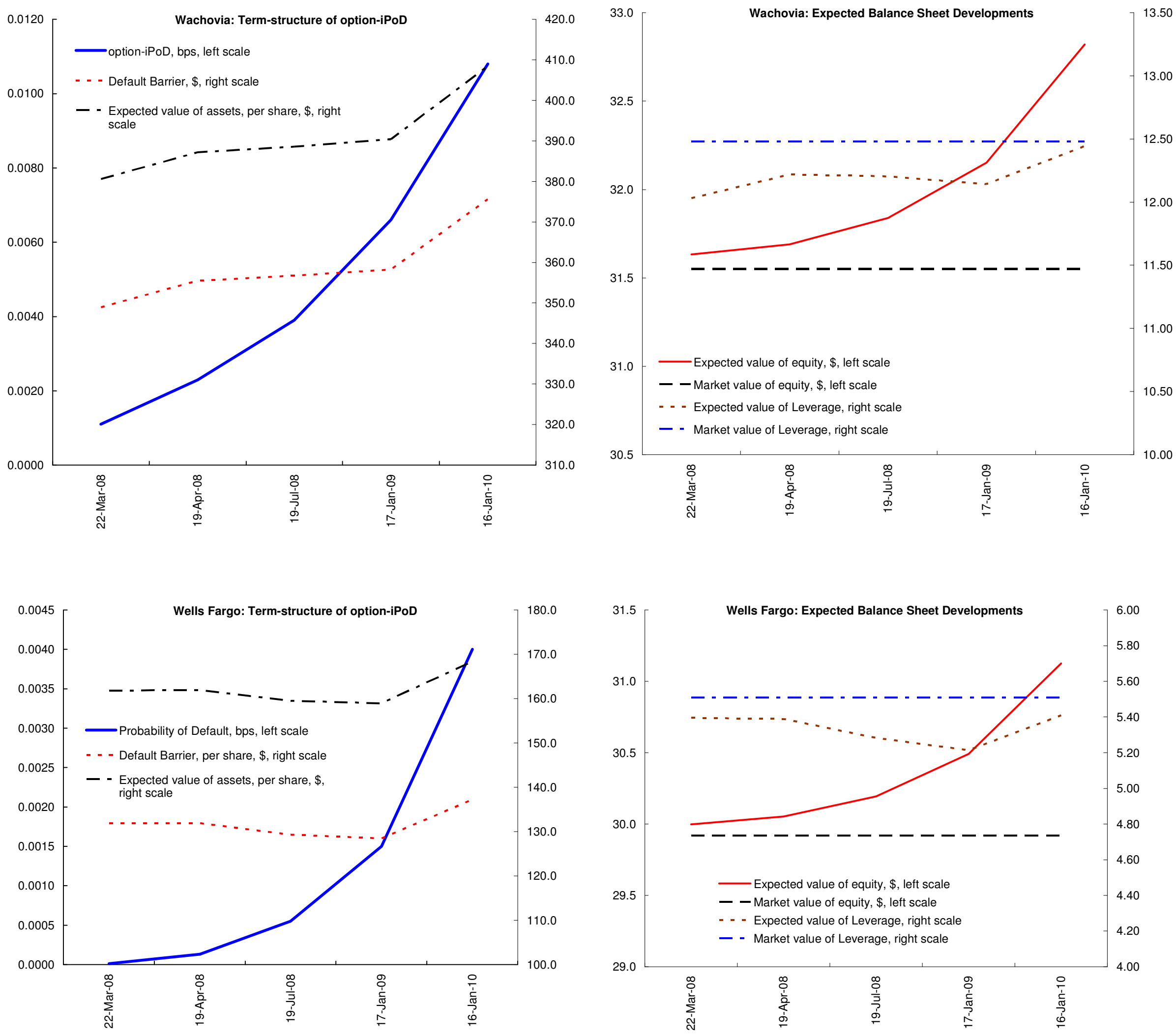






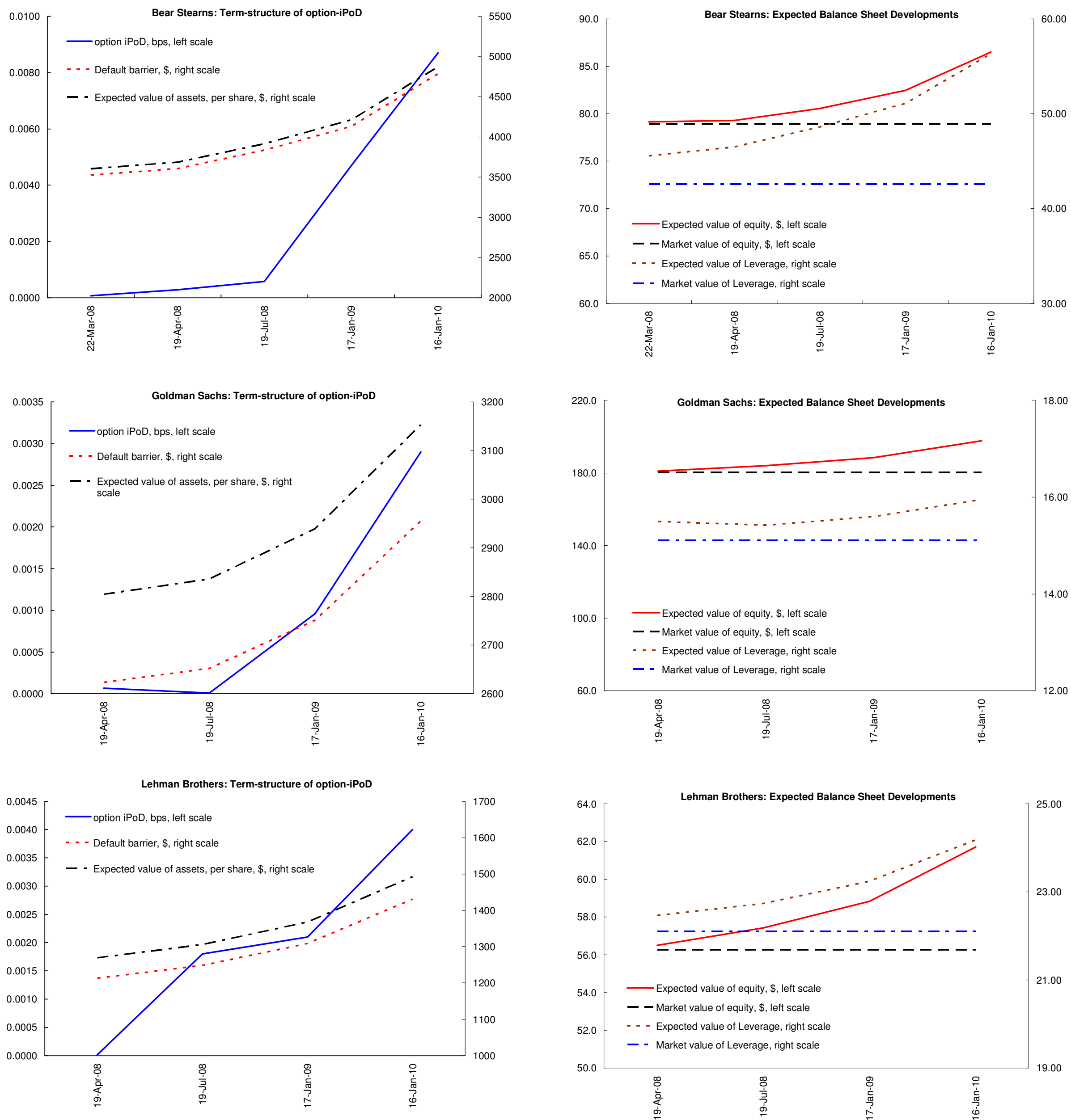
Source: Author's calculations.

Figure 2.A. Bank Holding Companies: *option*-iPoDs and Balance Sheet Developments on February 12, 2008



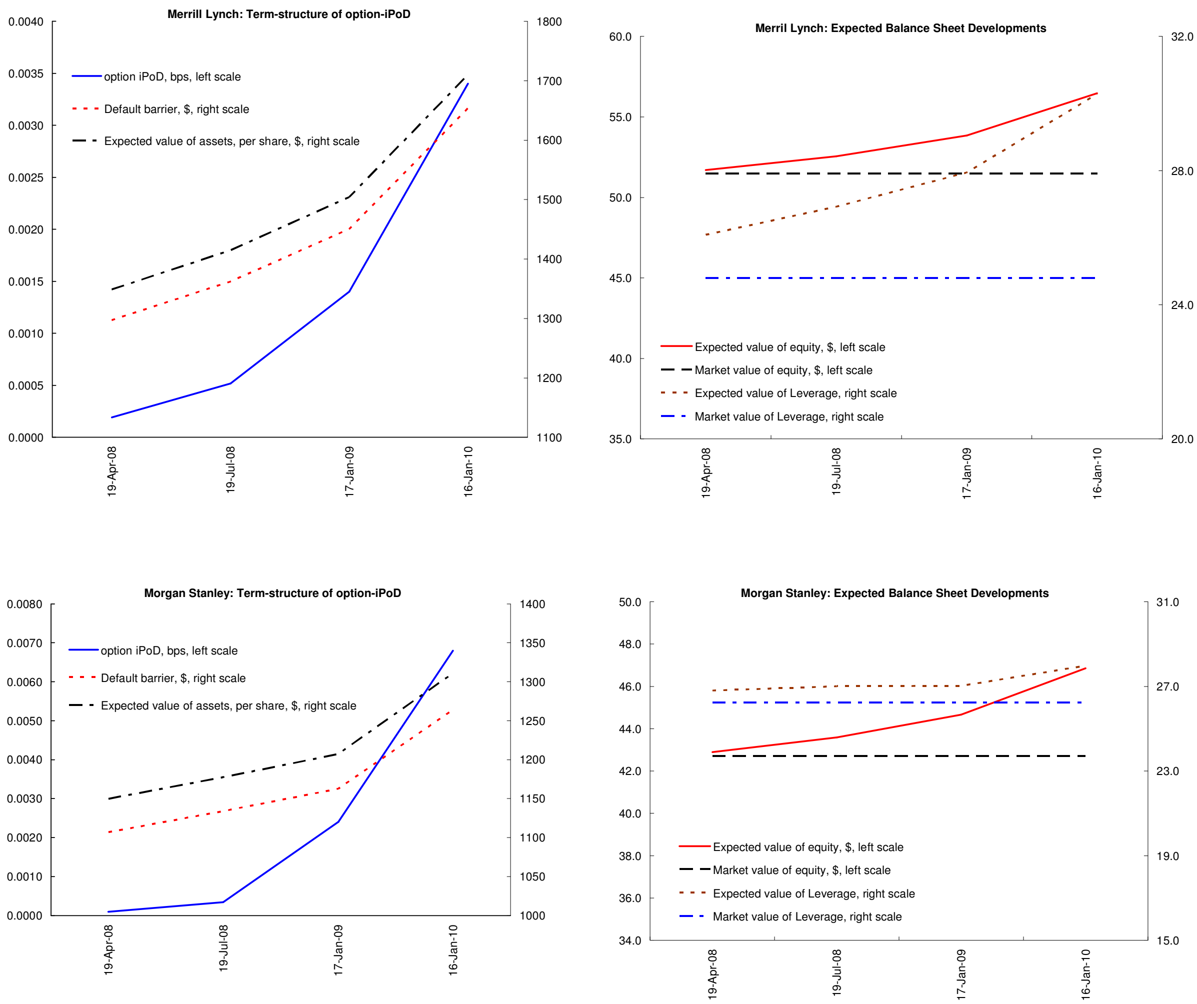
Source: Author's calculations

Figure 3.A. Investment Banks: *option*-iPoDs and Balance Sheet Developments on February 12, 2008



Source: Author's calculations

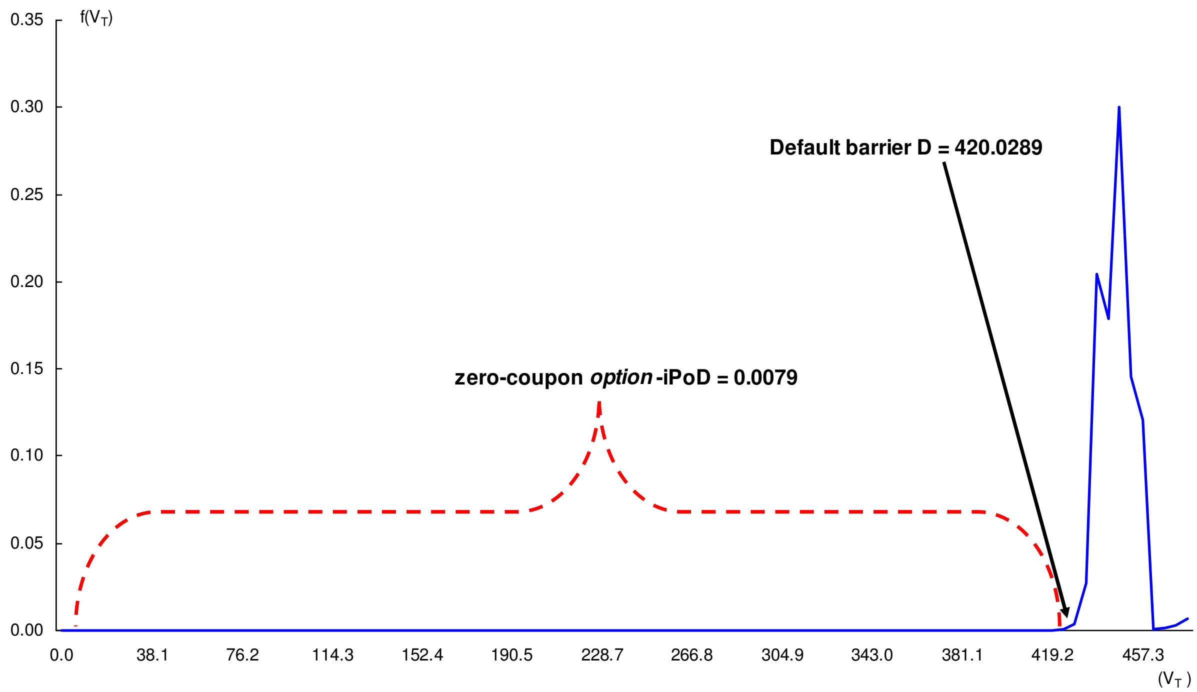
Figure 4.A. Investment Banks: *option*-iPoDs and Balance Sheet Developments on February 12, 2008



Source: Author's calculations

Appendix II. Extension with Zero-Coupon Bond

Figure 1.B. Citigroup. Zero-coupon *option*-iPoD on February 12, 2008 1/



1/ June 21, 2008 expiration. f(V) is estimated imposing the CDS spread quoted by the market on February 12, as explained in Section IX.