A New Simple Square Root Option Pricing Model

António Câmara[1]

Yaw-Huei Wang[2]

August 25, 2008

[1]Watson Family Chair in Commodity and Financial Risk Management and Associate Professor of Finance, Oklahoma State University, Spears School of Business, Stillwater, OK-74078, Tel: 405-744 1818, Fax: 405-744 5180, E-Mail: ⟨ acamara@okstate.edu ⟩, Internet: ⟨ http://spears.okstate.edu/∼ acamara ⟩.

[2]Assistant Professor of Finance, National Taiwan University, Department of Finance, No. 1, Sec. 4, Roosevelt Road, Taipei 106, Taiwan. Tel.: 886-2-33661092. Email: ⟨ yhwang@management.ntu.edu.tw⟩, Internet: ⟨http://www.fin.ntu.edu.tw/∼fm/yhwang.htm ⟩.

Abstract

A New Simple Square Root Option Pricing Model

This paper derives a simple square root option pricing model (SSROPM) using a general equilibrium approach in an economy where the representative agent has a generalized logarithmic utility function. Our option pricing formulae, like the Black-Scholes model, do not depend on the preference parameters of the utility function of the representative agent. While the Black-Scholes model introduces limited liability in asset prices by assuming that the logarithm of the stock price has a normal distribution, our basic square root option pricing model introduces limited liability by assuming that the square root of the stock price has a normal distribution. The empirical tests on the S&P500 index options market show that our model has smaller fitting errors than the Black-Scholes model, and that it generates volatility *skews* with similar shapes to those observed in the marketplace.

1. Introduction

The option pricing model of Black and Scholes (1973) is among the most important works in finance theory, and their model was recognized since its inception by many researchers including Merton (1973). According to Merton (1973), the option pricing "theory developed by Black and Scholes (1973) is particularly attractive because it is a complete equilibrium formulation of the problem and because the final formula is a function of observable variables, making the model subject to direct empirical tests". During the last 35 years many empirical tests have been done to the Black-Scholes model and many alternative models have been proposed.[1]

In this paper, we introduce the simple square root option pricing model with a closed-form solution of the Black-Scholes type. While the Black-Scholes model assumes that the logarithm of the stock price follows a normal distribution, our basic square root option pricing model assumes that the square root of the stock price follows a normal distribution. These are two different ways to introduce the limited liability property in asset prices, and they produce two simple and competitive option pricing models. There are many other option pricing models including the pure jump model of Cox and Ross (1976), the jump-diffusion model of Merton (1976), the displaced diffusion model of Rubinstein (1983), the stochastic volatility model of Heston (1993), and the affine jump-diffusion of Duffie, Pan, and Singleton (2000). These models extend the Black-Scholes theory in several other directions and have much more complicated formulae. Our model complements such existing option pricing literature with a fairly simple formula.

We price options in a simple general equilibrium economy with a representative agent who has a generalized logarithm utility function. Rubinstein (1976) advocates the use of the generalized logarithm utility function as the primer model in finance. To the best of our knowledge this utility function was not used previously in the pricing of derivatives. We assume that in the real world

---

[1]For a review of the literature on empirical tests of the Black-Scholes model and alternative option pricing models see McDonald (2003) and Hull (2009).

the stock price has, in general, a square root distribution with four parameters. In the risk-neutral world, the distribution implicit in the square root option pricing formula only depends on three parameters. The parameter μ does not affect our option pricing formula, as it does not affect the Black-Scholes formula. Both ours and the Black-Scholes option pricing formulas are preference-free, i.e. none of the formulas depends on preference parameters. In order to eliminate preferences from the option pricing formula we extend the technique of pricing by substitution in equilibrium of Brennan (1979), Stapleton and Subrahmanyam (1984), Camara (2003, 2008), and Schroder (2004).

The empirical tests of our model are conducted using the S&P 500 index options. Similar to the Black-Scholes model, our model is also fairly easy to implement although we have two more parameters. These two parameters are the minimum possible value of the index and the rescaling parameter. Nonetheless, our model is superior to the Black-Scholes model in at least two aspects. Our model not only has significantly smaller fitting errors of option prices, but also generates a negatively sloped implied volatility function that is much closer to the stylized fact observed in the real equity market. Hence, the SSROPM is able to solve some of the empirical biases of the Black-Scholes model. These pricing biases have been reported by many authors including Rubinstein (1994), Bakshi, Cao, and Chen (1997), Ait-Sahalia and Lo (1998), and Dumas, Fleming, and Whaley (1998).

The remainder of this paper is organized as follows. Section 2 derives a new simple square root option pricing model. Section 3 describes the data used to empirically test the model. Section 4 presents the empirical results for the S&P 500 index options. We provide some concluding remarks in Section 5.

2. The Economic model

The model assumes that markets open at the beginning and at the end of the economy and, then, that there is no trade between these two dates. There is a representative agent who is endowed,

at time 0, with the aggregate wealth W_0. The representative agent invests his endowment in both risky assets and riskless assets. The initial portfolio of risky assets consists of n_s shares of stock with a price S_0, n_c call options on the stock with strike price K and current price C_0, and n_p put options on the stock with strike price K and current price P_0. Therefore, the representative agent invests in riskless assets an amount of $W_0 - (n_s \cdot S_0 + n_c \cdot C_0 + n_p \cdot P_0)$ at the continuously compounded interest rate r. At the end of the economy, time T, his total wealth consists in $n_s \cdot S_T$ of stock, $n_c \cdot C_T = n_c \cdot (S_T - K)^+$ of call options, and $n_p \cdot P_T = n_p \cdot (K - S_T)^+$ of put options, and $(W_0 - n_s \cdot S_0 - n_c \cdot C_0 - n_p \cdot P_0)e^{rT}$ of bonds. That is:

$$
\begin{aligned}
W_T &= (W_0 - n_s \cdot S_0 - n_c \cdot C_0 - n_p \cdot P_0)e^{rT} + n_s \cdot S_T + n_c \cdot C_T + n_p \cdot P_T \\
&= W_0 e^{rT} + n_s(S_T - S_0 e^{rT}) + n_c(C_T - C_0 e^{rT}) + n_p(P_T - P_0 e^{rT}).
\end{aligned}
\tag{1}
$$

The representative agent maximizes his expected utility of terminal wealth with respect to (n_s, n_c, n_p) to find his optimal portfolio and equilibrium prices:

$$
\underset{n_s, n_c, n_p}{Max} \qquad E\left[U(W_T)\right].
\tag{2}
$$

The first order conditions for a maximum are:

$$
\begin{aligned}
\frac{dE\left[U(W_T)\right]}{dn_s} &= 0 \Longrightarrow E\left[U'(W_T)\left(S_T - S_0 e^{rT}\right)\right] = 0, \\
\frac{dE\left[U(W_T)\right]}{dn_c} &= 0 \Longrightarrow E\left[U'(W_T)\left(C_T - C_0 e^{rT}\right)\right] = 0, \\
\frac{dE\left[U(W_T)\right]}{dn_p} &= 0 \Longrightarrow E\left[U'(W_T)\left(P_T - P_0 e^{rT}\right)\right] = 0.
\end{aligned}
$$

Solving these three equations for the current prices of the stock, call, and put yields the following equilibrium pricing relationships:

$$
\begin{aligned}
S_0 &= e^{-rT} E\left[\frac{U'(W_T)}{E[U'(W_T)]} S_T\right], \\
C_0 &= e^{-rT} E\left[\frac{U'(W_T)}{E[U'(W_T)]}(S_T - K)^+\right],
\end{aligned}
$$

3

$$P_0 = e^{-rT} E\left[\frac{U'(W_T)}{E[U'(W_T)]}(K - S_T)^+\right],$$

since $C_T = (S_T - K)^+$ and $P_T = (K - S_T)^+$.

Using the law of iterated expectations in the previous equations yields:

$$S_0 = e^{-rT} E\left\{E\left[\frac{U'(W_T)}{E[U'(W_T)]}S_T \mid S_T\right]\right\} = e^{-rT} E\left\{S_T E\left[\frac{U'(W_T)}{E[U'(W_T)]} \mid S_T\right]\right\},$$

$$C_0 = e^{-rT} E\left\{E\left[\frac{U'(W_T)}{E[U'(W_T)]}(S_T - K)^+ \mid S_T\right]\right\} = e^{-rT} E\left\{(S_T - K)^+ E\left[\frac{U'(W_T)}{E[U'(W_T)]} \mid S_T\right]\right\},$$

$$P_0 = e^{-rT} E\left\{E\left[\frac{U'(W_T)}{E[U'(W_T)]}(K - S_T)^+ \mid S_T\right]\right\} = e^{-rT} E\left\{(K - S_T)^+ E\left[\frac{U'(W_T)}{E[U'(W_T)]} \mid S_T\right]\right\}.$$

We define, as in Camara (2003), the asset specific pricing kernel as:

$$\psi(S_T) = E\left[\frac{U'(W_T)}{E[U'(W_T)]} \mid S_T\right], \tag{3}$$

and use the definition in the previous equations yielding:

$$S_0 = e^{-rT} \left[S_T \psi(S_T)\right], \tag{4}$$

$$C_0 = e^{-rT} \left[(S_T - K)^+ \psi(S_T)\right], \tag{5}$$

$$P_0 = e^{-rT} \left[(K - S_T)^+ \psi(S_T)\right]. \tag{6}$$

We are going to use equations (4), (5), and (6) to obtain closed-form solutions for our simple square root option pricing model (SSROPM). These three equations depend on the asset specific pricing kernel $\psi(S_T)$. In order to evaluate the asset specific pricing kernel we need to make assumptions on the utility function of the representative agent, the distribution of aggregate wealth, and the distribution of the stock price.

Definition 1. (The generalized logarithmic utility function) *The generalized logarithmic utility function of wealth is given by:*

$$U(W_T) = \ln(A + CW_T), \tag{7}$$

4

with $W_T > -A/C$. *Nonsatiation implies that $C > 0$. Aggregate preferences are characterized by decreasing proportional risk aversion (DPRA), constant proportional risk aversion (CPRA), or increasing proportional risk aversion (IPRA) if $A < 0$, $A = 0$, or $A = 0$.*

Definition 2. (The displaced lognormal distribution of aggregate wealth) *Let z_w be a standard normal random variable, i.e. $z_w \sim N(0,1)$. Then aggregate wealth W_T has a displaced lognormal distribution defined by:*

$$z_w = -\mu_w T \left(\sigma_w \sqrt{T}\right)^{-1} + \left(\sigma_w \sqrt{T}\right)^{-1} \ln\left(\frac{A}{C} + W_T\right), \tag{8}$$

with probability density function

$$f(W_T) = \frac{1}{\sqrt{2\pi\sigma_w^2 T}\left(\frac{A}{C} + W_T\right)} exp\left\{-\frac{1}{2\sigma_w^2 T}\left[ln\left(\frac{A}{C} + W_T\right) - \mu_w T\right]^2\right\}. \tag{9}$$

We say that aggregate wealth has a displaced lognormal distribution $W_T \sim \Lambda\left(-\frac{A}{C}, \mu_w T, \sigma_w^2 T\right)$ where Λ denotes the lognormal distribution, $-\infty < -A/C < \infty$ is the lower bound, i.e. $W_T > -A/C$, $-\infty < \mu_w < \infty$ is the drift, and $\sigma_w > 0$ is the volatility of W_T.

The displaced lognormal distribution implies that $W_T > -A/C$. The utility function of wealth is only defined for values of wealth greater than $-A/C$, meaning that the distribution of wealth and the utility of wealth agree on the range of values that wealth can take. The generalized logarithmic utility function of wealth and the displaced lognormal distribution of wealth are, therefore, consistent.[2] In this paper, there is only one role for the displaced lognormal distribution of wealth. We link the distribution of aggregate wealth to the utility function of wealth in order to derive a pricing kernel. The role of the pricing kernel is to discount the risky payoffs of the assets. Hence the only displaced lognormal distribution that is relevant for this paper is the actual or objective displaced lognormal distribution. Several authors have derived option prices assuming that the stock price has a displaced lognormal distribution (See e.g. Rubinstein (1983) and Camara (1999)),

---

[2]In this sense the power utility and the normal distribution are inconsistent since the utility is only defined for positive wealth while the normal distribution is defined over the real line.

and in those papers both the objective displaced lognormal and the risk-neutral displaced lognormal are important. This is clearly distinct from our paper since we do not price options when the stock price has a displaced lognormal distribution. Hence the risk-neutral displaced lognormal distribution is unimportant for our paper. We price assets on a stock which has a square root distribution. In this paper we set up the objective square root distribution for an underlying stock price, and we will derive the risk-neutral square root distribution implicit in option prices.

Definition 3. (The square root distribution of the stock price) *Let z_s be a standard normal random variable, i.e. $z_s \sim N(0,1)$. Then the stock price S_T has a square root distribution defined by:*

$$z_s = -\mu T \left(\sigma\sqrt{T}\right)^{-1} + \left(\sigma\sqrt{T}\right)^{-1}\sqrt{\frac{S_T - \alpha}{\beta}}, \tag{10}$$

with probability density function

$$f(S_T) = \frac{1}{\sqrt{2\pi}2\sigma\sqrt{T}\sqrt{\beta(S_T - \alpha)}}exp\left\{-\frac{1}{2\sigma^2 T}\left[\sqrt{\frac{S_T - \alpha}{\beta}} - \mu T\right]^2\right\}. \tag{11}$$

We say that the stock price has a square root distribution $S_T \sim \Theta\left(\alpha, \mu T, \sigma^2 T, \beta\right)$ where Θ denotes the square root distribution, $-\infty < \alpha < \infty$ is the lower bound, i.e. $S_T > \alpha$, $-\infty < \mu < \infty$ is the drift, $\sigma > 0$ is the volatility, and $\beta > 0$ is the rescale of S_T.

We assume that z_w and z_s have correlation ρ. Now we can evaluate the asset specific pricing kernel (3), whose result is presented in the next Lemma.

Lemma 1. (The asset specific pricing kernel) *Assume that the representative agent has a generalized logarithmic utility function, that aggregate wealth has a displaced lognormal distribution, and that the stock price has a square root distribution given by definitions 1, 2, and 3 respectively. Then asset specific pricing kernel defined by (3) is given by:*

$$\psi(S_T) = exp\left\{-b\mu T + b\sqrt{\frac{S_T - \alpha}{\beta}} - \frac{1}{2}b^2\sigma^2 T\right\}, \tag{12}$$

where $b = -\sigma_w\rho/\sigma$.

Proof: See the Appendix.

We are now in conditions to price the stock, the call, and the put. We start with the stock. Equation (4) can be rewritten as:

$$
\begin{aligned}
S_0 &= e^{-rT} \int_{-\infty}^{\infty} S_T \psi(S_T) f(S_T) dS_T \\
&= e^{-rT} \int_{-\infty}^{\infty} S_T \frac{1}{\sqrt{2\pi} 2\sigma \sqrt{T} \sqrt{\beta(S_T - \alpha)}} exp \left\{ -\frac{1}{2\sigma^2 T} \left[\sqrt{\frac{S_T - \alpha}{\beta}} - \left(\mu T + b\sigma^2 T \right) \right]^2 \right\} dS_T \\
&= e^{-rT} \left[\alpha + \beta \sigma^2 T + \beta \left(\mu T + b\sigma^2 T \right)^2 \right],
\end{aligned}
\tag{13}
$$

where the second equality follows from substituting both the asset specific pricing kernel $\psi(S_T)$ from (12) and the square root p.d.f. of the stock price $f(S_T)$ from (11) into the first equality and simplifying the resulting expression, and the third equality follows from the properties of the normal distribution. Equation (13) yields the current stock price in equilibrium. Later we will use this expression to simplify the price of the call and put options.

The equilibrium price of the call option can be rewritten from (5) as:

$$
\begin{aligned}
C_0 &= e^{-rT} \int_{-\infty}^{\infty} (S_T - K)^+ \psi(S_T) f(S_T) dS_T \\
&= e^{-rT} \int_{-\infty}^{\infty} (S_T - K)^+ \frac{1}{\sqrt{2\pi} 2\sigma \sqrt{T} \sqrt{\beta(S_T - \alpha)}} exp \left\{ -\frac{1}{2\sigma^2 T} \left[\sqrt{\frac{S_T - \alpha}{\beta}} - \left(\mu T + b\sigma^2 T \right) \right]^2 \right\} dS_T \\
&= e^{-rT} \left[\alpha + \beta \sigma^2 T + \beta \left(\mu T + b\sigma^2 T \right)^2 - K \right] N(d) \\
&+ e^{-rT} \beta \sigma \sqrt{T} \left[\sqrt{\frac{K - \alpha}{\beta}} + \left(\mu T + b\sigma^2 T \right) \right] n(-d),
\end{aligned}
\tag{14}
$$

where $d = \frac{\mu T + b\sigma^2 T - \sqrt{\frac{K - \alpha}{\beta}}}{\sigma \sqrt{T}}$, $N(\cdot)$ is the cumulative distribution function of a standard normal random variable, and $n(\cdot)$ is the density function of a standard normal random variable. Here the second equality follows from substituting both the asset specific pricing kernel $\psi(S_T)$ from (12) and the square root p.d.f. of the stock price $f(S_T)$ from (11) into the first equality and simplifying the resulting expression, and the third equality uses the moments of the truncated normal random variable. Equation (14) yields the current price of the call in equilibrium.

The equilibrium price of the put option can be rewritten from (6) as:

$$
\begin{aligned}
P_0 &= e^{-rT} \int_{-\infty}^{\infty} (K - S_T)^+ \, \psi(S_T) f(S_T) dS_T \\
&= e^{-rT} \int_{-\infty}^{\infty} (K - S_T)^+ \frac{1}{\sqrt{2\pi} 2\sigma \sqrt{T} \sqrt{\beta(S_T - \alpha)}} exp \left\{ -\frac{1}{2\sigma^2 T} \left[\sqrt{\frac{S_T - \alpha}{\beta}} - \left(\mu T + b\sigma^2 T \right) \right]^2 \right\} dS_T \\
&= e^{-rT} \left[K - \alpha - \beta\sigma^2 T - \beta \left(\mu T + b\sigma^2 T \right)^2 \right] N(-d) \\
&\quad + e^{-rT} \beta\sigma\sqrt{T} \left[\sqrt{\frac{K - \alpha}{\beta}} + \left(\mu T + b\sigma^2 T \right) \right] n(-d),
\end{aligned}
\tag{15}
$$

where the steps are similar to those used to price the call. Equation (15) yields the current price of the put in equilibrium.

Equations (14) and (15) depend implicitly, but not explicitly, on the current stock price (13).[3] The next proposition presents the equilibrium prices of the call and the put as explicit functions of the current stock price.

Proposition 1. (The Simple square root option pricing model) *Assume that the representative agent has a generalized logarithmic utility function, that aggregate wealth has a displaced lognormal distribution, and that the stock price has a square root distribution given by definitions 1, 2, and 3 respectively. Then the simple square root option pricing model (SSROPM) is given by:*

$$
\begin{aligned}
C_0 &= \left(S_0 - Ke^{-rT} \right) N(d) \\
&\quad + e^{-rT} \beta\sigma\sqrt{T} \left[\sqrt{\frac{K - \alpha}{\beta}} + \sqrt{\frac{S_0 e^{rT} - \alpha}{\beta} - \sigma^2 T} \right] n(-d),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
P_0 &= \left(Ke^{-rT} - S_0 \right) N(-d) \\
&\quad + e^{-rT} \beta\sigma\sqrt{T} \left[\sqrt{\frac{K - \alpha}{\beta}} + \sqrt{\frac{S_0 e^{rT} - \alpha}{\beta} - \sigma^2 T} \right] n(-d),
\end{aligned}
\tag{17}
$$

where:

$$
d = \frac{\sqrt{\frac{S_0 e^{rT} - \alpha}{\beta} - \sigma^2 T} - \sqrt{\frac{K - \alpha}{\beta}}}{\sigma\sqrt{T}},
$$

---

[3]If we use (14) and (15) in the call-put parity we also obtain the current equilibrium stock price (13). Hence (13) is implicit in (14) and (15).

$N(\cdot)$ *is the cumulative distribution function of a standard normal random variable, $n(\cdot)$ is the density function of a standard normal random variable, K is the strike price, and T the maturity of the options.*

Proof: See the Appendix.

Equations (16) and (17) yield the equilibrium prices of the call and put options as functions of the current stock price S_0, the strike price K, the maturity of the options T, the interest rate r, the lower bound of the stock α, the volatility of the stock σ, and the rescale parameter of the stock β. It is interesting to observe that these option pricing equations depend neither on the drift of the stock μ nor on any parameter of the utility function or distribution of wealth. In this sense the formulas (16) and (17) are preference-free like the Black-Scholes option pricing model (BSOPM).

Equation (11) yields the objective pdf of the square root distribution of the stock price. We made assumptions about the objective pdf (11) to derive option prices. However, this is not the pdf implicit in the option pricing equations (16) and (17). The risk-neutral pdf implicit in equations (16) and (17) is given by:

$$f(S_T) = \frac{1}{\sqrt{2\pi}\,2\sigma\sqrt{T}\sqrt{\beta(S_T - \alpha)}} exp\left\{ -\frac{1}{2\sigma^2 T}\left[\sqrt{\frac{S_T - \alpha}{\beta}} - \sqrt{\frac{S_0 e^{rT} - \alpha}{\beta}} - \sigma^2 T \right]^2 \right\}.$$

The risk-neutral square root distribution implicit in option prices places a restriction on parameters i.e. $S_0 > e^{-rt}\left[\alpha + \beta\sigma^2 T\right]$. Option prices also place the restriction $K > \alpha$.

3. Data

We use the options written on the S&P 500 index to empirically investigate the properties of our option pricing model, the SSROPM. Therefore, the primary data are the daily option prices (closing mid-quotes) for the period from 1996 to 2005, which are collected from OptionMetrics Ivy DB.

As inputs of our option pricing model, the daily closing levels and dividend yields of the

9

S&P 500 index are directly provided along with the option prices. The risk-free interest rates are calculated from linearly interpolating the zero curves formed by a collection of continuously-compounded zero-coupon interest rates; that is, the horizons of the risk-free interest rates exactly match the time-to-maturities of the options. We then get rid of the dividend effect from our option prices by using the ex-dividend S&P 500 index levels as the underlying asset prices in our model.

Although call and put option prices for the same strike and maturity should essentially contain the same level of information, only out-of-money options are used since out-of-money options are usually traded more heavily than in-the-money options. Moreover, the option prices are filtered using some additional criteria. Firstly, we exclude those option prices that violate any arbitrage-free bounds. Secondly, we exclude prices of options with time-to-maturities less than seven calendar days in order to avoid the liquidity issue imposed on the short-maturity options. In addition, we also ignore those options with prices less than 0.5 as these option prices could be insensitive to the information flow.

In order to simplify the analysis, firstly the prices of out-of-money put options are converted into prices of call options by the put-call parity, and then all (equivalent) call options are grouped by the moneyness sectioned at 0.97 and 1.03 and/or by the time-to-maturity sectioned at 90 and 180 days. The averaged option prices and the numbers of observations for all kinds of groups are presented in Table 1. In total we have 383,971 observations and short-term out-of-money put options take the largest proportion. Since we have filtered out those unreasonable option prices by the criteria mentioned earlier, the pattern of our price surface in terms of call options is consistent with the general sense that the call price is negatively related to the moneyness and positively associated with the time-to-maturity.

4. Empirical Results

In this section, the empirical properties of our SSROPM are compared with those of the Black-Scholes model. Firstly, we investigate the properties of parameters estimated from the option market prices by minimizing the sum of squared differences between theoretical and market prices of (equivalent) call options with the same trading and maturity dates. Secondly, given the estimated parameters, we analyze the fitting errors of all option prices. Finally, we investigate the patterns of implied volatility functions converted from the fitted theoretical option prices.

4.1. Estimation and Properties of Parameters

With the market prices of (equivalent) call options with the same trading and expiration dates, the three parameters of our SSROPM (16), α, β and σ, are estimated for each set of market data by minimizing the following loss function:

$$\sum_{i=1}^{n}(C(K_i) - c(K_i|\alpha, \beta, \sigma))^2,$$

where C_i and c_i denote the market and theoretical call prices, respectively, and n is the number of option contracts in the set of market data with different strike prices K_i . The only unknown parameter of the Black-Scholes model, σ-BS, is also estimated by the same approach.

The summary statistics of all parameter estimates across time-to-maturities are presented in Table 2. The parameter α is the minimum price level possible at the expiration day of an option. Hence, the estimates of α for medium and long maturities are identical to 0 without any exception; that is, the underlying asset has a true lower bound at zero for a longer horizon although the chance of approaching that value is very small. By contrast, since the possible price change interval for a short horizon is much narrower, the α estimates for some very short maturities such as 7 or 8 days can be far from 0, with a maximum as high as 1260.18.

Because β is a rescaling parameter, essentially its value depends on the price level of the

11

underlying asset. Therefore, regardless of which maturities are considered, the β estimates are tightly related to the corresponding S&P 500 index levels and show no significant difference across time-to-maturities.

The variation parameter in our model, σ, describes the volatility of the distribution of the underlying asset price at expiration, but does not equal the so-called standard deviation. Therefore, essentially it is different from that variation parameter in the Black-Scholes model, σ-BS, although both are volatility measures. In general, the σ estimate is lower than the σ-BS estimate by about 0.04; the mean levels for σ and σ-BS estimates are about 0.16 and 0.20, respectively. Moreover, we find no significant difference among the results for different time-to-maturities, which is consistent with the general sense for an asset price dynamic that the price volatility should not rely on the modeling horizon.

In summary, the three parameters have their own economic meaning and their estimates generated from the S&P 500 index options are consistent with our intuition.

4.2. Fitting Errors of Option Prices

Given the parameter estimates obtained in the previous step, we firstly calculate the theoretical prices of the SSROPM (16) and the Black-Scholes models for all (equivalent) call options observed in the market, and then compute the price fitting errors measured by the absolute differences between the theoretical and market prices for the two models respectively. In addition, we also measure the error in terms of the Balck-Scholes implied volatility. The fitting errors for the two models are summarized across moneynesses and time-to-maturities with the results in terms of price errors and implied volatility errors being presented in Table 3, Panels A and B respectively.

For both models, the price errors are positively associated with the time-to-maturity. Also, for both models, the price errors are smallest for the at-the-money options. However, regardless

of which moenyness and/or time-to-maturity is considered, the price fitting errors of our model are much lower than those of the Black-Scholes model. For example, for the whole sample, the averaged price errors for our and the Black-Scholes models are 3.45 and 4.08, respectively. All these results are also obtained with errors measured by the implied volatility. For example, the averaged volatility errors for our and the Black-Scholes models are 0.0401 and 0.0469, respectively.

Although all the standard deviations of our errors are smaller than those of the Black-Scholes model errors, this might look insufficient to support the robustness of the superiority of our model. Therefore, we further utilize the Wilcoxon's Signed-Rank Test to examine the null hypothesis that the median of our errors (in terms of both price and volatility) is equal to or larger than that of the Black-Scholes model to statistically support our model. The test statistic is:

$$S = \sum_{i=1}^{m} I_+(d_i) rank(|d_i|),$$

where d_i is the fitting error of our model minus that of the Black-Scholes model for the ith option in the sample. $I_+(d_i) = 1$ if $d_i > 0$, otherwise $I_+(d_i) = 0$, and m denotes the number of options in the sample. Moreover, its standardized version is asymptotically standard normal,

$$S_a = \frac{S - \frac{m(m+1)}{4}}{\sqrt{\frac{m(m+1)(2m+1)}{24}}} \sim^a N(0, 1).$$

The test statistics for all intersections of the moeynesses and time-to-maturities are reported in Table 4. In general, all results strongly support the superiority of our model; all statistics are significant at the 1% of significance level. This strong evidence does not rely on the error measure, moeyness and time-to-maturity.

Like the Black-Scholes model, our option pricing model also has an elegant formula and is simple to empirically implement. With two more parameters, our model has much smaller pricing errors.

4.3. Patterns of Implied Volatility Functions

We select four horizons, 30, 60, 120, and 240 days, to investigate the patterns of implied volatility functions of the SSROPM and the Black-Scholes model. Firstly, we use the averages of those parameters estimated in the first step as the inputs of the two alternative models. Secondly, we use the parameter averages to compute the theoretical call prices with various strike prices in terms of moneyness defined by the strike price over the forward price. Finally, all theoretical prices are converted into Black-Scholes implied volatilities, which are plotted against their moneynesses for the four different time-to-maturities in Figure 1.

Without any surprise, the volatility functions generated by the Black-Scholes model are always a flat line as this is one of the assumptions for the model. By contrast, the implied volatility curves produced by our model lead to negatively sloped lines, which are much closer to the volatility smirk observed in the marketplace. The reasonableness of our implied volatility function holds across different time-to-maturities. Authors that show that the implied volatilities across strike prices in the S&P500 index option market prices form a skew or a smirk include Rubinstein (1994), Ait-Sahalia and Lo (1998), and Dumas, Fleming, and Whaley (1998).

In summary, the more reasonable implied volatility function of the SSROPM and the significantly smaller option price fitting error of the SSROPM support the superiority of our model over the Black-Scholes model. Hence, we propose an option pricing model which is also fairly easy to implement and much more realistic than the Black-Scholes model.

5. Concluding Remarks

The attractiveness of the Black-Scholes model comes not only from the fact that it is a complete equilibrium formulation of the option pricing problem, but also that the final formula is a function of

observable variables. Rather than assuming that the logarithm of the stock price follows a normal distribution, we assume that the square root of the stock price follows a normal distribution, and price options in a simple general equilibrium economy with a representative agent who has a generalized logarithmic utility function. We therefore derive a new simple square root option pricing model (SSROPM). Compared to the Black-Scholes model, our model has two more parameters, but is still easy to implement. Moreover, our model produces significantly smaller fitting errors of option prices and generates a negatively sloped implied volatility function. Hence, the SSROPM solves some pricing biases present in the Black-Scholes option pricing model. Although many studies have extended the Black-Scholes model in several other directions, their option pricing formulae are much more complicated and hard to implement in practice. Our paper contributes to literature by proposing a realistic alternative to the Black-Scholes model for option pricing.

Appendix A

Proof of Lemma 1: The marginal utility function is $U'(W_T) = \left(W_T + \frac{A}{C}\right)^{-1}$. Since W_T has a displaced lognormal distribution $W_T \sim \Lambda\left(-\frac{A}{C}, \mu_w T, \sigma_w^2 T\right)$ then the marginal utility function has a standard lognormal distribution $U'(W_T) = \left(W_T + \frac{A}{C}\right)^{-1} \sim \Lambda\left(-\mu_w T, \sigma_w^2 T\right)$. This means that $\ln U'(W_T) \sim N\left(-\mu_w T, \sigma_w^2 T\right)$ with $E\left[\ln U'(W_T)\right] = -\mu_w T$ and $Var\left[\ln U'(W_T)\right] = \sigma_w^2 T$. Therefore:

$$E\left[U'(W_T)\right] = exp\left(-\mu_w T + \frac{1}{2}\sigma_w^2 T\right). \tag{18}$$

By definition 3, equation (10), $\sqrt{\frac{S_T - \alpha}{\beta}}$ has a normal distribution $\sqrt{\frac{S_T - \alpha}{\beta}} \sim N\left(\mu T, \sigma^2 T\right)$. Since $\ln U'(W_T)$ and $\sqrt{\frac{S_T - \alpha}{\beta}}$ are correlated they are necessarily joint normal, and we can write the linear regression:

$$-\ln\left(W_T + \frac{A}{C}\right) = a + b\sqrt{\frac{S_T - \alpha}{\beta}} + \epsilon,$$

where $\sqrt{\frac{S_T - \alpha}{\beta}}$ is independent of ϵ with $\epsilon \sim N(0, \sigma_\epsilon^2)$. Hence:

$$E\left[-\ln\left(W_T + \frac{A}{C}\right)\right] = a + b\mu T \quad and \quad Var\left[-\ln\left(W_T + \frac{A}{C}\right)\right] = b^2\sigma^2 T + \sigma_\epsilon^2.$$

Since $E\left[-\ln\left(W_T + \frac{A}{C}\right)\right] = -\mu_w T$ then $a = -\mu_w T - b\mu T$, and we can write the conditional expectation of the marginal utility function:

$$\begin{aligned} E\left[-\ln\left(W_T + \frac{A}{C}\right) \mid S_T\right] &= a + b\sqrt{\frac{S_T - \alpha}{\beta}} \\ &= -\mu_w T - b\mu T + b\sqrt{\frac{S_T - \alpha}{\beta}}. \end{aligned}$$

Since $Var\left[-\ln\left(W_T + \frac{A}{C}\right)\right] = \sigma_w^2 T$ then $\sigma_\epsilon^2 = \sigma_w^2 T - b^2\sigma^2 T$, and we can write the conditional variance of the marginal utility function:

$$\begin{aligned} Var\left[-\ln\left(W_T + \frac{A}{C}\right) \mid S_T\right] &= \sigma_\epsilon^2 \\ &= \sigma_w^2 T - b^2\sigma^2 T. \end{aligned}$$

Then the expected value of the conditional marginal utility function is:

$$E\left[U^{'}(W_T) \mid S_T\right] = exp\left[-\mu_w T - b\mu T + b\sqrt{\frac{S_T - \alpha}{\beta}} + \frac{1}{2}\left(\sigma_w^2 T - b^2 \sigma^2 T\right)\right]. \tag{19}$$

Substituting equations (18) and (19) into the definition of the asset specific pricing kernel given by equation (3) yields the desired result (12). Moreover, since $Cov\left[-\ln\left(W_T + \frac{A}{C}\right), \sqrt{\frac{S_T - \alpha}{\beta}}\right] = b\sigma^2 T$ then $b = -\rho\sigma_w/\sigma.\square$

Proof of Proposition 1: We write from (13):

$$S_0 e^{rT} = \alpha + \beta\sigma^2 T + \beta\left(\mu T + b\sigma^2 T\right)^2. \tag{20}$$

Substituting (20) into (14) and (15) yields:

$$
\begin{aligned}
C_0 &= \left[S_0 - e^{-rT}K\right]N(d) \\
&+ e^{-rT}\beta\sigma\sqrt{T}\left[\sqrt{\frac{K - \alpha}{\beta}} + \left(\mu T + b\sigma^2 T\right)\right]n(-d), \\
P_0 &= e^{-rT}\left[Ke^{-rT} - S_0\right]N(-d) \\
&+ e^{-rT}\beta\sigma\sqrt{T}\left[\sqrt{\frac{K - \alpha}{\beta}} + \left(\mu T + b\sigma^2 T\right)\right]n(-d),
\end{aligned}
\tag{21}
$$
$$\tag{22}$$

where $d = \frac{\mu T + b\sigma^2 T - \sqrt{\frac{K-\alpha}{\beta}}}{\sigma\sqrt{T}}$. From (20) write:

$$\mu T + b\sigma^2 T = \sqrt{\frac{S_0 e^{rT} - \alpha}{\beta} - \sigma^2 T}.$$

Then use this last expression into (21) and (22) in order to obtain the desired result. $\square$

References

Ait-Sahalia, Y. and A. Lo, 1998, "Nonparametric estimation of state-price densities implicit in financial asset prices", *Journal of Finance*, 53, 499-547.

Bakshi, G., C. Cao, and Z. Chen, 1997, "Empirical performance of alternative option pricing models", *Journal of Finance*, 52, 2003-2049.

Black, F. and M. Scholes, 1973, "The pricing of options and corporate liabilities", *Journal of Political Economy*, 81, 637-654.

Brennan, M., 1979, "The pricing of contingent claims in discrete time models", *Journal of Finance*, 34, 53-68.

Camara, A., 1999, "An extended set of risk neutral valuation relationships for the pricing of contingent claims", *Review of Derivatives Research*, 3, 67-83.

Camara, A., 2003, "A generalization of the Brennan-Rubinstein approach for the pricing of derivatives", *Journal of Finance*, 58, 805-820.

Camara, A., 2008, "Two counters of jumps", *Journal of Banking and Finance*, Forthcoming.

Cox, J. and Ross, S. 1976, "The valuation of options for alternative stochastic processes", *Journal of Financial Economics*, 3, 145-166.

Duffie, D., J. Pan, and K. Singleton, 2000, "Transform analysis and asset pricing for affine jump diffusions", *Econometrica*, 68, 1343-1376.

Dumas, B., J. Fleming, and R. Whaley, 1998, "Implied volatility functions: empirical tests", *Journal of Finance*, 53, 2059-2106.

Heston, S., 1993, "A closed-form solution of options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, 6, 327-343.

Hull, J., 2009, "Options, futures, and other derivatives", *Prentice Hall*, New jersey.

McDonald, R., 2003, "Derivatives markets", *Addison Wesley*, Boston.

Merton, R., 1973, "Theory of rational option pricing", *Bell Journal of Economics and Management*

Science, 4, 141-183.

Merton, R., 1976, "Option pricing when underlying stock returns are discontinuous", *Journal of Financial Economics*, 3, 125-144.

Rubinstein, M., 1976, "The strong case for the generalized logarithmic utility model as the premier model of financial markets", *Journal of Finance*, 31, 551-571.

Rubinstein, M., 1983, "Displaced diffusion option pricing, *Journal of Finance*, 38, 213-217.

Rubinstein, M., 1994, "Implied binomial trees, *Journal of Finance*, 49, 771-818.

Schroder, M., 2004, "Risk-neutral parameter shifts and derivatives pricing in discrete time", *Journal of Finance*, 59, 2375-2402.

Stapleton, R. and S. Subrahmanyam, 1984, "The valuation of multivariate contingent claims in discrete time models", *Journal of Finance*, 39, 207-228.

Table 1: Summary statistics of option market prices

This table consists of the averaged market prices of the S&P 500 call options across moneynesses and time-to-maturities. The figures in parentheses are the numbers of option prices observed in the market for the corresponding moenynesses and time-to-maturities. The prices of in-the-money call options are converted from those of out-of-the-money put options by the put-call parity.

Time-to-maturity / Moneyness	T<90	90≤T≤180	T>180	Total
m<0.97	135.85	215.40	249.41	192.36
	(86668)	(41207)	(69182)	(197057)
0.97≤m≤1.03	27.06	49.45	75.27	39.18
	(52575)	(10391)	(14694)	(77660)
m>1.03	6.32	12.91	22.64	14.36
	(42476)	(21800)	(44978)	(109254)
Total	74.10	131.77	150.40	110.73
	(181719)	(73398)	(128854)	(383971)

* Moneyness is defined as the ratio of a strike price to the forward price of the underlying asset.

Table 2: Summary statistics of the parameter estimates

This table presents the summary statistics of the parameters estimated by minimizing the sum of the differences between the theoretical and market call option prices with the same trading and expiration dates across strike prices. The parameters of both the SSROPM (16) and the Black-Scholes model are estimated and all estimates are summarized across time-to-maturities. σ-BS denotes the Black-Scholes volatility parameter.

Parameter	Statistics	T<90	90≤T≤180	T>180	Total
α	Mean	1.05	0.00	0.00	0.49
	Std Dev	30.51	0.00	0.00	20.96
	Max	1260.18	0.00	0.00	1260.18
	Min	0.00	0.00	0.00	0.00
β	Mean	445.90	453.24	449.07	448.33
	Std Dev	119.94	125.98	129.72	124.54
	Max	772.96	774.48	766.41	774.48
	Min	158.09	174.29	171.07	158.09
σ	Mean	0.15	0.16	0.16	0.16
	Std Dev	0.04	0.03	0.03	0.03
	Max	0.48	0.30	0.29	0.48
	Min	0.09	0.10	0.11	0.09
σ-BS	Mean	0.20	0.20	0.20	0.20
	Std Dev	0.06	0.05	0.04	0.05
	Max	0.45	0.39	0.38	0.45
	Min	0.09	0.11	0.11	0.09

Table 3: The summary statistics of fitting errors of option prices and implied volatilities

This table consists of the mean and standard deviation (in parentheses) of fitting errors of option prices (Panel A) and implied volatilities (Panel B) for our square root (SR) and the Black-Scholes (BS) option pricing models. All the errors are summarized across moneynesses and time-to-maturities.

Panel A: Errors of Option Prices

Time-to-maturity / Moneyness	T<90		90≤T≤180		T>180		Total	
	SR	BS	SR	BS	SR	BS	SR	BS
m<0.97	2.46	2.69	4.07	4.74	5.68	7.15	3.92	4.69
	(1.73)	(1.96)	(2.75)	(3.30)	(3.52)	(4.47)	(3.04)	(3.85)
0.97≤m≤1.03	1.22	1.28	1.77	1.89	2.33	2.57	1.50	1.61
	(1.03)	(1.08)	(1.47)	(1.53)	(1.98)	(2.17)	(1.39)	(1.50)
m>1.03	2.30	2.57	4.17	4.83	5.52	6.76	4.00	4.75
	(1.67)	(1.81)	(2.61)	(2.87)	(3.45)	(4.06)	(3.07)	(3.64)
Total	2.06	2.25	3.77	4.36	5.24	6.49	3.45	4.08
	(1.64)	(1.82)	(2.69)	(3.14)	(3.51)	(4.36)	(2.97)	(3.66)

Panel B: Errors of Implied Volatilities

Time-to-maturity / Moneyness	T<90		90≤T≤180		T>180		Total	
	SR	BS	SR	BS	SR	BS	SR	BS
m<0.97	0.0699	0.0772	0.0622	0.0745	0.0471	0.0606	0.0603	0.0708
	(0.0501)	(0.0551)	(0.0453)	(0.0540)	(0.0341)	(0.0430)	(0.0452)	(0.0515)
0.97≤m≤1.03	0.0102	0.0107	0.0071	0.0076	0.0061	0.0068	0.0090	0.0096
	(0.0086)	(0.0089)	(0.0054)	(0.0057)	(0.0047)	(0.0051)	(0.0078)	(0.0081)
m>1.03	0.0269	0.0301	0.0278	0.0326	0.0240	0.0298	0.0259	0.0305
	(0.0180)	(0.0189)	(0.0152)	(0.0174)	(0.0125)	(0.0159)	(0.0155)	(0.0175)
Total	0.0426	0.0470	0.0442	0.0526	0.0343	0.0437	0.0401	0.0469
	(0.0449)	(0.0494)	(0.0410)	(0.0490)	(0.0300)	(0.0382)	(0.0399)	(0.0460)

Table 4: The results of Wilcoxon's Signed-Rank Test

This table consists of the statistics of Wilconxon's Signed-Rank Test for the null hypothesis that the median of fitting errors from our square root (SR) model is equal to or larger than that from the Black-Scholes (BS) model. The errors are measured in terms of both price and implied volatility (in parentheses). The results are summarized across moneynesses and time-to-maturities.

Time-to-maturity \ Moneyness	T<90	90≤T≤180	T>180	Total
m<0.97	-253.41	-174.31	-226.11	-380.87
	(-254.41)	(-175.72)	(-227.66)	(-383.96)
0.97≤m≤1.03	-100.04	-23.03	-30.19	-86.34
	(-110.78)	(-26.30)	(-31.77)	(-106.60)
m>1.03	-177.20	-126.65	-180.62	-279.80
	(-174.47)	(-125.67)	(-180.83)	(-280.97)
Total	-326.15	-216.26	-294.29	-484.46
	(-350.22)	(-227.58)	(-303.32)	(-515.47)

* All test statistics are significant under the 1% significance level.

Figure 1: Black-Scholes implied volatility of fitted option prices

This figure presents the Black-Scholes implied volatility converted form the fitted option prices across four time-to-maturities. The implied volatility functions of our model (squares) are compared to those of the Black-Scholes model (triangles).