The Term Structure of Interest Rates in an Equilibrium Economy with Short Term and Long Term Investments

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Abstract

This paper develops an equilibrium model in which agents' heterogeneous investment horizons determine the dynamics of the real term structure of interest rates. The model endogenizes agents' decisions on consumption and investment with short and long term horizons. There are two production technologies that generate a time-varying market price of risk, one that is short term and fully reversible and one that is a long term time-to-build technology. The model is calibrated with U.S. data from 1970 to 2007 using Simulated Method of Moments and captures several results and stylized facts, such as: (i) the excess returns on short and long term investments together with a low volatility of consumption using a reasonably low risk aversion parameter; (ii) a low correlation between long term investments (e.g. direct investments in real estate) and short term investments; and (iii) the slightly positive slope of the real term structure of interest rates.

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1 Introduction

Similar to all the prices in a market economy, interest rates are determined by an equilibrium of supply and demand forces. Particularly, interest rates are the rates that equilibrate the supply and the demand for credit. If the demand of credit rises relative to the supply, the interest rates will tend to increase as borrowers compete for obtaining scarce funds. The main source of the demand for credit arises from the willingness of individuals, firms, and governments to consume and invest in the short run. If the supply of credit from lenders rises relative to the demand from borrowers, interest rates will tend to decrease as lenders compete for allocating these funds. The main source of the supply of credit arises from savings, or the willingness to delay consumption and investments[1].

The time to maturity of credit adds a new dimension to this analysis. The equilibrium interest rates for lending and borrowing at short maturities can be different from the rates for lending and borrowing at long maturities because the main sources of demand and supply of credit at different maturities may be different. The term structure of interest rates is the curve of interest rates with respect to the time to maturity of the debt and, therefore, it contains information on the main sources of demand and supply of credit (e.g. consumption, and investments at different time horizons).

This paper proposes and develops a general equilibrium model to study the relationship among consumption, investments and the real term structure of interest rates. In this model, the role of heterogeneous investment horizons (e.g. short versus long term investment) is an essential determinant of the joint dynamics of consumption, investments and the term structure of interest rates. Specifically, the term structure of interest rates is endogenously determined from the representative agent's decisions on consumption of real assets and investment (e.g. production of real assets) with short and long term horizons.

1.1 Introduction to the model and the economic intuition

The model developed in this paper is a fundamental extension of the Cox, Ingersoll and Ross (CIR, 1985a and 1985b) model. It introduces several production technologies associated with different investment horizons. For simplicity, I consider the problem with just two investment horizons: short and long term. Consequently, there are two production technologies in this economy[2]: (i) a *short term* fully reversible technology that offers constant returns to scale as in the CIR model; and (ii) a *long term* technology that requires multiple periods (say τ years) to build new capital goods, as in Kydland and Prescott (1982). There is one real asset that can be produced by any of

[1]The Federal Reserve and depository institutions (e.g. banks, credit unions) have an important influence in the supply of credit. In this paper, I will abstract from this influence and focus on the real interest rates that comes from the equilibrium between supply and demand for credit providing the set of consumption and investment opportunities that the economic agents face.

[2]Hirshleifer (1972) and Diamond and Dybvig (1983) are two antecedents to my model that present a similar two production technologies setup.

these two production technologies[3].

I assume that capital is the only input used in both production technologies [4]. Consequently, I define short (long) term investments as the investments made in the production of the assets through the short (long) production technologies. I also define returns on short (long) term investments as the returns associated with short (long) term investments. Note that short term investments are fully reversible. However, the capital allocated to long term investment at time t is illiquid for τ years and, therefore, this capital cannot be consumed or reinvested in any of the technologies until time $t + \tau$.

The source of illiquidity in the model is related to the length of the period to maturity τ of the assets produced using the long term technology. I assume that these assets present a "point-input point-output" payment pattern [5]. Prototypical examples of these assets are trees grown for timber, but not trees grown for fruit. Several types of assets (e.g. assets that require time to build such as buildings) can be simplified to a "point-input point-output" payment pattern[6]: there is an investment made at time t or "point-input" (e.g. payment of the construction costs on the first day of construction of the building) and all the stochastic returns on the investment are collected at time $t + \tau$ or "point-output" (e.g. the new building pays off the present value of all the future rents). As a result, in my model, illiquidity is the inability to rebalance past consumption and investment plans when more accurate information becomes available.

I assume an infinitely lived representative agent with preferences for consumption. At each period, the agent has to reassess how much capital he consumes, how much capital he allocates to short term investments (e.g. short term "reversible" production) and how much capital he allocates to long term investments (e.g. long term "time to build" production). I study the equilibrium implications of these decisions on the endogenously generated term structure of interest rates. The model and supporting empirical calibration suggest that investors demand high risk premia for holding long term investments for three reasons. First, physical irreversibility in the long term production technology prevents the agents from incorporating new available information about the economy over time. Therefore, undertaking a long term production project requires that the agents

[3] Therefore, I do not consider the research and development (R&D) associated to the short term technology, but the production of the good given the available technology. In my model, R&D would be a long term technology that provides a new asset. R&D needs τ periods to develop this new asset. Although R&D is an important part of the economy, I am not accounting for it because of the unavailability of consistent data.

[4] I make two important assumptions regarding the short term and the long term production technologies. Firstly, I consider constant returns to scale. Secondly, I assume that there is a permanent labor surplus rate or, alternatively, that labor is unnecessary in production. The study can be expanded to include nonlinear technologies or labor inputs without any fundamental complications.

[5] Imagine that we are studying real estate assets. A building is a long term asset that can be "physically" illiquid but "market" liquid at the same time. A building under construction can be a very "physically" illiquid asset in the sense that the physical benefits (e.g. capital from rents or from selling apartments or offices) will be obtained several years from now. However, the owner of this building can issue securities on the building and sell them for cash. The total price of these securities is the discounted value of the future value of the building at maturity, that is, the present value of all the future rents. When these securities exist, the long term asset is "market" liquid.

[6] The concept of equity duration developed in Cornell (2000), Dechow, Sloan, and Soliman (2004), Santa-Clara (2004) and Lettau and Wachter (2007a, 2007b) may be applied here. They define *zero-coupon equity* as equity with the cash flow pattern of a zero-coupon bond. Equity duration is the duration (maturity) of its equivalent zero-coupon equity. This literature uses this concept to study the response of stock returns to interest rate shocks.

give up the option value of delaying the decision, and the rate of return must be high enough to compensate them for giving up that option value. Second, returns on investments in the long term production technology are higher than returns on investments in the short term production technology. Hence, the representative agent is willing to pay a higher interest rate on long term borrowing because the returns on long term investments are higher. Third, investors are risk averse and care about consumption in the long run.

Finally, I focus on the main type of long term investments: real estate investments. The capital invested in real estate (both residential and nonresidential) represents, on average, 84.1% of the total long term investments in the U.S. in the period 1970-2007[7]. Real estate depreciates slowly, has a long service life and represents the canonical time-to-build production technology[8]. The model proposed in this paper provides a setup to study the dynamics of real estate investments as long term investments. It is different from previous models in the literature because consumption, investments in real estate and the term structure of interest rates are all endogenously determined.

1.2 Calibration of the model and the results

The model is calibrated with U.S. data from 1970 to 2007 using the Simulated Method of Moments to obtain estimates of short and long term fixed assets and investments, the level of consumption, and the term structure of interest rates. I use economic aggregate U.S. data for: (i) fixed assets (stocks of capital) from the Fixed Asset Tables of the National Economic Accounts; (ii) consumption and investments (flows of capital) from the National Income and Product Accounts (NIPA); (iii) the yield curve from the Federal Reserve; and (iv) data on inflation measured by the CPI in order to deflate nominal units to real units.

I separate the fixed assets and the investment accounts provided by the Fixed Asset Tables and the NIPA accounts into short and long term accounts by using the estimations of economic depreciation in Hulten and Wykoff (1981a, 1981b) and Fraumeni (1997)[9]. Then, I aggregate the various classes of short accounts into a short term capital account and the various classes of long accounts into a long term capital account. In the model, the short term capital account contains the liquid capital (the capital that is available for consumption and any kind of investment) and the long term capital account contains the illiquid capital (the capital locked into long term investments).

[Compare empirical results to Ang, Bekaert and Wei(2008) here!]

The calibration of the model captures several stylized facts of the literature.

Firstly, the model captures the fact that long term interest rates are comprised of investors' assumptions about future interest rates and a premium for holding long-term bonds, called the liquidity premium. This premium rewards investors for the additional risk of having their money

[7] It ranges from 75% in 1981Q4 to 88.6% in 2005Q3.

[8] These physical characteristics make the elasticity of real estate supply low in response to a decline in real estate demand (see Glaeser and Gyourko (2005)).

[9] Private investments in fixed assets with high rates of depreciation (e.g. information processing equipment and software) have been included in the short term accounts. Private investments in fixed assets with low rates of depreciation (e.g. residential structures) have been included in the long term accounts. Note that this classification implicitly assumes that there is a strong relation between time to build and depreciation

tied up for a longer period. Investors show a preference for liquidity over illiquidity and, therefore, short term rates tend to be lower than long term rates and the term structure slopes upward on average[10].

Secondly, the model provides estimated excess returns on investments in short and long term technologies that are comparable to historical excess returns on investments using a reasonably low risk aversion parameter. Traditionally, equilibrium models of consumption need very high values of the risk aversion parameter to generate the high excess returns that we see in the data. However, recently, Bekaert, Engstrom and Grenadier (2004), Wachter (2006) and Buraschi and Jiltov (2007) developed consumption-based term structure models that produce realistic moments when they are calibrated to real data on both bond and stock markets. Their models are driven by the concept of external habit persistence introduced in Campbell and Cochrane (1999), which generates a time-varying market price of risk[11].

Furthermore, there are other significant issues that the model addresses. First, in contrast to the reduced-form term structure models (See Dai and Singleton (2002); Duffee (2002); Ang and Piazzesi (2003); and Lettau and Wachter (2007)) that impose statistical structures on the market price of risk, in this model the time-varying market price of risk is solved for endogenously without imposing any arbitrary structure on the stochastic discount factor[12]. By endogenizing the market price of risk, my model provides rich economic intuition concerning the channels by which short and long term investments and consumption jointly determine the dynamics of the term structure of interest rates.

Finally, the model suggests that the short term and long term capital outstanding in the economy plays an important role in determining the dynamics of the term structure of interest rates. Figures 1 and 2 show why it might be interesting to use short and long term investments as well as consumption in order to study the dynamics of the term structure of interest rates. Panel A of Figure 1 shows that quarterly changes in consumption are very smooth compared to the real short interest rate suggesting that purely consumption based models would be unlikely to generate reasonable excess returns and interest rates patterns using low risk aversion parameters. Panel B of Figure 1 shows the dynamics of the quarterly changes in short term investments compared to the real short interest rate. Comparing panels A and B of Figure 1, we can see that the short term investment channel is more volatile than consumption[13]. Panel A of Figure 2 shows that periods of low consumption are related to recessions and decreases of the slope of the term structure. Re-

[10]Note this is a supply-side argument due to the uncertainty in aggregate consumption. Lenders prefer to lend short term at lower rates so they can adjust to consumption shocks. This is consistent with the earlier demand-side argument: borrowers get higher rates of return by investing in long term real projects, and thus are willing to pay higher interest rates for long term borrowing. The equilibrium model accounts for both supply and demand arguments.

[11]For more details on a related literature that attempts to simultaneously explain bond and equity returns, see also Bakshi and Chen (1996), Gabaix (2007) and Bansal and Shaliastovich (2007).

[12]The affine term structure models introduced by Duffie and Kan (1996) are the most widely used type of reduced-form model. The common assumption in the affine term structure literature is that the market price of risk is affine.

[13]Note that all the periods of decreasing short term investments (e.g the periods with negative percentage change in short term investments) precede recession periods and/or anticipate decreases of the real short interest rate.

markably, periods of decreasing long term investments appear to be strongly related to periods in which the term structure is downward sloping (see Panel B of Figure 2). Furthermore, Panel B of Figure 2 suggests that periods of decreasing long term investments anticipate recession periods and usually last until the recession is over. Note also that long term investment appears to be the most volatile channel of the model.

1.3 Literature related to short and long term investments

The idea of distinguishing among different investment horizons goes back to Marshall (See Marshall (1926) and Schumpeter (1941)). Based on this distinction, Culbertson (1957) proposed the market segmentation hypothesis that posited that the market supply and demand for short and long terms instruments is determined independently. Although this theory provided an intuitive setup to illustrate the dynamics of the slope of the term structure, it failed to explain the empirical regularity that short and long term interest rates tend to move together.

Modigliani and Sutch (1965) proposed the preferred habitat theory that was formalized by Wachter (2003) and Vayanos and Vila (2007). This approach requires that, in addition to interest rate expectations, investors have different investment horizons and require a meaningful premium to buy bonds outside their preferred habitat or maturity. Different from the above, the investors in my model do not decide whether they are committed to either short or long terms investments, nor ex ante must they select their preferred maturity affiliation. Instead, they make their decisions about how much capital they allocate to short and long term investments over time[14].

There is also a large body of literature that studies illiquidity and asset pricing[15]. Although it does not focus on the term structure of interest rates, it is related to the key assumption of different investment horizons, in the sense that either long term investments cannot be liquidated in certain periods of time or liquidation is costly. My model is built upon the idea of liquid short term investments and illiquid long term investments, and uses some of the intuition provided by this literature.

The remainder of the paper is organized as follows. Section 2 specifies the model. First, a simple version of the central planner's problem for $\tau = 2$ is presented in order to illustrate the main mechanics of the model. The general model will be developed in the second part of this section. Section 3 studies the equilibrium first order conditions (FOCs) of the model and their economic implications. Section 4 uses the equilibrium results in Section 3 to provide intuition behind the dynamics of the term structure. Section 5 explains the details about the data and the calibration

[14] Diamond and Dybvig (1983) exploit a similar mechanism based on short and long term technologies in a model to explain bank runs. Subsequent literature such as Postlewaite and Vives (1987), Chari and Jagannathan (1988), Wallace (1988) and Jacklin (1993) adapted Diamon and Dybvig's model to the study of why bank runs, information in the banking industry, illiquidity and the bank deposits, respectively. Although the goals of this literature are different than the objectives of my paper, the setup of their models provides interesting insights to any equilibrium model with short and long term technologies.

[15] For example, Amihud and Mendelson (1986) used bid-ask spreads as a measure of illiquidity and found that agents with different investment horizons trade assets with different bid-ask spreads. Similarly, Brennan and Subrahmanyam (1996) and Novy-Marx (2007) considered different levels of illiquidity of assets to price the compensation for illiquidity in stock returns. Their empirical results show that this compensation is not very high on average.

and shows the empirical results. Finally, section 6 concludes.

2 The Model

The goal of this section is to set up the central planner's problem, which will be the basic framework to model the term structure of interest rates. This problem is formally motivated by the general equilibrium production economy presented in Appendix A1. The definition of competitive equilibrium for this economy is shown in Appendix A2. This general equilibrium setup is necessary because, in models with perfect financial markets, we can use the solution of the central planner's problem to build upon the competitive equilibrium in a decentralized production economy as in Lucas and Prescott [51], and Prescott and Mehra [52], as shown in Appendix A3.

2.1 General model of the central planner's problem

First, I present the model in discrete time that illustrates the mechanics of the model. I will characterize the problem of the optimal allocation of resources subject to technological constraints, the central planner's problem, for short and long term investments. In this model, it takes τ periods to produce goods using the long term (time to build) technology and obtain the returns on these long term investments. Alternatively, it takes just one period for goods to be produced by the short term (reversible) technology and obtain the returns on these short term investments. Therefore, there are two production technologies to produce the capital good: (i) a short term production technology K that allows the agent to instantaneously produce the good, and (ii) a long term "time-to-build" production technology I that requires τ units of time to complete production.

Let us consider a representative agent who maximizes his expected utility of intertemporal consumption, and has time separable utility $U(c_t)$ and a patience rate of time preference parameter given by ρ. At each period, he must decide: (i) how much capital stock c_t he consumes[16], and (ii) how much capital stock Ψ_t he allocates to long term investments. The capital that is not consumed or invested in the long term is allocated to short term investments. The agent solves the following problem:

$$\max_{c_t, x_{1t}, x_{2t}, \Psi_t} \left\{ E_0 \left[\sum_{t=0}^{\infty} e^{-\rho t} U(c_t) \right] \right\} \tag{1}$$

such that:

$$\Delta K_t = x_{1t} f_K\left(K_t\right) + x_{2t} K_t r_t \Delta t + (1 - x_{1t} - x_{2t}) K_t \frac{\Delta B_t}{B_t} - c_t \Delta t - \Psi_t \Delta t + \Psi_t^{(0)} \Delta t \tag{2}$$

[16] In this model, consumption goods are capital goods. In classical economics, capital is usually one of the three traditional factors of production. The others are land and labor. Goods are considered capital if: (i) they can be used in the production of other goods (they are a factor of production), and (ii) they were produced (e.g. they are not natural resources such as land and minerals).

$$\Psi_t^{(\tau)} = \Psi_t \tag{3}$$

$$\Psi_t^{(\tau-i)} = f_I\left(\Psi_{t-1}^{(\tau-i+1)}\right) \quad \text{for} \quad i = 1, ..., \tau \tag{4}$$

Furthermore, the following non-negativity constraints are imposed:

$$K_t > 0, \quad c_t > 0, \quad \text{and} \quad \Psi_t^{(\tau-i)} > 0 \quad \text{for all } t \text{ and } i = 1, ..., \tau \tag{5}$$

and the irreversibility constraint for long-term investments is:

$$\Psi_t \geq 0 \quad for\ all\ t\ . \tag{6}$$

There are two types of capital accounts in this economy. First, an account K_t for the short term capital that follows the process that is shown in equation (2). This is a liquid account that contains the capital stock that is ready to be consumed or invested. Second, a set of $\tau + 1$ capital subaccounts $\Psi_t^{(\tau-i)}$ for the long term capital that follows the processes shown in equation (3) and the set on τ equations in (4). Both the short and the long term production technologies only need capital K_t and Ψ_t, respectively, as input. Production that comes from short and long term technologies provides returns given by $f_K(K_t)$ and $f_I\left(\Psi_t^{(\tau-i+1)}\right)$, respectively[17].

The long term investment mechanism operates as follows. When the agent invests an amount of capital Ψ_t into the long term investment, then Ψ_t is transferred from the short term capital account in equation (2) to the first long term capital account $\Psi_t^{(\tau)}$ in equation (3). The account $\Psi_t^{(\tau)}$ contains the capital that will mature τ periods from now. The capital allocated into the long term investment one period ago, Ψ_{t-1}^{τ} is moved from the capital account $\Psi_t^{(\tau)}$ in equation (3) to the capital account $\Psi_t^{(\tau-1)}$ in equation (4). Each account $\Psi_t^{(\tau-i)}$ contains the capital that will mature $\tau - i$ periods from now. Finally the capital $\Psi_t^{(0)}$ that was allocated to the long term capital investment τ periods ago as $\Psi_{t-\tau}^{(\tau)}$ has gone through the set of accounts in (4) and it is returning now to the short term capital account K_t which process is shown in (2).

Figure 3 schematically shows how consumption c_t, short term investments in equation (2) and long term investments in (3) and the set of equations (4) evolve over time. To solve this problem, it is necessary to break-up short term capital (top of Figure 3) and long term capital (bottom of Figure 3) into two different types of capital accounts. The aggregated short term capital is represented by the process K_t in equation (2) and the top half of Figure 3.

In addition to the real investment options there are financial securities. The agent is allowed to decide how to allocate capital among the real and the financial assets. In particular, the agent can decide the proportion of short term capital x_{1t} to allocate to the short term production technology $f_K(K_t)$, the proportion of short term capital x_{2t} to allocate to the short risk free rate r_t and, hence,

[17]For simplicity, in the continuous-time general model, I will assume that f_K and f_I follow geometric Brownian motions.

the remaining $(1 - x_{1t} - x_{2t})$ is allocated to securities (e.g. bonds at different maturities) that have price B_t. Since the financial assets are in zero net supply, equilibrium will require setting the rate of return on the financial assets a the right level, so that the representative agent will choose not to invest in them. This argument implies that, because of the zero net supply of the risk free bonds, in equilibrium the following must hold: $x_{1t} = 1$ and $x_{2t} = 0$.

2.2 Simplified model of the central planner's problem

In this second part of the section, I will consider three issues in order to simplify the general model in section 2.1 and make it more tractable.

Firstly, I denote I_t the total amount of capital allocated in long term investments at any time t. Therefore, $I_t = \sum_{i=1}^{\tau} \Psi_t^i$. Using (3) and (4), and assuming that f_I follows the same geometric brownian motion $f_I\left(\Psi_{t-1}^{(\tau-i+1)}\right) = \Psi_{t-1}^{(\tau-i+1)}(1+\mu_I \Delta t + \sigma_I \Delta W_t^I)$ for all maturities of capital i, then it can be shown that the changes in the aggregated account I_t are determined by[18]

$$\Delta I_t = I_t - I_{t-1} = I_{t-1}(\mu_I \Delta t + \sigma_I \Delta W_t^I) + \Psi_t \Delta t - \Psi_t^{(0)} \Delta t \tag{7}$$

Secondly, I assume that there exists a constant λ such that $\Psi_t^{(0)} = \lambda I_t$. This means that the capital amount $\Psi_t^{(0)}$ can be approximated as a fraction λ of the total amount of capital allocated in long term investments I_t.

Thirdly, I generalize the discrete-time setup in section 2.1 to a continuous-time model. Consider an infinite horizon production economy with just one type of nondurable good.

When taking into account these three issues, the model described by (1)-(6) becomes the following simple model in which the representative agent solves the problem:

$$\max_{c_t, x_{1t}, x_{2t}, \Psi_t} \left\{ E_0 \left[\int_0^\infty e^{-\rho t} U(c_t) dt \right] \right\} \tag{8}$$

such that

$$dK_t = x_{1t} K_t \left(\mu_K dt + \sigma_K dW_t^K\right) + x_{2t} K_t r_t dt + (1 - x_{1t} - x_{2t}) K_t \frac{dB_t}{B_t} - c_t dt - \Psi_t dt + \lambda I_t dt \tag{9}$$

$$dI_t = I_t \left(\mu_I dt + \sigma_I dW_t^I\right) + \Psi_t dt - \lambda I_t dt \tag{10}$$

with the non-negativity and irreversibility constraints for all t:

$$K_t > 0, \quad c_t > 0, \quad I_t > 0, \quad and \quad \Psi_t \geq 0. \tag{11}$$

Notice that with this simplification, the model is pure Markov and, therefore, I do not have to

[18]For simplicity, I assume constant returns-to-scale risky production technologies and that capital (but not labor) is needed as an input to produce the good. Therefore, f_K and f_I are characterized by their drifts, μ_K and μ_I and volatilities, σ_K and σ_I, respectively. This simple characterization will be useful later in the paper in order to provide economic intuition.

keep track of all the investments made in the past τ periods. I just have to keep track of the two state variables K_t and I_t.

3 Equilibrium

This section studies the equilibrium of the representative agent problem described by (8)-(11). The term structure of the economy described in Section 2 will be obtained in equilibrium. In this section, I will calculate the first order conditions (FOCs) and provide economic intuition about consumption, investment decisions, and the term structure of interest rates

3.1 Preliminary implications in equilibrium

The central planner's problem that I develop in section 2 is motivated by the general equilibrium production economy developed in Appendixes A1 and A2. Appendix A3 sketches the implications for the competitive equilibrium. The main equilibrium results in this appendix are summarized in the following two lemmas.

Lemma 1 *There exists a competitive equilibrium with dynamically complete markets. The set of stochastic processes* $\{K_t^*, I_t^*, c_t^*, \Psi_t^*\}$ *is determined as the solution of the central planner's problem in (8)-(11).*

Lemma 2 *The equilibrium of the economy described above by (8)-(11) is such that the optimal portfolio of the representative agent in equilibrium is determined by* $x_{1t} = 1$ *and* $x_{2t} = 0$.

Lemma 1 uses the findings in Anderson and Raimondo (2007) who give conditions to ensure that equilibrium is dynamically complete in a related, but not identical model. Lemma 2 states the market clearing condition in equilibrium. Therefore, the sum of the total amount of lending and borrowing must be zero. This implication from the lemma means that all the bonds in the economy are in zero net supply.

3.2 The value function

The problem for the economy described by (8)-(11) is Markov because the setup includes state variables for the past τ units of time. At each time t, the problem includes τ state variables related to long term investments from time $t-\tau$ to time t, plus one state variable related to short term investments. Therefore, $\tau + 1$ state variables are involved in this problem.

The simplified model described in Section 2.2 by by (8)-(11) leads to a two state variable problem since the τ state variables related to long term investments collapse into one. Under the simplified model setup, define the value function J for this problem as:

$$J = J(K_t, I_t, t) = \max_{c_t, x_{1t}, x_{2t}, \Psi_t} \left\{ E_0 \left[\int_0^\infty e^{-\rho t} U(c_t) dt \right] \right\}. \tag{12}$$

Note that the value function J just depends on: (i) the short term capital account K, and (ii) the long term capital account I. The state space of the problem $\{K_t, I_t\}$ is divided into two regions: the no-investment and the investment region[19]. When the pair $\{K_t, I_t\}$ is inside the no-investment region, the agent consumes from the good, but makes no new long term investment. When the pair $\{K_t, I_t\}$ is inside the investment region, the agent consumes from the good and makes a new long term investment. In the no-investment region, I have that $J_K > J_I$, while in the investment region the equality $J_K = J_I$ holds.

3.3 The Hamilton-Jacobi-Bellman equation

The solution of the agent's optimal control problem defined by (8)-(11) satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max \Bigg\{ \underbrace{\sup_{c_t, x_{1t}, x_{2t}, \Psi_t} \left\{ E_0 \left[\widehat{dJ^*} + e^{-\rho t} U(c_t) dt \right] \right\}}_{No-investment\ region}, \underbrace{J_K - J_I}_{\substack{Investment \\ region}} \Bigg\} = 0 \tag{13}$$

where J_K and J_I are partial derivatives of $J(K_t, I_t, t)$ with respect to K_t and I_t respectively and $\widehat{dJ^*}$ is represented by the following expression:

$$\widehat{dJ^*} = J_t + \left[x_{1t} K_t \mu_K + x_{2t} K_t r_t + (1 - x_{1t} - x_{2t}) K_t \frac{dB_t}{B_t} - c_t + \lambda I_t \right] J_{K_t} + \frac{1}{2} \left[x_{1t}^2 K_t^2 \sigma_K^2 \right] J_{K_t K_t} +$$

$$+ \left[I_t \mu_I - \lambda I_t \right] J_I + + \frac{1}{2} \left[I_t^2 \sigma_I^2 \right] J_{I_t I_t} + \left[x_{1t} \sigma_L \sigma_I \rho_{LI} K_t I_t \right] J_{KI}$$

These no-investment and investment regions introduced in the last subsection correspond to the two parts within the maximization function in equation (13). When the first part of this maximization problem binds, then the pair $\{K_t, I_t\}$ is inside the no-investment region and, therefore, $\Psi_t = 0$. Alternatively, when the second part of this maximization problem binds, then the pair $\{K_t, I_t\}$ is inside the investment region and, consequently, $\Psi_t > 0$.

3.4 Obtaining the FOCs (1): Envelope condition

By taking derivatives of the HJB (Hamilton-Jacobi-Bellman) equation with respect to c_t, I obtain the following FOC:

$$J_{K_t} = e^{-\rho t} U_{c_t}(c_t) \tag{14}$$

This condition establishes the equilibrium tradeoff between consumption today and consumption next period, meaning that there is an equilibrium between:

- the discounted marginal gain or loss in utility $e^{-\rho t} U_{c_t}(c_t)$ from consuming one more unit at time t, and

[19]See Kogan (2001), Mamaysky (2001), Casassus, Collin-Dufresne and Routledge (2005) and Kogan, Livdan and Yaron (2006) for similar problems based on irreversible investments.

- the marginal gain or loss J_{K_t} from investing this unit either in the short term or in the long term account and therefore, increasing the short term capital account K_t or the long term capital account I_t, which will have an effect on future utilities.

Consequently, in equilibrium, the representative agent must be indifferent among the following decisions: (i) consuming an extra unit of capital today; (ii) investing an extra unit of capital into the short term technology (and, therefore, being able to consume some extra capital next period which amount will depend on the shock dW_t^K and the parameters μ_K, and σ_K); and (iii) investing an extra unit of capital into the long term technology (and, therefore, being able to consume some extra capital next period which amount will depend on the shock dW_t^I and the parameters μ_I, σ_I, and λ).

Using the expression of stochastic discount factor (SDF) of this economy studied in Appendix A3 and given by $M_{t,s} = e^{-\rho(s-t)}\frac{U_c(c_s^*)}{U_c(c_t^*)}$, for $s > t$, we can obtain the following Euler equation:

$$E_t\left\{e^{-\rho(s-t)}\frac{U_{c_t}(c_s)}{U_{c_t}(c_t)}R_{t,s}\right\} = 1 \ . \tag{15}$$

where $R_{t,s}$ denotes the return at time t of a security that matures at time s, with $s > t$. Note that this equation is equivalent to the condition $e^{-\rho s}E_t\left\{U_{c_t}(c_s)R_{t,s}\right\} = e^{\rho t}U_{c_t}(c_t)$, which means that there is an equilibrium between:

- the marginal gain or loss in utility $e^{-\rho t}U_{c_t}(c_t)$ from consuming one more unit at time t, discounted at time zero, and
- the expected discounted marginal gain or loss in utility $e^{-\rho s}E_t\left\{U_{c_t}(c_s)R_{t,s}\right\}$ from investing this unit in a security associated to a maturity of $t - s$ and, therefore, being able to enjoy extra consumption at time s when this long term investment pays off.

3.5 Obtaining the FOCs (2): Equilibrium short term interest rate

By taking derivatives of the HJB (Bellman) equation with respect to x_{1t}, I obtain the following FOC:

$$0 = J_{K_t}K_t\left(\mu_K - \frac{dB_t}{B_t}\right) + J_{K_tK_t}x_{1t}K_t^2\sigma_K^2 + J_{K_tI_t}K_tI_t\sigma_K\sigma_I\rho_{KI}. \tag{16}$$

Considering that in equilibrium $x_{1t} = 1$ and $x_{2t} = 0$ for all t (from Lemma 2), then I can obtain the following expression for the equilibrium short term interest rate:

$$r_t = \mu_K + \sigma_K^2\underbrace{\left[\frac{J_{K_tK_t}}{J_{K_t}}K_t\right]}_{\text{Risk aversion}} + \underbrace{\sigma_K\sigma_I\rho_{KI}\left[\frac{J_{K_tI_t}}{J_{K_t}}I_t\right]}_{\substack{\text{Hedging term for}\\ \text{long term investments}}} \ . \tag{17}$$

Let us define the following variables for notational purposes:

$$\Theta_K = J_{K_t}K_t$$

$$\Theta_{KK} = J_{K_tK_t}K_t^2$$

$$\Theta_{KI} = J_{K_tI_t}K_tI_t \ .$$

Therefore, the expression (17) for the short interest rate (level of the term structure) becomes:

$$r_t = \underbrace{\mu_K}_{\substack{\text{Drift in the short}\\ \text{term process.}}} + \sigma_K^2 \underbrace{\frac{\Theta_{KK}}{\Theta_K}}_{\substack{\text{Risk}\\ \text{aversion}}} + \underbrace{\sigma_K\sigma_I\rho_{KI}\frac{\Theta_{KI}}{\Theta_K}}_{\substack{\text{Hedging term for}\\ \text{long term investments}}} \ . \tag{18}$$

This result is a generalization of the equilibrium interest rate in the one-dimensional Cox, Ingersoll and Ross (CIR 1985a) term structure model. Although the variables Θ_K and Θ_{KK} in the CIR (1985a) model are similar to the ones in this model, the variable Θ_{KI} in CIR (1985a) does not account for the long term investment.

In addition, this result is a generalization of the equilibrium interest rate in the Mamaysky (2001) model of durable goods. Mamaysky (2001) presents just one source of uncertainty: shocks in the capital stock of nondurable goods. The depreciation of the durable good in that model is deterministic, that is, $\sigma_I = \rho_{KI} = 0$. Therefore, the hedging term for long term investments in (18) is zero in that model.

From equation (18), the short interest rate r_t presents the following characteristics:

- r_t is a function of the drift (mean of the growth) μ_K of short term process, such that an increase in μ_K produces an increase of the same magnitude in r_t.

- r_t is related to the volatility σ_K^2 of the short term process and the amount of short term (liquid) capital K_t according to the hedging term $\sigma_K^2 \frac{J_{K_tK_t}}{J_{K_t}}K_t$.

- An increase in the uncertainty in the short run σ_K^2 produces a decrease in r_t, because $J_{K_tK_t} < 0$ by concavity of the value function.

- $\frac{J_{K_tK_t}}{J_{K_t}}K_t$ has a negative sign and it is a measure of risk aversion towards short term capital.

- r_t is related to the standard deviations σ_K and σ_I of the short and long term processes, respectively, and the correlation ρ_{KI} between these processes, through the following term: $\frac{J_{K_tI_t}}{J_{K_t}}I_t$. This is a hedging term for the short term versus long term risk. This term is zero when short and long term processes are uncorrelated ($\rho_{KI} = 0$), positive for a negative correlation ($\rho_{KI} < 0$) and negative for positive correlation ($\rho_{KI} > 0$).

- $\frac{J_{K_tI_t}}{J_{K_t}}I_t$ has a negative sign and it is a measure of the risk aversion towards long term capital investments I_t.

3.6 Obtaining the FOCs (3): Equilibrium long term investment

By taking derivatives of the HJB (Bellman) equation with respect to Ψ_t, the following FOC will be obtained:

$$J_{K_t} \geqslant J_{I_t} \tag{19}$$

Note again that the state space of the problem $\{K_t, I_t\}$ is divided into two regions: the no-investment region ($\Psi_t = 0$) in which $J_K > J_I$, and the investment region ($\Psi_t > 0$) in which the equality $J_K = J_I$ holds. Initially, in the investment region, the following inequality holds: $J_I > J_L$. However, the agent allocates capital into long term investments and, therefore, the agents make the ratio $\frac{K_t}{I_t}$ decrease, until J_I is equal to J_K. Hence, agents allocate capital into long term investments when the ratio $\frac{K_t}{I_t}$ of long to short term capital becomes sufficiently high.

3.7 Equilibrium ODE and reduction in the number of state variables

Using the left part of the HJB (Hamilton-Jacobi-Bellman) equation in (13), the FOCs, and Lemma 2, I obtain the following two-dimensional ordinary differential equation (ODE) that applies to the no-investment region:

$$\begin{aligned} 0 = {} & e^{-\rho t} U_{c_t}(c_t) + J_t + J_{K_t}\left[K_t \mu_K - c_t + \lambda I_t\right] + J_{I_t}\left[I_t \mu_I - \lambda I_t\right] \\ & + 0.5 J_{K_t K_t}\left[K_t^2 \sigma_K^2\right] + 0.5 J_{I_t I_t}\left[I_t^2 \sigma_I^2\right] + J_{K_t I_t}\left[K_t I_t \sigma_K \sigma_I \rho_{KI}\right] \end{aligned} \tag{20}$$

such that the conditions in equations (14) and the equality in (19) hold.

I also introduce a further reduction in the number of state variables from two to one. First, let us consider the utility function $U(c_t) = \frac{(c_t)^{1-\gamma}}{1-\gamma}$. Because the numeraire good production function is homogeneous of degree one and the utility function is homogeneous of degree $1-\gamma$, then the value function is homogeneous of degree one as well. This implies that the ratio of short term to long term investment is sufficient to characterize this economy. Let us define $g(\omega_t)$ as part of the value function such that:

$$J(K_t, I_t, t) = \frac{{I_t}^{1-\gamma}}{1-\gamma} g(\omega_t) \tag{21}$$

where ω_t is defined as $\omega_t = \log\left(\frac{K_t}{I_t}\right)$. Given this state variable ω_t, the no-investment region will be given by $(-\infty, \omega^*]$, where ω^* is determined as part of the agent's control problem[20]. Therefore, agents allocate capital into long term investments when the ratio $\frac{K_t}{I_t}$ of short to long term capital becomes sufficiently high.

The process for ω_t is obtained using the two-dimensional version of the Ito′s Lemma:

$$d\omega_t = \left[(\mu_k - \mu_I) - 0.5(\sigma_K^2 - \sigma_I^2) + \lambda(e^{\omega_t} - 1) - \frac{c_t}{K_t}\right] dt + +\sigma_K dW_t^K - \sigma_I dW_t^I + \Lambda_t dt \tag{22}$$

[20]In the investment region the following holds: $J_I > J_K$. Then the agent would allocate capital into long term investments until J_I is equal to J_K. The trigger ω^* is the value of the state variable ω_t such that $J_I = J_K$.

where $\Lambda_t = -\left[\frac{\Psi_t}{K_t}(e^{\omega_t}+1)\right]$. Therefore Λ_t is a function of the ratios $\widehat{\Psi_t} = \Psi_t/K_t$ and $e^{\omega_t} = K_t/I_t$ or, equivalently, $\Lambda_t = \Lambda_t(\widehat{\Psi_t}, \omega_t)$. Note that when $\Psi_t = 0$ (inside the no-investment region), then $\Lambda_t = 0$.

3.8 Equilibrium conditions in terms of the long to short term ratio K_t/I_t

From (14) and (19), I respectively obtain the following expressions for the optimal consumption policy c_t and the smooth pasting condition at the boundary ω^* of the investment region:

$$c_t = I_t \left(\frac{g'(\omega_t)}{(1-\gamma)e^{\omega_t}} \right)^{\frac{-1}{\gamma}} \tag{23}$$

$$(e^{-\omega^*}+1)g'(\omega^*) = (1-\gamma)g(\omega^*) \; . \tag{24}$$

Besides, the super-contact condition $J_{KI} = J_{KK}$ or, equivalently $J_{IK} = J_{II}$, introduced in Dumas (1991) is another boundary condition that must hold at the boundary of the investment region. When the form of the value function in (21) is take in to account, then the super-contact condition becomes:

$$((1-\gamma)e^{\omega^*}-1)g'(\omega^*) = (e^{\omega^*}+1)g''(\omega^*) \; . \tag{25}$$

Remarkably, the ratio c_t/K_t that can be obtained from (23), the smooth pasting condition in (24), and the super-contact condition in (25) only depend on the state variable ω_t or the realization of the state value at $\omega_t = \omega^*$.

Finally, we need to find the form of the function $g(\omega)$ in order to determine the equilibrium conditions. The following theorem describes the functional form for $g(\omega)$.

Theorem 3 *The function $g(\omega)$ is the solution of the following ODE:*

$$0 = (0.5\beta_1)\, g''(\omega) + \left(\lambda(1+e^{-\omega}) + \beta_2\right) g'(\omega) +$$

$$+ (2-\gamma)\left(\frac{g'(\omega)}{(1-\gamma)e^{\omega}} \right)^{\frac{\gamma-1}{\gamma}} + (\beta_3 - \lambda(1-\gamma))g(\omega) \tag{26}$$

where β_1, β_2, β_3, and β_4 are the following constants that do not depend on λ nor δ:

$$\beta_1 = \sigma_K^2 + \sigma_I^2 - 2\sigma_K\sigma_I\rho_{KI} \tag{27}$$

$$\beta_2 = \left(\mu_K - 0.5\sigma_K^2\right) - \left(\mu_I - 0.5(2\gamma-1)\sigma_I^2\right) + (1-\gamma)\sigma_K\sigma_I\rho_{KI} \tag{28}$$

$$\beta_3 = -\rho + (1-\gamma)(\mu_I - 0.5\gamma\sigma_I^2) \tag{29}$$

under the conditions shown in equations (24) and (25), which are conditions that must hold at the optimal boundary ω^*. Additionally, the following boundary condition is needed in order to *account for the states in which ω becomes very small:*

$$\lim_{\omega\to-\infty} g(\omega) = +\infty. \tag{30}$$

Proof. See Appendix A4. ■

The solution of the ODE in (26) subject to (24), (25), and (30) for the constants defined in (27)-(29) is needed in order to solve the optimal control problem. To the best of my knowledge, there is no closed-form solution for this problem. Therefore, I will solve this ODE numerically[21].

3.9 The stochastic discount factor and the short interest rate

Let D_t denote the price of the risk-free money-market account, which consequently follows the process $\frac{dD_t}{D_t} = -r_t dt$. The stochastic discount factor (SDF) of this economy $M_{0,t}$ is driven by the process:

$$\frac{dM_{0,t}}{M_{0,t}} = \frac{dD_t}{D_t} - \left[\sigma_K \frac{J_{KK}}{J_K} K_t\right] d\omega_{K,t} - \left[\sigma_I \frac{J_{KI}}{J_K} I_t\right] d\omega_{I,t} \tag{31}$$

with $M_{0,0} = 1$. Furthermore, in equilibrium, the SDF is determined by the following equation:

$$M_{t,s} = \frac{J_K(K_s, I_s)}{J_K(K_t, I_t)} = e^{-\rho(s-t)} \frac{U_c(c_s)}{U_c(c_t)} = e^{-\rho(s-t)} \left(\frac{I_s^*}{I_t^*}\right)^{-\gamma} \frac{g'(\omega_s^*)e^{\omega_t^*}}{g'(\omega_t^*)e^{\omega_s^*}} \tag{32}$$

for $s > t$. Taking into account the definition of ω_t as the ratio of short term to long term capital, then the expression for the SDF in (32) becomes:

$$M_{t,s} = e^{-\rho(s-t)} \frac{K_t^*}{K_s^*} \left(\frac{I_s^*}{I_t^*}\right)^{1-\gamma} \frac{g'(K_s^*/I_s^*)}{g'(K_t^*/I_t^*)} \tag{33}$$

By expressing the equation (18) for the short interest rate in terms of the state variable $g(\omega_t)$, we obtain that:

$$r_t = \mu_K + \underbrace{\sigma_K^2 \left[\frac{g''(\omega_t) - g'(\omega_t)}{g'(\omega_t)}\right]}_{\textit{Risk aversion}} + \underbrace{\sigma_K \sigma_I \rho_{KI} \left[\frac{(1-\gamma)g''(\omega_t) + g'(\omega_t)}{g'(\omega_t)}\right]}_{\substack{\textit{Hedging term for} \\ \textit{long term investments}}}. \tag{34}$$

The following proposition shows the differential form for the short term interest rate. As it is discussed in Duffie and Kan (1994), most of the parametric models are of the form described in this proposition.

Proposition 4 *The short rate process for this problem is the solution of the stochastic differential equation of the form*

[21] By defining $g_1 = g(\omega_t)$ and $g_2 = g_1'$, I will transform this second order ODE into a system of two one dimensional ODEs. Then, I will solve this system using the Runge-Kutta iterative method. In particular, I use the forth-order explicit Runge-Kutta method as developed in Dormand and Prince (1980), which will provide an approximation solution of the system of ODEs. Fourth-order means that the error per iteration step is on the order of ϵ^5 and the total accumulated error has order ϵ^4. By solving the system for g_1 and g_2, we simultaneously obtain the functions $g(\omega_t)$ and $g'(\omega_t)$ since $g(\omega_t) = g_1$ and $g'(\omega_t) = g_2$.

$$dr_t = [\alpha_1 + \alpha_2 r_t]\, dt + \left[\alpha_1^K + \alpha_2^K r_t\right] dW_t^K + \left[\alpha_1^I + \alpha_2^I r_t\right] dW_t^I \tag{35}$$

where

$$\alpha_1 = \alpha_1(\widehat{\Psi_t}, \omega_t, t) = \sigma_K \Upsilon_1(\widehat{\Psi_t}, \omega_t, t)\Upsilon_2(\omega_t, t) + \Gamma_{KI}\Upsilon_3(\omega_t, t)$$

$$\alpha_2 = \alpha_2(\widehat{\Psi_t}, \omega_t, t) = \sigma_K \Upsilon_1(\widehat{\Psi_t}, \omega_t, t)\Upsilon_2(\omega_t, t) + \Gamma_{KI}\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2$$

$$\alpha_1^K = \alpha_1^K(\omega_t, t) = \sigma_K \Upsilon_2(\omega_t, t)$$

$$\alpha_2^K = \alpha_2^K(\omega_t, t) = -\sigma_K \frac{g''(\omega_t)}{g'(\omega_t)}$$

$$\alpha_1^I = \alpha_1^I(\omega_t, t) = \sigma_I \Upsilon_2(\omega_t, t)$$

$$\alpha_2^I = \alpha_2^I(\omega_t, t) = -\sigma_I \frac{g''(\omega_t)}{g'(\omega_t)}$$

and where $\Upsilon_1(\widehat{\Psi_t}, \omega_t, t)$, $\Upsilon_2(\omega_t, t)$, $\Upsilon_3(\omega_t, t)$, and Γ_{KI} have been defined, for notational purposes, as follows:

$$\Upsilon_1(\widehat{\Psi_t}, \omega_t, t) = (\mu_k - \mu_I) - 0.5(\sigma_K^2 - \sigma_I^2) + \lambda(e^{\omega_t} - 1) - e^{-\omega_t}\left(\frac{g'(\omega_t)}{(1-\gamma)e^{\omega}}\right)^{\frac{-1}{\gamma}} + \Lambda_t(\widehat{\Psi_t}, \omega_t)$$

$$\Upsilon_2(\omega_t, t) = \mu_k \frac{g''(\omega_t)}{g'(\omega_t)} + \sigma_K^2 \frac{g'''(\omega_t) - g''(\omega_t)}{g'(\omega_t)} + \sigma_K \sigma_I \rho_{KI} \frac{(1-\gamma)g'''(\omega_t) - g''(\omega_t)}{g'(\omega_t)}$$

$$\Upsilon_3(\omega_t, t) = -\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2 \mu_K + 0.5\left(\frac{g^{(iv)}(\omega_t)}{g'(\omega_t)} - 3\frac{g'''(\omega_t)g''(\omega_t)}{(g'(\omega_t))^2} + 2\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2\right)\sigma_K +$$

$$+0.5\left((1-\gamma)\left(\frac{g^{(iv)}(\omega_t)}{g'(\omega_t)} - 3\frac{g'''(\omega_t)g''(\omega_t)}{(g'(\omega_t))^2}\right) - 2\left(\frac{g''(\omega_t)}{g'(\omega_t)}\right)^2\right)\sigma_K \sigma_I \rho_{KI}$$

$$\Gamma_{KI} = \sigma_K^2 + \sigma_I^2 - 2\sigma_K \sigma_I \rho_{KI}.$$

Proof. Apply Ito's Lemma to equation (34) and rearrange terms. ■

Table 1 compares the forms of the short rate process for different classic term structure models. The short rate process obtained by the model developed in this paper and shown in proposition 4 has a similar form than the two-factor Cox, Ingersoll and Ross (1985b) model and the set of two-factor affine term structure models (ATSM) specified in Dai and Singleton (2000). However, there are two main differences between these models and my model that must be taken into account:

1. the power θ of the diffusion terms is 0.5 in the Cox-Ingersoll-Ross model and in the ATSM, while is 1.0 in my model as it is in Merton (1973), Vasicek (1977), Brennan and Schwartz (1979), and Black, Derman and Toy (1990); and

2. the time-dependent coefficients α_1, α_2, α_3, α_1^K, α_2^K, α_1^I, and α_2^I are not deterministic nor defined by an affine structure in my model; alternatively, these coefficients are provided by

the optimization problem described in section 2 and solved in section 3 in order to obtain the functional form of $g(\omega_t)$ and $\widehat{\Psi_t}$.

3.10 Bond prices and the term structure of interest rates

Denote the equilibrium bond prices $B(\omega_t, t, T)$ in this economy as the date-t securities that deliver one unit of the consumption good at date T. The following theorem shows how to calculate the price of any bond $B(\omega_t, t, T)$.

Theorem 5 *The equilibrium price at time t of a zero-coupon bond that expires at time T, $B(\omega_t, t, T)$, is the solution of the following partial differential equation (PDE):*

$$0 = B_t - r_t B + \left[(\mu_K - \mu_I) - 0.5(\sigma_K^2 - \sigma_I^2) - \frac{c_t}{K_t} + \lambda(e^{\omega_t} - 1) + \Lambda_t\right] B_\omega +$$

$$+0.5\left[\sigma_K^2 + \sigma_I^2 + \rho_{KI}\sigma_K\sigma_I\right] B_{\omega\omega} \tag{36}$$

subject to the following boundary conditions:

$$B(\omega_T, T, T) = 1 \tag{37}$$

$$B_\omega(\omega_T^*, t, T) = 0 \tag{38}$$

$$B_\omega(-\infty, t, T) = 0 \tag{39}$$

Proof. See Appendix A4. ■

There are five remarks that arise from Theorem 5. First, the term $\frac{c_t}{K_t}$ in (36) is a function of just ω_t because $\frac{c_t}{K_t} = e^{-\omega_t}\left(\frac{g'(\omega_t)}{(1-\gamma)e^{\omega}}\right)^{\frac{-1}{\gamma}}$. Second, $\Lambda_t = -\left[\frac{\Psi_t}{K_t}(e^{\omega_t} + 1)\right]$ is zero in the no-investment region ($\Psi_t = 0$). Third, the boundary condition (37) is necessary to impose that $B(\omega_t, t, T)$ is the price of a security that pays \$1 at time T. Fourth, the conditions (38) and (39) are necessary to rule out arbitrage opportunities and come from imposing that the bond returns at the reflecting boundary should be equal to the risk-free rate. Finally, the term structure of interest rates at time t for different maturities s, with $s \geqslant 0$, is given by $y(\omega_t, t, s)$ and the following expression:

$$y(\omega_t, t, s) = -\frac{log\left(B(\omega_t, t, t+s)\right)}{s}. \tag{40}$$

4 Intuition behind the dynamics of the term structure

In this section, I present the details about the data and the intuition about the dynamics of the term structure that the model may provide. Section 4.1 describes the data and how the capital accounts have been constructed. In section 4.2, I discuss the cyclicality of the short term interest rate and the slope of the term structure in terms of the cyclicality of the state variable $\omega_t = log(\frac{K_t}{I_t})$.

4.1 Data

I use data on: (i) fixed assets (stocks of capital), (ii) consumption and investments (flows of capital), (iii) short and long interest rates, and (iv) inflation (in order to deflate nominal units to real units). I consider data from the first quarter of 1970 to the first quarter of 2007.

My dataset for *fixed assets* consists of the net stock of fixed assets from the Fixed Asset Tables (FAT) of the National Economic Accounts provided by the Bureau of Economic Analysis (BEA). I consider both private (FAT Section 2, Tables 2.1 and 2.2) and government assets (FAT Section 7, Tables 7.1.A, 7.1.B, 7.2.A, and 7.2.B). Residential assets are considered to be long term investments. Non residential assets are separated into equipment and software (short term) and structures (long term). These data provide the stock of the short term capital account K_t and the long term capital accounts I_t in the model.

For *consumption and investments*, I use quarterly data from the National Income and Product Accounts (NIPA) on real consumption of nondurable goods from the personal consumption expenditures (NIPA Section 2, Tables 2.3.4 and 2.3.5), real gross private domestic fixed investment (NIPA Section 5, Tables 5.3.4 and 5.3.5) and real government consumption expenditures and gross investment (NIPA Section 3, Tables 3.9.4 and 3.9.5) provided by BEA.

I link each NIPA account to its equivalent type of account identified in BEA Table 3 in Fraumeni (1997). I then use the rates of economic depreciation reported in Hulten and Wykoff (1981a, 1981b) and Fraumeni (1997) to split the different NIPA accounts between short and long term using the following conditions[22]:

- if the average rate of depreciation is more than 0.1, the investment account will be considered a short term investment, otherwise it will be considered a long term investment;
- if the average service years is less than 20 years, the investment account will be considered a short term investment, otherwise it will be considered a long term investment; and
- if the average declining balance rate is greater than 1, the investment account will be considered a short term investment, otherwise it will be considered a long term investment.

Table 2 shows the results of this classification of the private investment NIPA accounts between short and long term accounts, and their corresponding rates of depreciation, service life, declining-balance rates and Hulten-Wykoff categories. Table 3 shows these same variables for the aggregated NIPA accounts in Table 2. As an example of the difference between short and long term accounts, note that information processing equipment and software (e.g. computers) has a high rate of depreciation and a short service life, while residential structures (e.g. residential real estate) has a low rate of depreciation and a long service life. Nonresidential structures related to real estate (e.g. commercial and health care or manufacturing) also show low rates of depreciation and a long service life.

[22] Although there are some minor divergences in accounts related to nonresidential structures, the Hulten-Wykoff categories A (long term), B (intermediate term) and C (short term) confirm this categorization.

The NIPA data on government consumption and investment is provided as an aggregate of consumption expenditures, gross investment in structures and gross investment in equipment and software. I consider gross investments in structures as long term investments and gross investments in equipment and software as short term investments. I include government consumption expenditures as short term investments because the agent does not have utility for these goods. Accordingly, I use the data on real consumption of nondurable goods as c_t, data on long term investments as Ψ_t, and data on short term investments as the non-invested in long term and non-consumed shared of the account K_t.

Figure 4 shows the derivable flows of capital obtained from the NIPA accounts for the period 1970Q1 to 2007Q1. Panel A of Figure 4 shows the real consumption, short term investments and long term investments in absolute terms. Panel B of Figure 4 shows the same time series in relative terms. Notice that consumption has increased in real absolute terms over time (Panel A), but it shows a slight increase over time relative to investments (Panel B). Notice also that the flow of short term investment has increased at a lower rate than consumption (Panel A) and, in relative terms, it remains almost constant. Finally, long term investments have remained stable over time (Panel A) but have decreased relative to consumption and short term investments (Panel B).

Figure 5 disaggregates short and long term investments in Figure 4 into their components. Real estate investment (residential and nonresidential) is the most important channel within long term investments. It represents an average of 84.1% of the total long term investments in the U.S. for the study period. Figure 5 shows that real estate drives the dynamics of long term investments, since the remaining (non-real estate) long term investments present a flat pattern. Furthermore, Figure 5 provides evidence of the following facts: (i) Real estate investments decrease during recessions; (ii) government consumption expenditures (GCE) increase during recessions; and (iii) short term investments (excluding GCE) decrease during recessions. These three variables present an increasing trend over the period 1970Q1 to 2007Q1.

Although long term investments have decreased relative to consumption and short term investments (Panel B of Figure 4), the capital stock of long term investments has increased even though the flow of new long term investments has been stable. Figure 6 shows the net capital stock of fixed assets from the FAT tables and consumption from NIPA. This figure shows that the capital stock of long term investments has increased more sharply than consumption and the capital stock of short term investments over the period 1970-2007. The calibration of the model will show that this is due to the higher returns provided by long with respect to short term investments, the high rate of reinvestment of long term capital and the high depreciation of short term assets compared to long term assets.

For the *short and the long term interest rates*, I use data for the term structure of interest rates from the Federal Reserve Board. I use the 3-month rate as the short rate r_t and the 5-year rate as the long rate R_t. Note that the 5-year rate $y(\omega_t, t, 5)$ in the model in equation (40) is equivalent to this long rate R_t.

Finally, I need data on *inflation* in order to deflate the data on the term structure of interest rates. Notice that deflated data on the NIPA accounts can be directly obtained from the BEA. Hence, I use real and not nominal data in this paper. One question may arise at this point: Is the deflated nominal interest rate a good proxy for the real interest rate? We need long time series to calibrate the model but the U.S. Treasury started issuing Treasury Inflation-Protected Securities (TIPS) in 1997. Figure 7 compares the data on TIPS and real rates calculated as deflated nominal rates for the period 2003Q1-2007Q4. It shows that the rates between TIPS and the calculated real rates may differ (Panel A) but that the patterns of their slopes are very similar (Panel B).

4.2 The cyclical fluctuations of the short rate, the slope of the term structure, and the state variable $\omega_t = log\left(\frac{K_t}{I_t}\right)$ are jointly determined

When we analyze the time series of the real short term interest rate and the % quarterly change in the ratio of real short term to long term investments K_t/I_t, we observe that they both follow cyclical patterns. Figure 8 shows the time series for these two variables from 1970 to 2007. These time series have been smoothed using exponential smoothing in order to show a more clear graph. The dashed boxes represent the periods in which the % in the ratio K_t/I_t increases, that is the periods in which the long term capital I_t decreases with respect to the short term capital K_t. The shaded areas show the NBER recessions. The cycles of the real short interest rate and the cycles of the % change in K_t/I_t are sketched below the graph and their increasing and decreasing periods have been denoted in the graph by (+) and (-), respectively.

Similarly, we also observe that the time series of the slope of the term structure of interest rates and the % quarterly change in the ratio of real short term to long term investments K_t/I_t follow cyclical patters (see Figure 9). We may observe the following findings from the analysis of these time series in Figures 8 and 9:

- All the NBER recessions in the period of analysis (1970-2007) happened in phases in which the long term capital I_t decreased with respect to the short term capital K_t (e.g. the NBER recessions are in the dashed areas of the graph);

- in the periods of low long term investment activity (e.g. increasing % change in K_t/I_t) the real short term investment is more volatile than in the periods of high long term investment activity (e.g. decreasing % change in K_t/I_t) as shown in Figure 8;

- the real slope of the term structure of interest rates presents a "V" shape (e.g. decreasing first, and then increasing) within the time intervals of low long term investment activity (e.g. increasing % change in K_t/I_t) as shown in Figure 9;

- the time series for the real short rate and the % change in K_t/I_t show periodic patterns which present an harmonic motion with the real short term rate following that of % change in K_t/I_t by some delay, as shown in Figure 8; and

- the time series for the real slope of the term structure of interest rates and the % change in

K_t/I_t show periodic patterns which present an harmonic motion with the real short term rate following that of % change in K_t/I_t by some delay, as shown in Figure 9.

The first finding shows evidence of the link between real investments and the business cycle. In particular, it confirms that right before or during recessions the amount of capital allocated to long term investments with respect to the capital allocated to short term investments decreases.

The second finding suggests that there may exist two different investment regions as predicted by the model in section 3. The model accounts for a distinction between no-investment and investment regions. Similarly, the aggregate data in Figures 8 and 9 suggest the distinction between regions of low levels of long term investments and regions of high levels of long term investments.

The third finding shows that the short rate and the long rate may behave differently in periods of low long term investment activity. The slope of the real term structure of interest rates decreases during the first quarters within this interval, that is, the real long rate decreases with respect to the real short rate in response to a decreasing long term investments with respect to short term investments.

Finally, the fourth and fifth findings suggest that the real investments and the first two components of the real term structure of interest rates (e.g. real short interest rate and real slope) follow the dynamics of a predator-prey system. Proposition 6 states the connection between this empirical finding and the model proposed in section 3 through a system of first order, non-linear, differential equations that is equivalent to the predator-prey equations.

Proposition 6 *The dynamics of the state variable of the problem $\omega_t = log\left(\frac{K_t}{I_t}\right)$, the real short term rate r_t, and the real long term (s-year) rate $y_{t,s} = y(\omega_t, t, s)$ are described by a system of three non-linear differential equations of the form:*

$$E\left[\frac{d\omega_t}{dt}\right] = f_1\left(\omega_t, r_t, y_{t,s}\right)\omega_t \tag{41}$$

$$E\left[\frac{dr_t}{dt}\right] = f_2\left(\omega_t, r_t, y_{t,s}\right)r_t \tag{42}$$

$$E\left[\frac{dy_{t,s}}{dt}\right] = f_3\left(\omega_t, r_t, y_{t,s}\right)y_{t,s} \tag{43}$$

Proof. See Appendix A4. ■

This system is equivalent to a generalized version of the predator-prey equations, also known as the Lotka-Volterra equations[23], for one prey (ω_t) and two predators (r_t and $y_{t,s}$). The analogy of the system of non-linear equations in Proposition 6 and the generalized predator-prey equations to any number of species competing against each other is described as follows:

[23] Lotka and Volterra proposed a pair of differential equations to describe the dynamics of interactions between two biological species in an ecosystem. One of the species is a predator (e.g. wolves) and the other one is a prey (e.g. rabbits). In their model, rabbits reproduce exponentially unless wolves eat them. The more rabbits, the more the number of rabbits will grow. The more wolves, the more predation and, therefore, less increase in the number of rabbits. Finally, the model accounts for a natural death of the wolves in order to close the model.

- The state variable of the problem $\omega_t = log\left(\frac{K_t}{I_t}\right)$ is equivalent to the prey population;
- the short term rate r_t is equivalent to a predator population; and
- the long term (s-year) rate $y_{t,s} = y(\omega_t, t, s)$ is equivalent to another predator population.

The system of differential equations has cyclical (periodic) solutions that do not present a simple expression. If this problem is solved numerically, we obtain a simple harmonic pattern for both populations (predators and prey) with the population of prey leading the two populations of predators by some time interval.

5 Empirical results

In this section, I present the details about the empirical analysis of the model. Section 5.1 shows how the model has been calibrated using the Simulated Method of Moments (SMM). Section 5.2 reports the empirical results of this calibration.

5.1 Calibration of the model

In this subsection I will study the empirical properties of the model. I will implement the Simulated Method of Moments (SMM) of Duffie and Singleton (1993) to estimate the parameters of the model. I want to estimate $\Psi = \{\mu_K, \mu_I, \sigma_K, \sigma_I, \rho_{KI}, \lambda, A, \rho\}$, the set of structural parameters of the model. However, due to the high computational burden of the simulated-based approach that I implement, I estimate only the following subset of parameters: $\widetilde{\psi} = \{\mu_K, \mu_I, \sigma_K, \sigma_I, \lambda\} \subset \Psi$. Because I am mostly interested in estimating the characteristics of investments in short and long term capital, the other parameters (ρ_{KI}, A, and ρ) are set to reasonable values according to existing studies.

I will implement the Simulated Method of Moments (SMM) of Duffie and Singleton (1993) to estimate the parameters of the model. I have to pick parameters that minimize the weighted distance between a set of model unconditional moments $F_Z(\widetilde{\psi})$, and their moment conditions from the empirical data F_T.

I use a vector f_t of time series of the following variables as described in subsection 4.1: (i) the real short term interest rate, (ii) the real long term interest rate, (iii) the real aggregate consumption of nondurable goods, (iv) the real short term investments, and (v) the real long term investments. The sample has size $T = 149$ quarters from 1970Q1 to 2007Q1. The summary statistics of this set of variables is shown in Table 4. Let the set of unconditional moments be the sample averages in the dataset: $F_T = \frac{1}{T}\sum_{t=1}^{T} f_t$.

Then, I simulate the economy described by my model for a particular set of parameters $\widehat{\psi}$. This economy is uniquely determined by its state variable ω_t defined in subsection 3.7 and endogenously determined by the optimal consumption and investment strategy of the agent. Hence, I firstly have to solve the optimal control problem, that is, the HJB equation (26) subject to (24), (25) and (30). Since no analytical solution exists for this equation, I solve it numerically following the approach in Casassus, Collin-Dufresne and Routledge (2006). Using Monte Carlo simulations, I estimate the implied density function of the state variable, $h(\omega; \widehat{\psi})$ and I calculate the implied moments of the model: $F_Z(\widehat{\psi}) = E\left[f(\omega; \widehat{\psi})\right] \approx \int f(\omega; \widehat{\psi}) h(\omega; \widehat{\psi}) d\omega$.

Finally, the SMM requires solving the problem:

$$\widehat{\psi}^* = \underset{\psi \in \widehat{\psi}}{\arg\min} \left[F_Z(\psi) - F_T\right]' W_T \left[F_Z(\psi) - F_T\right] \tag{44}$$

where W_T is the weighting matrix[24]. I assume $W_T = V^{-1}$, where V is the asymptotic of the unbiased estimate covariance matrix of the sample averages F_T. My numerical procedure is similar in spirit from Casassus, Collin-Dufresne and Routledge (2006) and Kogan, Livdan and Yaron (2006), although the goals of their papers and the setup of their models are different[25].

5.2 Results from the calibration

Table 5 reports the parameter estimates obtained from the calibration exercise and the values for the parameters that have been fixed. Since I focus on the first moments of the five variables in vector f_t described above, only five parameters can be independently estimated from the data. The growth of the long term capital process μ_I is higher than the growth of the short term capital process μ_K. This result is consistent with the fact that long term (illiquid) assets exist if and only if the investment in these long assets provide a compensation in productivity, in this case $\mu_I > \mu_K$. In my model, illiquidity is the loss of flexibility that arises from the inability to rebalance past consumption and investment plans when more accurate information becomes available. Therefore, long term assets (illiquid assets) provide a higher rate of return than the short term assets (liquid assets).

The modeled volatilities of the short term and the long term processes are 27.05% and 38.09%, respectively. The range of volatilities for returns on stocks, indexes of stocks and real assets is wide with a range of values from 8% to 45%. Table 5 also reports that the 13%, $(1 - \lambda)$, of the returns on the long term capital that mature at each period are reinvested in the long term technology. This means that 87%, (λ), of the long term returns revert to the short term capital account and are able to be used for short term investments, long term investments or consumption.

Table 6 shows the simulated (modeled) moments for the aggregate economy and compares them to the moments of the data. Note that the fit of the first moments of the real short and long term interest rates is very accurate. The model indicates that on average the term structure is upwarding, that is, $E(R_t) > E(r_t)$. The modeled expected excess returns from short and long term investments, $\mu_K - E(r_t)$ and $\mu_I - E(R_t)$, are 10.35% and 9.66%, respectively. Although these values are high,

[24] This matrix gives more weight to less volatile moments, which drives the scale of each moment condition to a similar level.

[25] The first paper studies spot and future oil prices, while the second one studies the term structure of futures volatility. Like my model, these papers develop equilibrium models in which prices and/or interest rates are determined endogenously in a production economy with investments constraints (e.g. irreversibility, capacity constraint or illiquidity). Consequently, these models can be calibrated using similar approaches.

they are consistent with historical excess returns of 8%-10% observed in the data for indexes of stocks and, remarkably, they are obtained with a low value of the risk aversion parameter $\gamma = 5$ (equivalently, $A = 1 - \gamma = -4$). Furthermore, the model accounts for a low correlation between direct long term investments (e.g. direct investments in real estate) and short term investments ($\rho_{KI} = 0.15$), which is consistent with the low historical correlation between the NCREIF Index and the S&P 500 Index.

Note also that the model slightly underestimates consumption and investments. Figure 10 shows a possible explanation for why it is hard to obtain a better estimation. This figure shows the ratio of optimal consumption over the capital stock of long term investment c_t/I_t, as a function of the ratio K_t/I_t. First, note that, although the model slightly underestimates the value of the ratio c_t/I_t, it does a good job of capturing the shape of the function. Second, note that since the beginning of the study period (1970Q1) the values of both c_t/I_t and K_t/I_t have been decreasing over time. This fact is due to the strong increase on the aggregate capital stock account I_t compared to moderate increases in c_t and K_t (see Figure 6). As I_t has increased faster than c_t and K_t, then the ratios c_t/I_t and K_t/I_t have consequently decreased over time (see Figure 10). Third, note that the curve for the real data shows evidence that there may have been at least two structural changes and, consequently, three periods during the period of analysis 1970Q1-2007Q1: (i) period 1970-1973, (ii) period 1974-1982, (iii) period 1983-2007. The period 1974-1982 includes three major recessions and years with high inflation, which dropped consumption and the ratio c_t/I_t (see Figure 10).

6 Conclusions

This paper has presented a formal theory of the real term structure of interest rates based on the preferences of a representative agent and his decisions on how to allocate capital in investments at heterogeneous investment horizons (e.g. short and long term). I formally study the assumption of heterogeneous investment horizons, and show that it is a crucial component to explain the dynamics of the term structure of interest rates as well as the dynamics of aggregate consumption and investments.

I have developed an equilibrium model to study these dynamics and provide economic intuition. The model endogenizes agents' decisions on consumption and investment with short and long term horizons. Its driving forces are a short term fully reversible and a long term time-to-build production technologies that generate a time-varying market price of risk for this economy. In the model, investors demand high risk premia for holding long term investments because (i) returns on production of long term assets are higher than returns on production of short term assets, (ii) physical irreversibility in production of long term assets prevents the agents from incorporating new available information about the economy over time, and (iii) investors are risk averse and care about consumption in the long run.

The model has been calibrated to U.S. data from 1970 to 2007 and accounts for several stylized facts such as the ability to jointly generate excess returns on short and long term investments comparable to historical excess returns and low volatility of consumption using a reasonably low risk aversion parameter; a low correlation between direct long term investments (e.g. direct investments in real estate) and short term investments; and the slightly positive slope of the real term structure of interest rates.

Finally, the paper's finding on the role of different investment horizons has important consequences for the interpretation of the dynamics of macroeconomic variables. The model relates the dynamics of the production of short and long term assets to macroeconomic variables such as aggregate consumption, investments and the term structure of interest rates. It is consistent with the findings of other production models in the literature such as Cochrane (1988, 1991), Jermann (1998, 2007) and Tallarini (2000) that production-based factors have explanatory power for asset prices dynamics.

Appendix

A1. Setup of the general equilibrium economy

Production technologies

Let us consider two production technologies in the productive side of this economy: a short term technology K and a long term technology I. The only input for production is capital (e.g. not labor). K is a perfectly reversible technology which has constant returns to scale (with growth μ_K and volatility σ_K) as in the CIR model. Let K_t denote the amount of short term capital good in this economy. The amount K_t is the liquid capital available for consumption and investment in any of the two technologies at any time t. Between times t and $t+dt$, $c_t dt$ units of capital are converted into consumption good, $\Psi_t dt$ units are invested in the long term technology I, and $L_{t,\tau}\Psi_{t-\tau}$ units are the output from investments made at time $t-\tau$ in the long term technology I. Note that $\Psi_{t-\tau}dt$ units were invested in the long term technology I at time $t-\tau$ and therefore, $L_{t,s}$ can be seen as the compound risky return on the long term technology at time t of an investment made at time $t-s$. The stock of capital K_t evolves according to the following process:

$$dK_t = K_t\left(\mu_K dt + \sigma_K dW_t^K\right) - c_t dt - \Psi_t dt + L_{t,\tau}\Psi_{t-\tau} \tag{A-1}$$

Technology I is a time to build technology in the sense that a non-negative amount $\Psi_t dt$ that is invested in this production technology at time t will not provide the output provided from this investment until time $t+\tau$. Therefore, it takes τ units of time for technology I to produce the good. This technology has constant returns to scale (with growth μ_I and volatility σ_I). Let I_t denote the stock of long term capital in this economy. The stock of capital I_t is the sum of the capital subaccounts $I_t^{(j)}$, being $I_t^{(j)}$ the stock of capital "under construction" (through the time to build technology) that will mature j units of time from now. Therefore, $I_t = \int_0^\tau I_t^{(\tau-s)}ds$. The stock of capital in each of these subaccounts $I_t^{(\tau-s)}$ evolves according to the following process:

$$dI_t^{(\tau-s)} = I_t^{(\tau-s)}\left(\mu_I dt + \sigma_I dW_t^I\right) + \Psi_{t-s}L_{t,s} - \Psi_{t-(s+\varepsilon)}L_{t,(s+\varepsilon)} \tag{A-2}$$

where s takes all the values of the interval $[0,\tau-\varepsilon]$ and ε is such that $\varepsilon > 0$ and $\varepsilon \to 0$. Note that (A-2) is a continuum of equations equivalent to the set of equations (3)-(4) in the simple model.

Furthermore, the production technologies carry the following non-negativity constraints:

$$K_t > 0, \quad c_t > 0, \quad \text{and} \quad I_t^{(j)} > 0 \quad \text{for all } t \text{ and all } j\ , \tag{A-3}$$

and the irreversibility constraint for long-term investments:

$$\Psi_t \geq 0 \quad \textit{for all } t\ . \tag{A-4}$$

For simplicity, assume that there is only one capital good in this economy that can be produced by any of the two technologies described above. Hence, at each time t there are K_t units of capital good available for consumption and for investment production in sector K, and I_t "illiquid" units of capital good that are involved in the technology I's time to build production processes. Finally, assume that there is no storage technology in this economy.

Firms

Let us assume that households own the entire stock of the capital good K_t and invest into the production technology K. Therefore, firms will be not defined for this production sector. Meanwhile, the production sector I is formed by a large number of competitive firms with identical technology and differ only in size. In the aggregate, firms own the entire stock of capital I_t.

Firms decide when and how much capital Ψ_t to invest. Note that if we assume that markets are dynamically complete, then it is irrelevant if the firms finance their investments with equity or debt. For simplicity, let us assume that investments are financed with equity. Households purchase equity of the firms in the production sector I in order to invest in the production technology of this sector.

Firms make investment decisions $\{\Psi_t\}$ that maximize their stock price. The stock price is the value of the output and the investment expenses. Let us consider the representative firm in the production sector I. This firm maximizes the present value of its future cash flows according to the following optimization problem:

$$\max_{\Psi_t} \left\{ \int_0^\infty M_{0,t} I_t dt - \int_0^\infty M_{0,t} d\Psi_t dt \right\} \tag{A-5}$$

subject to equation (A-2), $I_t > 0$, and $\psi_t \geqslant 0$. Let $M_{0,t}$ denote the stochastic discount factor (SDF) that the representative firm uses to discount the present value of its future cash flows. I assume that the firm is rational. Therefore, the SDF has to be consistent with the prices observed in the competitive market. We also assume that markets are dynamically complete, which implies that there exists a unique SDF for this problem.

Financial markets

Let us introduce three assets of this economy. First, assume that there exists an asset that generates an identical return to the sector K's production technology[26]. If ζ_t^K is the amount invested in this asset at time t, then its process is determined by

$$\frac{d\zeta_t^K}{\zeta_t^K} = \mu_K dt + \sigma_K dW_t^K. \tag{A-6}$$

Second, assume that there is an asset that is a claim on the cash flows generated by sector I. This asset is the stock (equity) on the sector I's representative firm. Let $\widehat{P}$ be the ex-dividend

[26]Note that there is just one good in this economy and, therefore, the price of this good in terms of itself is one at all times. Hence, the return of this good is equivalent to the amount invested ζ_t^K.

stock price. Note that the flow of dividends generated by this stock at time t is given by I_t minus investment Ψ_t. Third, assume that there is a bond that pays off an instantaneous risk-free rate r_t.

Households

The economy is populated by identical competitive households. The problem can be modeled as a single representative agent economy. The household has utility for consumption of the capital good c_t. We assume that the utility function of the agent has the following form:

$$U(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}. \tag{A-7}$$

The household solves the following maximization problem:

$$\max_{c_t, \zeta_t^L, \alpha_r, \theta_P} \left\{ \int_0^\infty e^{-\rho t} U(c_t) dt \right\} \tag{A-8}$$

subject to non-negativity consumption constraint $c_t \geqslant 0$ and the budget constraint:

$$dH_t = \zeta_t^L + \alpha_r r_t + \theta_t (d\widehat{P}_t) - c_t dt. \tag{A-9}$$

Let H_t denote the wealth process of the individual representative agent (household), α_r be the amount of capital invested in the risk-free bond and θ_t be the stock of capital invested in the shares of the production sector I's representative firm. At each time, the wealth process of the agent is given by:

$$H_t = \zeta_t^L + \alpha_r + \theta_t \widehat{P}_t \tag{A-10}$$

where $H_t \geqslant 0$ for each time t in order to avoid arbitrage opportunities.

A2. Definition of competitive equilibrium

Under the setup of the economy in subsection A.1 of the Appendix, a competitive equilibrium with dynamically complete markets is a set of processes $\{c_t^*, \zeta_t^{K*}, \alpha_r^*, \theta_t^*, K_t^*, I_t^*, \Psi_t^*, r_t, \widehat{P}_t, M_{t,s}\}$ such that the following statements hold:

1. Given $M_{t,s}$, then Ψ_t^* is the aggregate investment that solves the representative firm's problem in equation (A-5);
2. Given r_t, $\widehat{P}_t$ and I_t^* , then the set $\{c_t^*, \zeta_t^{K*}, \alpha_r^*, \theta_t^*\}$ solves the representative household's problem defined in equation (A-8);
3. Given c_t^*, Ψ_t^* and the initial amounts of capital goods in the economy K_0 and I_0, then the amounts of short term capital K_t^* and long term capital I_t^* solve the budget constraints in equations (A-1) and (A-2); and
4. Markets clear, therefore the following clearing conditions hold at each time t:

$$\alpha_t^* = 0$$

$$\zeta_t^{K*} = K_t^*$$

$$\theta_t^* = 1$$

5. The stochastic processes r_t, $\widehat{P}_t$, I_t^*, and Ψ_t^* are such that $M_{t,s}^*$ is the unique stochastic discount factor (SDF) of this economy and the following equations hold for each time t and s, with $s > t$:

$$\widehat{P}_t = E_t\left[\int_t^s M_{t,x} I_x^* dx - \int_t^s M_{t,x}\Psi_x^* dx + M_{t,s}\widehat{P}_s\right] \tag{A-11}$$

$$\frac{1}{1+r_t} = \lim_{dt\to 0} E_t\left[M_{t-dt,t}\right] \tag{A-12}$$

$$E_t\left[M_{t,s}e^{(\mu_K-\sigma_K/2)(s-t)+\sigma(W_s^K-W_t^I)}\right] = 1 \tag{A-13}$$

A3. Solving for the competitive equilibrium

There exists a competitive equilibrium with dynamically complete markets[27] in which the following statements hold:

1. The optimal portfolio of the household in equilibrium is determined by $\{\zeta_t^{K*}, \theta_t^*, \alpha_r^*\} = (K_t^*, 1, 0)$. Note that this implication is equivalent to the market clearing conditions in Appendix A2.

2. The set of stochastic processes $\{K_t^*,\ I_t^*,\ c_t^*,\ \Psi_t^*\}$ is determined as the solution of the central planner's problem developed in section 2.

3. The ex-dividend stock price is given by

$$\widehat{P}_t = I_t^* \frac{J_{I_t}}{J_{K_t}} = I_t^* \frac{J_{I_t}\left(K_t, K_{t+\tau}, I_t^{(\tau)}, I_{t+\varepsilon}^{(\tau)}, I_{t+\varepsilon}^{(\tau-\varepsilon)}, I_{t+2\varepsilon}^{(\tau-\varepsilon)}, ..., I_{t+\tau-\varepsilon}^{(\varepsilon)}, I_{t+\tau}^{(\varepsilon)}, t\right)}{J_{K_t}\left(K_t, K_{t+\tau}, I_t^{(\tau)}, I_{t+\varepsilon}^{(\tau)}, I_{t+\varepsilon}^{(\tau-\varepsilon)}, I_{t+2\varepsilon}^{(\tau-\varepsilon)}, ..., I_{t+\tau-\varepsilon}^{(\varepsilon)}, I_{t+\tau}^{(\varepsilon)}, t\right)} \tag{A-14}$$

4. The stochastic discount factor (SDF) of this economy is given by

$$M_{t,s} = e^{-\rho(s-t)} \frac{U_c(c_s^*)}{U_c(c_t^*)} \tag{A-15}$$

[27] Under the definition of complete markets in Appendix A2

A4 Proof of Theorems

Proof of Theorem 3

The ordinary differential equation (ODE) shown in (26) comes from the left part of the maximization function in equation (13) evaluated at the optimal consumption shown in (23). We know from the HJB equation in (13) that the following ODE holds in the no-investment region:

$$\sup_{c_t, x_{1t}, x_{2t}, \Psi_t} \left\{ E_0 \left[\widehat{dJ^*} + e^{-\rho t} U(c_t) dt \right] \right\} = 0$$

where $\widehat{dJ^*}$ is represented by the following expression:

$$\widehat{dJ^*} = J_t + \left[x_{1t} K_t \mu_K + x_{2t} K_t r_t + (1 - x_{1t} - x_{2t}) K_t \frac{dB_t}{B_t} - c_t + \lambda I_t \right] J_{K_t} + \frac{1}{2} \left[x_{1t}^2 K_t^2 \sigma_K^2 \right] J_{K_t K_t} +$$

$$+ \left[I_t \mu_I - \lambda I_t \right] J_I + + \frac{1}{2} \left[I_t^2 \sigma_I^2 \right] J_{I_t I_t} + \left[x_{1t} \sigma_L \sigma_I \rho_{LI} K_t I_t \right] J_{KI}$$

and J_K, J_{KK}, J_I, J_{II}, and J_{KI} are the first and second order partial derivatives of the value function $J(K_t, I_t, t)$ with respect to K_t and I_t. When taking into account Lemma 2 and the fact that in the no-investment region $\Psi_t = 0$, we obtain the following ODE:

$$0 = e^{-\rho t} U_{c_t}(c_t) + J_t + \left[K_t \mu_K - c_t + \lambda I_t \right] J_{K_t} + \frac{1}{2} \left[K_t^2 \sigma_K^2 \right] J_{K_t K_t} +$$

$$+ \left[I_t \mu_I - \lambda I_t \right] J_I + + \frac{1}{2} \left[I_t^2 \sigma_I^2 \right] J_{I_t I_t} + \left[\sigma_L \sigma_I \rho_{LI} K_t I_t \right] J_{KI}.$$

Let us consider the utility function $U(c_t) = \frac{(c_t)^{1-\gamma}}{1-\gamma}$ and the functional form of the value function defined by $J(K_t, I_t, t) = \frac{{I_t}^{1-\gamma}}{1-\gamma} g(\omega_t)$. When we plug the corresponding derivatives of $U(c_t)$ and J in the two dimensional ODE, we obtain the following one dimensional form of the ODE:

$$0 = (0.5\beta_1)\, g''(\omega) + \left(\lambda(1 + e^{-\omega}) + \beta_2 \right) g'(\omega) +$$

$$+ (2 - \gamma) \left(\frac{g'(\omega)}{(1-\gamma) e^{\omega}} \right)^{\frac{\gamma-1}{\gamma}} + (\beta_3 - \lambda(1-\gamma)) g(\omega)$$

where β_1, β_2, β_3, and β_4 are the following constants that do not depend on λ nor δ:

$$\beta_1 = \sigma_K^2 + \sigma_I^2 - 2\sigma_K \sigma_I \rho_{KI}$$

$$\beta_2 = \left(\mu_K - 0.5\sigma_K^2 \right) - \left(\mu_I - 0.5(2\gamma - 1)\sigma_I^2 \right) + (1-\gamma)\sigma_K \sigma_I \rho_{KI}$$

$$\beta_3 = -\rho + (1-\gamma)(\mu_I - 0.5\gamma\sigma_I^2)$$

This ODE must hold under the boundary conditions shown in equations (24) and (25). Besides, a third boundary condition is needed in order to account for the states in which ω becomes very small. We must impose that when the amount of short term capital K_t becomes very small, then the value function is zero[28], because $c_t = 0$ when L_t goes to zero by the non-negativity constraint for the illiquid investment Ψ_t. This final boundary condition is equivalent to impose that

$$\lim_{\omega \to -\infty} g(\omega) = +\infty$$

Q.E.D.

Proof of Theorem 5

The price at time t of a zero-coupon bond paying one unit at time T, $T > t$ is

$$B(t,T) = E_t \left[e^{-\int_t^T r_s ds} | \Im_t \right]$$

with $0 \leqslant t \leqslant T$ in the probability space $(\Omega, \Im, \widetilde{P})$. Because this problem is Markov (the processes dK_t and dI_t are Markov processes) and the equilibrium interest rate r_t given by (34) is a function of the factor ω_t, there must be a function $f(t, \omega_t)$ such that $B(t,T) = f(t, \omega_t)$.

The price of the risk-free money-market account or discount factor D_t follows the process $\frac{dD_t}{D_t} = -r_t dt$. Iterated conditioning implies that the discounted bond price $D_t B(t,T)$ is a martingale under the probability measure $\widetilde{P}$. I calculate the differential of $D_t B(t,T)$ by using the Ito′s lemma:

$$\begin{aligned}
d(D_t, B(t,T)) &= d(D_t, f(t, \omega_t)) \\
&= -r_t D_t f(t, \omega_t) dt + D_t df(t, \omega_t) \\
&= D_t \left[-r_t f dt + f_t dt + f_\omega d\omega_t + 0.5 f_{\omega\omega} d[\omega_t, \omega_t] \right] \\
&= D_t \left[-r_t f dt + f_t dt + f_\omega d\omega_t + 0.5 f_{\omega\omega} [\sigma_K^2 dt + \sigma_I^2 dt + \rho_{KI} \sigma_K \sigma_I dt] \right]
\end{aligned}$$

where ω_t is given by equation (22).

Because $D_t B(t,T)$ is a martingale under the probability measure $\widetilde{P}$, the dt term of $D_t B(t,T)$ must be zero. If we set the dt term equal to zero, and we take into account that $f = B(t,T)$, then we obtain the following PDE:

$$\begin{aligned}
0 = B_t - r_t B &+ \left[(\mu_K - \mu_I) - 0.5(\sigma_K^2 - \sigma_I^2) - \frac{c_t}{K_t} + \lambda(e^{\omega_t} - 1) + \Lambda_t \right] B_\omega + \\
&+ 0.5 \left[\sigma_K^2 + \sigma_I^2 + \rho_{KI} \sigma_K \sigma_I \right] B_{\omega\omega}.
\end{aligned}$$

Q.E.D.

[28]Note that we want to impose that the value function $J(K_t, I_t, t)$ goes to $-\infty$ when the ratio K_t/I_t goes to zero (or equivalently, when $\omega \to -\infty$). Therefore, according to equation (21), and because $I_t \geqslant 0$ and $1 - \gamma < 0$, then $g(\omega_t)$ must go to $+\infty$ in order to make $J(K_t, I_t, t)$ go to $-\infty$

Proof of Proposition 6

Recall the following expressions from (22) and (34), respectively:

$$E\left[\frac{d\omega_t}{dt}\right] = \left[(\mu_k - \mu_I) - 0.5(\sigma_K^2 - \sigma_I^2) + \lambda(e^{\omega_t} - 1) - \frac{c_t}{K_t}\right] + \Lambda_t \tag{A-16}$$

$$r_t = \mu_K + \sigma_K^2 \left[\frac{g''(\omega_t) - g'(\omega_t)}{g'(\omega_t)}\right] + \sigma_K \sigma_I \rho_{KI} \left[\frac{(1-\gamma)g''(\omega_t) + g'(\omega_t)}{g'(\omega_t)}\right] \tag{A-17}$$

If we combine equations (36) and (40) we obtain the following equation:

$$\begin{aligned} y_t = -\frac{1}{s} log\Bigg[\frac{1}{r_t}\Bigg[B_t + \left[(\mu_K - \mu_I) - 0.5(\sigma_K^2 - \sigma_I^2) - \frac{c_t}{K_t} + \lambda(e^{\omega_t} - 1) + \Lambda_t\right] B_\omega \\ + 0.5\left[\sigma_K^2 + \sigma_I^2 + \rho_{KI}\sigma_K\sigma_I\right] B_{\omega\omega}\Bigg]\Bigg] \end{aligned} \tag{A-18}$$

Combine equation (A-16) and (A-18), and note that $B = e^{-s \cdot y}$ from (40) to find the following expression for $d\omega_t/dt$:

$$E\left[\frac{d\omega_t}{dt}\right] = \left[r_t \frac{B}{\omega_t B_\omega} - \frac{B_t}{\omega_t B_\omega} - 0.5\left[\sigma_K^2 + \sigma_I^2 + \rho_{KI}\sigma_K\sigma_I\right]\frac{B_{\omega\omega}}{\omega_t B_\omega}\right]\omega_t = f_1(\omega_t, r_t, B)\omega_t \tag{A-19}$$

Note that r_t is a function of t and ω_t and, therefore, if we take expectations to the *chain rule*:

$$E\left[\frac{dr_t}{dt}\frac{dt}{dt} + \frac{dr_t}{d\omega_t}\frac{d\omega_t}{dt}\right] = 0 \tag{A-20}$$

and we combine it to equation (A-19) then

$$E\left[\frac{dr_t}{dt}\right] = -E\left[\frac{dr_t}{d\omega_t}\frac{d\omega_t}{dt}\right] = f_2(\omega_t, r_t, B)r_t \tag{A-21}$$

Finally, recall (40), take the derivative with respect to t, and combine it to (36) in order to obtain:

$$\begin{aligned} E\left[\frac{dy_t}{dt}\right] &= \left[\frac{r_t}{log(B)} - E\left[\frac{d\omega_t}{dt}\right]\frac{B_\omega}{B \cdot log(B)} - 0.5\left[\sigma_K^2 + \sigma_I^2 + \rho_{KI}\sigma_K\sigma_I\right]\frac{B_{\omega\omega}}{B \cdot log(B)}\right] y_t \\ &= f_3(\omega_t, r_t, B)y_t \end{aligned} \tag{A-22}$$

Q.E.D.

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$$dr_t = [\alpha_1 + \alpha_2 r_t + \alpha_3 r_t log(r_t)]\, dt + \left[\alpha_1^K + \alpha_2^K r_t\right]^\theta dW_t^K + \left[\alpha_1^I + \alpha_2^I r_t\right]^\theta dW_t^I$$

	α_1	α_2	α_3	α_1^K	α_2^K	α_1^I	α_2^I	θ
Merton (1973)	**o**	---	---	**o**	---	---	---	1.0
Vasicek (1977)	**o**	**o**	---	**o**	---	---	---	1.0
Brennan and Schwartz (1979)	**o**	**o**	---	---	**o**	---	---	1.0
Cox, Ingersoll and Ross (1985b), 1-factor	**o**	**o**	---	---	**o**	---	---	0.5
Cox, Ingersoll and Ross (1985b), 2-factor	**o**	**o**	---	---	**o**	---	**o**	0.5
Pearson and Sun (1990)	**o**	**o**	---	**o**	**o**	---	---	0.5
Black, Derman and Toy (1990)	---	**o**	**o**	---	**o**	---	---	1.0
Affine Term Structure Models, 2-factor	**o**	**o**	---	**o**	**o**	**o**	**o**	0.5
This paper	**o**	**o**	---	**o**	**o**	**o**	**o**	1.0

Table 1: Comparison to classic term structure models, indicating with "**o**" the coefficients different than zero for each case, and indicating the value of the power θ.

Type of asset (NIPA)	Type of asset (BEA Table 3)	Rate of depreciation	Service life (years)	Declining-balance rate	Hulten-Wykoff categories	Long/Short term
Nonresidential structures	**Private nonresidential structures**					
Commercial and health care	Mobile offices	0.0556	16	0.8892	A	**Long**
	Office buildings	0.0247	36	0.8892	A	
	Commercial warehouses	0.0222	40	0.8892	A	
	Other commercial buildings	0.0262	34	0.8892	A	
Manufacturing	Industrial buildings	0.0314	31	0.9747	A	**Long**
Power and communication	Telecommunications	0.0237	40	0.9480	C	**Long**
	Electric light and power	0.0211	45	0.9480	C	
	Gas	0.0237	40	0.9480	C	
	Petroleum pipelines	0.0237	40	0.9480	C	
Mining exploration, shafts, and wells	Petroleum and natural gas (before 1973)	0.0563	16	0.9008	C	**Long**
	Petroleum and natural gas (1973 and later)	0.0751	12	0.9008	C	
	Other	0.0450	20	0.9008	C	
Other structures	Religious buildings	0.0188	48	0.9024	C	**Long**
	Educational buildings	0.0188	48	0.9024	C	
	Hospital and institutional buildings	0.0188	48	0.9024	B	
	Hotels and motels	0.0281	32	0.8990	B	
	Amusement and recreational buildings	0.0300	30	0.8990	B	
	All other nonfarm buildings	0.0249	38	0.8990	B	
	Railroad replacement track	0.0275	38	0.9480	C	
	Other railroad structures	0.0166	54	0.9480	C	
	Farm	0.0239	38	0.9100	C	
	Local transit	0.0237	38	0.8990	C	
	Other	0.0225	40	0.8990	C	
Nonresid. equipment and software	**Private nonresidential equipment**					
Information processing equipment and software	Office, computing, and accounting machinery (before 1978)	0.2729	8	2.1832	B	**Short**
	Office, computing, and accounting machinery (1978 and later)	0.3119	7	2.1832	B	
	Communications equipment (business services)	0.1500	11	1.6500	C	
	Communications equipment (other industries)	0.1100	15	1.6500	C	
	Instruments	0.1350	12	1.6203	C	
	Photocopy and related equipment	0.1800	9	1.6203	C	
Industrial equipment	Nuclear fuel	---	4	---	---	**Short**
	Other fabricated metal products	0.0917	18	1.6500	C	
	Steam engines and turbines	0.0516	32	1.6500	C	
	Internal combustion engines	0.2063	8	1.6500	C	
	Metalworking machines	0.1225	16	1.9600	A	
	Special industrial machinery	0.1031	16	1.6500	C	
	General industrial, including materials handling equipment	0.1072	16	1.7150	A	
	Electrical transmission, distribution, and industrial apparatus	0.0500	33	1.6500	C	
	Farm tractors	0.1452	9	1.3064	A	
	Construction tractors	0.1633	8	1.3064	A	
	Agricultural machinery, except tractors	0.1179	14	1.6500	C	
	Construction machinery, except tractors	0.1550	10	1.5498	A	
	Mining and oil field machinery	0.1500	11	1.6500	C	
	Service industry machinery (Wholesale and retail trade)	0.1650	10	1.6500	C	
	Service industry machinery (Other industries)	0.1500	11	1.6500	C	
Transportation equipment	Trucks, buses, and truck trailers (Local and interurban passeng.)	0.1232	14	1.7252	A	**Short**
	Trucking and warehousing, auto repair, services, and parking	0.1725	10	1.7252	A	
	Trucks, buses, and truck trailers (Other industries)	0.1917	9	1.7252	A	
	Aircraft (Transport. by air, depository instit., and bus. services)	0.0825	20	1.6500	C	
	Aircraft (Other industries)	0.1100	15	1.6500	C	
	Ships and boats	0.0611	27	1.6500	B	
	Railroad equipment	0.0589	28	1.6500	C	
Other equipment	Household furniture and fixtures	0.1375	12	1.6500	C	**Short**
	Other furniture	0.1179	14	1.6500	C	
	Household appliances	0.1650	10	1.6500	C	
	Other electrical equipment	0.1834	9	1.6500	C	
	Other	0.1473	11	1.6230	C	
Residential	**Residential capital (private and government)**					
Structures	1-to-4-unit structures-new	0.0114	80	0.9100	A	**Long**
	1-to-4-unit structures-additions and alterations	0.0227	40	0.9100	A	
	1-to-4-unit structures-major replacements	0.0364	25	0.9100	A	
	5-or-more-unit structures-new	0.0140	65	0.9100	A	
	5-or-more-unit structures-additions and alterations	0.0284	32	0.9100	A	
	5-or-more-unit structures-major replacements	0.0455	20	0.9100	A	
	Mobile homes	0.0455	20	0.9100	A	
	Other structures	0.0227	40	0.9100	A	
Equipment	Equipment	0.1500	11	1.6500	C	**Short**

Table 2: Classification of the private investments NIPA accounts between short and long term assets using BEA Table 3.

	Rate of depreciation	Service life (years)	Declining-balance rate	Hulten-Wykoff category	Long/Short term
Nonresidential structures					
Commercial and health care	0.0295	34.8000	0.8918	4A 1B	**Long**
Manufacturing	0.0314	31.0000	0.9747	A	**Long**
Power and communication	0.0232	41.0000	0.9480	5C	**Long**
Mining exploration, shafts, and wells	0.0588	20.0000	0.9008	3C	**Long**
Other structures	0.0235	40.4000	0.9106	3B C7	**Long**
Nonresid. equipment and software					
Information processing equipment and software	0.1933	10.3333	1.8178	2B 4C	**Short**
Industrial equipment	0.1271	15.1429	1.6205	5A 9C	**Short**
Transportation equipment	0.1156	16.7778	1.6751	3A 1B 5C	**Short**
Other equipment	0.1502	11.2000	1.6446	5C	**Short**
Residential					
Structures	0.0264	43.6667	0.9100	6A	**Long**
Equipment	0.1500	11.0000	1.6500	1C	**Short**

Table 3: Classification of the private investments NIPA accounts between short and long term assets using BEA Table 3.

	Mean (annual)	Standard dev. (annual)	Maximum	Minimum
Real short rate (3-month), r_t	1.33%	2.319%	6.63%	-5.68%
Real long rate (5-year), R_t	2.48%	2.606%	8.69%	-4.87%
Nominal short (3-month) rate	6.06%	2.901%	15.36%	0.93%
Nominal long (5-year) rate	7.22%	2.512%	14.68%	2.57%
Consumption growth, $\Delta c_t/c_t$	0.65%	0.645%	2.46%	-1.42%
Short term fixed capital growth, $\Delta K_t/K_t$	0.76%	0.886%	3.81%	-0.29%
Long term fixed capital growth, $\Delta I_t/I_t$	1.28%	0.809%	3.86%	-0.44%
Ratio K_t/I_t	0.2678	0.0469	0.3444	0.1617

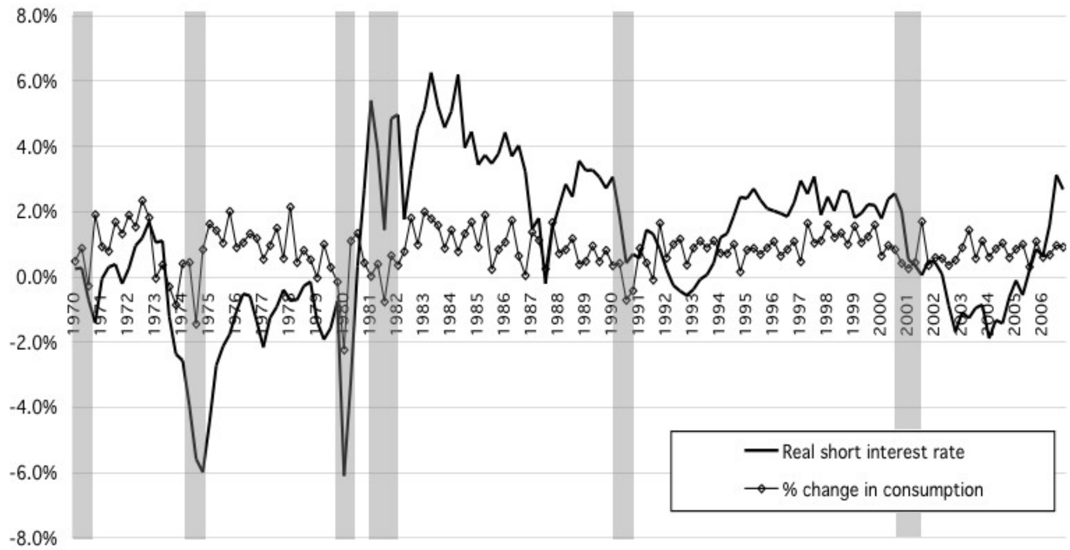
Table 4: Summary statistics of the US data. Period 1970Q1-2007Q1.

Parameter	Value	
μ_K	12.80%	Estimated
μ_I	16.72%	Estimated
σ_K	27.05%	Estimated
σ_I	38.09%	Estimated
λ	0.87	Estimated
ρ_{KI}	0.15	Fixed
A	-4	Fixed
ρ	0.005	Fixed

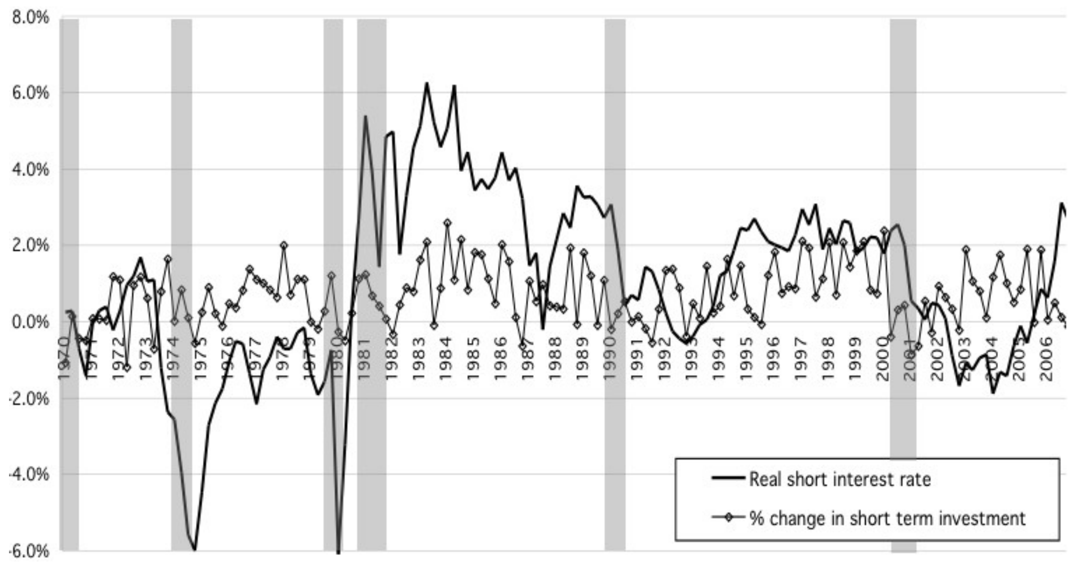
Table 5: Parameter estimates. This table reports our parameter estimates. We use SMM to estimate the following vector of five structural parameters $\hat{\Psi} = \{\mu_K, \mu_I, \sigma_K, \sigma_I, \lambda\}$. Note: Means and standard deviations in annual units. The model is simulated at a quarterly frequency using data from 1970Q1 to 2007Q1.

	Data	Model	Description of the moment
$E(r_t)$	2.45%	2.70%	Short (3-month) real interest rate
$E(R_t)$	6.21%	7.06%	Long (5-year) real interest rate
$E(\Delta c_{t+1}/c_t)$	0.65%	0.79%	Mean of consumption growth
$E(\Delta K_{t+1}/K_t)$	0.76%	0.78%	Mean of the short term fixed capital account growth
$E(\Delta I_{t+1}/I_t)$	1.29%	0.81%	Mean of the long term fixed capital account growth

Table 6: Simulated moments for the aggregate economy. This table compares the moments from the data to the modeled moments.

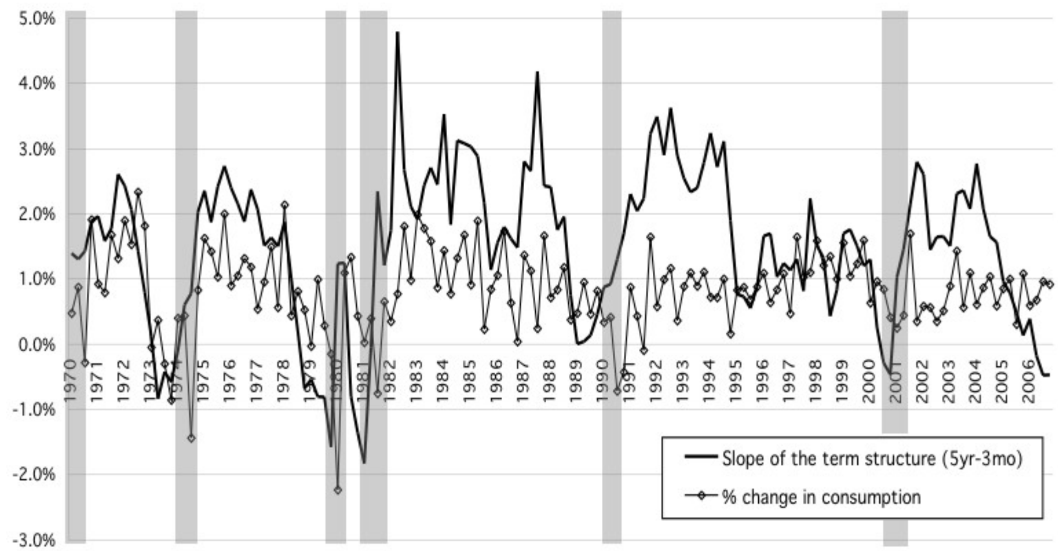


(a) Panel A: Quarterly change in consumption vs. the ex-post real short interest rate.

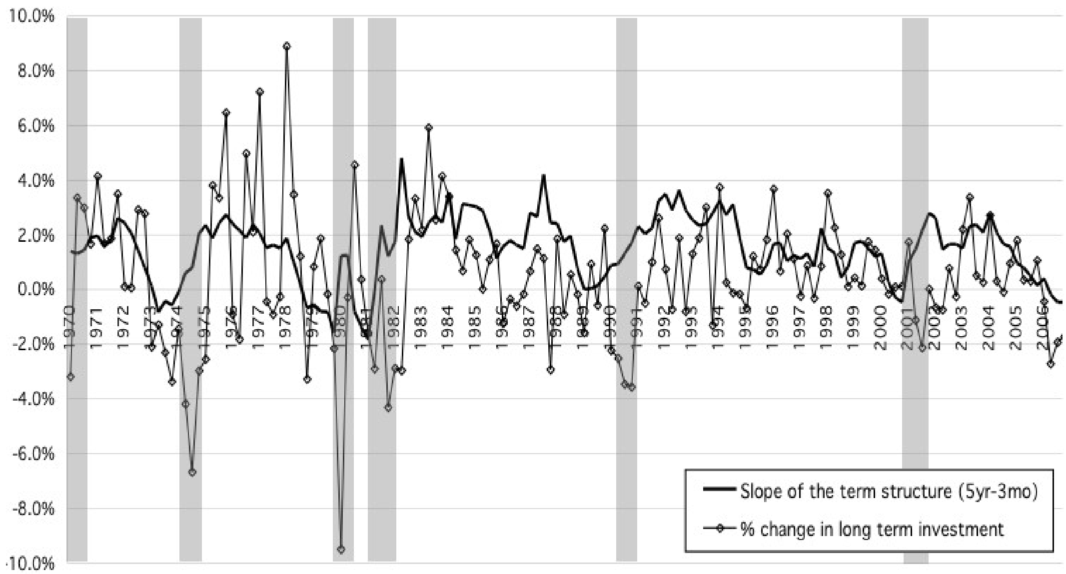


(b) Panel B: Quarterly change in short term investments vs. the ex-post real short interest rate.

Figure 1: The short interest rate versus % changes in consumption and % changes in short term investments. Quarterly data from 1970 to 2007. The shaded areas represent the NBER recessions.



(a) Panel A: Quarterly change in consumption vs. the slope of the term structure.



(b) Panel B: Quarterly change in short term investments vs. the slope of the term structure.

Figure 2: The slope of the term structure of interest rates versus % changes in consumption and % changes in long term investments. Quarterly data from 1970 to 2007. The shaded areas represent the NBER recessions.

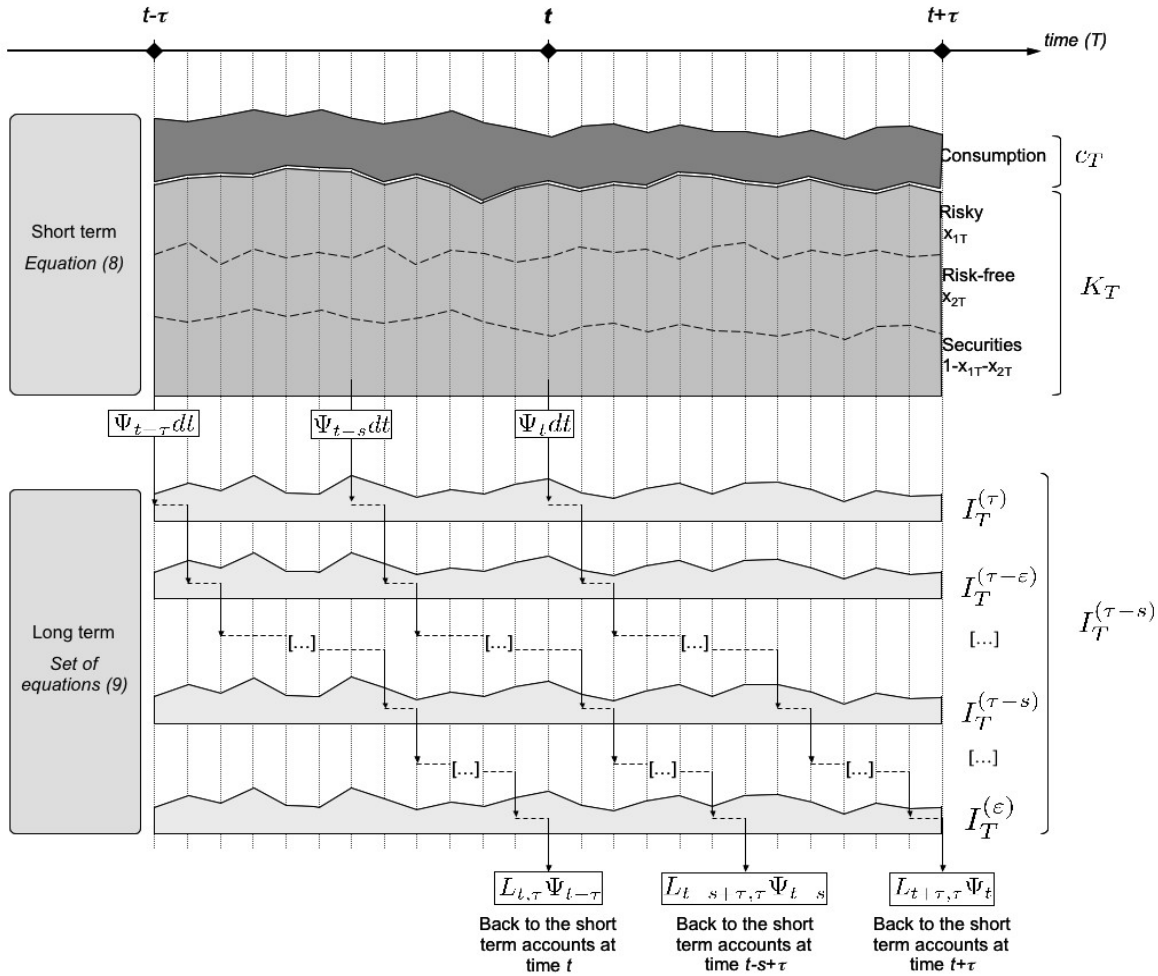
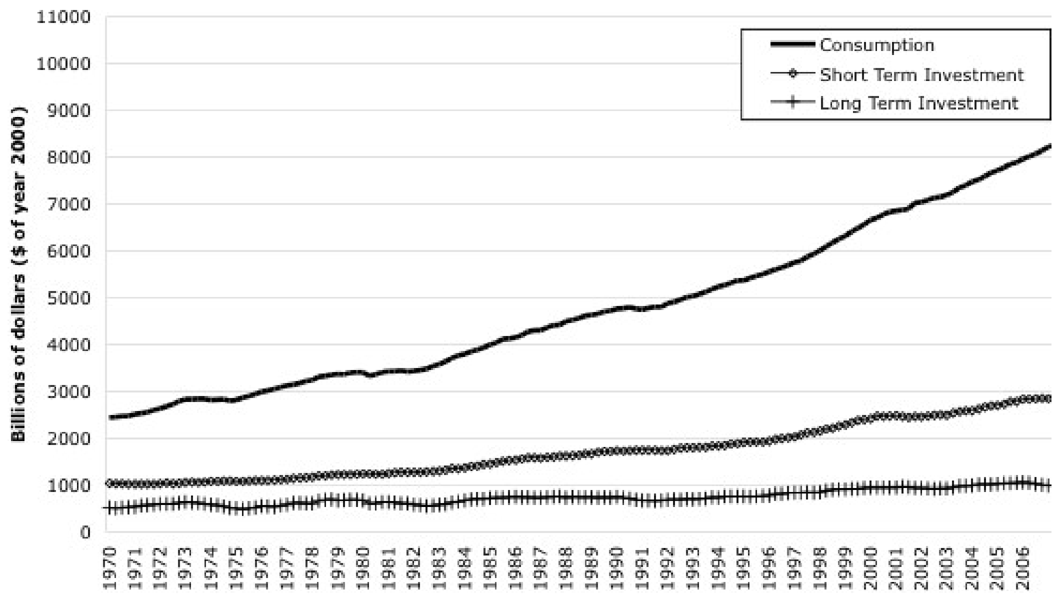
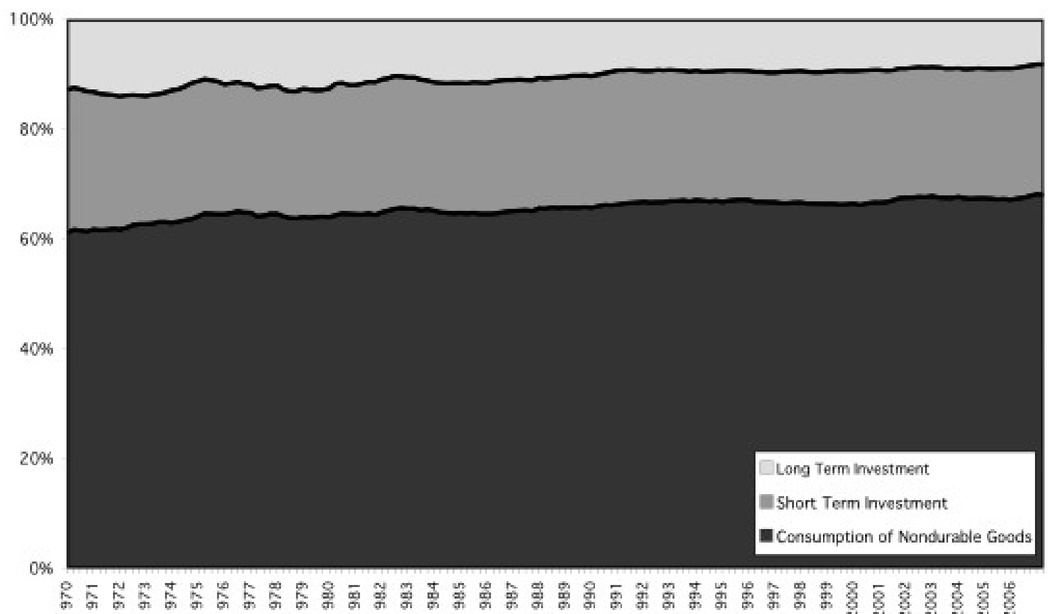


Figure 3: Scheme of capital allocation at any time T. At time T=t, the agent must choose the amount of capital : (i) c_t that he consumes, and (ii) $\Psi_t dt$ that he allocates to the long term investment. The rest is allocated in the short term investment account K_t. The long term investment I_t follows a similar process than the time-to-build channel in Kydland and Prescott (1982). First, $\Psi_t dt$ goes to the long term subaccount $I_t^{(\tau)}$, which is the subaccount at time $T = t$ for the capital that will mature τ periods from time t, that is, at time $T = t + \tau$ At time $T = t + \epsilon$, this amount $\Psi_t dt$ together with the return earn in the interval $[t, t + \epsilon]$ will go to the next subaccount $I_{t+\epsilon}^{(\tau-\epsilon)}$. This long term investment moves to next subaccount over time until it gets the account $I_\epsilon^{(\tau+\epsilon)}$ at time $T = t + \epsilon$. Therefore, at time $T = t + \epsilon$, the amount $\Psi_t dt$ will have become $L_{t-\tau,\tau}\Psi_t$, where $L_{t-\tau,\tau}$ its compounded risky long term return. This amount $L_{t-\tau,\tau}\Psi_t$ will revert (return) to the short term account $K_{t+\tau}$ at time $T = t + \tau$. Note that the amount $\Psi_{t-\tau}dt$ that he invested at time $T = t - \tau$ comes back with its accrued return as $L_{t,\tau}\Psi_{t-\tau}$. Note also that all the amounts Ψ_{t-s} that he has invested in the long term technology at any time T within the interval $(t - \tau, t)$ will come back to the short term account with their accrued return as $L_{t-s+\tau,\tau}\Psi_{t-s}$ after τ units of time, where s scans all the values of the interval $[0, \tau - \epsilon]$.



(a) Panel A. Absolute terms.



(b) Panel B. Relative terms.

Figure 4: Historical performance of real consumption of nondurable goods (c_t), short term investment and long term investment from 1970Q1 to 2007Q1 in absolute and relative terms.

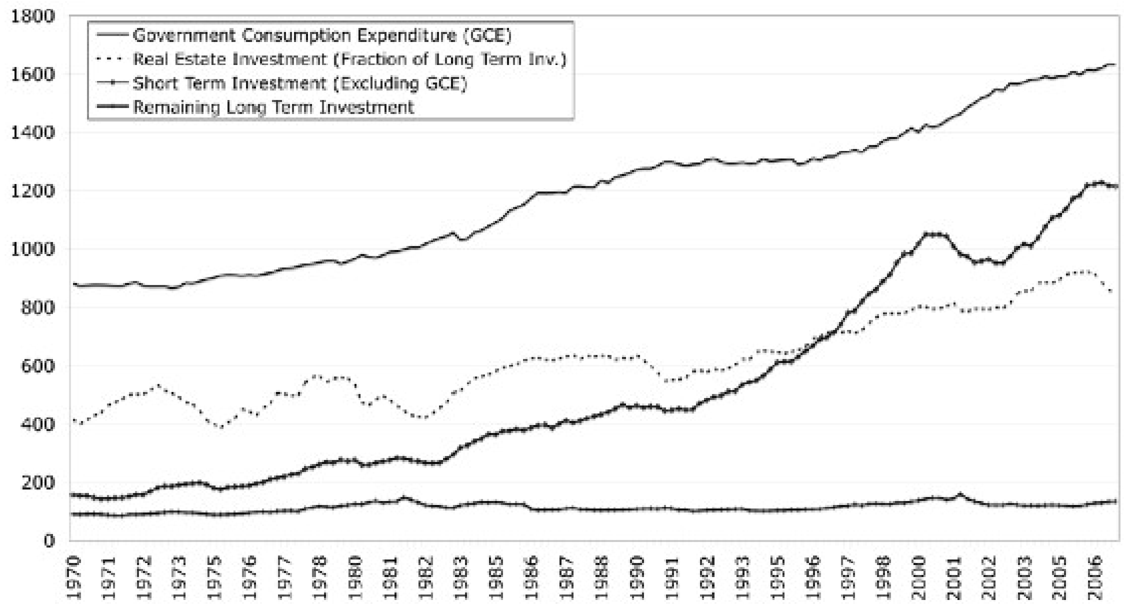


Figure 5: Historical performance in real terms of government consumption expenditure (GCE), short term investment (excluding GCE), real estate investment (fraction of long term investment) and the remaining non-real estate long term investment. The shaded areas represent the NBER recessions.

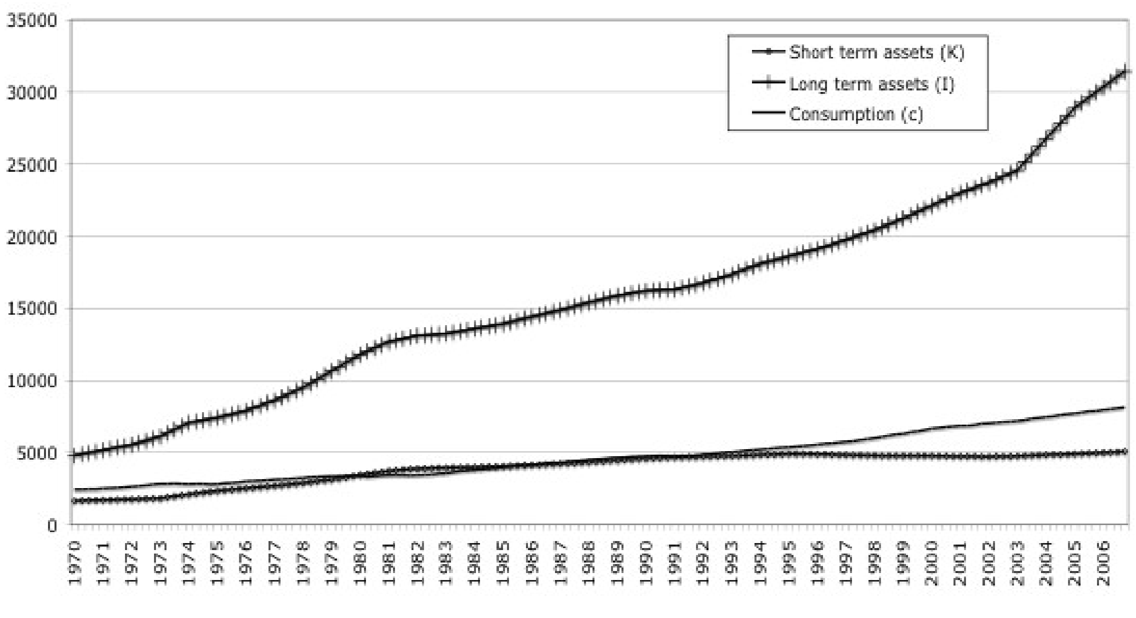
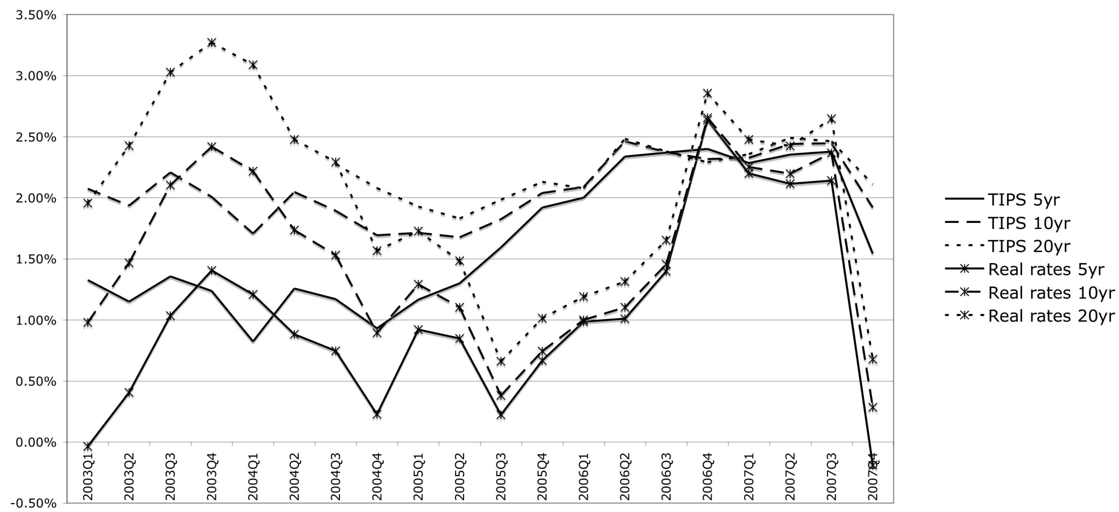
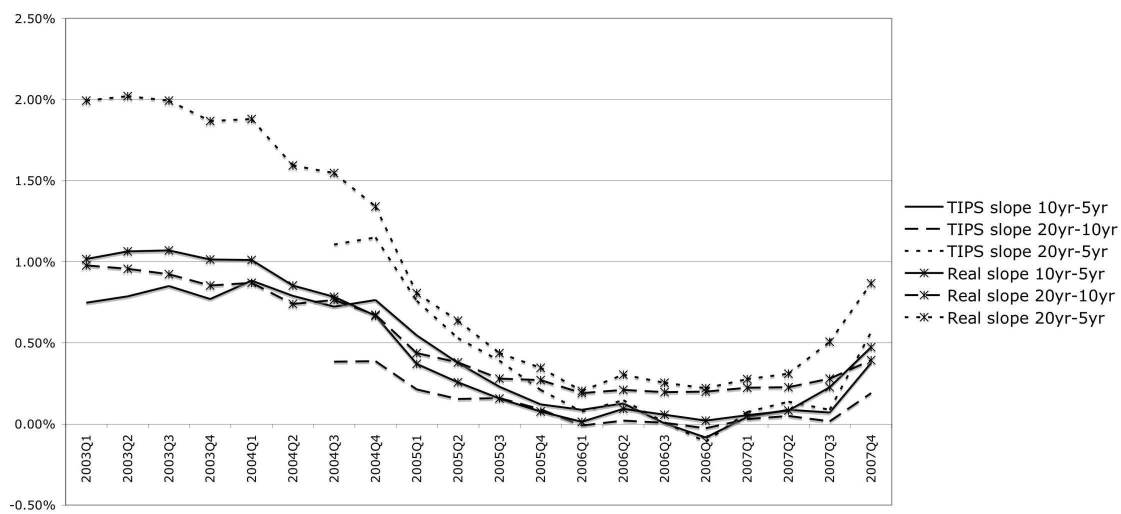


Figure 6: Historical real aggregate short term assets (K_t) and long term assets (I_t) and real consumption (c_t) in absolute terms (in Billions of $ of year 2000).



(a) Panel A. Comparison between TIPS rates and real rates.



(b) Panel B. Comparison between the slope of the term structures of TIPS rates and real rates.

Figure 7: Comparison between data on TIPS and real rates (calculated as deflated nominal rates). Period 2003-2007.

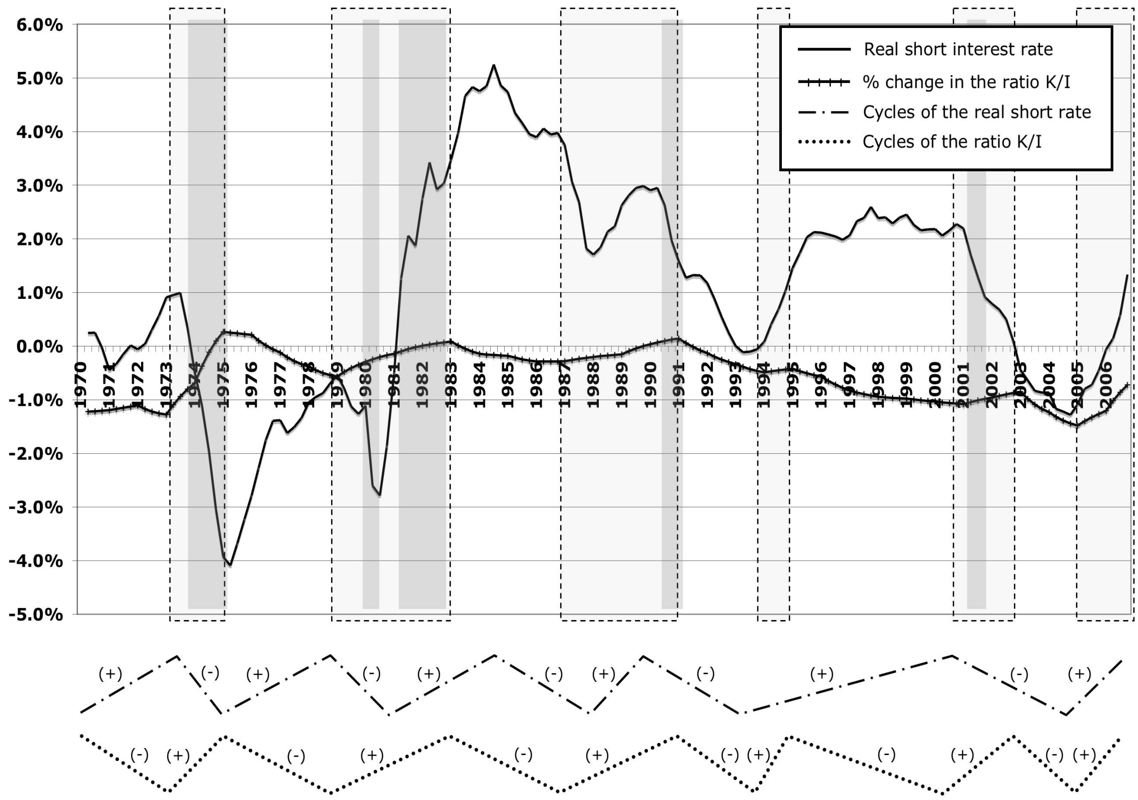


Figure 8: Historical real short term interest rate and % quarterly change in the ratio K_t/I_t. The dashed areas remark the periods in which the % change in the ratio K_t/I_t increases. The shaded areas represent the NBER recessions. The cycles of the real short interest rate and the cycles of the % change in K_t/I_t are shown below the graph.

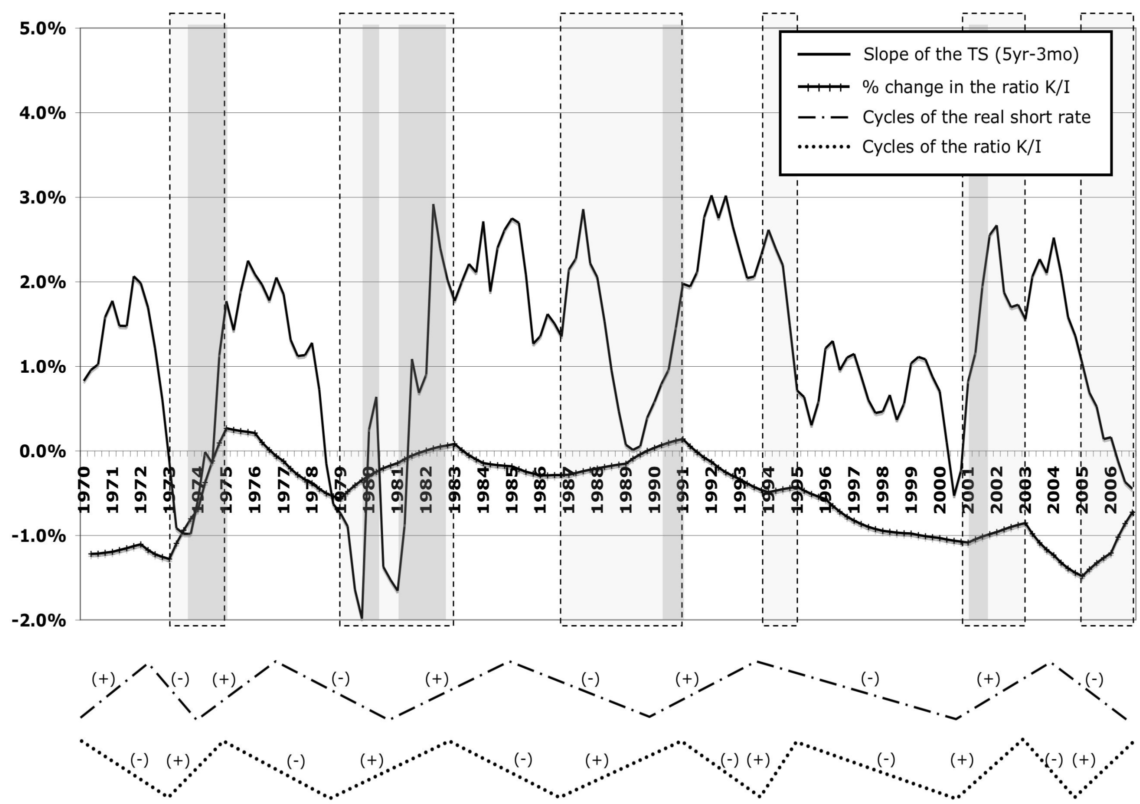


Figure 9: Historical performance of the slope of the term structure of interest rates and % quarterly change in the ratio K_t/I_t. The dashed areas remark the periods in which the % change in the ratio K_t/I_t increases. The shaded areas represent the NBER recessions. The cycles of the slope of the term structure and the cycles of the % change in K_t/I_t are shown below the graph.

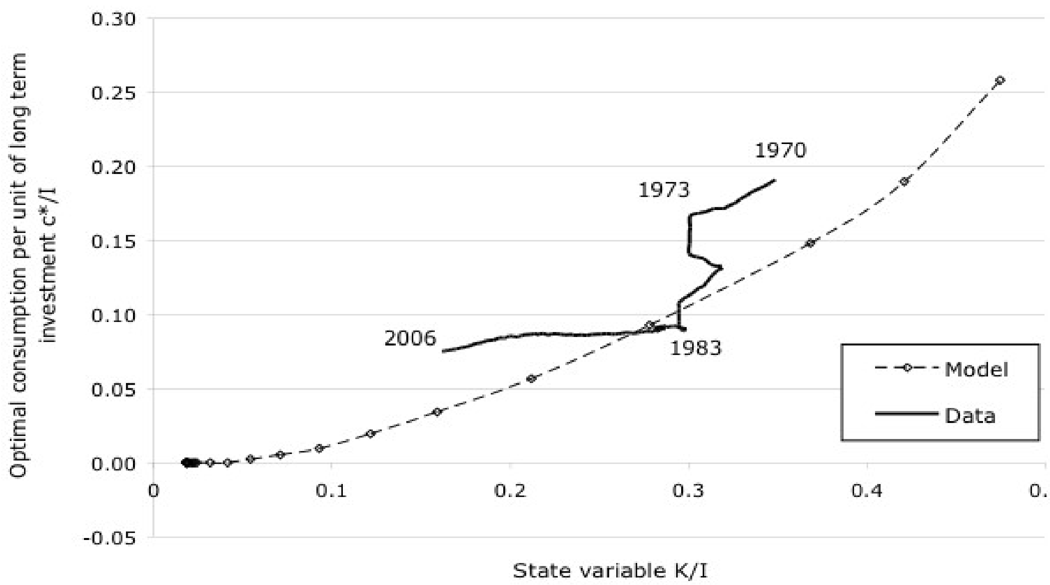


Figure 10: Optimal consumption over capital stock of long term investment c_t/I_t, as a function of the ratio K_t/I_t. Model vs. real data.