Intertemporal Asset Allocation with Habit Formation in Preferences: An Approximate Analytical Solution*

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Abstract

In this paper we derive an approximate analytical solution to the optimal consumption and portfolio choice problem of an infinitely-lived investor with power utility defined over the difference between consumption and an external habit. The investor is assumed to have access to two tradable assets: a risk free asset with constant return and a risky asset with a time-varying premium. We extend the approach proposed by Campbell and Viceira (1999), which builds on log-linearizations of the Euler equation, intertemporal budget constraint, and portfolio return, to also contain the log-linearized surplus consumption ratio. The 'difference habit model' implies that the relative risk aversion is time-varying which is in line with recent evidence from the asset pricing literature. We show that accounting for habit affects both the myopic and intertemporal hedge component of optimal asset demand, and introduces an additional component that works as a hedge against changes in the investor's habit level. In an empirical application, we calibrate the model to U.S. data and show that habit formation has significant effects on both the optimal consumption and portfolio choice compared to a standard CRRA utility function.

JEL Classification: C32, G11, G12
Keywords: Intertemporal consumption and portfolio choice, habit formation, time-varying expected returns, time-varying risk aversion

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1 Introduction

One of the most promising findings in finance in recent years is the fact that habit formation appears to be able to resolve the equity premium puzzle.[1] A highly used assumption when examining the puzzle is that investors' utility is separable over time, i.e. the utility of investors' consumption choice today is not affected by consumption in previous periods. Habit formation breaks with this somewhat unrealistic assumption and introduces time-nonseparability in investors' consumption choice by letting utility be defined over a function of investors' consumption and habit level. The intuition here is that investors do not derive utility from the level of their consumption, but from their consumption relative to a given habit level. In the literature on habit formation there are two types of models that govern the specification of the habit level: Internal habit models where investors' habit is a function of their own previous consumption, and external habit models where habit is a function of the consumption of a peer group or aggregate consumption.[2] Another modelling issue in the habit literature revolves around how investors' consumption is related to the habit level. Abel (1990) and Chan and Kogan (2002) are examples of so-called ratio models in which utility is defined over the ratio of consumption to habit, while Constantinides (1990) and Campbell and Cochrane (1999) are examples of so-called difference models in which utility is defined over the difference between consumption and habit. The main difference between these two types of models is that the relative risk aversion is constant in ratio models but time-varying in difference models. Time-varying relative risk aversion implies that the price of risk is time-varying, and hence expected returns will also vary over time. This feature is in line with the ever growing literature documenting time-varying expected stock returns, see e.g. Fama and French (1988, 1989), Campbell and Shiller (1988a,b), Campbell (1991), and Lettau and Ludvigson (2001).

A field of finance where time-varying expected returns have great importance is intertemporal asset allocation. As first documented by Samuelson (1969) and Merton (1969, 1971), the optimal portfolio of an investor with a multiperiod investment horizon differs from the optimal portfolio of a myopic investor if returns are not independent and identically distributed (i.i.d.) over time. Following the documentation of time-varying expected returns in the late 1980's this field of finance has received a lot of attention resulting in a growing body of literature. One string of the literature delivers solutions for the optimal asset allocation using numerical methods based on discrete-state approximations (see e.g. Brennan et al. (1997), Balduzzi and Lynch (1999), Barberis (2000), and Lynch (2001)), while another derives exact closed-form solutions in a continuous-time setting (see e.g. Kim and Omberg (1996) and Wachter (2002)). Finally, approximate analytical solutions in discrete time have been developed by Campbell and Viceira (1999)

[1] Mehra and Prescott (1985) coined the frase *Equity Premium Puzzle* with their seminal paper in which they documented that the average excess return on U.S. stocks over U.S. Tresury Bills is too high to be explained by reasonable values of the relative risk aversion. Kocherlakota (1996) reexamined their results using a different approach but still arrived at the same conclusion.

[2] External habit formation is also referred to as "keeping/catching up with the Joneses" in the literature, cf. Abel (1990). In this paper, we will use the phrase "external habit formation".

in a setup with only one risky asset and Campbell et al. (2003) in a setup that allows for multiple risky assets. However, a standard assumption in this literature is that investors' relative risk aversion is constant over time, which is not in accordance with the recent evidence from the asset pricing literature.

Despite its success in explaining asset returns and intuitive sensible feature of time-varying risk aversion when choosing difference models, habit formation has received surprisingly little attention in the intertemporal asset allocation literature. An early proponent of including habit in the intertemporal asset allocation model is Rubinstein (1976a,b) who works with models in which the habit or subsistence level is constant over time. In continuous time, Sundaresan (1989) and Ingersoll (1992) provide closed-form solutions to the portfolio and consumption rule in a setup with habit and constant investment opportunities, while Munk (2007) allows for time-varying investment opportunities under the assumption of complete markets. In the case with mean-reverting stock returns and constant interest rates, Munk (2007) derives a closed-form solution, and furthermore he gives numerical examples that illustrate the magnitude with which habit formation influences the optimal portfolio choice. Other examples from the continuous-time literature are Hindy et al. (1997) and Bodie et al. (2004), who also rely on the complete markets assumption.[3] In discrete time, habit formation has primarily been used in the literature on optimal portfolio choice over the life-cycle; see e.g. Lax (2002), Gomes and Michaelides (2003), and Polkovnichenko (2007). Gomes and Michaelides (2003) consider an investor with risky nontradable labor income and access to a constant investment opportunity set and solve the problem numerically, while Lax (2002) derives an analytical solution in a setup without labor income but also with i.i.d. returns. Polkovnichenko (2007) allows for risky assets with stochastic returns in the form of an i.i.d. Markov process with two outcomes and characterize admissible habit-wealth regions for every age analytically before he solves the problem numerically within each region. Heaton and Lucas (1997) use the same assumption regarding the return on the risky asset and solve the optimal consumption and portfolio problem numerically in an infinite-horizon setting. Heaton and Lucas (1997) and Polkovnichenko (2007) both allow the investor to have risky nontradable labor income. All these papers specify habit to be formed internally.

However, the current literature gives no analytical solutions to the discrete-time intertemporal asset allocation problem when the investor faces time-varying expected returns and is endowed with external habit formation in his preferences. In this paper, we fill this gap in the literature by using the approach suggested by Campbell and Viceira (1999) to derive the optimal portfolio and consumption choice of an infinitely-lived investor. This approach is set in discrete time and by using a number of approximate relations, Campbell and Viceiria are able to derive an approximate analytical solution to the intertemporal consumption and portfolio problem.[4] The difference between their

[3] In a slightly different setting, Shore and White (2006) use external habit formation to explain the equity home bias puzzle by assuming that a small group of agents is forced to hold domestic stocks. They derive an exact result for the portfolio and consumption choice in continuous time and calibrate the model using consumption and asset return moments from several countries.

[4] A common assumption in the continuous-time literature is the existence of complete markets, i.e.

result and the result given in this paper lies solely in the formulation of the investor's preferences: Campbell and Viceira use recursive preferences proposed by Epstein and Zin (1989, 1991) and Weil (1989), while we use standard power utility with an incorporated habit effect. Recursive preferences imply that investor's utility is state-nonseparable, while habit formation in preferences imply time-nonseparability. We choose a difference habit model and hence the investor's risk aversion varies over time in this framework in contrast to the framework by Campbell and Viceira and the vast majority of papers on this topic. Futhermore, in contrast to the current literature, we assume an external habit that evolves with the rate of change in per capita consumption. By extending the approximate framework in Campbell and Viceira (1999) to also include a log-linearization of the surplus consumption ratio as defined by Campbell and Cochrane (1999), we are able to derive an approximate analytical solution to the intertemporal consumption and portfolio choice problem. We show that both the optimal myopic and intertemporal hedge demand is affected by the distance between investor's consumption and habit level. This distance works as an adjustment factor to the utility curvature parameter. Hence, the optimal asset allocation can be very different for investors with the same utility curvature parameter if their distance to habit is different. Besides affecting the myopic and intertemporal hedge demand, habit formation also gives rise to a new component in the optimal portfolio choice. This component works as a hedge against changes in investor's habit level and carries most weight when the investor consumes close to his habit level. In an empirical application, we calibrate the model to U.S. quarterly data and show that habit formation has very large effects on both the optimal myopic and intertemporal hedge demand as well as the optimal consumption choice.

It is important to note that the approach in this paper is - as most approaches in this area are - partial equilibrium in nature. The model is solved for an investor with a given utility function and who takes the return process as exogenously given. There is nothing in this model that makes this particular return process consistent with general equilibrium. As noted by Cochrane (1999), in a general equilibrium model the average investor will always hold the market portfolio and not be engaged in strategic or tactical asset allocation. Thus, this model gives the optimal consumption and portfolio choice for an investor who somehow deviates from the average investor, for example because of higher or lower risk aversion.

The paper is organized as follows. Section 2 explains the setup and derives the approximate analytical solution, and Section 3 presents an empirical application to U.S. quarterly data. Section 4 contains some concluding remarks.

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asset returns and state variables governing investment opportunities are driven by the same stochastic process, which implies that innovations to investment opportunities are perfectly hedgeable using financial assets. The discrete-time models by Campbell and Viceira (1999) and Campbell et al. (2003) sidestep this assumption and allow for incomplete markets, which is an important empirical advantage.

2 The asset allocation model

2.1 Investment assets and state variables

The investor is assumed to have access to two tradable assets: A risky asset with gross return $R_{1,t+1}$ and a riskless asset with constant gross return R_f. The portfolio return from time t to time $t+1$ is given by

$$R_{p,t+1} = \alpha_t \left(R_{1,t+1} - R_f\right) + R_f,$$

where α_t denotes the proportion of total wealth invested in the risky asset at time t. The log (continuously compounded) return is given as $r_{p,t+1} = \log\left(R_{p,t+1}\right)$, $r_{1,t+1} = \log\left(R_{1,t+1}\right)$, and $r_f = \log\left(R_f\right)$, respectively.

The expected excess log return on the risky asset depends on one state variable x_t

$$E_t r_{1,t+1} - r_f = x_t, \tag{1}$$

which follows an AR(1)

$$x_{t+1} = \mu + \phi\left(x_t - \mu\right) + \eta_{t+1}, \tag{2}$$

where $\eta_{t+1} \sim N\left(0, \sigma_\eta^2\right)$.

The unexpected log return on the risky asset, $u_{t+1} = r_{1,t+1} - E_t r_{1,t+1}$, is also conditionally homoskedastic and normally distributed, i.e. $u_{t+1} \sim N(0, \sigma_u^2)$. Furthermore, it is correlated with the innovations in the state variable

$$Cov_t(u_{t+1}, \eta_{t+1}) = \sigma_{u\eta}.$$

This simplified setup with only one risky asset, a riskless asset with constant return and one state variable is adopted for expositional purposes. Following Campbell et al. (2003) the approach in this paper can by generalized to multiple risky assets and state variables at the cost of greater complexity.

2.2 Investor preferences

We want the investor to account for habit when evaluating his utility. This requires some considerations, since habit formation can be specified in a number of ways. First, habit can be formed internally or externally, meaning that either the investor forms his habit level based on his own previous consumption level, or he uses the consumption of others such as a peer group or the economy as a whole as benchmark. The choice here is a matter of belief. By letting the habit be formed internally you are also automatically saying that the investor does not care how much people around him are consuming. In this paper, we let the habit be formed externally. Second, we need to distinguish between difference and ratio models. Ratio models have the advantage of ensuring that utility is defined whenever consumption and habit are positive, while difference models require that consumption is always above habit in order for utility to be defined. We will use a difference model, since, as opposed to a ratio model, this implies time-varying risk aversion. Finally, we will let the utility function be of the power form and assume that the investor has an infinite investment horizon, which implies that the utility maximization problem is given by

$$\max \ E_t \sum_{j=0}^{\infty} \delta^j \frac{(C_{t+j} - H_{t+j})^{1-\gamma} - 1}{1-\gamma}, \qquad C_{t+j} > H_{t+j}$$

where C_t is the investor's real consumption, H_t is the level of external habit, $\gamma > 0$ is the utility curvature parameter, and δ is the discount factor. The investor is only endowed with financial wealth, and hence maximizes his utility subject to the standard intertemporal budget constraint

$$W_{t+1} = R_{p,t+1}(W_t - C_t).$$

where W_t denotes (financial) wealth.

The habit level is assumed to evolve according to the following law of motion

$$H_t = H_{t-1} \left(\frac{\overline{C}_t}{\overline{C}_{t-1}} \right)^{\lambda}, \tag{3}$$

where $\overline{C}_t$ denotes per capita consumption in the economy. According to this law of motion, the change in the individual specific habit level depends on the change in per capita consumption. The parameter λ determines how much the individual investor responds to per capita consumption growth in the economy. We will refer to this parameter as 'per capita consumption sensitivity'. For $\lambda = 1$, the growth in the investor's habit level is exactly equal to the growth in per capita consumption, while for $\lambda = 0$, the habit level is constant over time. For $0 < \lambda < 1$, the growth rate in investor's habit is less than the growth rate in per capita consumption, while for $\lambda > 1$, the growth rate is larger. For $\lambda < 0$, the investor's habit level responds negatively (positively) to increases (decreases) in per capita consumption, which clearly does not seem reasonable, and hence only non-negative values of λ will be considered in the empirical application. The main reason for choosing this law of motion is to link the change in habit, which is unobservable, to the change in an observable variable which is a good proxy for how the habit level evolves, namely per capita consumption. The choice of proxy is by no means restrictive, since this approach allows for an arbitrary choice of proxy as long as it can be characterised as an external variable. The use of λ is intended to mimic the fact that investors belong to different social groups, and hence might not form their habit based on per capita consumption in the economy as a whole, but rather on per capita consumption in their peer group. Investors with a high social status might choose a high λ, while investors with low social status might choose a low λ.

The growth in per capita consumption is modelled as an independently and identically distributed (i.i.d.) lognormal process

$$\Delta \bar{c}_{t+1} = g + v_{t+1}, \tag{4}$$

where Δ is the first difference operator and lowercase letters denote the variables in logs. The innovations in equation (4) are assumed to be conditionally homoskedastic and normally distributed , i.e. $v_{t+1} \sim N\left(0, \sigma_v^2\right)$, and correlated with the unexpected log return on the risky asset and the innovations in the state variable, respectively

$$Cov_t\left(u_{t+1}, v_{t+1}\right) = \sigma_{uv},$$

$$Cov_t\left(\eta_{t+1}, v_{t+1}\right) = \sigma_{\eta v}.$$

This specification implies that the expected change in log habit is constant.

Following Campbell and Cochrane (1999), we define the *surplus consumption ratio* as

$$S_t = \frac{C_t - H_t}{C_t}, \tag{5}$$

which implies that the relative risk aversion is measured as γ / S_t. The relative risk aversion is time-varying and counter-cyclical: In periods where consumption is close to habit, S_t is low and the relative risk aversion is high and when consumption is well above habit, S_t is high and relative risk aversion is low.

With this utility function, the optimal consumption and portfolio choice must satisfy the following Euler equation for any asset i:

$$1 = E_t\left[\delta\left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} R_{i,t+1}\right].$$

2.3 An approximate framework

If we assume that consumption growth, growth in the surplus consumption ratio, and return on wealth have a joint conditional lognormal distribution, we can write the log Euler equation as follows

$$
\begin{aligned}
0 = &\log \delta - \gamma E_t \Delta s_{t+1} - \gamma E_t \Delta c_{t+1} + E_t r_{i,t+1} \\
&+ \frac{1}{2} Var_t \left(-\gamma \Delta s_{t+1} - \gamma \Delta c_{t+1} + r_{i,t+1} \right).
\end{aligned} \tag{6}
$$

Alternatively, we can use a second-order Taylor approximation around the conditional mean of $\{\Delta s_{t+1}, \Delta c_{t+1}, r_{i,t+1}\}$ and the approximation $\log(1+x) \approx x$ for small x to justify this log version of the Euler equation. Subtracting the log Euler equation for the riskless asset from the general log Euler equation, we get the following

$$
E_t r_{i,t+1} - r_f + \frac{1}{2} Var_t \left(r_{i,t+1} \right) = \gamma Cov_t \left(\Delta s_{t+1}, r_{i,t+1} \right) + \gamma Cov_t \left(\Delta c_{t+1}, r_{i,t+1} \right). \tag{7}
$$

The optimal portfolio and consumption choice must satisfy this equation.

Following Campbell (1993, 1996) we also log-linearize the budget constraint around the mean consumption-wealth ratio which yields

$$
\Delta w_{t+1} = r_{p,t+1} + \left(1 - \frac{1}{\rho} \right) (c_t - w_t) + k_1. \tag{8}
$$

where $\rho = 1 - \exp\left[E\left(c_t - w_t \right) \right]$ and $k_1 = \log \rho + \frac{1-\rho}{\rho} \log(1-\rho)$. The details of this result are given in Appendix 1. As noted by Campbell and Viceira (1999), ρ is endogenous in the sense that it depends on the average consumption-wealth ratio which is unknown until the model has been solved.

One of the elements in this result is the log portfolio return. The aim is to determine the optimal portfolio choice, i.e. the fraction of wealth invested in the risky and the riskless asset, respectively. Hence, we need to replace the log portfolio return with the corresponding function of log return on the risky and the riskless asset. However, taking logs of the simple return on the portfolio yields a nonlinear relation. In order to get a linear relation we perform a second-order Taylor expansion around $r_{1,t+1} - r_f = 0$ to arrive at

$$
r_{p,t+1} - r_f = \alpha_t (r_{1,t+1} - r_f) + \frac{1}{2} \alpha_t (1 - \alpha_t) Var_t \left(r_{1,t+1} \right). \tag{9}
$$

The details are given in Appendix 2.

The approximate relations shown thus far are all identical to those used by Campbell and Viceira (1999). In this paper we need to push the approximate framework a bit further due to the use of habit formation. One of the elements of the log Euler equation is the change in log surplus consumption ratio, which can be written as follows

$$\Delta s_{t+1} = \log\left(1 - \exp\left(h_{t+1} - c_{t+1}\right)\right) - \log\left(1 - \exp\left(h_t - c_t\right)\right).$$

In order to get a linear relation we perform a first-order Taylor expansion around the mean habit-consumption ratio of each of the two nonlinear relations. This yields the following result

$$\Delta s_{t+1} = \left(1 - \frac{1}{\theta}\right)\left(\Delta h_{t+1} - \Delta c_{t+1}\right), \tag{10}$$

where $\theta = 1 - \exp\left[E\left(h_t - c_t\right)\right]$. The details are given in Appendix 3. The change in log surplus consumption ratio is now a function of the change in log consumption and log habit, which can be rewritten as a function of change in log per capita consumption

$$\Delta h_{t+1} = \lambda \Delta \bar{c}_{t+1}. \tag{11}$$

This follows directly from the assumed law of motion of the habit level as seen in (3).

While the log-linearization parameter from the budget constraint ρ is endogenous as mentioned above, the log-linearization parameter from the surplus consumption ratio θ will in this model be treated as an exogenous parameter. We will solve the model using different prefixed values of θ. For utility to be well defined, habit must always be below consumption, which implies that $0 < \theta < 1$. By choosing values of θ close to 1, we examine the optimal portfolio- and consumption choice of an investor who on average consumes well above his habit level, and by choosing θ close to 0, we examine the optimal choice of an investor who on average consumes close to his habit level.

2.4 The surplus consumption ratio

When the habit-consumption ratio is constant, θ can be interpreted as $\left(C - H\right)/C$, i.e. the surplus consumption ratio, which again implies that we can interpret γ/θ as the 'average' relative risk aversion. For fixed values of γ this measure of relative risk aversion varies between investors if their average distances between consumption and habit level are different. Hence, for the investor who consumes well above his habit level on average, the 'average' relative risk aversion is approximately equal to γ, while it will increase for investors with consumption closer to habit. Note that the time-varying relative risk aversion is still given by γ/S_t, and thus depends on the investor's actual consumption choice relative to his habit level. Hence, it is endogenous in this model. The true average relative risk aversion is the average of γ/S_t. In order to avoid misunderstandings, we will in the following refer to θ as 'distance to habit' as opposed to surplus consumption ratio.

If we assume a starting value for the surplus consumption ratio, we will be able to analyze how the relative risk aversion varies over time. We can rewrite (10) as follows using (11)

$$S_{t+1} = S_t \exp\left[\left(1 - \frac{1}{\theta}\right)(\lambda \Delta \bar{c}_{t+1} - \Delta c_{t+1})\right]. \tag{12}$$

Furthermore, if we combine the log-linear budget constraint (8) with the trivial equality

$$\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}, \tag{13}$$

we can write

$$\Delta c_{t+1} = r_{p,t+1} + (c_{t+1} - w_{t+1}) - \frac{1}{\rho}(c_t - w_t) + k_1. \tag{14}$$

Since per capita consumption is exogenously given and we are going to solve the model for the consumption-wealth ratio and the portfolio return, we are thus able to calculate the surplus consumption ratio. Another approach would be to assume a starting value for the habit level, calculate the habit over time using (3), and then use (5) together with the optimal consumption choice to get the surplus consumption ratio. However, the investor's optimal consumption and portfolio choice is based on the approximate framework, and hence on the log-linearized surplus consumption ratio. In order to be consistent with the setup in this model, we use (12) to calculate the surplus consumption ratio and then derive the habit using (5). This approach also has the advantage of ensuring a positive surplus consumption ratio (if the assumed starting value is positive), which implies that consumption is larger than habit, and hence that utility is well defined.[5]

2.5 Consumption implications

Using the previous stated results we can rewrite the log Euler equation for the portfolio return in the following way

$$E_t \Delta c_{t+1} = \frac{\theta}{\gamma} \log \delta + \lambda (1 - \theta) E_t \Delta \bar{c}_{t+1} + \frac{\theta}{\gamma} E_t r_{p,t+1} + \upsilon_{p,t}, \tag{15}$$

where $\upsilon_{p,t}$ is given as

$$\upsilon_{p,t} = \frac{\theta\gamma}{2} Var_t\left(\left(\frac{1}{\theta} - 1\right)\lambda \Delta \bar{c}_{t+1} - \frac{1}{\theta}\Delta c_{t+1} + \frac{1}{\gamma} r_{p,t+1}\right). \tag{16}$$

[5] The sparse literature on optimal consumption and portfolio choice with habit formation in the utility function contains different approaches to ensuring that consumption stays above habit. In the discrete-time life-cycle literature Lax (2002) and Polkovnichenko (2007) derive maximum habit-wealth regions above which the investors cannot solve the problem, while Gomes and Michaelides (2003) only solve the problem for parameter values for which the probability of consumption falling below habit is either zero or negligible.

From (15) we can identify four forces acting on individual specific consumption. First, the more patient, the investor is, i.e. the higher δ is, the more he is willing to postpone consumption, which implies a high consumption growth. Second, since $\lambda(1-\theta)$ is non-negative an increase in expected growth in per capita consumption implies that the investor will lower his consumption today in order to ensure that his consumption tomorrow will be able to match the increase in his habit level. Note that this effect drops out if the investor has a constant habit level, i.e. $\lambda = 0$, or if the investor on average consumes well above his habit level, i.e. $\theta \approx 1$, and thus does not take habit into account. Third, a high expected portfolio return gives the investor an incentive to postpone consumption. The first and the third term are adjusted by θ/γ. As opposed to the standard CRRA case, this adjustment factor depends on the distance to habit, which implies that the closer to habit, the investor consumes, the less emphasis he will put on these factors in determining his consumption path. In other words, decreasing distance to habit works in the same way as increasing risk aversion. The intution here is that the closer the investor is to his habit level, the more he is interested in maintaining a consumption growth, that will match the growth in per capita consumption, and thus in his habit level. Hence, the weight on the second term in (15) is high, while the weights on the first and third term are low. Finally, the fourth term captures the consumption uncertainty. A risk averse investor responds to uncertainty by increasing precautionary savings, i.e. lower his consumption today. Compared to the case without habit this effect is adjusted by the distance to habit. The explanation here is the same as for the first and the third term: When the investor's consumption is close to his habit level, he is only interested in matching consumption to habit in order to ensure that consumption will not fall below habit. Hence, in this case decreasing distance to habit works in the opposite direction as increasing risk aversion.

We can obtain additional insight into the model by examining the consumption-wealth ratio. If we combine the log-linear budget constraint (8) with the trivial equality (13), solve forward assuming $\lim_{j\to\infty} \rho^j (c_{t+j} - w_{t+j}) = 0$, and take conditional expectations, we get

$$c_t - w_t = E_t \sum_{j=1}^{\infty} \rho^j (r_{p,t+j} - \Delta c_{t+j}) + \frac{\rho k_1}{1-\rho}. \tag{17}$$

This equation says that a high consumption-wealth ratio must be followed by either high returns or low consumption growth in the future. If we in addition to lognormality assume homoskedasticity in deriving the log Euler equation (6), we can write (15) as

$$E_t \Delta c_{t+1} = \lambda(1-\theta) E_t \Delta \overline{c}_{t+1} + \frac{\theta}{\gamma} E_t r_{p,t+1} + \chi, \tag{18}$$

where χ includes the rate of time preference and the effects of risk on consumption. Substituting (18) into (17) we get

$$c_t - w_t = \left(1 - \frac{\theta}{\gamma}\right) E_t \sum_{j=1}^{\infty} \rho^j r_{p,t+j} - \lambda\left(1-\theta\right) E_t \sum_{j=1}^{\infty} \rho^j \Delta \bar{c}_{t+j} + \frac{\rho\left(k_1 - \chi\right)}{1-\rho}. \qquad (19)$$

Since $\lambda\left(1-\theta\right)$ is non-negative and $1-\theta/\gamma$ is non-negative for investors with $\gamma \geq 1$, this equation says that for these types of investors a high consumption-wealth ratio today must be followed by either high returns or low growth in per capita consumption, i.e. the habit level. For investors with $\gamma < 1$ a high consumption-wealth ratio can actually be followed by negative returns given that $\theta/\gamma > 1$. From this equation we also see that in order for the consumption-wealth ratio to be constant, it must hold that $\theta/\gamma = 1$ and either $\lambda = 0$ or $\theta = 1$. Hence, the consumption-wealth ratio will only be constant for investors with $\gamma = 1$, who does not take habit into account and for investors with $\gamma < 1$, $\theta = \gamma$, and a constant habit level. Thus, opposed to the standard CRRA case, it is now possible for an investor with $\gamma = 1$ to experience a time-varying consumption-wealth ratio.

Note that we model $E_t \Delta \bar{c}_{t+1}$ to be costant and equal to g as seen from (4).

2.6 Characterizing the optimal portfolio rule

Now we are ready to characterize the optimal portfolio rule. We do this by characterizing the covariance terms entering (7). We start with $Cov_t\left(\Delta c_{t+1}, r_{1,t+1}\right)$, where we use the following result that appears when inserting the log portfolio return (9) in the log budget constraint (8)

$$\Delta w_{t+1} = \alpha_t(r_{1,t+1} - r_f) + r_f + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) + k_1 + \frac{1}{2}\alpha_t(1-\alpha_t)Var_t\left(r_{1,t+1}\right),$$

as well as the trivial equality (13). Thus we get

$$Cov_t\left(\Delta c_{t+1}, r_{1,t+1}\right) = Cov_t\left(r_{1,t+1}, c_{t+1} - w_{t+1}\right) + \alpha_t Var_t\left(r_{1,t+1}\right).$$

Regarding $Cov_t\left(\Delta s_{t+1}, r_{1,t+1}\right)$ we get the following result

$$\begin{aligned} Cov_t\left(\Delta s_{t+1}, r_{1,t+1}\right) = &\left(1 - \frac{1}{\theta}\right)\lambda Cov_t\left(r_{1,t+1}, \Delta \bar{c}_{t+1}\right) - \left(1 - \frac{1}{\theta}\right) Cov_t\left(r_{1,t+1}, c_{t+1} - w_{t+1}\right) \\ &- \left(1 - \frac{1}{\theta}\right)\alpha_t Var_t\left(r_{1,t+1}\right). \end{aligned}$$

Substituting these results into (7) and rearranging the equation we can characterize the optimal portfolio rule in the following way

$$
\begin{aligned}
\alpha_t = & \frac{\theta}{\gamma} \frac{E_t r_{1,t+1} - r_f + \frac{1}{2} Var_t \left(r_{1,t+1}\right)}{Var_t \left(r_{1,t+1}\right)} \\
& + (1-\theta) \lambda \frac{Cov_t \left(r_{1,t+1}, \Delta \bar{c}_{t+1}\right)}{Var_t \left(r_{1,t+1}\right)} - \frac{Cov_t \left(r_{1,t+1}, c_{t+1} - w_{t+1}\right)}{Var_t \left(r_{1,t+1}\right)}.
\end{aligned}
$$

Comparing this result with the optimal portfolio rule when using standard CRRA utility we see that the second term is new compared to the case without habit. The first term captures the myopic component of asset demand, while the last term captures the intertemporal hedge component of asset demand. The new term in the optimal portfolio rule can be interpretated in the following way: As stated previously, $\theta = 1 - \exp\left[E\left(h_t - c_t\right)\right]$ and for utility to be well defined, it must hold that $0 < \theta < 1$. Hence, the sign of the new term depends only on $Cov_t\left(r_{1,t+1}, \Delta \bar{c}_{t+1}\right)$. If $Cov_t\left(r_{1,t+1}, \Delta \bar{c}_{t+1}\right) < 0$, the investor will reduce his holdings of the risky asset, while he will increase his holdings if $Cov_t\left(r_{1,t+1}, \Delta \bar{c}_{t+1}\right) > 0$ compared to the case without habit. The intuition here is, that investor's habit depends on per capita consumption, and the investor wants to hedge himself against habit changes. Or in other words, the investor wants to hold assets that pay off when per capita consumption increases, i.e. when the habit level increases. When distance to habit is large, $\theta \approx 1$, and the term drops out (or becomes very small), while when the distance is short, $\theta \approx 0$. This implies that the hedge element carries most weight when consumption is close to habit. Finally, λ also determines the size of the new term. The larger the λ, the more the investor responds to growth in per capita consumption.

Compared to the case without habit, θ is also multiplied on the myopic component of the optimal portfolio weight. As mentioned before, when distance to habit is large, $\theta \approx 1$, and so the myopic component stays unchanged, but when the distance is short, $\theta \approx 0$, and the myopic component drops out (or becomes very small). The explanation here is, that when the investor's consumption is very close to his habit level, he dislikes the risk associated with the myopic component and is only interested in assets that will ensure a consumption level above habit, i.e. the investor shifts his asset holdings to the riskless asset.

2.7 Solving for the optimal policies

The solution presented in the previous section is not a complete solution of the model, since the consumption-wealth ratio is a function of expected future returns and consumption growth rates, which can be seen from (17). The solution is thus a function of future portfolio and consumption decisions, which are endogenous to the problem. Following Campbell and Viceira (1999), we proceed by guessing a functional form for the optimal consumption and portfolio policies and then determine the coefficients of these policies using the method of undetermined coefficients. We guess that

$$(i) \quad \alpha_t = a_0 + a_1 x_t,$$

$$(ii) \quad c_t - w_t = b_0 + b_1 x_t + b_2 x_t^2,$$

where $\{a_0, a_1, b_0, b_1, b_2\}$ are fixed coefficients to be determined. In order to determine these coefficients, we need a few results, which Campbell and Viceira (1999) state as lemmas in their appendix. A number of these lemmas are reproduced in Appendix 4 for completeness and two are modified slightly due to the use of power utility with habit formation.

Appendix 5 verifies the guess that the optimal portfolio choice is linear in x_t with the coefficients

$$a_0 = \frac{\theta}{2\gamma} + (1-\theta)\,\lambda\frac{\sigma_{uv}}{\sigma_u^2} - b_1\frac{\sigma_{u\eta}}{\sigma_u^2} - b_2 2\mu\,(1-\phi)\,\frac{\sigma_{u\eta}}{\sigma_u^2}, \tag{20}$$

$$a_1 = \frac{\theta}{\gamma}\frac{1}{\sigma_u^2} - b_2 2\phi\frac{\sigma_{u\eta}}{\sigma_u^2}. \tag{21}$$

The first term in a_0 is the myopic component of asset demand, while the second term is the hedge component due to habit formation. The final two terms represent the intertemporal hedge component. In a_1 the first term is also the myopic component, while the second term is the intertemporal hedge component. Compared the standard CRRA case, the introduction of habit gives rise to the new term $(1-\theta)\,\lambda\,(\sigma_{uv}/\sigma_u^2)$, which we will denote the 'habit hedge component', and it also affects the myopic component through θ. Furthermore, it has an indirect effect on the intertemporal hedge component through b_1 and b_2. The magnitude of these individual effects and the total effect of incorporating habit in the model will be analyzed in the empirical application. Besides the cases where habit level is constant ($\lambda = 0$) and habit is not taken into account ($\theta = 1$), we note that the habit hedge component drops out if the covariance of the risky asset with changes in per capita consumption is zero, i.e. the risky asset provides no hedge against changes in the investor's habit level. This is similar to the case where the covariance of the risky asset with revisions in the expected future return is zero, which implies no intertemporal hedge demand.

In order to solve for $\{a_0, a_1\}$, we need to know $\{b_1, b_2\}$. Appendix 6 shows that these coefficients can be found by solving the following recursive nonlinear equation system

$$0 = \Lambda_{10} + \Lambda_{11}b_0 + \Lambda_{12}b_1 + \Lambda_{13}b_2 + \Lambda_{14}b_1^2 + \Lambda_{15}b_2^2 + \Lambda_{16}b_1 b_2 \tag{22}$$
$$0 = \Lambda_{20} + \Lambda_{21}b_1 + \Lambda_{22}b_2 + \Lambda_{23}b_2^2 + \Lambda_{24}b_1 b_2 \tag{23}$$
$$0 = \Lambda_{30} + \Lambda_{31}b_2 + \Lambda_{32}b_2^2, \tag{24}$$

where Λ_{ij} are known constants. Comparing this result to the result by Campbell and Viceira (1999), we see that the equation system has the same structure. The difference lies in the definition of Λ_{ij}, which now contains parameters stemming from incorporating habit into the utility function $\{g, \lambda, \theta, \sigma_v^2, \sigma_{uv}, \sigma_{\eta v}\}$ in addition to the parameters in Campbell and Viceiras model with exception of the intertemporal elasticity of substitution, which enters their model as a separate parameter due to the use of Epstein-Zin recursive preferences.

In solving this equation system, we follow the procedure outlined by Campbell and Viceira (1999, p. 447). This procedure entails that we first solve for b_2 and then b_1 and b_0, choose the positive root of the equation discriminant in (24), and iterate on the system until the difference between two consecutive values of ρ is less than 10^{-4}. ρ is a function of the consumption-wealth ratio and hence of $\{b_0, b_1, b_2\}$. By choosing the positive root of the discriminant we ensure that as $\theta \to 1$ and $\gamma \to 1$, then $b_2 \to 0$, $b_1 \to 0$, $\rho \to \delta$, $b_0 \to \log(1-\delta)$, $a_0 \to \frac{1}{2}$, and $a_1 \to 1/\sigma_u^2$. This represents the known exact solution for risk neutral investors (log utility) in which both the optimal consumption rule and the optimal portfolio rule are myopic. Choosing instead the negative root would imply that we would not arrive at the known exact solution since the approximate solutions would diverge in this case as $\theta \to 1$ and $\gamma \to 1$. As noted by Campbell and Viceira, for some parameter values, ρ might converge to one in infinite-horizon optimization problems. In this model, this is for example the case for very high values of $(\lambda\gamma)/\theta$.

Note that if we in the solution of the model set $\theta = 1$, the habit hedge component drops out, the myopic demand and $\{b_0, b_1, b_2\}$ are unaffected, and we are back in the standard case with CRRA utility. This makes it easy to analyze the effect of incorporating habit in the utility function simply by choosing different values of θ and comparing the results to the case where $\theta = 1$.

3 Empirical application

3.1 Data and estimation

In order to investigate the importance of accounting for habit in the intertemporal consumption and portfolio choice, we calibrate the model to quarterly U.S. financial data for the sample period 1947.1-2007.4. In the application, the risky asset is the U.S. stock market and the state variable is the log dividend-price ratio. We use the quarterly return on the CRSP value-weighted market portfolio inclusive of the NYSE, AMEX, and NASDAQ markets as the return on the stock market. The dividends are constructed in the usual way from the return and the price series using the same data source. As is standard in the literature, the log dividend-price ratio is measured as the log of the total dividend over the last four quarters minus log of the price at the end of the period. The three-month yield from Fama Risk Free Rates on the treasury tape of CRSP is the source of the risk free return. In order to calculate the real log risk free return, we deflate the beginning-of-quarter log nominal yield by the end-of-quarter log rate of change in the Consumer Price Index from the Treasury and Inflation tape on CRSP. As per capita consumption we use the chained per capita personal consumption expenditures from the National Income and Product Accounts supplied by the Bureau of Economic Analysis.

Using these variables, we estimate the following model by OLS

$$\begin{bmatrix} r_{1,t+1} - r_f \\ d_{t+1} - p_{t+1} \\ \Delta \bar{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d_t - p_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \\ \varepsilon_{3,t+1} \end{bmatrix},$$

where $[\varepsilon_{1,t+1}, \varepsilon_{2,t+1}, \varepsilon_{3,t+1}] \sim N(0, \Omega)$. The results are given in Table 1, Panel A, which shows that the dividend-price ratio forecasts excess returns with a positive sign and that the covariance between the innovations in excess returns and the dividend-price ratio is negative. This implies that excess stock returns are mean-reverting: a negative shock to excess returns will through the negative covariance coincide with a positive shock to the dividend-price ratio, which results in an increase in the excess return in the next period due to the positive coefficient on the dividend-price ratio. This feature implies that the intertemporal hedge demand for stocks will be positive since the asset works as a hedge against adverse changes in its own investment opportunities. Panel A also shows that the R^2 on the excess return equation is fairly low, but the dividend-price ratio is still significant.[6]

From this estimated model, we can recover the stochastic structure of the model, which is given in Table 1, Panel B.[7] The growth rate in per capita consumption is estimated to be 2.4 percent per year (0.6 percent per quarter), the unconditional expected log excess return to be 6.3 percent per year (1.6 percent per quarter), and the log real risk free rate to be 1.2 percent per year (0.3 percent per quarter).[8] The covariance between innovations in excess returns and growth in per capita consumption is positive implying that the habit hedge component of asset demand is positive, i.e. the investor can hedge himself against increases in his habit level by holding stocks.

In order to solve the model we need to choose values for a number of parameters. We set the time discount factor δ equal to 0.94 in annual terms, while we consider utility curvature parameters $\gamma = \{1, 2, 4, 10, 20\}$, distance to habit ratios $\theta = \{1, 0.8, 0.6, 0.4, 0.2\}$,

[6] The table also shows that the dividend-price ratio follows a near unit-root process, which implies that finite-sample bias might seriously distort the estimates in both the excess return and the dividend-price ratio equation (see e.g. Bekaert et al., 1997). Engsted and Pedersen (2008) examine this potential problem in a multivariate asset allocation model and find that bias-adjustment can have significant effect on the optimal portfolio choice. We will not pursue the issue further in this paper, but instead assume that the investor takes the estimated coefficients as given.

[7] We recover the stochastic structure as follows: $\mu = \beta_{11} + \beta_{12}\left(\overline{d_t - p_t}\right)$, $\phi = \beta_{22}$, $g = \beta_{31}$, $\sigma_u^2 = \Omega_{11}$, $\sigma_\eta^2 = \beta_{12}^2 \Omega_{22}$, $\sigma_v^2 = \Omega_{33}$, $\sigma_{u\eta} = \beta_{12}\Omega_{12}$, $\sigma_{uv} = \Omega_{13}$, and $\sigma_{\eta v} = \beta_{12}\Omega_{23}$. Campbell and Viceira (1999) state that the expected excess return can be recovered as $\mu = \beta_{11} + \beta_{12}\beta_{21}/\left(1 - \beta_{22}\right)$. However, this requires that the constant in the dividend-price ratio equation is fitted to match the arithmetic average.

[8] We have also estimated the model and derived the stochastic structure for the sample period used by Campbell and Viceira (1999, 2000): 1947.1-1995.4. Overall, we get fairly similar results, but we estimate both the unconditional expected log excess return and the log real risk free rate to be somewhat higher than Campbell and Viceira.

and per capita consumption sensitivity $\lambda = \{0, 0.5, 1, 1.5, 2\}$. These values allow us to analyze a number of special cases: the investor who disregards habit ($\theta = 1$), the investor with log utility in the standard CRRA case ($\gamma = 1$), and the investor with a constant habit level ($\lambda = 0$).

3.2 Optimal portfolio choice

Table 2 summarizes the mean optimal portfolio choice for different values of γ and θ while holding λ fixed at 1, which represents an investor whose habit level evolves with the rate of change in per capita consumption. From Panel A in this table it is clear that accounting for habit has large effects on the optimal portfolio choice: e.g. for $\gamma \leq 2$, the total demand for stocks by an investor that on average has consumed close to his habit level ($\theta = 0.2$) is less than half of the demand by an investor that disregards habit ($\theta = 1$). In general, Panel A shows that the total optimal demand is not only decreasing in γ, but also increasing in θ, i.e. investors who consume close to their habit will have lower stock holdings than investors who consume well above their habit. In order to explain the mechanisms underlying this result, Panel B, C, and D decompose the total demand into the myopic, habit hedge, and intertemporal hedge component. From (20) and (21) we can rewrite the mean myopic component as $(\theta/\gamma)\left(\frac{1}{2} + \mu/\sigma_u^2\right)$. Hence, besides being linear in $1/\gamma$, the myopic demand is also linear in the distance to habit θ. As Panel B shows, this implies that the optimal myopic demand for a given value of the utility curvature parameter γ can be much smaller than dictated by the standard CRRA utility function if the investor accounts for habit. We also notice that the myopic component carries much less weight for an investor that accounts for habit, e.g. for $\gamma = 4$, the myopic fraction of total demand goes from 49.7 percent for $\theta = 1$ to 18.1 percent for $\theta = 0.2$. The habit hedge component, $(1 - \theta)\,\lambda\,(\sigma_{uv}/\sigma_u^2)$, which is constant in γ, is shown in Panel C. Since σ_{uv} is positive this component is also positive, but the very small covariance implies that this component is by and large negligible. The component carries most weight if the investor is very risk averse ($\gamma = 20$) and consumes close to his habit level ($\theta = 0.2$). In this case the habit hedge component accounts for 2.2 percent of the total demand. Eventhough the habit hedge component is very small in this application of the model, it does not imply that we can disregard this component. Section 3.5 contains a sensitivity analysis of the magnitude of the habit hedge component, which shows that the importance of this component depends to a great extent on the porporties of the variable chosen to proxy for changes in the habit level. Finally, Panel D shows that habit also has large effects on the intertemporal hedge component, and that the effect depends on the value of γ. First, even with $\gamma = 1$ the intertemporal hedge demand is positive when the investor accounts for habit, and furthermore it increases as θ decreases, resulting in an intertemporal hedge fraction of total demand of 59 percent when $\theta = 0.2$. For values of γ equal to 2 and 4, the hedge demand is hump-shaped in θ, while for $\gamma > 4$, it decreases as θ decreases. The explanation of these effects is that distance to habit works as an adjustment factor to the utility curvature parameter, i.e. to the risk aversion. In other words, for a given utility curvature parameter the investors who consume close to their habit is more risk averse than investors who consume well above their habit. We see the same effect, when increasing γ and holding θ fixed.[9]

Table 3, Panel A summarizes the total mean habit effect. Despite the effect from the intertemporal hedge component being positive in certain cases, we overall see a negative effect of accounting for habit on the mean optimal demand for stocks. Furthermore, Panel A shows that habit demand is decreasing in γ in absolutte terms. Total demand is also decreasing in γ, but as Panel B shows this decline is not as large as the decline in habit demand, since the habit fraction also falls with γ. Hence, the higher the value of the utility curvature parameter, the less important it is to account for habit. In other words, the relative importance of the adjustment to risk aversion induced by distance to habit declines as γ increases.

The model's feature of lower stock holdings for a given utility curvature parameter γ when accounting for habit compared to the standard CRRA case is in line with evidence from the asset pricing literature. In general equilibrium models, the literature shows that a much lower value of γ is needed to obtain equilibrium when allowing for habit formation in preferences than when not. Campbell and Viceira (1999) find very high stock holdings for moderate values of the relative risk aversion, or stated otherwise, they find that a high relative risk aversion is needed to obtain reasonable optimal stock holdings. They state that this is a manifestation of the equity premium puzzle. Following this statement, the results in this paper is a manifestation of the ability of habit formation in preferences to resolve the puzzle.

In order to investigate the importance of the per capita consumption sensitivity λ, Table 4 shows the mean optimal allocation to stocks for different values of λ and θ while holding γ fixed at 4. The table is organized in the same way as Table 2. First we notice that in the standard CRRA case ($\theta = 1$), the sensitivity has no effect on the optimal portfolio choice as should be the case. Likewise, the myopic component is not affected by λ, which is verified by examining (20) and (21). The habit hedge component, shown in Panel C, depends on λ but as is clear from the table, the total effect is very small due to this components negligible size in the present application. Hence, in this application the effect of changing λ comes mainly from the intertemporal hedge component. As is clear from Panel D (and Panel A), the lower θ is, the more sensitive the optimal portfolio choice is to changes in λ: The investor who on average consumes well above his habit level ($\theta = 0.8$) is relatively insensitive to changes in λ, while the per capita consumption sensitivity is very important for the investor who on average consumes close to his habit level ($\theta = 0.2$). The case of constant habit level ($\lambda = 0$) is seen not to deviate from the case of time-varying habit level in a noticeable fashion. In general, Table 4 shows that the more sensitive the investor is to changes in per capita consumption, the more of his wealth he will allocate to stocks. The explanation here is that in order to maintain a consumption level above a habit level that experiences very high growth rates, the

[9] In a setup with internal habit, Munk (2007) finds that optimal myopic and intertemporal hedge demand are dampened by the presence of habit, and that this result is robust to changes in the utility curvature parameter γ. This is in contrast to our results that show that the effect of accounting for habit on the intertemporal hedge demand can be both positive and negative depending on the value of γ.

investor is forced to tilt his investment portfolio towards the risky asset that on average yields a higher return than the riskless asset and thus enables the investor to increase his consumption.

The same picture can be found in Table 5, Panel A, which shows total habit demand. As λ increases, the negative effect from accounting for habit decreases. This also implies that the habit fraction of total demand is decreasing in λ as shown in Panel B. This panel also reveals that habit actually has the largest effect on optimal portfolio choice if the habit level is constant over time.

3.3 Optimal consumption choice

Table 6 summarizes the mean optimal consumption choice for different values of γ and θ while holding λ fixed at 1, i.e. the setup matches that presented in Table 2. Panel A shows that for $\gamma > 1$, the optimal consumption-wealth ratio is increasing in the distance to habit, while it is hump-shaped in θ for $\gamma = 1$. Hence, a risk averse investor that accounts for habit consumes a smaller fraction of his wealth compared to the investor with standard CRRA utility. The intuition here is that the investor with habit formation in his preferences saves a larger fraction of wealth in order to maintain a buffer of savings from which he can consume if the habit level experiences unexpected increases, and hence enables him to consume above his habit level. We can also interpret the result using (19) and Panel B. According to (19), a decrease in the distance to habit will partly lead to an increase in consumption through the expected portfolio return (the first term) and partly to a decrease through the expected habit growth (the second term). However, the portfolio return is endogenous and as shown in Panel B, it is strictly increasing in distance to habit. This can be explained by the same arguments as presented in the previous section: Compared to an investor who does not take habit into account, an investor who has habit formation in his preferences tilts his portfolio allocation towards the riskless asset in order to ensure a consumption level above habit, and this implies a lower average return on wealth. Hence, the investor who consumes close to his habit obtains a low return on wealth, which offsets the increase in $1 - \theta/\gamma$, and thus the consumption-wealth ratio declines. For $\gamma = 1$, the consumption first increases and then decreases as θ decreases. The explanation is that initially the decrease in expected return is not large enough to offset the increase in $1 - \theta/\gamma$ in this case. Only an investor who consumes very close to habit will decrease consumption.

A very risk averse investor that does not account for habit wants to maintain a constant expected consumption growth over time regardless of current investment opportunities, cf. (15). This can be accomplished by consuming the long-term average return on wealth adjusted by a precautionary savings term. By comparing the results in Panel A and Panel B, we see that the consumption-wealth ratio (1.69) in fact is very close to the long-term average return (1.71). When a very risk averse investor accounts for habit, (15) shows that the expected consumption growth is no longer constant since the investor wants to match consumption growth to the growth in his habit level. Hence, this investor will decrease his consumption today in order to ensure a future consumption growth that will match the growth in habit. This implies that the optimal consumption-wealth ratio will deviate from the long-term average return on wealth for a very risk averse investor if he accounts for habit, and that this difference is largest for investors who consume close to their habit. This is verified by the results in the last row of Panel A and Panel B.

Table 7 shows the results of the same variables for different values of λ and θ while holding γ fixed at 4. As we already know from (19), the sensitivity to per capita consumption λ has no effect on optimal consumption choice when the investor does not account for habit, which is also clear from the first column in Panel A. Furthermore, Table 7 shows that consumption is decreasing in λ, i.e. the more the investor responds to growth in per capita consumption, the less he wants to consume today in order to be able to match future habit growth. As was the case for optimal portfolio choice, changes in λ has the greatest influence when consumption is close to habit. By examining the columns of Table 7 individually, we see that for increasing λ, the consumption-wealth ratio is decreasing while the long-term average return on wealth is increasing. Equation (19) shows that the consumption-wealth ratio depends negatively on λ and positively on the expected return on wealth. However, the results in Table 7 shows that the negative effect from λ more than offsets the positive effect from the expected return.

3.4 Portfolio allocation, consumption, and relative risk aversion over time

The main feature of this model is that it allows investors' relative risk aversion to vary over time. Figure 1 shows the relative risk aversion in the lower plot along with the optimal asset allocation to stocks (upper plot) and the optimal consumption-wealth ratio (middle plot). For illustrative purposes the plots are made for investors with moderate utility curvature ($\gamma = 4$) and whose habit levels evolve with the rate of change in per capita consumption ($\lambda = 1$). To demonstrate the effect of accounting for habit, the plots hold the results for an investor with a moderate distance to habit ($\theta = 0.6$) and an investor that disregards habit ($\theta = 1$), i.e. a standard CRRA utility investor. The optimal portfolio allocation and consumption-wealth ratio are functions of the state variable whether the investor accounts for habit or not, implying that these evolve similarly over time for $\theta = 0.6$ and $\theta = 1$. The difference between the optimal choices for $\theta = 0.6$ and $\theta = 1$ lies in the size of the coefficients as determined by (20)-(24). According to Table 2, the mean optimal allocation to stocks is higher for the investor that disregards habit than for the investor that accounts for habit. The upper plot in Figure 1 shows that this result depends on the signal the state variable is giving about future returns. In the late 1990's and the beginning of the new millennium the optimal allocation to stocks was actually higher for the investor that accounts for habit than for the investor that disregards habit, and since then the optimal allocation has been more or less identical for these two investors. The explanation is again that the distance to habit works as an adjustment factor to the utility curvature parameter, implying that the investor with $\theta = 0.6$ is more risk averse than the investor with $\theta = 1$, and the more risk averse the investor is, the less sensitive he is to the signals given by the state variable. The same explanation holds for the difference in the optimal consumption-wealth ratio shown in the middle plot.

In order to make the plot of relative risk aversion, we have to choose a starting value for the surplus consumption ratio S_t, cf. (12). We have fitted this starting value to ensure that $1 - \exp\left[E\left(h_t - c_t\right)\right]$ is equal to θ. Examining the lower plot in Figure 1, we first notice that as expected relative risk aversion is constant over time and equal to 4 if $\theta = 1$. However, if the investor accounts for habit, his relative risk aversion varies over time. In this case, the relative risk aversion decreased until the mid 1950's and in the 1980's, while it increased in the 1970's and the 1990's. Finally, it has remained relatively stable since the turn of the millennium. Figure 2, which reproduces the relative risk aversion and plots it together with the surplus consumption ratio for different values of λ, shows this pattern more clearly. It also shows how the surplus consumption ratio in general increases as λ decreases, which in turn implies a decline in relative risk aversion. Thus, the more sensitive the investor is to changes in per capita consumption, the more risk averse he is. Comparing this observation to the results in Table 4, we see that higher risk aversion does not necessarily imply lower stock holdings when you account for habit. Note, the lower λ is, the less volatile the habit level is, and when $\lambda = 0$ it is constant. Hence, this illustrates how (in the case of positive consumption growth) the habit level becomes increasingly irrelevant as it approaches a constant.

In order to obtain additional insight into the behavior of the surplus consumption ratio, Figure 3 plots the model-implied investor consumption and the per capita consumption for different values of θ, while holding γ and λ fixed. The consumption series are indexed to start at 100. Figure 3 clearly illustrates how the before mentioned pattern in relative risk aversion, and hence surplus consumption ratio, can be explained by differences in the model-implied and the per capita consumption growth. In general, we observe a steady increase in per capita consumption, while the model-implied consumption moves more irregularly. The investor consumption growth depends in this model solely on the investor's choice of consumption-wealth ratio and returns obtained in the financial market through the portfolio choice, cf. (14). Hence, if the investor obtains negative returns on his investments and maintains a relatively stable consumption-wealth ratio, he will experience negative consumption growth. This is what happened in the 1970's, where the dividend-price ratio indicated increasing future returns, which implied a larger share of the risky asset in the investor's portfolio and a slightly increasing consumption-wealth ratio. However, in this period the return on stocks were negative and quite large in absolute value, which in combination with the relatively stable consumption-wealth ratio implied a large decrease in the investor's consumption as seen in Figure 3. The decline in the surplus consumption ratio in the 1990's is not due to negative returns but is instead a result of the decline in consumption-wealth ratio and stock holdings, which results in a negative model-implied consumption growth. Combined with a large growth in per capita consumption, this implies a decline in the surplus consumption ratio. In general, we observe a large consumption decline, and hence a large decrease in the surplus consumption ratio in periods where the dividend-price ratio does not forecast future stock returns so well.

Figure 3 also shows the model-implied habit level calculated using (5).[10] In general, consumption is closer to habit in the second half of the sample than in the first. An investor with $\theta = 0.2$ has a consumption level virtually equal to his habit level in the second half of the sample, while the investor with $\theta = 0.8$ consumes well above his habit level in the whole sample period. Furthermore, the consumption level of an investor with $\theta = 0.2$ is much less volatile than the consumption level of an investor with $\theta = 0.8$, which among other things imply that the investor with $\theta = 0.2$ has a much lower consumption level in the first part of the sample and experience a much smaller drop in consumption in the 1970's than the investor with $\theta = 0.8$. These differences in consumption can be explained be reexamining (15). The investor with a very short distance to habit primarily uses the expected growth in per capita consumption to guide his choice of consumption in order to secure a consumption level above habit, while the investor who consumes well above habit puts more weight on the expected portfolio return. Hence, the investor with $\theta = 0.8$ responds much more to the high expected returns in the beginning of the sample than the investor with $\theta = 0.2$, and thus also experiences a much larger drop in consumption when the portfolio return decreased in the 1970's.

It is important to note that the asset allocation model presented in this paper is a partial equilibrium model. More specifically it is an asset allocation model for an investor who only has financial wealth (i.e. no labor income) and who only invests in stocks and the riskless asset. In other words, the investor must rely on returns on stocks and the riskless asset to sustain a consumption level above habit. If he does not do well on the financial market, he will inevitable experience a decrease in his surplus consumption ratio and hence an increase in his risk aversion. Thus, the surplus consumption ratio shown in Figure 2 must not be confused with surplus consumption ratios derived in the asset pricing literature, which deals with general equilibrium models.

3.5 Habit hedge demand

According to the empirical application, the habit hedge component is of negligible size. However, this result depends on the properties of the data. In this section, we examine how sensitive the result is to changes in the properties of the variable used to govern the law of motion of the habit. In the empirical application, we assume that the investor uses changes in per capita consumption as proxy for changes in habit, but this assumption is by no means conclusive. Another investor might find that a different variable is a better proxy for the development of his habit. The question then is how a different choice of proxy affects the habit hedge demand?[11] Assuming all else equal, Figure 4 shows the habit hedge demand and habit hedge demand's fraction of total demand as a function of correlation between the risky asset and the proxy used to govern the change in habit as well as of the variance of this proxy. The figure holds the results for $\gamma = 4$,

[10] The model-implied habit deviates slightly from the habit derived directly from (3) due to approximation error.

[11] Changing σ_v^2 and $corr\,(u, v)$ does not affect the myopic component. It does affect the intertemporal hedge component, but the effect is small, and thus we only focus on the habit hedge component.

$\theta = 0.6$, and $\lambda = 1$. In the upper plot, the variance of the proxy is set equal to the estimated variance, $\sigma_v^2 = 6.953E-5$, and in the lower plot it is set equal to the estimated variance of the risky asset, $\sigma_v^2 = 6.323E-3$. The upper plot illustrates that even though the correlation between the risky asset and the proxy variable is high, the habit hedge demand is still of negligible size; e.g. if the correlation is 0.5 the habit hedge demand is only around 2 percent, which corresponds to approximately 2 percent of the total demand. The explanation is that the demand is not determined by the correlation, but by the covariance and hence by the relative size of the variances of the risky asset and the proxy. To illustrate this point, we set the variance of the proxy variable equal to the variance of the risky asset in the lower plot. Now if the correlation is 0.5, habit hedge demand is around 20 percent corresponding to 13 percent of total demand. Hence, the size and importance of the habit hedge component depend to a great extend on the properties of the variable used to govern the law of motion of the habit relative to the properties of the risky asset, implying that we can not in general disregard this component despite its negligible size in the empirical application. This observation is also relevant if the investor finds that changes in per capita consumption is the best proxy for changes in his habit level, but is interested in another risky asset than stocks, e.g. bonds, commodities, real estate etc. In this application the correlation between the risky asset and the proxy might be different as might also be the case for the relative size of the variances.

4 Concluding remarks

One of the most influential advances in finance in recent years has been the acknowledgement of habit formation in investors' preferences. This implies that investors' preferences are no longer assumed to be time-separable as a rule. In this paper, we explore how time-nonseparable preferences affect the optimal consumption and portfolio choice and compare it to the standard case with CRRA preferences. We derive an approximate analytical solution for an infinitely-lived investor with external habit in his preferences, who have access to a riskless asset with a constant return and a risky asset with a time-varying premium using the discrete-time approach by Campbell and Viceira (1999). The solution reveals several effects on the optimal portfolio choice due to habit formation. First, the myopic component is adjusted by the distance between investor's consumption and habit level. Second, the intertemporal hedge component is also affected by this distance through the coefficients on the optimal consumption choice. Third, habit formation give rise to a habit hedge component, which works as a hedge against changes in investor's habit level and carries most weight for an investor who consumes close to his habit level.

An empirical application calibrated to U.S. quarterly data with stocks as the risky asset shows that habit formation in preferences generally reduces the optimal stock holdings. Campbell and Viceira (1999) note that the very large stock holdings they find using moderate values of the relative risk aversion is a manifestation of the equity premium puzzle. The asset pricing literature suggests that habit formation in investors' preferences can be a potential explanation of the high equity premium. The lower stock holdings found in this paper is a manifestation of habit formation's ability to resolve the equity premium puzzle.

An important limitation in this model is that we have disregarded the state-nonseparability induced by the recursive preferences used by Campbell and Viceira (1999). A natural extension of the results in this paper would be to incorporate both time- and state-nonseparability into the model. Furthermore, it would be relatively straightforward to extend the model to the multivariate setup along the lines of Campbell et al. (2003). Finally, it would be highly relevant to incorporate labor income into the model. As this model is formulated, the investor must rely on returns on the financial market to sustain a consumption level above habit. The inclusion of labor income would modify this result.

5 Appendix

5.1 Appendix 1: Budget constraint

The budget constraint can be written in the following way

$$\frac{W_{t+1}}{W_t} = R_{p,t+1}\left(1 - \frac{C_t}{W_t}\right),$$

and taking logs we get

$$\Delta w_{t+1} = r_{p,t+1} + \log\left(1 - \exp\left(c_t - w_t\right)\right).$$

In order to get a linear relation we perform a first-order Taylor expansion of $f_t(c_t - w_t) = \log\left(1 - \exp\left(c_t - w_t\right)\right)$ around the mean consumption-wealth ratio $E\left(c_t - w_t\right)$. The first derivative is given as

$$f_t'(c_t - w_t) = \frac{-\exp\left(c_t - w_t\right)}{1 - \exp\left(c_t - w_t\right)}.$$

The Taylor approximation yields the following result

$$\begin{aligned}
f_t(c_t - w_t) &\approx \log\left(1 - \exp\left[E\left(c_t - w_t\right)\right]\right) - \frac{\exp\left[E\left(c_t - w_t\right)\right]}{1 - \exp\left[E\left(c_t - w_t\right)\right]}\left[\left(c_t - w_t\right) - E\left(c_t - w_t\right)\right] \\
&= \log\rho + \frac{1-\rho}{\rho}E\left(c_t - w_t\right) - \frac{1-\rho}{\rho}\left(c_t - w_t\right) \\
&= \log\rho + \frac{1-\rho}{\rho}\log(1-\rho) + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) \\
&= k_1 + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right),
\end{aligned}$$

where $\rho = 1 - \exp\left[E\left(c_t - w_t\right)\right]$ and $k_1 = \log\rho + \frac{1-\rho}{\rho}\log(1-\rho)$. Thus we get

$$\Delta w_{t+1} = r_{p,t+1} + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) + k_1.$$

5.2 Appendix 2: Portfolio return

The portfolio return can be written in the following way

$$\frac{R_{p,t+1}}{R_f} = \alpha_t \left(\frac{R_{1,t+1}}{R_f} - 1 \right) + 1,$$

and taking logs we get

$$r_{p,t+1} - r_f = \log \left[\alpha_t \left(\exp(r_{1,t+1} - r_f) - 1 \right) + 1 \right].$$

In order to get a linear relation we perform a second-order Taylor expansion of $f_t(r_{1,t+1} - r_f) = \log \left[\alpha_t \left(\exp(r_{1,t+1} - r_f) - 1 \right) + 1 \right]$ around the expansion point $r_{1,t+1} - r_f = 0$. The first and second derivative are given as

$$f_t'(r_{1,t+1} - r_f) = \frac{\alpha_t \left(\exp(r_{1,t+1} - r_f) \right)}{1 + \alpha_t \left(\exp(r_{1,t+1} - r_f) - 1 \right)},$$

and

$$f_t''(r_{1,t+1} - r_f) = \frac{(\alpha_t - \alpha_t^2) \left(\exp(r_{1,t+1} - r_f) \right)}{\left[1 + \alpha_t \left(\exp(r_{1,t+1} - r_f) - 1 \right) \right]^2},$$

respectively. Evaluating $f_t(r_{1,t+1} - r_f)$ and its first and second derivative at the expansion point, we get $f_t(0) = 0$, $f_t'(0) = \alpha_t$, and $f_t''(0) = (\alpha_t - \alpha_t^2) = \alpha_t(1 - \alpha_t)$. Thus, the Taylor approximation yields the following result

$$r_{p,t+1} - r_f \approx \alpha_t(r_{1,t+1} - r_f) + \frac{1}{2}\alpha_t(1 - \alpha_t)(r_{1,t+1} - r_f)^2.$$

Replacing $(r_{1,t+1} - r_f)^2$ with its conditional expectation $Var_t\left(r_{1,t+1}\right)$ and neglecting the approximation error, we get

$$r_{p,t+1} - r_f = \alpha_t(r_{1,t+1} - r_f) + \frac{1}{2}\alpha_t(1 - \alpha_t) Var_t\left(r_{1,t+1}\right).$$

5.3 Appendix 3: Surplus consumption ratio

The change in log surplus consumption ratio Δs_{t+1} can be written in the following way

$$\begin{aligned}
\Delta s_{t+1} &= \log \left(\frac{C_{t+1} - H_{t+1}}{C_{t+1}} \right) - \log \left(\frac{C_t - H_t}{C_t} \right) \\
&= \log \left(1 - \frac{H_{t+1}}{C_{t+1}} \right) - \log \left(1 - \frac{H_t}{C_t} \right) \\
&= \log \left(1 - \exp \left(h_{t+1} - c_{t+1} \right) \right) - \log \left(1 - \exp \left(h_t - c_t \right) \right).
\end{aligned}$$

In order to get a linear relation we perform a first-order Taylor expansion of $f_t(h_t - c_t) = \log(1 - \exp(h_t - c_t))$ around the mean habit-consumption ratio $E(h_t - c_t)$. The first derivative is given as

$$f_t'(h_t - c_t) = \frac{-\exp(h_t - c_t)}{1 - \exp(h_t - c_t)}.$$

The Taylor approximation yields the following result

$$\begin{aligned}
f_t(h_t - c_t) &\approx \log(1 - \exp[E(h_t - c_t)]) - \frac{\exp[E(h_t - c_t)]}{1 - \exp[E(h_t - c_t)]}[(h_t - c_t) - E(h_t - c_t)] \\
&= \log\theta + \frac{1-\theta}{\theta}E(h_t - c_t) - \frac{1-\theta}{\theta}(h_t - c_t) \\
&= \log\theta + \frac{1-\theta}{\theta}\log(1-\theta) + \left(1 - \frac{1}{\theta}\right)(h_t - c_t) \\
&= k_2 + \left(1 - \frac{1}{\theta}\right)(h_t - c_t),
\end{aligned}$$

where $\theta = 1 - \exp[E(h_t - c_t)]$ and $k_2 = \log\theta + \frac{1-\theta}{\theta}\log(1-\theta)$. Thus we get

$$\begin{aligned}
\Delta s_{t+1} &= k_2 + \left(1 - \frac{1}{\theta}\right)(h_{t+1} - c_{t+1}) - k_2 - \left(1 - \frac{1}{\theta}\right)(h_t - c_t) \\
&= \left(1 - \frac{1}{\theta}\right)(h_{t+1} - c_{t+1} - h_t + c_t) \\
&= \left(1 - \frac{1}{\theta}\right)(\Delta h_{t+1} - \Delta c_{t+1}).
\end{aligned}$$

5.4 Appendix 4: Some useful lemmas

Lemma 1 *The conditional expectation of future values of the state variable is a linear function of its current value, while the conditional expectation of future values of the squared state variable is a quadratic function of the current state variable:*

$$E_{t+1}x_{t+j} = \mu + \phi^j(x_t - \mu) + \phi^{j-1}\eta_{t+1},$$

$$\begin{aligned}
E_{t+1}x_{t+j}^2 = {}& \mu^2\left(1 - \phi^j\right)^2 + \frac{1 - \phi^{2(j-1)}}{1 - \phi^2}\sigma_\eta^2 + 2\mu\phi^j\left(1 - \phi^j\right)x_t \\
&+ \phi^{2j}x_t^2 + \phi^{2(j-1)}\eta_{t+1}^2 + 2\phi^{1+2(j-1)}(x_t - \mu)\eta_{t+1} \\
&+ 2\mu\phi^{j-1}\eta_{t+1}.
\end{aligned}$$

Proof of Lemma 1. By simple forward recursion of x_t and x_t^2 in (2), we get

$$x_{t+j} - \mu = \phi^j (x_t - \mu) + \sum_{l=0}^{j-1} \phi^l \eta_{t+j-l}$$

and

$$(x_{t+j} - \mu)^2 = \phi^{2j} (x_t - \mu)^2 + \sum_{l=0}^{j-1} \phi^{2l} \eta_{t+j-l}^2 + 2\phi \sum_{l=0}^{j-1} \phi^{2l} (x_{t+j-1-l} - \mu) \eta_{t+j-l}. \tag{25}$$

The results stated in the lemma follow from the expressions above, after taking conditional expectations at time $(t+1)$, E_{t+1}, and the martingale assumption concerning the state variable, which implies that $E_{t+l-1}\eta_{t+l} = 0$, $E_{t+1}\eta_{t+l} = 0$, and $E_{t+1}[x_{t+l-1}\eta_{t+l}] = 0$, $\forall l > 1$.■

Lemma 2 *The innovation in next period's squared state variable is linear in the current state variable:*

$$x_{t+1}^2 - E_t x_{t+1}^2 = \left(\eta_{t+1}^2 - \sigma_\eta^2\right) + (2\mu(1-\phi) + 2\phi x_t)\, \eta_{t+1}.$$

Proof of Lemma 2. The proof for this lemma is similar to that for Lemma 1. From (25), find x_{t+1}^2 by setting $j = 1$:

$$x_{t+1}^2 = \mu^2 (1-\phi)^2 + 2\mu\phi (1-\phi) x_t + \phi^2 x_t^2 + \eta_{t+1}^2 + [2\phi (x_t - \mu) + 2\mu]\, \eta_{t+1}.$$

Lemma 2 then follows by applying the conditional expectations operator E_t to this expression under the martingale assumption concerning the state variable.■

Lemma 3 *The expected portfolio return next period is quadratic in the current state variable, and the unexpected portfolio return is linear in the current state variable:*

$$E_t r_{p,t+1} = r_f + p_0 + p_1 x_t + p_2 x_t^2,$$

and

$$r_{p,t+1} - E_t r_{p,t+1} = (a_0 + a_1 x_t)\, u_{t+1},$$

where

$$\begin{aligned}
p_0 &= a_0 (1 - a_0) \frac{\sigma_u^2}{2}, \\
p_1 &= a_0 + a_1 (1 - 2a_0) \frac{\sigma_u^2}{2}, \\
p_2 &= a_1 - a_1^2 \frac{\sigma_u^2}{2}.
\end{aligned}$$

Proof of Lemma 3. From (9) and guess (i) on the optimal portfolio rule, we have that

$$\begin{aligned}
E_t r_{p,t+1} &= \alpha_t E_t (r_{1,t+1} - r_f) + r_f + \alpha_t (1 - \alpha_t) \frac{\sigma_u^2}{2} \\
&= (a_0 + a_1 x_t)\, x_t + r_f + \left((a_0 + a_1 x_t) - (a_0 + a_1 x_t)^2\right) \frac{\sigma_u^2}{2},
\end{aligned}$$

where the last line follows from (1). Reordering terms we get a quadratic expression in x_t whose coefficients are those given in Lemma 3. Subtracting this result from the log-linear approximation of the portfolio return (9), we arrive at

$$\begin{aligned}
r_{p,t+1} - E_t r_{p,t+1} &= \alpha_t \left((r_{1,t+1} - r_f) - E_t (r_{1,t+1} - r_f)\right) \\
&= (a_0 + a_1 x_t)\, u_{t+1}.
\end{aligned}$$

■

Lemma 4 *Expected optimal consumption growth over the next period is quadratic in the current state variable, and unexpected consumption growth is linear in the current state variable:*

$$\begin{aligned}
E_t \Delta c_{t+1} &= E_t r_{p,t+1} + E_t (c_{t+1} - w_{t+1}) - \frac{1}{\rho} (c_t - w_t) + k_1 \\
&= c_0 + c_1 x_t + c_2 x_t^2,
\end{aligned}$$

$$\begin{aligned}
\Delta c_{t+1} - E_t \Delta c_{t+1} &= (a_0 + a_1 x_t)\, u_{t+1} + b_1 \eta_{t+1} \\
&\quad + b_2 \left(2\mu (1 - \phi) + 2\phi x_t\right) \eta_{t+1} + b_2 \left(\eta_{t+1}^2 - \sigma_\eta^2\right),
\end{aligned}$$

where

$$\begin{aligned}
c_0 &= r_f + a_0(1-a_0)\frac{\sigma_u^2}{2} + k_1 + b_0\left(1-\frac{1}{\rho}\right) + b_1(\mu(1-\phi)) \\
&\quad + b_2\left(\mu^2(1-\phi)^2 + \sigma_\eta^2\right), \\
c_1 &= a_0 + a_1(1-2a_0)\frac{\sigma_u^2}{2} + b_1\left(\phi-\frac{1}{\rho}\right) + b_2(2\mu\phi(1-\phi)), \\
c_2 &= a_1 - a_1^2\frac{\sigma_u^2}{2} + b_2\left(\phi^2-\frac{1}{\rho}\right).
\end{aligned}$$

Proof of Lemma 4. Using Lemma 1, 2, and 3 together with guess (ii) and (14), we can write

$$\begin{aligned}
E_t\Delta c_{t+1} &= E_t r_{p,t+1} + E_t(c_{t+1}-w_{t+1}) - \frac{1}{\rho}(c_t-w_t) + k_1 \\
&= r_f + p_0 + p_1x_t + p_2x_t^2 \\
&\quad + b_0\left(1-\frac{1}{\rho}\right) + b_1\left(E_tx_{t+1}-\frac{1}{\rho}x_t\right) + b_2\left(E_tx_{t+1}^2-\frac{1}{\rho}x_t^2\right) + k_1 \\
&= r_f + a_0(1-a_0)\frac{\sigma_u^2}{2} + \left(a_0 + a_1(1-2a_0)\frac{\sigma_u^2}{2}\right)x_t + \left(a_1 - a_1^2\frac{\sigma_u^2}{2}\right)x_t^2 \\
&\quad + b_0\left(1-\frac{1}{\rho}\right) + b_1\left(\mu(1-\phi) + \left(\phi-\frac{1}{\rho}\right)x_t\right) \\
&\quad + b_2\left(\mu^2(1-\phi)^2 + \sigma_\eta^2 + 2\mu\phi(1-\phi)x_t + \left(\phi^2-\frac{1}{\rho}\right)x_t^2\right) + k_1 \\
&= \left\{r_f + a_0(1-a_0)\frac{\sigma_u^2}{2} + b_0\left(1-\frac{1}{\rho}\right) + b_1\mu(1-\phi) + b_2\left(\mu^2(1-\phi)^2+\sigma_\eta^2\right) + k_1\right\} \\
&\quad + \left\{a_0 + a_1(1-2a_0)\frac{\sigma_u^2}{2} + b_1\left(\phi-\frac{1}{\rho}\right) + b_2 2\mu\phi(1-\phi)\right\}x_t \\
&\quad + \left\{a_1 - a_1^2\frac{\sigma_u^2}{2} + b_2\left(\phi^2-\frac{1}{\rho}\right)\right\}x_t^2.
\end{aligned}$$

To verify the result on unexpected consumption growth we first note that guess (ii) implies the following for the unexpected consumption-wealth ratio

$$\begin{aligned}
(c_{t+1}-w_{t+1}) - E_t(c_{t+1}-w_{t+1}) &= b_1(x_{t+1}-E_tx_{t+1}) + b_2\left(x_{t+1}^2 - E_tx_{t+1}^2\right) \\
&= b_1\eta_{t+1} + b_2\left(\left(\eta_{t+1}^2-\sigma_\eta^2\right) + (2\mu(1-\phi)+2\phi x_t)\eta_{t+1}\right).
\end{aligned}$$

Now we can write

$$
\begin{aligned}
\Delta c_{t+1} - E_t \Delta c_{t+1} &= \left(r_{p,t+1} - E_t r_{p,t+1}\right) + \left(c_{t+1} - w_{t+1}\right) - E_t \left(c_{t+1} - w_{t+1}\right) \\
&= \left(a_0 + a_1 x_t\right) u_{t+1} + b_1 \eta_{t+1} \\
&+ b_2 \left(\left(\eta_{t+1}^2 - \sigma_\eta^2\right) + \left(2\mu\left(1-\phi\right) + 2\phi x_t\right)\eta_{t+1}\right). \qquad (26)
\end{aligned}
$$

■

Lemma 5 *Equation (16) is a quadratic function of the state variable:*

$$
\upsilon_{p,t} = \upsilon_0 + \upsilon_1 x_t + \upsilon_2 x_t^2,
$$

where

$$
\begin{aligned}
\upsilon_0 &= \frac{1}{2}\lambda^2\gamma\theta\left(\frac{1}{\theta}-1\right)^2\sigma_v^2 + a_0\left[\lambda\left(1-\theta\right)\left(1-\frac{\gamma}{\theta}\right)\sigma_{uv}\right] + b_1\left[\lambda\left(1-\frac{1}{\theta}\right)\gamma\sigma_{\eta v}\right] \\
&+ b_2\left[\lambda\left(1-\frac{1}{\theta}\right)\gamma 2\mu\left(1-\phi\right)\sigma_{\eta v}\right] + a_0^2\left[\frac{1}{2}\gamma\theta\left(\frac{1}{\gamma}-\frac{1}{\theta}\right)^2\sigma_u^2\right] \\
&+ a_0 b_1\left[\left(\frac{\gamma}{\theta}-1\right)\sigma_{u\eta}\right] + a_0 b_2\left[\left(\frac{\gamma}{\theta}-1\right)2\mu\left(1-\phi\right)\sigma_{u\eta}\right] + b_1^2\left[\frac{1}{2}\frac{\gamma}{\theta}\sigma_\eta^2\right] \\
&+ b_1 b_2\left[\frac{\gamma}{\theta}2\mu\left(1-\phi\right)\sigma_\eta^2\right] + b_2^2\left[\frac{\gamma}{\theta}\left(\sigma_\eta^2 + 2\mu^2\left(1-\phi\right)^2\right)\sigma_\eta^2\right], \\
\upsilon_1 &= a_1\left[\lambda\left(1-\theta\right)\left(1-\frac{\gamma}{\theta}\right)\sigma_{uv}\right] + b_2\left[\lambda\left(1-\frac{1}{\theta}\right)\gamma 2\phi\sigma_{\eta v}\right] \\
&+ a_0 a_1\left[\gamma\theta\left(\frac{1}{\gamma}-\frac{1}{\theta}\right)^2\sigma_u^2\right] + a_1 b_1\left[\left(\frac{\gamma}{\theta}-1\right)\sigma_{u\eta}\right] \\
&+ a_0 b_2\left[\left(\frac{\gamma}{\theta}-1\right)2\phi\sigma_{u\eta}\right] + a_1 b_2\left[\left(\frac{\gamma}{\theta}-1\right)2\mu\left(1-\phi\right)\sigma_{u\eta}\right] \\
&+ b_1 b_2\left[\frac{\gamma}{\theta}2\phi\sigma_\eta^2\right] + b_2^2\left[\frac{\gamma}{\theta}4\mu\left(1-\phi\right)\phi\sigma_\eta^2\right], \\
\upsilon_2 &= a_1^2\left[\frac{1}{2}\gamma\theta\left(\frac{1}{\gamma}-\frac{1}{\theta}\right)^2\sigma_u^2\right] + a_1 b_2\left[\left(\frac{\gamma}{\theta}-1\right)2\phi\sigma_{u\eta}\right] + b_2^2\left[\frac{\gamma}{\theta}2\phi^2\sigma_\eta^2\right].
\end{aligned}
$$

Proof of Lemma 5. *Using (26) and (4) we can rewrite* $\upsilon_{p,t}$ *as follows*

$$
\begin{aligned}
\upsilon_{p,t} &= \frac{\theta\gamma}{2}E_t[\left(\frac{1}{\theta}-1\right)\lambda\left(\Delta\overline{c}_{t+1}-E_t\Delta\overline{c}_{t+1}\right)-\frac{1}{\theta}\left(\Delta c_{t+1}-E_t\Delta c_{t+1}\right)+\frac{1}{\gamma}\left(r_{p,t+1}-E_t r_{p,t+1}\right)]^2 \\
&= \frac{\theta\gamma}{2}E_t[\left(\frac{1}{\theta}-1\right)\lambda\left(\Delta\overline{c}_{t+1}-E_t\Delta\overline{c}_{t+1}\right) \\
&-\frac{1}{\theta}\left(\left(r_{p,t+1}-E_t r_{p,t+1}\right)+\left(c_{t+1}-w_{t+1}\right)-E_t\left(c_{t+1}-w_{t+1}\right)\right)+\frac{1}{\gamma}\left(r_{p,t+1}-E_t r_{p,t+1}\right)]^2 \\
&= \frac{\theta\gamma}{2}E_t[\left(\frac{1}{\theta}-1\right)\lambda\left(\Delta\overline{c}_{t+1}-E_t\Delta\overline{c}_{t+1}\right)+\left(\frac{1}{\gamma}-\frac{1}{\theta}\right)\left(r_{p,t+1}-E_t r_{p,t+1}\right) \\
&-\frac{1}{\theta}\left(\left(c_{t+1}-w_{t+1}\right)-E_t\left(c_{t+1}-w_{t+1}\right)\right)]^2 \\
&= \frac{\theta\gamma}{2}E_t[\left(\frac{1}{\theta}-1\right)\lambda v_{t+1}+\left(\frac{1}{\gamma}-\frac{1}{\theta}\right)\left(a_0+a_1x_t\right)u_{t+1} \\
&-\frac{1}{\theta}(b_1\eta_{t+1}+b_2\left(\left(\eta_{t+1}^2-\sigma_\eta^2\right)+\left(2\mu\left(1-\phi\right)+2\phi x_t\right)\eta_{t+1}\right))]^2.
\end{aligned}
$$

Computing the conditional expectation and reordering the terms will yield the result stated in Lemma 5.■

Lemma 6 *The parameters defining the optimal consumption rule (ii) satisfy the following three-equation system:*

$$
\begin{aligned}
\upsilon_0 &= k_1-\frac{\theta}{\gamma}\log\delta+\left(1-\frac{\theta}{\gamma}\right)r_f-\lambda\left(1-\theta\right)g+a_0\left(1-\frac{\theta}{\gamma}\right)\frac{\sigma_u^2}{2}+b_0\left(1-\frac{1}{\rho}\right) \\
&+b_1\mu\left(1-\phi\right)+b_2\left(\mu^2\left(1-\phi\right)^2+\sigma_\eta^2\right)+a_0^2\left(\frac{\theta}{\gamma}-1\right)\frac{\sigma_u^2}{2}, \\
\upsilon_1 &= a_0\left(1-\frac{\theta}{\gamma}\right)+a_1\left(1-\frac{\theta}{\gamma}\right)\frac{\sigma_u^2}{2}+b_1\left(\phi-\frac{1}{\rho}\right)+b_2 2\mu\phi\left(1-\phi\right)+a_0a_1\left(\frac{\theta}{\gamma}-1\right)\sigma_u^2, \\
\upsilon_2 &= a_1\left(1-\frac{\theta}{\gamma}\right)+b_2\left(\phi^2-\frac{1}{\rho}\right)+a_1^2\left(\frac{\theta}{\gamma}-1\right)\frac{\sigma_u^2}{2}.
\end{aligned}
$$

Proof of Lemma 6. Using Lemma 3 and 5 together with (4) we can write (15) in the following way

$$
\begin{aligned}
E_t\Delta c_{t+1} &= \frac{\theta}{\gamma}\log\delta + \lambda\left(1-\theta\right)E_t\Delta\bar{c}_{t+1} + \frac{\theta}{\gamma}E_t r_{p,t+1} + \upsilon_{p,t} \\
&= \frac{\theta}{\gamma}\log\delta + \lambda\left(1-\theta\right)g + \frac{\theta}{\gamma}r_f + \frac{\theta}{\gamma}a_0\left(1-a_0\right)\frac{\sigma_u^2}{2} + \frac{\theta}{\gamma}\left(a_0 + a_1\left(1-2a_0\right)\frac{\sigma_u^2}{2}\right)x_t \\
&+ \frac{\theta}{\gamma}\left(a_1 - a_1^2\frac{\sigma_u^2}{2}\right)x_t^2 + \upsilon_0 + \upsilon_1 x_t + \upsilon_2 x_t^2 \\
&= \frac{\theta}{\gamma}\log\delta + \lambda\left(1-\theta\right)g + \frac{\theta}{\gamma}r_f + \frac{\theta}{\gamma}a_0\left(1-a_0\right)\frac{\sigma_u^2}{2} + \upsilon_0 + \\
&\left\{\upsilon_1 + \frac{\theta}{\gamma}\left(a_0 + a_1\left(1-2a_0\right)\frac{\sigma_u^2}{2}\right)\right\}x_t + \left\{\upsilon_2 + \frac{\theta}{\gamma}\left(a_1 - a_1^2\frac{\sigma_u^2}{2}\right)\right\}x_t^2,
\end{aligned}
$$

which is a quadratic function of the state variable. But from Lemma 4 we also have that $E_t\Delta c_{t+1}$ is a quadratic function in x_t but with coefficients $\{c_0, c_1, c_2\}$. Equating the two sets of coefficients, Lemma 6 follows immediately.$\blacksquare$

5.5 Appendix 5: Coefficients in optimal portfolio choice

Using Lemma 2 in Appendix 4 and guess (ii), we can rewrite $Cov_t\left(r_{1,t+1}, c_{t+1} - w_{t+1}\right)$ in the following way

$$
\begin{aligned}
Cov_t\left(r_{1,t+1}, c_{t+1} - w_{t+1}\right) &= Cov_t\left(r_{1,t+1}, b_0 + b_1 x_{t+1} + b_2 x_{t+1}^2\right) \\
&= Cov_t\left[r_{1,t+1} - E_t r_{1,t+1}, b_1\left(x_{t+1} - E_t x_{t+1}\right) + b_2\left(x_{t+1}^2 - E_t x_{t+1}^2\right)\right] \\
&= Cov_t\left[u_{t+1}, b_1\eta_{t+1} + b_2\left(\eta_{t+1}^2 - \sigma_\eta^2\right) + b_2\left(2\mu\left(1-\phi\right) + 2\phi x_t\right)\eta_{t+1}\right] \\
&= b_1\sigma_{u\eta} + b_2\left(2\mu\left(1-\phi\right) + 2\phi x_t\right)\sigma_{u\eta}.
\end{aligned}
$$

Using this result, we can rewrite the optimal portfolio choice as follows

$$
\begin{aligned}
\alpha_t &= \frac{\theta}{\gamma}\frac{x_t + \frac{1}{2}\sigma_u^2}{\sigma_u^2} + \left(1-\theta\right)\lambda\frac{\sigma_{uv}}{\sigma_u^2} - \frac{b_1\sigma_{u\eta} + b_2\left(2\mu\left(1-\phi\right) + 2\phi x_t\right)\sigma_{u\eta}}{\sigma_u^2} \\
&= \frac{\theta}{\gamma}\frac{x_t}{\sigma_u^2} + \frac{\theta}{2\gamma} + \left(1-\theta\right)\lambda\frac{\sigma_{uv}}{\sigma_u^2} - b_1\frac{\sigma_{u\eta}}{\sigma_u^2} - b_2\left(2\mu\left(1-\phi\right) + 2\phi x_t\right)\frac{\sigma_{u\eta}}{\sigma_u^2} \\
&= \left\{\frac{\theta}{2\gamma} + \left(1-\theta\right)\lambda\frac{\sigma_{uv}}{\sigma_u^2} - b_1\frac{\sigma_{u\eta}}{\sigma_u^2} - b_2 2\mu\left(1-\phi\right)\frac{\sigma_{u\eta}}{\sigma_u^2}\right\} \\
&+ \left\{\frac{\theta}{\gamma}\frac{1}{\sigma_u^2} - b_2 2\phi\frac{\sigma_{u\eta}}{\sigma_u^2}\right\}x_t,
\end{aligned}
$$

where we have used the fact that $E_t r_{1,t+1} - r_f = x_t$, $Var_t\left(r_{1,t+1}\right) = \sigma_u^2$, and $Cov_t\left(r_{1,t+1}, \Delta\bar{c}_{t+1}\right) = \sigma_{uv}$. As can be seen from this result, the optimal portfolio choice is linear in x_t and hence guess (i) is verified.

5.6 Appendix 6: Coefficients in optimal consumption choice

Lemma 5 in Appendix 4 defines a nonlinear equation system for $\{\upsilon_0, \upsilon_1, \upsilon_2\}$, $\{a_0, a_1\}$, and $\{b_0, b_1, b_2\}$:

$$
\begin{aligned}
\upsilon_0 &= V_{10} + V_{11}a_0 + V_{12}b_1 + V_{13}b_2 + V_{14}a_0^2 + V_{15}b_1^2 + V_{16}b_2^2 + V_{17}a_0b_1 + V_{18}a_0b_2 + V_{19}b_1b_2 \\
\upsilon_1 &= V_{21}a_1 + V_{22}b_2 + V_{23}b_2^2 + V_{24}a_0a_1 + V_{25}a_0b_2 + V_{26}a_1b_1 + V_{27}a_1b_2 + V_{28}b_1b_2 \\
\upsilon_2 &= V_{31}a_1^2 + V_{32}b_2^2 + V_{33}a_1b_2,
\end{aligned}
$$

where the coefficients V_{ij} are functions of the primitive parameters of the model and are immediately identifiable from Lemma 5.

Similarly, Lemma 6 in Appendix 4 defines a second system for $\{\upsilon_0, \upsilon_1, \upsilon_2\}$, $\{a_0, a_1\}$, and $\{b_0, b_1, b_2\}$:

$$
\begin{aligned}
\upsilon_0 &= B_{10} + B_{11}b_0 + B_{12}b_1 + B_{13}b_2 + B_{14}a_0 + B_{15}a_0^2 \\
\upsilon_1 &= B_{21}a_0 + B_{22}b_1 + B_{23}b_2 + B_{24}a_1 + B_{25}a_0a_1 \\
\upsilon_2 &= B_{31}a_1 + B_{32}b_2 + B_{33}a_1^2,
\end{aligned}
$$

where again the coefficients B_{ij} are functions of the primitive parameters of the model.

Finally, the result on the coefficients in the optimal portfolio choice defines another system for $\{a_0, a_1\}$ and $\{b_0, b_1, b_2\}$:

$$
\begin{aligned}
a_0 &= A_{10} + A_{11}b_1 + A_{12}b_2 \\
a_1 &= A_{20} + A_{21}b_2,
\end{aligned}
$$

where the coefficients A_{ij} also are functions of the primitive parameters of the model.

By equating the first and the second system and substituting in the third system for $\{a_0, a_1\}$ we obtain a recursive equation system for $\{b_0, b_1, b_2\}$:

$$
\begin{aligned}
0 = & \left\{V_{10} + (V_{11} - B_{14})\, A_{10} + (V_{14} - B_{15})\, A_{10}^2 - B_{10}\right\} + \{-B_{11}\}\, b_0 \\
& + \{(V_{11} - B_{14})\, A_{11} + 2\,(V_{14} - B_{15})\, A_{10}A_{11} + V_{12} + V_{17}A_{10} - B_{12}\}\, b_1 \\
& + \{(V_{11} - B_{14})\, A_{12} + 2\,(V_{14} - B_{15})\, A_{10}A_{12} + V_{13} + V_{18}A_{10} - B_{13}\}\, b_2 \\
& + \left\{(V_{14} - B_{15})\, A_{11}^2 + V_{15} + V_{17}A_{11}\right\} b_1^2 \\
& + \left\{(V_{14} - B_{15})\, A_{12}^2 + V_{16} + V_{18}A_{11}\right\} b_2^2 \\
& + \{2\,(V_{14} - B_{15})\, A_{11}A_{12} + V_{17}A_{12} + V_{18}A_{11} + V_{19}\}\, b_1 b_2 \\
0 = & \{(V_{21} - B_{24})\, A_{20} + (V_{24} - B_{25})\, A_{10}A_{20} - B_{21}A_{10}\} \\
& + \{(V_{24} - B_{25})\, A_{11}A_{20} + V_{26}A_{20} - B_{21}A_{11} - B_{22}\}\, b_1 \\
& + \{(V_{21} - B_{24})\, A_{21} + (V_{24} - B_{25})\,(A_{10}A_{21} + A_{12}A_{20}) + V_{22} + V_{25}A_{10} \\
& + V_{27}A_{20} - B_{23} - B_{21}A_{12}\} b_2 \\
& + \{(V_{24} - B_{25})\, A_{12}A_{21} + V_{23} + V_{25}A_{12} + V_{27}A_{21}\}\, b_2^2 \\
& + \{(V_{24} - B_{25})\, A_{11}A_{21} + V_{25}A_{11} + V_{26}A_{21} + V_{28}\}\, b_1 b_2 \\
0 = & \left\{(V_{31} - B_{33})\, A_{20}^2 - B_{31}A_{20}\right\} \\
& + \{2\,(V_{31} - B_{33})\, A_{20}A_{21} + V_{33}A_{20} - B_{31}A_{21} - B_{32}\}\, b_2 \\
& + \left\{(V_{31} - B_{33})\, A_{21}^2 + V_{32} + V_{33}A_{21}\right\} b_2^2.
\end{aligned}
$$

If we let Λ_{ij} be functions of A_{ij}, B_{ij}, and V_{ij} we can write the system as follows

$$
\begin{aligned}
0 &= \Lambda_{10} + \Lambda_{11}b_0 + \Lambda_{12}b_1 + \Lambda_{13}b_2 + \Lambda_{14}b_1^2 + \Lambda_{15}b_2^2 + \Lambda_{16}b_1b_2 \\
0 &= \Lambda_{20} + \Lambda_{21}b_1 + \Lambda_{22}b_2 + \Lambda_{23}b_2^2 + \Lambda_{24}b_1b_2 \\
0 &= \Lambda_{30} + \Lambda_{31}b_2 + \Lambda_{32}b_2^2.
\end{aligned}
$$

which identifies $\{b_0, b_1, b_2\}$.

6 References

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7 Tables and figures

(A) Estimated model:

$$\begin{bmatrix} r_{1,t+1} - r_f \\ d_{t+1} - p_{t+1} \\ \Delta\bar{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \begin{array}{c} 0.110 \\ (0.042) \end{array} & \begin{array}{c} 0.027 \\ (0.012) \end{array} \\ \begin{array}{c} -0.066 \\ (0.040) \end{array} & \begin{array}{c} 0.982 \\ (0.012) \end{array} \\ \begin{array}{c} 0.006 \\ (0.001) \end{array} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ d_t - p_t \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \\ \varepsilon_{3,t+1} \end{bmatrix}$$

$$\Omega = \begin{bmatrix} 6.323E-3 & -5.722E-3 & 7.631E-5 \\ -5.722E-3 & 5.720E-3 & -6.206E-5 \\ 7.631E-5 & -6.206E-5 & 6.953E-5 \end{bmatrix}, \qquad R^2 = \begin{bmatrix} 0.021 \\ 0.968 \\ 0.000 \end{bmatrix}$$

(B) Derived model:

$$x_{t+1} = 1.568E-2 + 0.982\,(x_t - \mu) + \eta_{t+1}$$

$$\Delta\bar{c}_{t+1} = 0.006 + v_{t+1}$$

$$\begin{bmatrix} \sigma_u^2 & \sigma_{u\eta} & \sigma_{uv} \\ \sigma_{u\eta} & \sigma_\eta^2 & \sigma_{\eta v} \\ \sigma_{uv} & \sigma_{\eta v} & \sigma_v^2 \end{bmatrix} = \begin{bmatrix} 6.323E-3 & -1.560E-4 & 7.631E-5 \\ -1.560E-4 & 4.255E-6 & -1.692E-6 \\ 7.631E-5 & -1.692E-6 & 6.953E-5 \end{bmatrix}$$

$$corr\,(u,\eta) = -0.951 \qquad corr\,(u,v) = 0.115 \qquad corr\,(\eta,v) = -0.098$$

$$r_f = 0.290E-2 \qquad \sigma_x^2/\sigma_u^2 = 2.129E-2$$

Notes: Standard errors are (in parentheses).

Table 1. Estimated and derived model for the stochastic structure of the model (1947.1-2007.4).

γ	θ				
	1	0.8	0.6	0.4	0.2
			(A) Total allocation to stocks		
1	298.0	276.5	243.8	200.7	145.8
2	217.0	194.8	170.7	142.5	108.4
4	149.9	136.5	121.9	104.8	82.4
10	95.2	88.7	80.8	71.4	58.0
20	68.8	64.5	60.2	53.7	44.0
			(B) Myopic component		
1	298.0	238.4	178.8	119.2	59.6
2	149.0	119.2	89.4	59.6	29.8
4	74.5	59.6	44.7	29.8	14.9
10	29.8	23.8	17.9	11.9	6.0
20	14.9	11.9	8.9	6.0	3.0
			(C) Habit hedge component		
1	0	0.2	0.5	0.7	1.0
2	0	0.2	0.5	0.7	1.0
4	0	0.2	0.5	0.7	1.0
10	0	0.2	0.5	0.7	1.0
20	0	0.2	0.5	0.7	1.0
			(D) Intertemporal hedge component		
1	0	37.9	64.5	80.8	85.3
2	68.0	75.4	80.8	82.1	77.7
4	75.4	76.7	76.7	74.3	66.6
10	65.4	64.6	62.4	58.7	51.1
20	53.9	52.3	50.7	47.0	40.0

Notes: Demand is in percent and $\lambda = 1$.

Table 2. Mean optimal allocation to stocks for different values of γ and θ.

γ	θ				
	1	0.8	0.6	0.4	0.2
	(A) Total habit demand for stocks				
1	0	-21.5	-54.2	-97.2	-152.2
2	0	-22.2	-46.2	-74.5	-108.5
4	0	-13.4	-28.0	-45.1	-67.4
10	0	-6.5	-14.4	-23.8	-37.1
20	0	-4.3	-8.6	-15.0	-24.8
	(B) Habit fraction of total demand				
1	0	-7.8	-22.2	-48.4	-104.0
2	0	-11.4	-27.1	-52.3	-100.1
4	0	-9.8	-23.0	-43.0	-81.8
10	0	-7.3	-17.8	-33.3	-64.0
20	0	-6.6	-14.3	-28.0	-56.4

Notes: Demand and fraction are in percent and $\lambda = 1$.

Table 3. Mean optimal habit demand for stocks for different values of γ and θ.

λ	θ				
	1	0.8	0.6	0.4	0.2
			(A) Total allocation to stocks		
0	149.9	133.5	115.3	95.2	68.8
0.5	149.9	135.0	118.7	99.4	75.1
1	149.9	136.5	121.9	104.8	82.4
1.5	149.9	138.1	124.7	109.8	91.1
2	149.9	139.6	128.2	116.3	100.7
			(B) Myopic component		
0	74.5	59.6	44.7	29.8	14.9
0.5	74.5	59.6	44.7	29.8	14.9
1	74.5	59.6	44.7	29.8	14.9
1.5	74.5	59.6	44.7	29.8	14.9
2	74.5	59.6	44.7	29.8	14.9
			(C) Habit hedge component		
0	0	0	0	0	0
0.5	0	0.1	0.2	0.4	0.5
1	0	0.2	0.5	0.7	1.0
1.5	0	0.4	0.7	1.1	1.4
2	0	0.5	1.0	1.4	1.9
			(D) Intertemporal hedge component		
0	75.4	73.9	70.6	65.4	53.9
0.5	75.4	75.3	73.7	69.2	59.7
1	75.4	76.7	76.7	74.3	66.6
1.5	75.4	78.1	79.3	78.9	74.8
2	75.4	79.6	82.5	85.0	83.9

Notes: Demand is in percent and $\gamma = 4$.

Table 4. Mean optimal allocation to stocks for different values of θ and λ.

λ	θ				
	1	0.8	0.6	0.4	0.2
	(A) Total habit demand for stocks				
0	0	-16.4	-34.6	-54.7	-81.1
0.5	0	-14.9	-31.2	-50.5	-74.8
1	0	-13.4	-28.0	-45.1	-67.4
1.5	0	-11.8	-25.1	-40.1	-58.7
2	0	-10.2	-21.7	-33.6	-49.2
	(B) Habit fraction of total demand				
0	0	-12.2	-30.0	-57.5	-118.0
0.5	0	-11.0	-26.3	-50.8	-99.6
1	0	-9.8	-23.0	-43.0	-81.8
1.5	0	-8.6	-20.2	-36.5	-64.4
2	0	-7.3	-16.9	-28.9	-48.8

Notes: Demand and fraction are in percent and $\gamma = 4$.

Table 5. Mean optimal habit demand for stocks for different values of θ and λ.

γ	θ				
	1	0.8	0.6	0.4	0.2
	(A) Consumption-wealth ratio				
1	1.53	1.89	2.16	2.26	2.08
2	2.48	2.44	2.35	2.15	1.79
4	2.46	2.31	2.11	1.85	1.43
10	2.05	1.87	1.65	1.37	0.91
20	1.69	1.51	1.30	1.00	0.46
	(B) Long-term expected return on wealth				
1	4.16	4.13	4.00	3.67	3.00
2	3.82	3.62	3.35	2.96	2.40
4	3.09	2.89	2.65	2.35	1.93
10	2.21	2.08	1.93	1.73	1.46
20	1.71	1.62	1.52	1.38	1.18

Notes: Consumption-wealth ratio and long-term expected return on wealth are in percent and $\lambda = 1$. Consumption-wealth ratios are calculated as $C_t/W_t = \exp\left[b_0 + b_1\mu + b_2\left(\mu^2 + \sigma_x^2\right)\right]$ and long-term expected returns are calculated as $E\left(r_{p,t+1}\right) = r_f + p_0 + p_1\mu + p_2\left(\mu^2 + \sigma_x^2\right)$, where $\{p_0, p_1, p_2\}$ are functions of $\{a_0, a_1, \sigma_u^2\}$ and given in Lemma 3 in Appendix 4.

Table 6. Mean optimal consumption-wealth ratio and long-term expected return on wealth for different values of γ and θ.

λ	θ				
	1	0.8	0.6	0.4	0.2
	(A) Consumption-wealth ratio				
0	2.46	2.38	2.25	2.05	1.69
0.5	2.46	2.34	2.18	1.95	1.57
1	2.46	2.31	2.11	1.85	1.43
1.5	2.46	2.27	2.04	1.75	1.29
2	2.46	2.23	1.97	1.65	1.12
	(B) Long-term expected return on wealth				
0	3.09	2.86	2.57	2.21	1.71
0.5	3.09	2.87	2.61	2.28	1.82
1	3.09	2.89	2.65	2.35	1.93
1.5	3.09	2.91	2.69	2.42	2.06
2	3.09	2.93	2.73	2.51	2.20

Notes: Consumption-wealth ratio and long-term expected return on wealth are in percent and $\gamma = 4$. Consumption-wealth ratios are calculated as $C_t/W_t = \exp\left[b_0 + b_1\mu + b_2\left(\mu^2 + \sigma_x^2\right)\right]$ and long-term expected returns are calculated as $E\left(r_{p,t+1}\right) = r_f + p_0 + p_1\mu + p_2\left(\mu^2 + \sigma_x^2\right)$, where $\{p_0, p_1, p_2\}$ are functions of $\{a_0, a_1, \sigma_u^2\}$ and given in Lemma 3 in Appendix 4.

Table 7. Mean optimal consumption-wealth ratio and long-term expected return on wealth for different values of θ and λ.

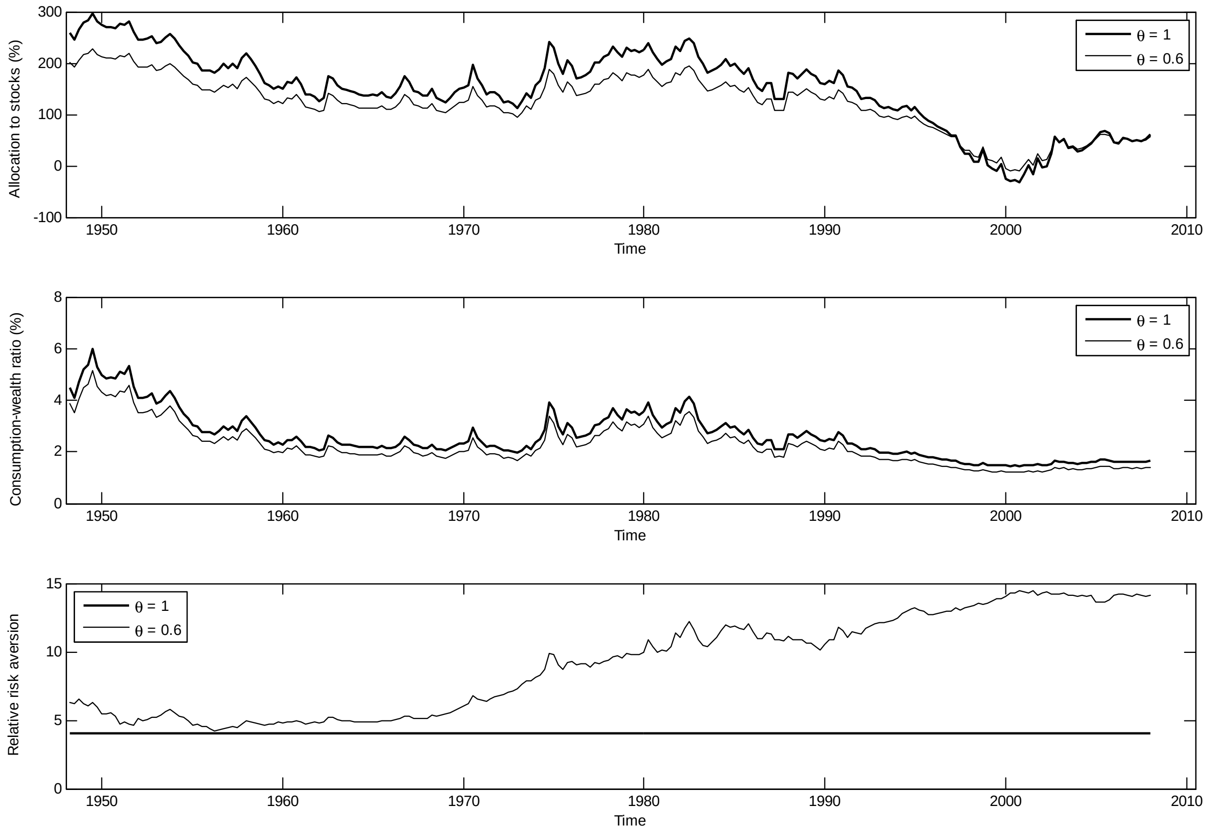

Figure 1: Portfolio allocation, consumption-wealth ratio, and relative risk aversion over time for $\gamma = 4$ and $\lambda = 1$.

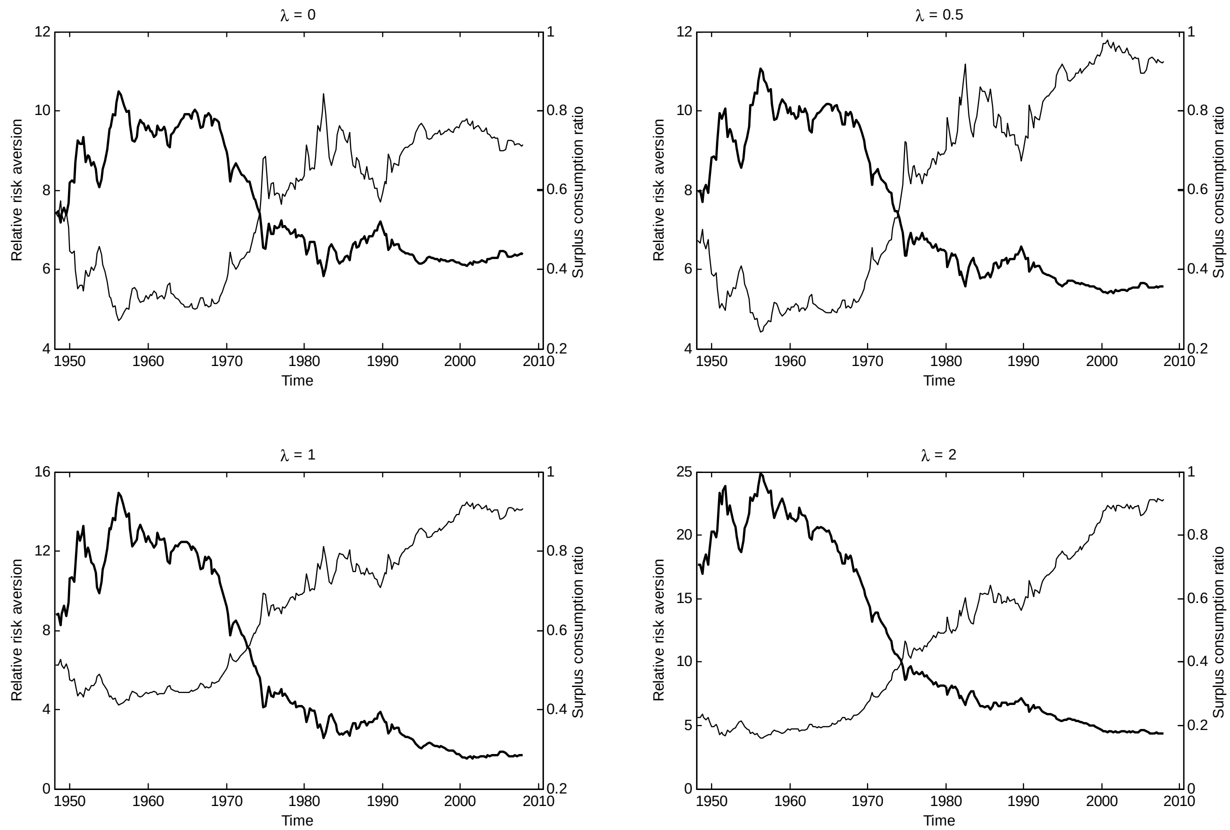

Figure 2: Relative risk aversion (solid line) and surplus consumption ratio (bold line) over time for $\gamma = 4$ and $\theta = 0.6$.

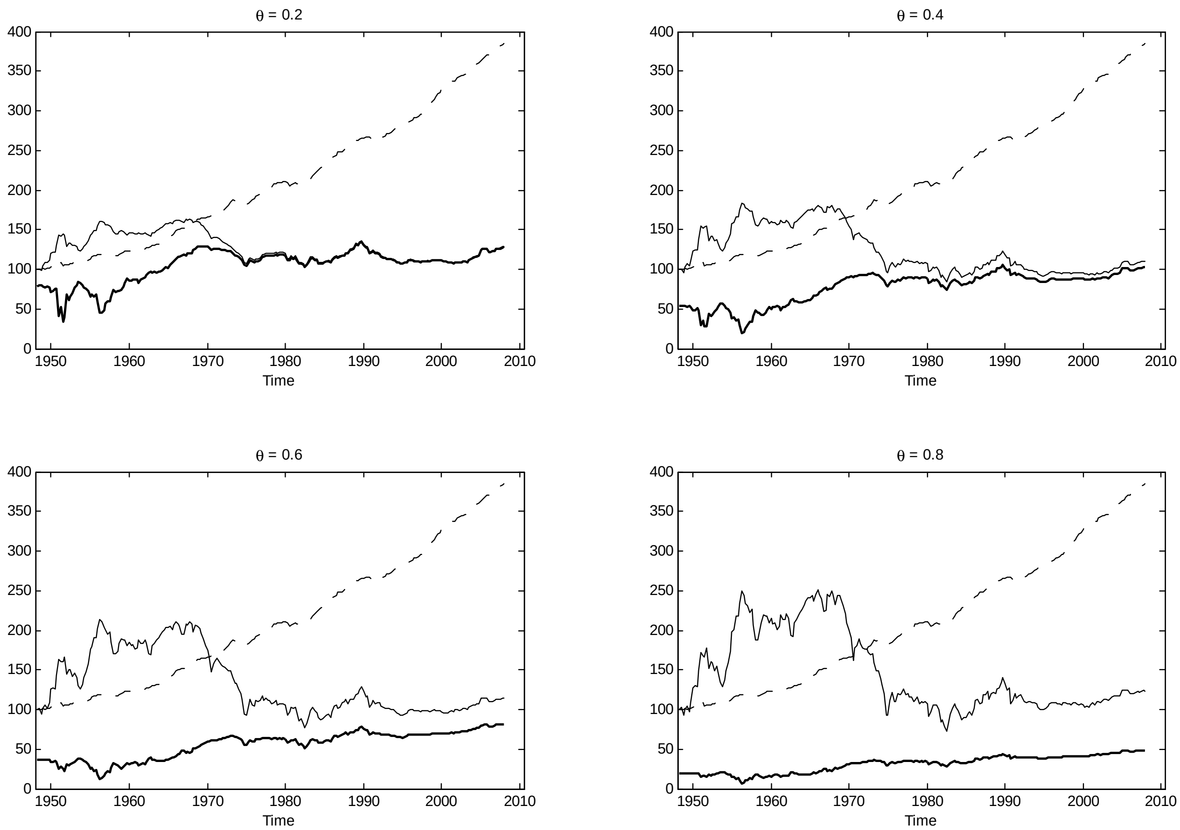

Figure 3: Indexed investor consumption (solid line), per capita consumption (dashed line), and model-implied habit (bold line) over time for $\gamma = 4$ and $\lambda = 1$.

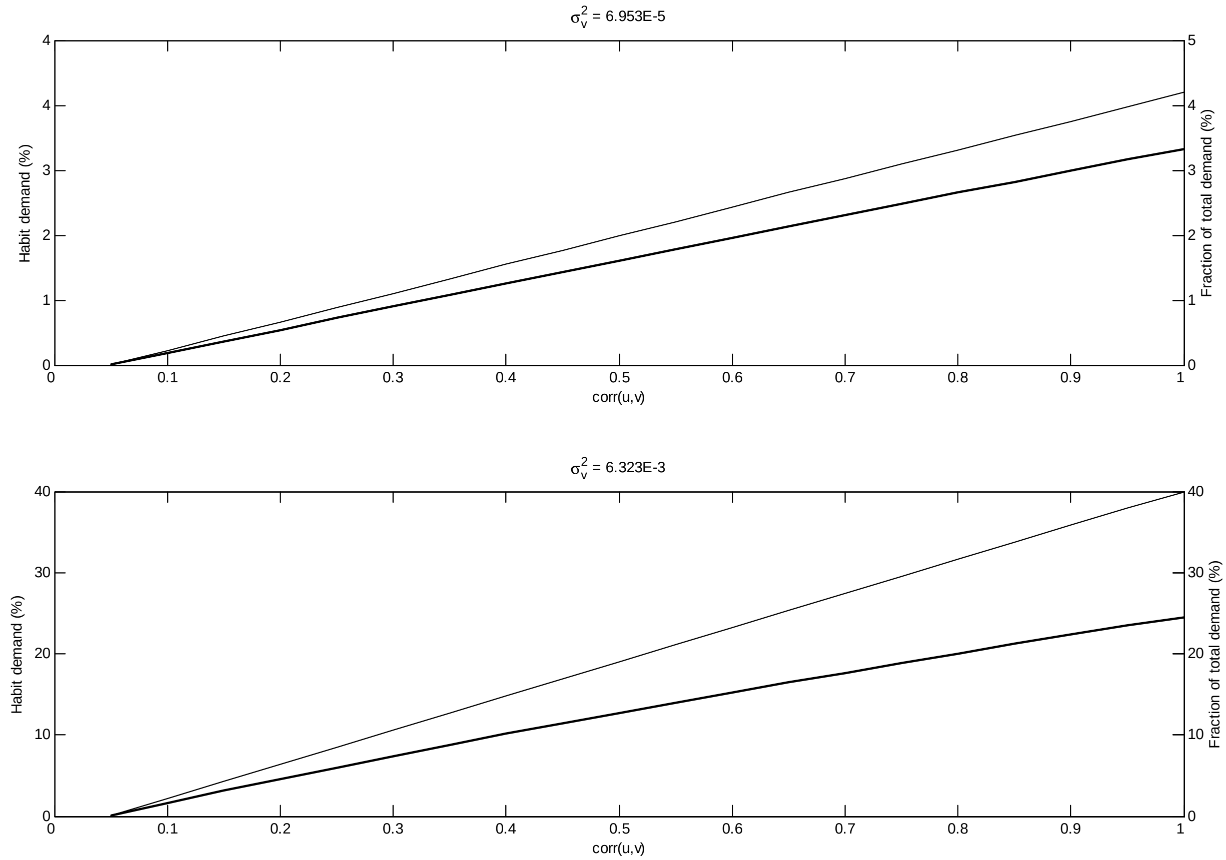

Figure 4: Habit hedge demand (solid line) and habit hedge demand's fraction of total demand (bold line) as a function of correlation between the risky asset and the proxy used to govern the change in habit for $\gamma = 4$, $\theta = 0.6$, and $\lambda = 1$.