How to Compute the Liquidity Cost in a Market Governed by Orders?

1 Introduction

Our particular focus in this paper is on following question. How to compute the liquidity cost in a market governed by orders under asymmetrical information? The answer of this question is related to another questions. First, how do informed and liquidity traders differ in their provision and use of market liquidity? Second, how do characteristics of the market, such as depth in the book or time left to trade, affect these strategies? And, third, how do characteristics of the underlying asset such as asset value volatility affect the provision of market liquidity? Numerous authors in finance have examined aspects of these questions both theoretically and empirically.

The choice between market orders and limit orders has been analyzed in various contexts, see, e.g., Chakravarty and Holden (1995), Cohen, Maier, Schwartz and Whitcomb (1981), Handa and Schwartz (1996), Kumar and Seppi (1993). Dynamic models of order-driven markets include Foucault (1999), Foucault, Kadan and Kandel (2005), Parlour (1998). The price behavior in limit order books has been analyzed theoretically by Biais, Martimort and Rochet (2000), Glosten (1994), OHara and Oldfield (1986), Rock (1990), and Seppi (1997). Models that analyze liquidity traders, the dynamics of prices and trades and the convergence of prices to the fundamental value include Glosten and Milgrom (1985), Kyle (1985), Admati and Pfleiderer (1988), Easley and O'Hara (1987).

Empirical studies of specific limit order markets include Biais, Hillion and Spatt (1995), Hollifield, Miller and Sandas (1999), Ahn, Bae and Chan (2001), Hasbrouck and Saar (2001) and Hollifield, Miller, Sandas and Slive (2006).

Financial markets microstructure theory can be adapted in order to analyze the equilibrium using time component. In this paper we used the stochastic calculus in order to describe the equilibrium on the financial markets taking into consideration their microstructure. Therefore, we consider that the transaction price (or observable price), S, is stochastic and follows a geometric Brownian motion defined by

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (1)$$

where μ, σ, t and W_t represent the instantaneous return of the financial asset, its volatility, time and a standard Brownian motion. Moreover, we consider that on the market two kind of agents exist: informed and uninformed agents. The informed agents know the differences between the equilibrium price and the transaction price. The differences between these prices (denoted by I) is defined by a martingale

$$dI_t = \sigma_i dB_t \quad (2)$$

where σ_i represents the volatility of the differences between the equilibrium price and the transaction price. The parameters μ, σ and σ_i are constants in time, and the standard Brownian motions, B_t and W_t, are instantaneous correlated, $dW_t dB_t = \rho dt$. The parameter ρ is constant in time and represents the instantaneous correlation coefficient. Also, we follow the hypothesis that on the market the equilibrium is not perfectly revealing the information, $\rho \neq \pm 1$. Thus, the presence of informed agents on the market is not perfectly revealed to uninformed agents.

Under these hypotheses, the equilibrium price, denoted by P, is defined by

$$Q_p P = QS + Q_i I \tag{3}$$

In the equation (3), Q_p represents the amount of financial asset traded at equilibrium, Q represents the amount of financial asset actually traded on the market and Q_i represents the additional amount of financial asset traded by the informed agents. Therefore, the market value at equilibrium is equal to the actually market value plus the market value due to transactions made by informed agents. In the situation in which informed agents don't exist on the market and the information is entirely public then the market value at equilibrium would be equal to the actually market value. In time, the transactions made by the informed agents will be discovered by the uniformed agents and consequently the transaction price tends to the equilibrium price.

The paper is organized as follows: section 2 analyses the equilibrium price of the financial asset in the stochastic environment, section 3 shows the expected value of the equilibrium price using the martingale restriction, section 4 presents the derivation of a liquidity cost formula when the market is governed by orders, section 5 shows the empirical results and section 6 summarizes and concludes.

2 Equilibrium Price

Equation (3) can be written as follow

$$P = \frac{Q}{Q_p} S + \frac{Q_i}{Q_p} I = \alpha S + \beta I \tag{4}$$

where α and β are constant parameters defined using the traded amounts. In dynamic, the equilibrium price will be the solution of the following stochastic differential equation.

$$dP_t = \alpha dS_t + \beta dI_t \tag{5}$$

Using the definitions of the stochastic processes followed by S and I, the equilibrium price is defined by the following stochastic differential equation:

$$dP_t = \alpha \mu S_t dt + \alpha \sigma S_t dW_t + \beta \sigma_i dB_t \tag{6}$$

Taking into consideration the fact that the equilibrium price is a function of S and I, its dynamic can be written based on Itô lemma

$$\begin{aligned} dP_t \quad = \quad & \left[\frac{\partial P}{\partial t} + \frac{\partial P}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 P}{\partial S^2} \sigma^2 S^2 + \frac{1}{2} \frac{\partial^2 P}{\partial I^2} \sigma_i^2 + \frac{\partial^2 P}{\partial S \partial I} \rho \sigma \sigma_i S \right] dt + \\ & \frac{\partial P}{\partial S} \sigma S dW_t + \frac{\partial P}{\partial I} \sigma_i dB_t \end{aligned} \tag{7}$$

Equalizing the drifts and the diffusion coefficients for the stochastic dynamics (6) and (7), the following relations are obtained:

$$\frac{\partial P}{\partial t}+\frac{\partial P}{\partial S}\mu S+\frac{1}{2}\frac{\partial^2 P}{\partial S^2}\sigma^2 S^2+\frac{1}{2}\frac{\partial^2 P}{\partial I^2}\sigma_i^2+\frac{\partial^2 P}{\partial S\partial I}\rho\sigma\sigma_i S=\alpha\mu S \tag{8}$$

$$\frac{\partial P}{\partial S}\sigma S=\alpha\sigma S \tag{9}$$

$$\frac{\partial P}{\partial I}\sigma_i=\beta\sigma_i \tag{10}$$

which means that $\alpha=\frac{\partial P}{\partial S}$ and $\beta=\frac{\partial P}{\partial I}$. Replacing α into equation (8), the equilibrium price is the solution of a partial derivatives equation

$$\frac{\partial P}{\partial t}+\frac{1}{2}\frac{\partial^2 P}{\partial S^2}\sigma^2 S^2+\frac{1}{2}\frac{\partial^2 P}{\partial I^2}\sigma_i^2+\frac{\partial^2 P}{\partial S\partial I}\rho\sigma\sigma_i S=0 \tag{11}$$

with final conditions $P_{T_a}=S_{T_a}$ and $I_{T_a}=0$. These conditions show that, at the optimum date T_a, the equilibrium price is equal to the transaction price. Taking the logarithm of the transaction price, the partial derivatives equation can be written

$$\frac{\partial P}{\partial t}-\frac{1}{2}\frac{\partial P}{\partial \ln S}\sigma^2+\frac{1}{2}\frac{\partial^2 P}{\partial\left(\ln S\right)^2}\sigma^2+\frac{1}{2}\frac{\partial^2 P}{\partial I^2}\sigma_i^2+\frac{\partial^2 P}{\partial \ln S\partial I}\rho\sigma\sigma_i=0 \tag{12}$$

with the final conditions $P_{T_a}=\exp\left(\ln S_{T_a}\right)$ and $I_{T_a}=0$.

Taking into the consideration the time period, $\tau_a=T_a-t$, the partial derivative of the equilibrium price with respect to t can be written

$$\frac{\partial P}{\partial t}=\frac{\partial P}{\partial \tau_a}\frac{\partial \tau_a}{\partial t}=-\frac{\partial P}{\partial \tau_a} \tag{13}$$

Considering that τ_a is a stopping time[1], the equilibrium price is defined by the following partial derivatives equation

$$\frac{\partial P}{\partial \tau_a}=-\frac{1}{2}\frac{\partial P}{\partial \ln S}\sigma^2+\frac{1}{2}\frac{\partial^2 P}{\partial\left(\ln S\right)^2}\sigma^2+\frac{1}{2}\frac{\partial^2 P}{\partial I^2}\sigma_i^2+\frac{\partial^2 P}{\partial \ln S\partial I}\rho\sigma\sigma_i \tag{14}$$

The solution of this partial derivatives equation is

$$P=\exp\left(M+NI+\ln S\right) \tag{15}$$

where I and S are known at the current date. Therefore, replacing this expression into the partial derivatives equation (14), the obtained results are

$$P\left(\frac{\partial M}{\partial \tau_a}+I\frac{\partial N}{\partial \tau_a}\right)=-\frac{1}{2}\sigma^2 P+\frac{1}{2}\sigma^2 P+\frac{1}{2}\sigma_i^2 N^2 P+\rho\sigma\sigma_i NP \tag{16}$$

[1] Let $(B_t, t\geq 0)$ and $a\in\mathbb{R}$. T_a is a stopping time if $T_a=\inf\{t\geq 0, B_t=a\}$. For $\lambda>0$, the Laplace transform of a stopping time is given by the formula: $E\left[e^{-\lambda T_a}\right]=e^{-|a|\sqrt{2\lambda}}$.

or

$$\frac{\partial M}{\partial \tau_a} - \frac{1}{2}\sigma_i^2 N^2 - \rho\sigma\sigma_i N = -I\frac{\partial N}{\partial \tau_a} \tag{17}$$

This relation is verified for all values of I. Thus, the term which is multiplied by I and the term which is independent of I must be zero. Therefore,

$$\frac{\partial M}{\partial \tau_a} = \frac{1}{2}\sigma_i^2 N^2 + \rho\sigma\sigma_i N \tag{18}$$

$$\frac{\partial N}{\partial \tau_a} = 0 \tag{19}$$

Consequently, N is a constant parameter and M is time dependent, $M = f(\tau_a)$. In fact, the equation (18) is an ordinary differential equation with the initial condition $M(0) = 0$.

$$\int_0^{\tau_a} dM(s) = \int_0^{\tau_a} \frac{1}{2}\sigma_i^2 N^2 ds + \int_0^{\tau_a} \rho\sigma\sigma_i N ds \tag{20}$$

The solution of this equation is:

$$M = \frac{1}{2}\sigma_i^2 N^2 \tau_a + \rho\sigma\sigma_i N \tau_a \tag{21}$$

On the other hand, the above equation allows us to obtain the parameter N as a solution of a quadratic equation. Thus,

$$N_{1,2} = -\rho\frac{\sigma}{\sigma_i} \pm \frac{1}{\sigma_i^2 \tau_a}\sqrt{\rho^2\sigma^2\sigma_i^2\tau_a^2 + 2\sigma_i^2\tau_a M} \tag{22}$$

Because N is not time dependent, $\rho^2\sigma^2\sigma_i^2\tau_a^2 + 2\sigma_i^2\tau_a M = 0$, the expressions of M and N are defined by

$$M = -\frac{1}{2}\rho^2\sigma^2\tau_a \tag{23}$$

$$N = -\rho\frac{\sigma}{\sigma_i} \tag{24}$$

Concluding, the equilibrium price is a random variable defined by certain parameters and the stopping time τ_a. The equilibrium price is given by

$$P = Se^{-\frac{1}{2}\rho^2\sigma^2\tau_a - \rho\frac{\sigma}{\sigma_i}I} \tag{25}$$

2.1 The Expected Value of the Equilibrium Price

In this paragraph we analyze the expected value of the equilibrium price, $E[P]$. In order to express analytically the expected value of the equilibrium price, we use the Laplace transform and its inverse. The Laplace transform is defined by

$$F(s) = \int_0^{\infty} e^{-sx} f(x)\, dx \tag{26}$$

For the function $F(s) = e^{-k\sqrt{s}}$, $k > 0$, the inverse of the Laplace transform[2] is

$$f(x) = \frac{k}{2\sqrt{\pi x^3}} e^{-\frac{k^2}{4x}} \tag{27}$$

If $k = |a|\sqrt{2}$ and $s = \lambda$, the density function of the stopping time, τ_a, is defined by

$$f(\tau_a) = \frac{|a|}{\sqrt{2\pi\tau_a^3}} e^{-\frac{|a|^2}{2\tau_a}} \tag{28}$$

The value of a can be obtained using the stochastic differential equation (2), knowing that $B_{\tau_a} = a$ and $I_{\tau_a} = 0$. Because $I_{\tau_a} - I = \sigma_i (B_{\tau_a} - B_0)$, the following result is true: $a = -\frac{I}{\sigma_i}$ and $|a| = \frac{|I|}{\sigma_i}$. Consequently, the density function of the stopping time becomes

$$f(\tau_a) = \frac{|I|}{\sigma_i\sqrt{2\pi\tau_a^3}} e^{-\frac{|I|^2}{2\sigma_i^2\tau_a}} \tag{29}$$

The expected value of the equilibrium price is defined by

$$E[P] = Se^{-\rho\frac{\sigma}{\sigma_i}I} \int_0^\infty e^{-\frac{1}{2}\rho^2\sigma^2\tau_a} \frac{|I|}{\sigma_i\sqrt{2\pi\tau_a^3}} e^{-\frac{|I|^2}{2\sigma_i^2\tau_a}} d\tau_a \tag{30}$$

This expression can be written in an easier way, such as

$$E[P] = S\frac{|I|}{\sigma_i} \int_0^\infty \frac{1}{\sqrt{2\pi\tau_a^3}} e^{-\frac{1}{2}\left(m\sqrt{\tau_a} + \frac{n}{\sqrt{\tau_a}}\right)^2} d\tau_a \tag{31}$$

where $m = \rho\sigma$ and $n = \frac{I}{\sigma_i}$. Into the above integral we change the variable $\tau_a = y^2$. Hence,

$$E[P] = 2S\frac{|I|}{\sigma_i} \int_0^\infty \frac{1}{y^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(my + \frac{n}{y}\right)^2} dy \tag{32}$$

Resolving the integral, we obtain the following result

$$\int_0^\infty \frac{1}{y^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(my + \frac{n}{y}\right)^2} dy = \frac{1}{2|n|} e^{-2mn} = \frac{1}{2}\frac{\sigma_i}{|I|} e^{-2\rho\frac{\sigma}{\sigma_i}I}, \text{ if } \begin{cases} I > 0; \rho > 0 \\ I < 0; \rho < 0 \end{cases} \tag{33}$$

$$\int_0^\infty \frac{1}{y^2\sqrt{2\pi}} e^{-\frac{1}{2}\left(my + \frac{n}{y}\right)^2} dy = \frac{1}{2|n|} = \frac{1}{2}\frac{\sigma_i}{|I|}, \text{ if } \begin{cases} I > 0; \rho < 0 \\ I < 0; \rho > 0 \end{cases} \tag{34}$$

Finally, the expected value of the equilibrium price is given by the following formula

$$E[P] = \begin{cases} Se^{-2\rho\frac{\sigma}{\sigma_i}I}, \text{ for } I > 0; \rho > 0 \text{ or } I < 0; \rho < 0 \\ S, \text{ for } I > 0; \rho < 0 \text{ or } I < 0; \rho > 0 \end{cases} \tag{35}$$

The expected value of the equilibrium price on financial asset market is given by the transaction price multiplied by a correction factor.

[2] See Abramowitz M. and Stegun I.A., (1970), "Handbook of Mathematical Functions", Dover, New York.

3 Martingale Restriction

Knowing the expression of the equilibrium price, defined by a nonlinear function of the stopping time, we can obtain the partial derivatives of the equilibrium price with respect to the transaction price and the differences between the equilibrium price and the transaction price. Therefore the parameters α and β are given by

$$\frac{\partial P}{\partial S} = e^{-\frac{1}{2}\rho^2\sigma^2\tau_a - \rho\frac{\sigma}{\sigma_i}I} = \frac{P}{S} \tag{36}$$

$$\frac{\partial P}{\partial I} = -\rho\frac{\sigma}{\sigma_i}Se^{-\frac{1}{2}\rho^2\sigma^2\tau_a - \rho\frac{\sigma}{\sigma_i}I} = -\rho\frac{\sigma}{\sigma_i}P \tag{37}$$

Consequently, the dynamic equation (6) of the equilibrium price becomes

$$dP_t = \mu S_t\frac{\partial P}{\partial S}dt + \sigma S_t\frac{\partial P}{\partial S}dW_t + \sigma_i\frac{\partial P}{\partial I}dB_t \tag{38}$$

or

$$dP_t = \mu P_t dt + \sigma P_t dW_t - \rho\sigma P_t dB_t \tag{39}$$

From now on, we transform the Brownian motion W into a Brownian motion Z independent from the Brownian motion B

$$Z = \frac{1}{\sqrt{1-\rho^2}}\left(W - \rho B\right) \tag{40}$$

or

$$dW_t - \rho dB_t = \sqrt{1-\rho^2}dZ_t \tag{41}$$

Thus, the dynamics of P and S can be written using the Brownian motion Z as follows

$$dP_t = \mu P_t dt + \sigma\sqrt{1-\rho^2}P_t dZ_t \tag{42}$$

$$dS_t = \mu S_t dt + \rho\sigma S_t dB_t + \sigma\sqrt{1-\rho^2}S_t dZ_t \tag{43}$$

Using Itô lemma, we obtain the following dynamic equations of the logarithms of P and S:

$$d\left(\ln P_t\right) = \left[\mu - \frac{1}{2}\sigma^2\left(1-\rho^2\right)\right]dt + \sigma\sqrt{1-\rho^2}dZ_t \tag{44}$$

$$d\left(\ln S_t\right) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \rho\sigma dB_t + \sigma\sqrt{1-\rho^2}dZ_t \tag{45}$$

By summation, we obtain:

$$d\left(\ln\frac{S_t}{P_t}\right) = -\frac{1}{2}\rho^2\sigma^2 dt + \rho\sigma dB_t \tag{46}$$

Using once again Itô lemma, the dynamic of the ratio between the two prices is given by

$$d\left(\frac{S_t}{P_t}\right) = \frac{S_t}{P_t}\rho\sigma dB_t \tag{47}$$

By integration, on the time period $t \to T_a$:

$$\frac{S_{T_a}}{P_{T_a}} - \frac{S}{P} = \int_t^{T_a} \frac{S_u}{P_u} \rho\sigma dB_u \tag{48}$$

Because the expected value of the stochastic integral is zero and at T_a the equilibrium price is equal to the transaction price, the expected value of the equilibrium price is given by the transaction price:

$$E[P] = S \tag{49}$$

This result is intuitively correct because it proves a fundamental reason of market mechanism: all the agents expect that the equilibrium price is the actually price.

4 Liquidity Cost

Taking into consideration the market microstructure, the liquidity has two alternative sources: prices negotiated by the market makers, if the market is governed by prices, or prices negotiated by the final investors, if the market is governed by orders. On the continuous market, a limit order is risky because its execution depends on the market conditions changes. Let's consider a situation where an agent gives a selling limit order at 100 €. If a new information arrives on market justifying a new price at 101 € and the agent is not willing to quickly change the order, then other agents would have the opportunity to gain 1 €. This phenomenon can be described using the option theory: the agent who gives a limit order offers an option to the rest of the market which can be exercised against him if the market goes contrary.

On the one hand, in the auction theory, the winner of an auction overestimates the value of the object to sell. Hence, the winner is "cursed" to pay a higher price. The agent who gives a limit order is faced with a similar problem. Due to its optional character, a buying limit order risks to be executed only if the real value of the asset becomes lower than the offered price (which means that the price overestimates the real value of the asset). Similarly, a selling limit order risks to be executed only if the price underestimates the real value of the asset.

On the other hand, the risk of a limit order can be explained by information asymmetry. Thus, an agent who gives a limit order is faced with the adverse selection risk. For example, a buying limit order allows an informed agent who knows that the real value of the asset is lower than the offered price to take advantage from his information against the buyer who gives the limit order.

Therefore, the agent will be less incited to give the limit orders and the market liquidity will decreases. This risk appears especially on markets with the automated execution of orders. Hereby, the market can quickly profit from the selling or buying limit orders which overestimate or underestimate the value of the financial assets before the agents have time to cancel or modify the limit orders.

From now on, we use the option theory in order to obtain a formula of the liquidity cost. Therefore, we consider a continuous market governed by orders with unlimited time execution of orders, such as French or Japanese stock exchange. A buying limit order gives to other agents the right but not the obligation to sell the financial asset at limit price offered for unlimited time. Therefore, the liquidity cost payable by the agent who gives the limit order is the price of a perpetual American put. The liquidity cost is defined by

$$L = \max_{\tau_l} E_Q \left[e^{-r\tau_l} \left(K - S_{\tau_l} \right) \right] \tag{50}$$

where K is the limit price offered by the buying limit order. $E_Q \left[e^{-r\tau_l} \left(K - S_{\tau_l} \right) \right]$ is the expected value under a risk neutral probability, Q, of the option payoff discounted at the risk free interest rate, r. τ_l is a stopping time. In a risk neutral world, the stochastic dynamics of the transaction price and the equilibrium price are given by

$$dS_t = rS_t dt + \sigma S_t dW_t^* \tag{51}$$

$$dP_t = rP_t dt + \sigma \sqrt{1-\rho^2} P_t dZ_t^* \tag{52}$$

where W_t^* and Z_t^* are the standard Brownian motions defined under a risk neutral probability, Q.

Let X a known positive level of the equilibrium price, P, so that $X < K$. If the current transaction price, S, is equal or lower than X, the buying limit order is executed instantly (or the put option is executed instantly). The value of the perpetual American put option will be $K - S$, because $\tau_l = 0$. If the current transaction price, S, is higher than X, the option will be executed at the stopping time τ_l defined by

$$\tau_l = \min \{ t \geq 0; S(t) = P(t) = X \} \tag{53}$$

where τ_l is ∞ if the price of the financial asset never reaches the value X. At exercise time, the value of the put option will be $K - S_{\tau_l} = K - X$. Hereby, the liquidity cost is

$$L = (K - X) E_Q \left[e^{-r\tau_l} \right] \quad \text{for all } S > X \tag{54}$$

Using Itô lemma, the solution of the stochastic differential equation (51) is given by

$$S(t) = S \exp \left[\sigma W_t^* + \left(r - \frac{\sigma^2}{2} \right) t \right] \tag{55}$$

The stopping time τ_l is the moment when the price reaches the level X. But $S(t) = X$, if and only if

$$-W_t^* - \frac{1}{\sigma} \left(r - \frac{\sigma^2}{2} \right) t = \frac{1}{\sigma} \ln \frac{S}{X} \tag{56}$$

In order to get $E_Q \left[e^{-r\tau_l} \right]$ we use the following theorem.

Theorem 1 *Let W_t^* a standard Brownian motion under the probability Q, let γ a real number and h a positive number. Let the stochastic process*

$$Y(t) = \gamma t + W_t^*$$

and the stopping time

$$\tau_h = \min\{t \geq 0; Y(t) = h\}$$

Then[3]

$$E_Q\left[e^{-\lambda\tau_h}\right] = e^{-h\left(-\gamma+\sqrt{\gamma^2+2\lambda}\right)} \quad \textit{for all } \lambda > 0$$

Replacing λ with r, γ with $-\frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$ and h with $\frac{1}{\sigma}\ln\frac{S}{X}$, we obtain

$$\begin{aligned} -\gamma + \sqrt{\gamma^2 + 2\lambda} &= \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) + \sqrt{\frac{1}{\sigma^2}\left(r - \frac{\sigma^2}{2}\right)^2 + 2r} \\ &= \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{\sigma}\sqrt{\left(r + \frac{\sigma^2}{2}\right)^2} \\ &= \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{\sigma}\left(r + \frac{\sigma^2}{2}\right) = \frac{2r}{\sigma} \end{aligned}$$

The enunciated theorem implies the following result

$$E_Q\left[e^{-r\tau_l}\right] = \exp\left[-\frac{1}{\sigma}\ln\frac{S}{X}\frac{2r}{\sigma}\right] = \left(\frac{S}{X}\right)^{-\frac{2r}{\sigma^2}} \tag{57}$$

Therefore, the liquidity cost payable by an agent who gives a buying limit order at the limit price K is:

$$L = \begin{cases} K - S, & \text{if } 0 \leq S \leq X \\ (K - X)\left(\frac{S}{X}\right)^{-\frac{2r}{\sigma^2}}, & \text{if } S > X \end{cases} \tag{58}$$

Until now, we treated the problem of the liquidity cost for an arbitrary value of the equilibrium price X. From now on, we analyze the liquidity cost for an optimum value of X. For S fixed, let X^* the optimum value of X which maximizes the amount:

$$g(X) = (K - X)\, X^{\frac{2r}{\sigma^2}} S^{-\frac{2r}{\sigma^2}} \tag{59}$$

Because $\frac{2r}{\sigma^2}$ is strictly positive, we get $g(0) = 0$ and $\lim_{X\to\infty} g(X) = -\infty$. More,

$$g'(X) = S^{-\frac{2r}{\sigma^2}}\left[K\frac{2r}{\sigma^2}X^{\frac{2r}{\sigma^2}-1} - \left(\frac{2r}{\sigma^2}+1\right)X^{\frac{2r}{\sigma^2}}\right] \tag{60}$$

[3]See the proof of the theorem in Shreve S., (2004), "Stochastic Calculus for Finance", Springer, New York, volume II, pages 346-347.

Using the first order condition, $g'(X^*) = 0$, we obtain

$$K\frac{2r}{\sigma^2}(X^*)^{\frac{2r}{\sigma^2}-1} = \left(\frac{2r}{\sigma^2}+1\right)X^{\frac{2r}{\sigma^2}} \tag{61}$$

which implies

$$X^* = \frac{2r}{2r+\sigma^2}K \tag{62}$$

The obtained result is a number between 0 and K, that is $X^* < K$. Consequently, the function $g(X^*)$ can be written

$$g(X^*) = \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{\frac{2r}{\sigma^2}} K^{\frac{2r+\sigma^2}{\sigma^2}} S^{-\frac{2r}{\sigma^2}} \tag{63}$$

Consequently, in the presence of the informed agents on the market, the final formula of the liquidity cost on a market governed by orders for a limit price K is given by

$$L = \begin{cases} K - S, & \text{if } 0 \le S \le \frac{2r}{2r+\sigma^2}K \\ \frac{\sigma^2}{2r+\sigma^2}\left(\frac{2r}{2r+\sigma^2}\right)^{\frac{2r}{\sigma^2}} K^{\frac{2r+\sigma^2}{\sigma^2}} S^{-\frac{2r}{\sigma^2}}, & \text{if } S > \frac{2r}{2r+\sigma^2}K \end{cases} \tag{64}$$

5 Empirical Results

In the literature, the empirical papers include Biais, Hillion, and Spatt (1995), who document the diagonal effect (positive autocorrelation of order flow) and the comovement effect (e.g., a downward move in the bid due to a large sell market order is followed by a smaller downward move in the ask – which increases the bid-ask spread); Sandas (2001), who uses data from the Stockholm exchange to reject the static conditions implied by the information model of Glosten (1994), and also finds that liquidity providers earn superior returns; Harris and Hasbrouck (1996) who obtain a similar result for the NYSE SuperDOT system; Hollifield, Miller and Sandas (2004) who test monotonicity conditions resulting from a dynamic model of the limit order book and provides some support for it; Hollifield, Miller, Sandas and Slive (2006) who use data from the Vancouver exchange to find that agents supply liquidity (by limit orders) when it is expensive and demand liquidity (by market orders) when it is cheap.

In this section we analyze empirically the liquidity cost formula. We used a database which includes the intraday transaction prices of the Carrefour Company negotiated on the French stock exchange, *Bourse de Paris*. The database contains the transaction prices from May 10, 2007 to July 31, 2007. The sample comprises 7076 records. The transaction prices evolution is shown in Figure 1. Also, the database contains the daily French Treasury-bill rates. This serves as a proxy for the current interest rate and is obtained from *Datastream*^{TM}. The average of the daily interest rates of the study period was 0.0187%.

For every trading day the volatility was computed as standard deviation of the intraday transaction prices. The average of the daily volatility over the period May 10, 2007 - July 31, 2007 was 0.4792%. The Figure 2 shows the evolution of the daily volatilities on the study period.

Figure 1: The Intraday Transaction Prices

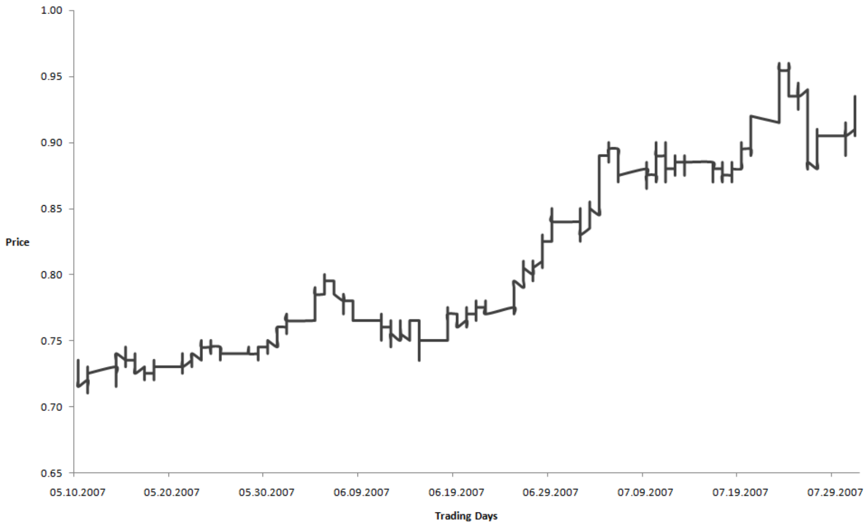
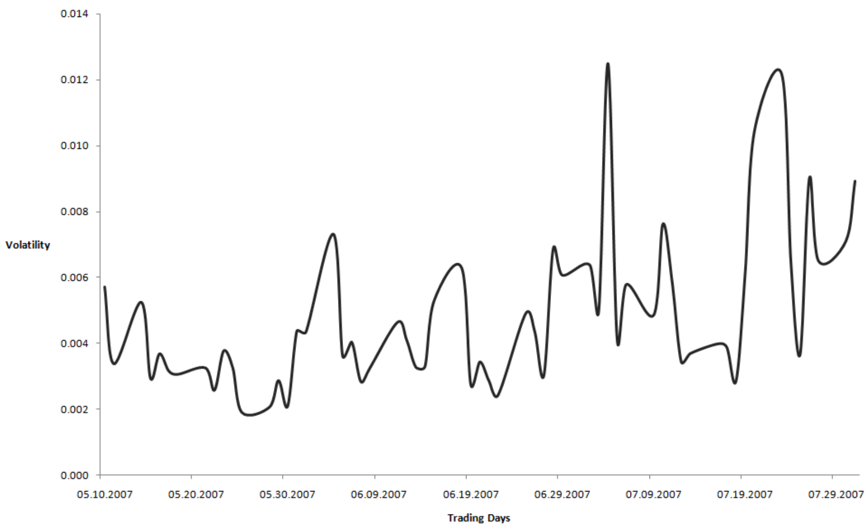


Figure 2: The Daily Volatility



Using the formula (64) of the liquidity cost, we suppose that the market depth is 0.5%, 1%, 3% or 5%. The market depth is computed as a percentage variation of the limit price with respect to the transaction price:

$$\text{Market Depth} = \frac{K - S}{S} \times 100 \tag{65}$$

The Figures from 3 to 6 present the evolution of the mean daily liquidity cost for four arbitrary values of the financial asset market depth.

Figure 3: The Mean Liquidity Cost for 0.5% Market Depth

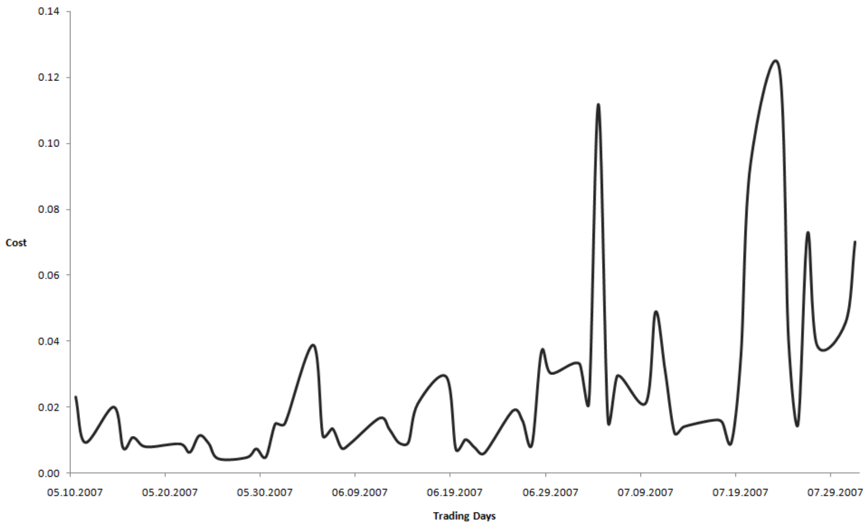


Figure 4: The Mean Liquidity Cost for 1% Market Depth

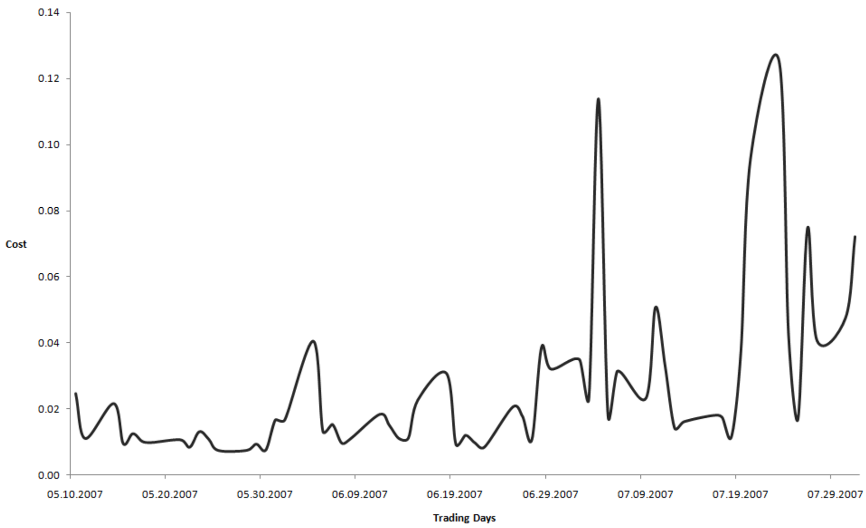


Figure 5: The Mean Liquidity Cost for 3% Market Depth

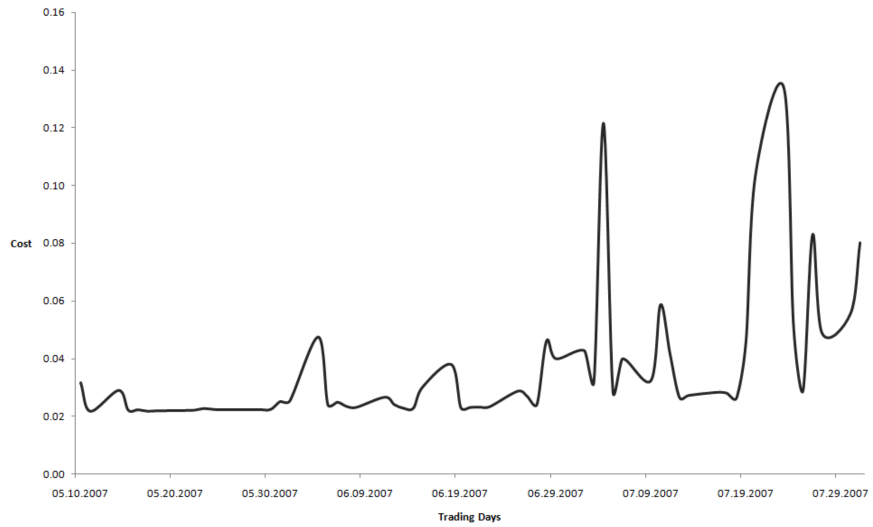
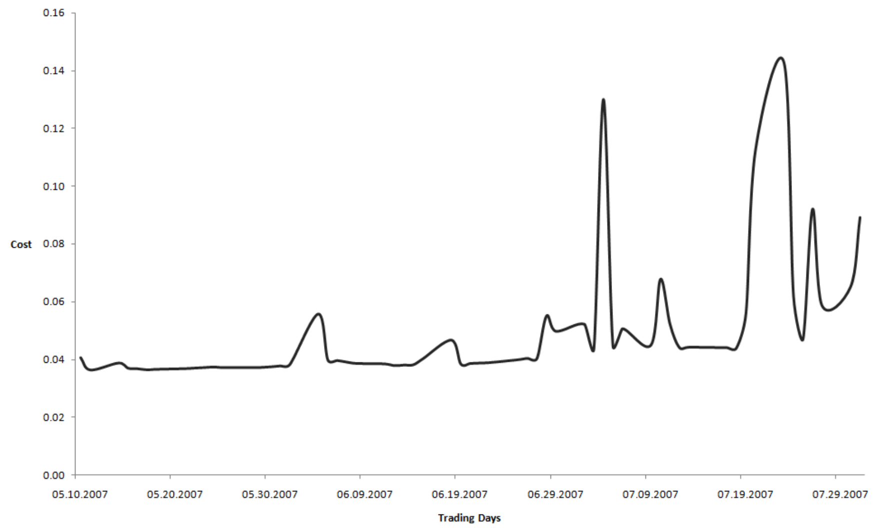


Figure 6: The Mean Liquidity Cost for 5% Market Depth



The Figures 7 and 8 compare the liquidity costs for different values of the asset market depth. The conclusion is that an increase of the market depth implies an increase of the liquidity cost of the financial asset market. The differences between the liquidity costs for 1% market depth and for 0.5% market depth are always positives and they vary to 70% maximum.

Figure 7: The Mean Liquidity Cost

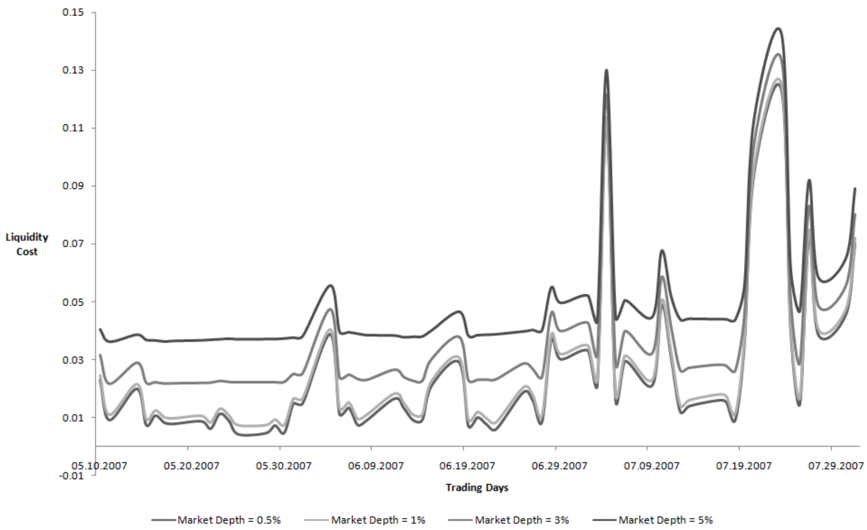
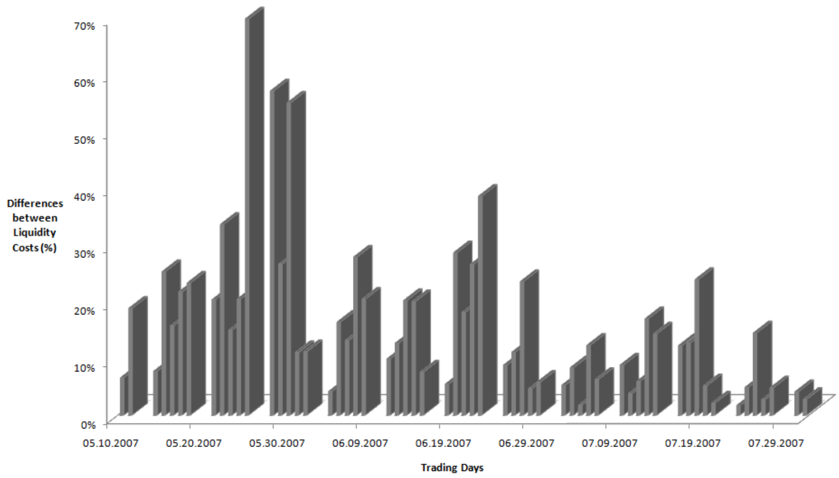


Figure 8: Differences between the Liquidity Costs with 1% and 0.5% Market Depth



The Figure 9 shows the evolution of the intraday liquidity cost for the study period, from May 10, 2007 to July 31, 2007. The liquidity cost is computed for 0.5% market depth. The Table I shows the descriptive statistics of the liquidity costs with 0.5%, 1%, 3% and 5% market depth. We notice that the mean value of the liquidity cost varies from 0.0256 euro for 0.5% market depth to 0.0500 euro for 5% market depth.

Figure 9: The Intraday Liquidity Cost for 0.5% Market Depth

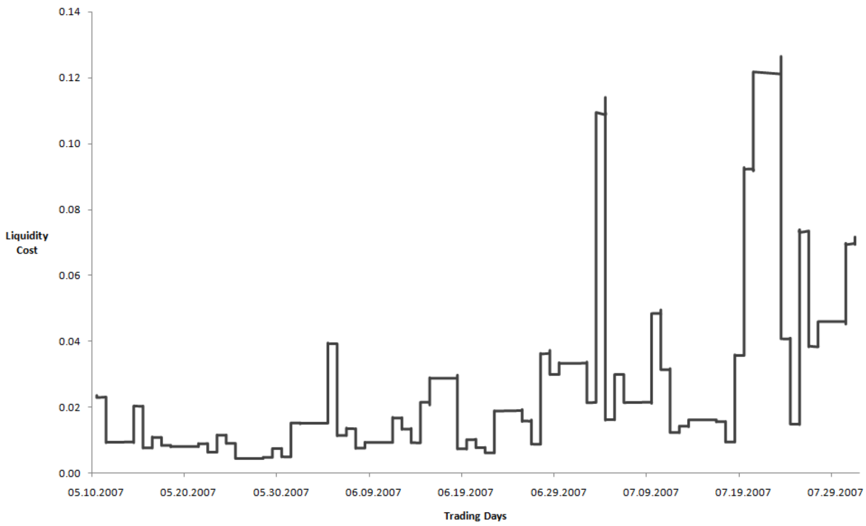


Table I: Descriptive Statistics for Liquidity Cost

	Mean	Standard Deviation	Skewness	Kurtosis	Min Value	Max Value	Range	Median
Market Depth = 0.5%	0.0256	0.0254	2.2421	7.8139	0.0043	0.1263	0.1219	0.0160
Market Depth = 1%	0.0274	0.0254	2.2572	7.8653	0.0073	0.1284	0.1210	0.0177
Market Depth = 3%	0.0374	0.0243	2.4434	8.6138	0.0215	0.1371	0.1156	0.0282
Market Depth = 5%	0.0500	0.0225	2.6398	9.5571	0.0357	0.1462	0.1104	0.0403

Also, the extreme values increase with the rising market depth. On the other hand, the standard deviation of the liquidity cost decreases with the rising market depth. Concluding, the mean liquidity cost of the financial asset market governed by orders represents about 3% of the transaction prices of the study period.

6 Conclusions

Based on classical hypotheses used in stochastic calculus applied in finance, the paper demonstrates the intuitive fact that under a market governed by information asymmetry the expected value of the equilibrium price is the current transaction price. Using these hypotheses, the paper proposes a measurement of the liquidity cost on a market governed by orders when the equilibrium is not perfectly revealed for all agents on the market. The proposed analytical formula of the liquidity cost of the financial asset market governed by orders depends on four parameters: the risk free interest rate, the transaction price of the financial asset, the volatility of the financial asset return and the limit price offered by the buying limit order.

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