EMPIRICAL TESTS OF DURATION SPECIFICATIONS

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ABSTRACT

We formulate AR(p) stochastic duration measures with constant volatility and risk premium. Numerical exercise shows that AR(p) stochastic duration overstate the weighted time to maturity compared to Modified duration. A closer look at the AR(p) model shows a need for model refinement in order to better fit empirical bond yield observations. Possible models include models with time-varying volatility and models for nominal interest rate.

1 INTRODUCTION

Interest rate risk is defined as an unexpected change in bond prices due to changes in the term structure; it is therefore essential in managing a bond portfolio to identify interest rate risk. Duration is the most commonly used measure of interest rate risk and accordingly is one of the most important tools available to bond portfolio managers. It is then not surprising that duration is of interest to both academic researchers and practitioners.

Our study attempts to contribute to the duration literature in two ways. First, our study attempts to place a common theoretical language on the various duration measures developed under the no-arbitrage or equilibrium-based bond pricing models. While a number of no-arbitrage or equilibrium-based bond pricing models have been developed following the seminal papers of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), a common theoretical language is missing. We develop our duration measure based on stochastic discount factor: a stochastic process that governs the pricing of state-contingent claims. We model the short rate as an autoregressive process of order p and we model our option-free discrete-time bond pricing model following those of Backus, Foresi, and Telmer (1998). To our knowledge, our model is the first duration measure derived under stochastic discount factor.

Second, we hope to bridge the gap between the highly mathematical no-arbitrage or equilibrium-based duration models and the discrete-time duration models based on bond mathematics similar in spirit to those of Macaulay (1938). Our aim is similar in spirit to those of Backus, Foresi, and Telmer (1998). Our duration model formulation uses continuous state variable in discrete time which eases the model implementation for practitioner. We hope that this study will add to the continuing work of bridging the gap between academics and practitioners.

2 LITERATURE REVIEW

Duration was first defined by Macaulay (1938) to compare loans with different payment schedules based on the weighted average of their income stream. The various duration measures that followed can generally be divided into durations based on traditional "bond mathematics" and durations based on no-arbitrage or equilibrium-based bond pricing models.

The first duration measure that does not explicitly model the underlying term structure behavior is Macaulay (1938). Subsequent studies include Cooper (1977), Bierwag (1977), Bierwag and Kaufman (1979), and Khang (1979) as discussed in Gultekin and Rogalski (1984). These early duration measures assume specific characteristics on the term structure movement such as changes in both the level and shape of the yield curve. For example, Khang (1979) proposes different duration measures for specific changes in the yield curve.

A more recent measures includes Chambers, Carleton, and McEnally (1988), Ho (1992), Nawalkha and Chambers (1996), Nawalkha and Chambers (1997) and Nawalkha, Soto, and Zhang (2003). Chambers, Carleton, and McEnally (1988) indicates that interest risk can only be measured by a vector of numbers as opposed to a single number by proposing the Duration Vector while Ho (1992) proposes the Key Rate Duration which associates the price sensitivity of a bond to multiple segments of the yield curve. Nawalkha and Chambers (1996) proposes M-Absolute duration measure which allows for superior immunization compared to traditional Fisher and Weil (1971) duration by selecting a bond portfolio clustered around its planning horizon date. Nawalkha and Chambers (1997) and Nawalkha, Soto, and Zhang (2003) extend the analysis of Nawalkha and Chambers (1996) into a multi-factor M-Vector.

The development of duration measures based on no-arbitrage or equilibrium-based bond pricing models coincides with Ingersoll, Skelton, and Weil (1978) appraisal of Macaulay (1938) duration and developments in studies of bond pricing models. The seminal papers by Vasicek (1977) and Cox, Ingersoll, and Ross (1985) initiate the no-arbitrage or equilibrium-based bond pricing models. Ingersoll, Skelton, and Weil (1978) provide an arbitrage-based criticism of the Macaulay (1938) duration. To resolve this criticism, durations measures that take into account an explicit random process driving models of the bond pricing were developed. Cox, Ingersoll and Ross (1978) formally derives a duration measure consistent with the general equilibrium conditions of the Cox, Ingersoll, and Ross (1985) term structure model.

Our study models the short rate as an AR(p) process similar in spirit to those of Vasicek (1977) and, unlike duration previous models, utilizes stochastic discount factor as the common element linking the bond pricing theory. The formulation of our discrete time model of bond pricing follows those of Backus, Foresi and Telmer (1998). A review of stochastic discount factor is provided by Cochrane (2005).

Empirical research of duration's effectiveness as an interest rate risk measure can be categorized into two school of thoughts. One school of thought looks at duration's explanatory power over a cross section of bond return while the other looks at duration's performance in an immunization strategy. Gultekin and Rogalski (1984) was the first to examine duration's explanatory power over a cross-sectional bond return. Their study of seven different duration measures show that duration explains about 50% of cross-sectional variation of bond return for U.S. treasury from 1947 – 1976. An exclusion of yield changes as an independent variable in the Gultekin and Rogalski (1984) statistical method however leads to underestimated R^2 values. Subsequent study by Ilmanen (1992) corrected the omission by including yield changes as an independent variable[1]. Ilmanen (1992) shows that duration explained 80% to 90% bond return variance from the period of 1959 – 1989.

3 THEORETICAL BACKGROUND

3.1 Stochastic Discount Factor

Stochastic discount factor is the rate at which an investor is willing to substitute consumption tomorrow for consumption today. In other words, stochastic discount factor is the investor's intertemporal marginal utility of substitution of consumption. As per Cochrane (2005), given an investor's one-period consumption-investment decision, the

[1] We would like to thank Assaf Eisdorfer for observing the omission on Gultekin and Rogalski (1984) which lead us to Ilmanen (1992)

marginal utility of consuming less and buying more of the asset today should be equal to marginal utility of consuming more of the asset in the future.

Following Cochrane (2005), let p_t be the asset price at time t, d_{t+1} be the dividend from the asset at time $t+1$, c_t be the consumption at time t, β be the subjective discount factor, U be the utility function, and x_{t+1} be the asset payoff at time $t+1$ where $x_{t+1} = p_{t+1} + d_{t+1}$. The first order condition for optimal consumption and portfolio choice is then

$$p_t U'(c_t) = E_t[\beta U'(c_{t+1}) x_{t+1}] \Leftrightarrow U'(c_t) = E_t[\beta U'(c_{t+1}) \frac{x_{t+1}}{p_t}] \tag{1}$$

The left hand side stands for the marginal utility cost of consuming one dollar less at time t. The right hand side stands for the expected marginal utility of investing one dollar at time t, selling the dollar at time $t+1$, and consuming the investment at time $t+1$. Dividing equation (4) by $U'(c_t)$, we obtain

$$1 = E_t[\beta \frac{U'(c_{t+1})}{U'(c_t)} \frac{x_{t+1}}{p_t}] = E_t[m_{k,t+1} \frac{x_{t+1}}{p_t}] \tag{2}$$

where m_{t+1} is the stochastic discount factor.

3.2 Interest Rate Risk and Macaulay Duration

Interest rate risk is defined as unexpected changes in bond prices due to changes in the term structure. Macaulay (1938) defined the duration measure D for a bond to measure bond price sensitivity to interest rate changes under infinitesimal parallel shift in term structure.

$$D = -\frac{\partial P}{\partial Y} \frac{1}{P} = -\sum_{t=1}^{T} (-t) \frac{C_t}{(1+Y)^{(t+1)}} \frac{1}{P} \tag{3}$$

where C_t is the cash flow at time t, Y is the yield-to-maturity, T is the time to maturity, and P is the bond price such that

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+Y)^t} \tag{4}$$

Equation (1) can be rewritten as follows

$$\frac{\partial P}{P} = D(-\partial Y) \tag{5}$$

As noted by Ilmanen (1992), extension to non-flat term structure would require the use of spot rates as opposed to yield-to-maturity. Fortunately, the use of spot rates as opposed to yield-to-maturity has only negligible impact on duration as observed by Ingersoll (1983). It is important to note as per Campbell, Lo, and MacKinlay (1997) that duration is sensitivity of n-period bond return to n-period yield and not sensitivity to a 1-period yield.

3.3 Duration Model based on p-Factor Stochastic Process

We model the short rate Z_t as an autoregressive process of order p, an *AR(p)* process, and the stochastic discount factor m_{t+1} as

$$-\log m_{t+1} = \frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + Z_t - \lambda \sigma_\varepsilon \varepsilon_{t+1} \tag{6}$$

where λ is the risk premium and ε ~ N(0,1). As shown in Appendix A, we can then recursively obtain a model for discount bond price with par value of \$1, $b_{n,t}$, at time t with n periods to maturity. We model discount bond price model assuming that the short rate Z_t follows AR(1), AR(2), and AR(3).

The discount bond price model assuming that the short rate Z_t follows an autoregressive process of order *1*, AR(1) is:

$$b_{n,t} = e^{-(A_n + B_n Z_t)} \tag{7}$$

where

$$\begin{aligned}
&A_0 = B_0 = 0, A_1 = 0, B_1 = 1 \\
&A_n = \frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + A_{n-1} + B_{n-1}(1-\varphi_1)\bar{Z} - \frac{1}{2}(\lambda\sigma_\varepsilon - B_{n-1}\sigma_\varepsilon)^2 \\
&B_n = 1 + B_{n-1}\varphi_1
\end{aligned}$$

while discount bond price model assuming that the short rate Z_t follows an autoregressive process of order *2*, AR(2) is:

$$b_{n,t} = e^{-(A_n + B_n Z_t + C_n Z_{t-1})} \tag{8}$$

where

$$\begin{aligned}
&A_0 = B_0 = C_0 = D_0 = 0, A_1 = C_1 = D_1 = 0, B_1 = 1 \\
&A_n = \frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + A_{n-1} + B_{n-1}(1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \frac{1}{2}(\lambda\sigma_\varepsilon - B_{n-1}\sigma_\varepsilon)^2 \\
&B_n = 1 + B_{n-1}\varphi_1 + C_{n-1} \\
&C_n = B_{n-1}\varphi_2 + D_{n-1} \\
&D_n = B_{n-1}\varphi_3
\end{aligned}$$

The discount bond price model assuming that the short rate Z_t follows an autoregressive process of order *3*, AR(3) is:

$$b_{n,t} = e^{-(A_n + B_n Z_t + C_n Z_{t-1} + D_n Z_{t-2})} \tag{9}$$

where

$$
\begin{aligned}
&A_0 = B_0 = C_0 = D_0 = 0, A_1 = C_1 = D_1 = 0, B_1 = 1 \\
&A_n = \frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + A_{n-1} + B_{n-1}(1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \frac{1}{2}(\lambda\sigma_\varepsilon - B_{n-1}\sigma_\varepsilon)^2 \\
&B_n = 1 + B_{n-1}\varphi_1 + C_{n-1} \\
&C_n = B_{n-1}\varphi_2 + D_{n-1} \\
&D_n = B_{n-1}\varphi_3
\end{aligned}
$$

As per Appendix B, duration measure for an AR(p) option-free discount bond with par value of \$F is

$$D_{n,t} = -\ln(Fb_{n,t}) \tag{10}$$

and duration measure for an AR(p) option-free coupon bond model $V_{n,t}$ is

$$D_{n,t} = \sum_{i=1}^{n} -\ln(C_i b_{i,t}) \frac{C_i b_{i,t}}{V_{n,t}} \tag{11}$$

4 NUMERICAL ANALYSIS

4.1 Parameter Estimation

4.1.1 Description of the Data for Estimation

We estimates the model parameters using CRSP (Center for Research in Security Prices) U.S. term structure of interest rate (discount bond yield) with maturities of 1, 3, 6, and 9 months in addition to 1, 2, 3, 4, and 5 years[2]. The estimation period runs from June 1964 to December 1995. The data spans non-overlapping 379 months. Summary statistics of the annualized data is presented in Table 1.

[2] The same dataset is used in Bansal and Zhou (2002). We would like to thank Ravi Bansal and Hou Zhou for providing us with the dataset.

4.1.2 Parameter Estimation Methods

We estimate the following parameters from our models: the long run mean ($\bar{Z}$), the constant time-series residual standard deviation (σ_ε), the n-th autocorrelation of the state variable (φ_n) and the constant risk premium coefficient (λ). The parameters are estimated using a two-step procedure. In the first step, we estimate the time series parameters of the models: the long run mean ($\bar{Z}$), the time-series residual standard deviation (σ_ε) and is the n-th autocorrelation of the state variable (φ_n) using linear regression. In the second step, we estimate the constant risk premium coefficient, λ, using a non-linear least squares. The estimation results are presented in Table 2.

4.2 Numerical Exercise

We calculated the duration for monthly non-callable United States Treasury Bills, Notes and Bonds from January 1996 to December 2006 obtained from Center for Research in Security Prices (CRSP). Bonds with special tax features, flower bonds, as well as bonds missing relevant data are excluded. The data spans non-overlapping 132 months with a total of 23,135 individual observations. The duration summary statistics are available on Table 3.

The results of our duration calculation shows that AR(p) duration performs worse as maturity increases compared to Modified duration. As shown in table 4, at maturity of 0.5 years, our AR(p) duration is comparable to Modified duration. As the maturity of the bond increase however, our AR(p) duration overstate weighted time to maturity compared to Modified duration.

5 CONCLUSION

Empirically AR(p) duration performs worse as a proxy of weighted time to maturity than Modified duration in our forecast period. A closer look at the AR(p) model shows a need for model refinement in order to better fit empirical bond yield observations. Possible model include models with time-varying volatility and models for nominal interest rate model.

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Table 1: Summary statistics for annualized U.S. Treasury term structure from June 1964 to December 1995

Maturity	1-month	3-months	6-months	9-months	1-year	2-years	3-years	4-years	5-years
Mean	6.446001	6.716742	6.944729	7.085367	7.129530	7.338939	7.495414	7.619863	7.689058
Std. Dev.	2.648370	2.712717	2.703928	2.685806	2.599354	2.521404	2.441905	2.403834	2.371477
Skewness	1.211099	1.211768	1.151771	1.101269	1.030724	0.977756	0.961475	0.926271	0.879131
Kurtosis	4.590163	4.523702	4.314679	4.160509	3.909804	3.661173	3.589720	3.506315	3.353150

Table 1 reports the mean, standard deviation, skewness, and kurtosis of annualized U.S. Treasury term structure from June 1964 to December 1995.

Table 2: Parameter estimation results for AR(p) discount bond model using U.S. Treasury term structure from June 1964 to December 1995

Model	φ_1	φ_2	φ_3	σ_ε	λ	$\bar{Z}$
AR(1)	0.957608	-	-	0.000148	0.510157	0.000118
AR(2)	0.879656	0.081137	-	0.000147	0.517970	0.000109
AR(3)	0.886068	0.154651	-0.082922	0.000146	0.492114	0.000117

Table 2 reports the estimation results for the AR(p) discount bond model.

Table 3: Summary statistics for AR(1), AR(2), AR(3), and Macaulay Duration Specification from March 1996 to December 2006

	AR(1) Duration	AR(2) Duration	AR(3) Duration	Macaulay
Mean	6.936376	6.759994	6.695676	4.287787
Std. Dev	18.84907	18.91014	18.81771	4.80017
Skewness	2.209346	2.196086	2.20906	1.254747
Kurtosis	7.155883	7.132464	7.202382	3.318123

Table 3 reports AR(1), AR(2), AR(3), and Macaulay Duration using the U.S. Treasury data from March 1996 to December 2006.

Table 4: Average Duration

Maturity (in years)	Modified	AR(1)	AR(2)	AR(3)
0.5	0.522	0.521	0.521	0.500
1	1.309	1.307	1.307	0.988
2	2.807	2.802	2.804	1.870
5	7.848	7.839	7.840	4.370
10	16.716	16.701	16.706	8.003

Table 4 reports AR(1), AR(2), AR(3), and Macaulay Duration based on maturity using the U.S. Treasury data from March 1996 to December 2006. The data is an average duration for securities with 10 days around 0.5, 1, 2, 5, and 10 years of maturity.

APPENDIX A

Similar in spirit to Backus, Foresi and Telmer (1998), we construct our option-free discount bond pricing model. The stochastic discount factor is

$$-\log m_{t+1} = \delta + Z_t - \lambda\sigma_\varepsilon \varepsilon_{t+1} \tag{1}$$

where m_{t+1} is the stochastic discount factor at time $t+1$, Z_t is a state variable at time t, λ is the risk premium, δ is a free variable and ε ~ N(0,1). We restrict the risk premium λ to be a positive number. The state variable Z_t follows a normal autoregressive process of order p, AR(p):

$$Z_t = (1 - \varphi_1 - \varphi_2 - ... - \varphi_p)\overline{Z} + \varphi_1 Z_{t-1} + \varphi_2 Z_{t-2} + ... + \varphi_p Z_{t-p} + \sigma_\varepsilon \varepsilon_t \tag{2}$$

where ε ~ N(0,1), $\overline{Z}$ is the long run mean, σ_ε is the residual standard deviation and φ_n is the n-th autocorrelation of the state variable. Let $b_{n,t}$ denote the value of a discount bond at time t with n periods to maturity with the following pricing relation

$$1 = E_t(m_{t+1}\frac{b_{n,t+1}}{b_{n+1,t}}) \Leftrightarrow b_{n+1,t} = E_t(m_{t+1}b_{n,t+1}) \tag{3}$$

Assume that at any time t, the value of a matured discount bond $b_{0,t}$ is \$1. The value of a discount bond with one-period (n = 1) to maturity at time t is then:

$$b_{1,t} = E_t(m_{t+1}b_{0,t+1}) = E_t(m_{t+1}) \tag{4}$$

Given m_{t+1} as a lognormal random variable and imposing no arbitrage opportunity, we obtain stochastic discount factor as

$$-\log m_{t+1} = \frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + Z_t - \lambda\sigma_\varepsilon \varepsilon_{t+1} \tag{5}$$

where state variable Z_t is the 1-month Treasury bill rate at time t. We can then recursively obtain theoretical price of discount bond. We will illustrate the process assuming that the state variable Z_t follows an AR(3) process.

We assume the general form for discount bond price as

$$b_{n,t} = e^{-(A_n + B_n Z_t + C_n Z_{t-1} + D_n Z_{t-2})} \tag{6}$$

where

$A_0 = B_0 = C_0 = D_0 = 0, A_1 = C_1 = D_1 = 0, B_1 = 1$

The value of a discount bond with two-period until maturity (n = 2) at time t is

$$\begin{aligned} b_{2,t} &= E_t(m_{t+1} b_{1,t+1}) = E_t(e^{-\frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 - Z_t + \lambda \sigma_\varepsilon \varepsilon_{t+1}} e^{-Z_{t+1}}) \\ &= E_t(e^{-\frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 - Z_t + \lambda \sigma_\varepsilon \varepsilon_{t+1}} e^{-(1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \varphi_1 Z_t - \varphi_2 Z_{t-1} - \varphi_3 Z_{t-2} - \sigma_\varepsilon \varepsilon_{t+1}}) \\ &= E_t(e^{-\frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 - Z_t - (1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \varphi_1 Z_t - \varphi_2 Z_{t-1} - \varphi_3 Z_{t-2} + (\lambda\sigma_\varepsilon - \sigma_\varepsilon)\varepsilon_{t+1}}) \\ &= e^{-\frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 - Z_t - (1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \varphi_1 Z_t - \varphi_2 Z_{t-1} - \varphi_3 Z_{t-2} + \frac{1}{2}(\lambda\sigma_\varepsilon - \sigma_\varepsilon)^2} \\ &= e^{-(\frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + (1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \frac{1}{2}(\lambda\sigma_\varepsilon - \sigma_\varepsilon)^2) - (1+\varphi_1) Z_t - \varphi_2 Z_{t-1} - \varphi_3 Z_{t-2}} \\ &= e^{-A_2 - B_2 Z_t - C_2 Z_{t-1} - D_2 Z_{t-2}} \end{aligned} \tag{7}$$

Generalizing the recursion above, the price of an option-free discount bond with n periods to maturity at time t, denoted by $b_{n,t}$ has the following general form:

$$b_{n,t} = e^{-(A_n + B_n Z_t + C_n Z_{t-1} + D_n Z_{t-2})} \tag{8}$$

where

$$
\begin{aligned}
&A_0 = B_0 = C_0 = D_0 = 0, A_1 = C_1 = D_1 = 0, B_1 = 1 \\
&A_n = \frac{1}{2}\lambda^2 \sigma_\varepsilon{}^2 + A_{n-1} + B_{n-1}(1-\varphi_1-\varphi_2-\varphi_3)\bar{Z} - \frac{1}{2}(\lambda\sigma_\varepsilon - B_{n-1}\sigma_\varepsilon)^2 \\
&B_n = 1 + B_{n-1}\varphi_1 + C_{n-1} \\
&C_n = B_{n-1}\varphi_2 + D_{n-1} \\
&D_n = B_{n-1}\varphi_3
\end{aligned}
$$

The price of the discount with par of $1 bond above can be viewed as a discount factor for option-free coupon bonds *V* where

$$V_{n,t} = \sum_{i=1}^{n} C_i b_{i,t} \tag{9}$$

where C_i is future cash flows associated with each period.

APPENDIX B

Duration is a point elasticity of bond price to yield-to-maturity. As per Campbell, Lo, and MacKinlay (1997), duration is sensitivity of *n*-period bond return to *n*-period yield and not sensitivity to a 1-period yield. Macaulay duration for discount bond with *n* period-to-maturity at time *t* is formulated as

$$D_{n,t} = -\frac{\partial b_{n,t}}{\partial y_{n,t}} \frac{y_{n,t}}{b_{n,t}} \tag{10}$$

As per Campbell, Lo, and MacKinlay (1997), the coupon bond can be seen as a portfolio of discount bonds and the Macaulay duration of a coupon bond is the present-value-weighted average of the underlying discount bond's duration. The duration for a discount bond with *n* period-to-maturity at time *t* is formulated as *n*. Given a coupon bond $V_{n,t}$, duration measure for $V_{n,t}$ is therefore

$$D_{n,t} = \sum_{i=1}^{n} i \frac{C_i b_{i,t}}{V_{n,t}} \tag{11}$$