Derivatives Hedging Errors and Volatility

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Abstract

This paper provides a general representation for the errors of delta-hedging derivatives contracts under mis-estimated volatility. A new option Greek η, 'eta', non-linear but easily computable for portfolios, is developed, which quantifies the dependence between the prospective hedging errors and the volatility forecast errors. The hedging errors are studied in more detail for a standard vanilla option, a geometric average rate option, and an up-and-out call option with a continuously monitored barrier. Two alternative approaches are provided for deriving the conditional and unconditional distribution of hedging errors: binomial tree and kernel estimation. The techniques developed enable us to quantify the absolute and relative difficulties of hedging different instruments or portfolios of instruments.

I. Introduction

Participants in derivatives market rely heavily on models to price and hedge derivative instruments. Most of the derivative models require that the underlying asset of the derivative follow a certain process. However, in practice, the process of the underlying asset cannot be completely specified because critical parameters of the process, such as volatility, are not known with certainty. As a result, financial institutions embark upon hedging strategies employing volatility estimates in order to cover their risk exposure. But how do they know that the estimates used are sufficiently good? What if the estimation errors lead to seriously wrong hedging techniques, large replication errors and, ultimately, great losses? The aim of this paper is to investigate hedging errors arising from incorporating the wrong hedge ratios and analyse the distribution of these errors. Furthermore, the paper provides techniques for obtaining information on the absolute and relative difficulties of hedging different instruments or portfolios of instruments, an issue of particular importance to both academics and practitioners.

The research literature on hedging techniques and their implications for risk management is comprised of two main strands based on the hedging methodology used. The first strand is established within the Black and Scholes (1973) paradigm and includes studies following Black and Scholes' (BS) methodology of an exact replication strategy under a completely specified price process. Boyle and Emanuel's (1980) study forms one of the main contributions in this stream. Their analysis investigates the return distribution of discretely adjusted hedge portfolios and shows that the excess hedge return is highly skewed and has its own variance, which is inversely related to the frequency of rebalancing. Other papers, which use a similar methodology, include those of Bhattacharaya (1980) and Leland (1985). It should be clear at the outset of this research that, the approach taken by papers in the first strand requires the parameters of the processes for the underlying traded and non-traded securities to be specified precisely. Option prices and hedges are then derived as functions of the parameters of the processes and the prices of the underlying securities.

The second strand of the literature includes relatively recent papers, which explore more robust approaches to hedging. Their analysis is characterised by a more general set of assumptions, and their objective is to explore the implications of misspecified asset price processes. In particular, Ahn *et al* (1999) provide optimal hedging strategies for misspecified volatilities. El Karoui *et al* (1995) examine the robustness of the Black and Scholes' (1973) valuation formula as well as the behaviour of the hedged portfolio with respect to misspecified volatility. Neuberger (1994) constructs volatility-hedging instruments eliminating volatility exposure. Gibson *et al* (1999) define model risk and identify its sources. Their approach is based on a methodology which forms part of our analysis in section II below. Rubinstein (2001) provides a decomposition of the dollar profit earned from an option. Bick (1995) generalizes Black and Scholes' (1973) results and provides a family of dynamic trading strategies independent of any assumptions on the price process. Having adopted a different perspective from the above papers, Chatfield (1996) examines the impact of model uncertainty on forecast accuracy. Although his study is based on the principles of time series analysis, his results are equally important for constructing option hedges (and derivatives hedges in general), since lack of precise knowledge of the stock price distribution is a major source of uncertainty. Finally, research in this field has been also conducted by Jacquier and Jarrow (2000). They conduct tests of specification errors arising from the omission of a relevant variable in a derivatives model.

This paper examines in detail the nature of hedging errors arising from the use of incorrect volatility estimates. It is well understood that if we had perfect knowledge of the quadratic variation and hedge more frequently, we could obtain arbitrarily smaller hedge errors. We can, therefore, use the difference between the actual strategy and an artificial one which assumes perfect knowledge of the quadratic variation, to characterise the form of hedge errors. We present a general decomposition of the replication errors and techniques measuring the sensitivity of delta-hedging a derivatives contract with respect to the error in the volatility forecast. Furthermore, a new Greek η, 'eta', is introduced, which measures the sensitivity of the replicating portfolio with respect to movements in volatility levels. 'Eta' is non-linear but easily computable for portfolios, and quantifies the difficulty of hedging different option positions and portfolios of options. Finally, our analysis provides alternative methods of deriving the exact conditional and unconditional distribution of expiry hedging errors resulting from misspecified volatility. Our findings can be applied to any contingent claim including contracts with complicated path dependencies under different assumptions for the process of the underlying asset.

It is necessary to distinguish between three different measures of volatility[1] as functions of time which are utilised throughout the paper. The first designated by $\sigma(t)$ and referred to as the 'underlying' volatility, drives the unknown dynamics of the underlying asset price process. The second provides the 'ex-post' volatility $\sigma_R(t)$, which is a discrete time realization and corresponds to the realised sample quadratic variation. $\sigma_R(t)$ and $\sigma(t)$ are the same in the

[1] Throughout the analysis, reference is made to a fourth 'selling' volatility, a scalar at which the option is evaluated at the beginning of the contract.

continuous limit. The third measure corresponds to the forecast volatility $\sigma_F(t)$, which is the volatility used in the replication strategy and is also called the 'hedging' volatility.

The rest of the paper is structured as follows. Section II provides a general decomposition of hedging errors. A method of quantifying likely delta hedging errors as a function of the volatility forecast error is obtained. Section III presents applications of the above tools on three different option contracts: a standard vanilla option, a geometric average rate call option, and an up-and-out call option with a continuously monitored barrier. Section IV describes two alternative approaches for deriving the conditional and unconditional distributions of hedging errors: binomial tree and kernel estimation. Finally, section VI concludes.

II. Decomposition of Hedging Errors

Section II shows how to decompose the dollar replication error from an option position into three basic components: i) the discretisation error, which is the error due to adjusting he dynamic portfolio at discrete points in time, ii) the volatility error, which is the error due to hedging at an incorrect volatility estimate, and iii) the premium error, which is the error resulting from mis-pricing the option at the time of purchase. To give greater insight to these three components, we consider three scenarios, all based on a simple but realistic example.

A. Assumptions

We assume that a writer sells an option contract of maturity T and with strike K on an underlying asset, a stock $S(t)$. He then embarks upon a delta-hedging strategy to eliminate his risk exposure, by taking a long position in the underlying stock. The dynamics of the underlying stock price is driven by the following diffusion equation under the risk neutral measure Q,

(1) $$dS(t) = rS(t)dt + \sigma(t)S(t)dW(t)$$

where $W(t)$ is a Wiener process under Q. There are no transaction costs, no differential taxes, and no borrowing or lending restrictions. Interest rates are nonstochastic and constant.

The sample quadratic variation to be realised at time t_i can be computed as:

(2) $$QV(t_i) = \sum_{j=i+1}^{N} [r(t_j) - E(r(t_j))]^2 \,,\, r(t_j) = \ln\frac{S(t_j)}{S(t_{j-1})}, E(r(t_j)) = 0\,.$$

The ex-post volatility $\sigma_R(t_i)$, which corresponds to the sample quadratic variation at time t_i, is then

(3) $$\sigma_R(t_i) = \sqrt{\frac{QV(t_i)}{T - t_i}},\ i = 0...N-1\,.$$

The price of the option, which is evaluated under the assumption of geometric Brownian motion and corresponds to the volatility used in delta hedging the option, is referred to as the 'forecast' price, $c^F(S(t_i), t_i)$[2]. The 'correct value' of the option contract corresponds to the option price

[2] The superscript ^{R} stands for 'realized'. The superscript ^{F} stands for 'forecast' and indicates that computations are carried out according to the writer's assumptions.

evaluated at the ex-post volatility $\sigma_R(t_k)$, under the same assumptions, and is denoted as $c^R(S(t_i),t_i)$[3].

According to (2), the definition of the sample quadratic variation is based on revision on a fine time grid. However, revision dates do not have to be equally spaced for the analysis to hold. We now derive a general decomposition of hedge errors. As the notation is rather complicated, Table1 summarises the symbols used throughout the analysis.

[Insert Table 1 here]

B. 1st Component: Discretisation Error

In this section, we obtain the first component of the hedging error, which is due to adjusting the hedge portfolio at discrete points in time. It is well understood that even if the writer had perfect foreknowledge of the quadratic variation, he would still experience replication errors, as he could not continuously rebalance his hedging position. If the writer adjusts his delta hedge N times during the life of the option, the time interval between the hedge rebalances will be $\Delta t = T / N$. Assuming without loss of generality that the risk-free rate is zero, the return on the writer's net position from time t_{i-1} to t_i is:

(4) $$H(t_i) = -\Delta c(t_i) + c_S^R(t_{i-1})\Delta S(t_i).$$

[3] Naturally this price is not known until the end, but computing it provides us with a useful decomposition for hedging errors.

If the BS partial differential equation (PDE) is applied and any high-order terms are ignored, $H(t_i)$ can be written in the form,

$$
\begin{aligned}
H(t_i) &= \tfrac{1}{2} c_{SS}(t_{i-1})\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t - \tfrac{1}{2} c_{SS}(t_{i-1})\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t u^2(t_i) \\
&= \tfrac{1}{2} c_{SS}(t_{i-1})\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t(1-u^2(t_i))
\end{aligned} \tag{5}
$$

where $u(t_i) \sim N(0,1)$. The aggregate final hedging error can then be calculated as follows:

$$HE(T) = \sum_{i=1}^{N} H(t_i)\,. \tag{6}$$

Combining equations (5) and (6), we show that the final hedging error will be,

$$HE(T) = \sum_{i=1}^{N} \tfrac{1}{2} c_{SS}(t_{i-1})\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t(1-u^2(t_i))\,,\ i = 1 \ldots N,\ t_0 = 0,\ t_N = T\,. \tag{7}$$

Equation (7) shows the replication error the writer experiences at expiry due to rebalancing the portfolio at discrete points in time. Boyle and Emanuel (1980) show that the magnitude of this error is inversely related to the frequency of rebalancing under the BS assumptions. Nevertheless, the smaller the adjustment period of the portfolio, the higher the excess kurtosis for asset price returns under non-Gaussian processes, leading to a larger potential replication error.

We now present the second component of the hedging error, which is due to lack of precise knowledge of the critical parameter of volatility.

C. 2nd Component: Volatility Error

In this section, we use the difference between the actual delta strategy used by the writer and an artificial one, which assumes perfect knowledge of the quadratic variation. We also assume that the option is sold at the 'correct' price, which corresponds to the sample realised quadratic variation. The return on the writer's net position from time t_{i-1} to t_i will now be,

(8) $$H(t_i) = -\Delta c(t_i) + c_S^F(t_{i-1})\Delta S(t_i)$$

where the delta of the hedging strategy c_S^F is calculated based upon the writer's volatility forecast σ_F . Ignoring the high-order terms and employing BS PDE, we can show that the formula for the intermediate hedging errors (Gibson *et al*, 1998) is,

(9) $$H(t_i) = (c_S^F(t_{i-1}) - c_S^R(t_{i-1}))\sigma(t_{i-1})S(t_{i-1})\sqrt{\Delta t}u(t_i) + \tfrac{1}{2}c_{SS}(t_{i-1})\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t(1-u^2(t_i)) .$$

The hedging error at maturity is now given by,

$$HE(T) = \sum_{i=1}^{N} H(t_i) = \sum_{i-1}^{N}(-\Delta c(t_i) + c_S^F(t_{i-1})\Delta S(t_i))$$

(10) $$= \sum_{i=1}^{N}\{(c_S^F(t_{i-1}) - c_S^R(t_{i-1}))\sigma(t_{i-1})S(t_{i-1})\sqrt{\Delta t}u(t_i) + \tfrac{1}{2}c_{SS}\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t(1-u^2(t_i))\} .$$

Equation (10) is a general representation for final hedging errors of contingent claims under misspecified hedging strategies. The replication error is expressed as the sum of two different terms. The first one reflects the effect of the volatility misspecification. If the writer's forecast is correct, then the two deltas c_S^R and c_S^F are equal and equation (10) reduces to equation (7). In this case, hedging errors result only from the second term, the discretisation error in gamma. We now present the third and final hedging error component.

D. 3rd Component: Premium Error

The two hedging error components presented come from rebalancing the hedge portfolio at discrete points in time and from imprecise knowledge of volatility. We now provide the third component, which comes from selling the option at the wrong premium. In this case, the final value of the hedging error will be given by,

$$HE(T) = (c^F(t_0) - c^R(t_0)) + \sum_{i=1}^{N}(-\Delta c(t_i) + c_S^F(t_{i-1})\Delta S(t_i)$$

$$= (c^F(t_0) - c^R(t_0)) +$$

$$\sum_{i=1}^{N}\{(c_S^F(t_{i-1}) - c_S^R(t_{i-1}))\sigma(t_{i-1})S(t_{i-1})\sqrt{\Delta t}u(t_i) + \tfrac{1}{2}c_{SS}\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t(1-u^2(t_i))\}\ .$$

The first term of the above equation represents the bias caused by the difference between the forecast price and the correct option value. Selling the option at the wrong volatility will shift the hedging portfolio upwards or downwards with respect to the option payoff by the price bias. Because this component is just a scalar, it is not included in the remaining of the analysis.

In the following section, we present a new option 'Greek' which quantifies the sensitivity of replication error to volatility misspecification.

E. Sensitivity Measure against Volatility Errors

Let $\delta(t_{i-1}) = (c_S^F(t_{i-1}) - c_S^R(t_{i-1}))\sigma(t_{i-1})S(t_{i-1})\sqrt{\Delta t}$ and $\gamma(t_{i-1}) = \tfrac{1}{2}c_{SS}(t_{i-1})\sigma^2(t_{i-1})S^2(t_{i-1})\Delta t$ denote the exposure from volatility misestimation and gamma discretisation respectively. Equations (9) and (10) then become,

(11) $$H(t_i) = \delta(t_{i-1})u(t_i) + \gamma(t_{i-1})(1-u^2(t_i))\ ,$$

$$HE(T) = \sum_{i=1}^{N} \delta(t_{i-1}) u(t_i) + \sum_{i=1}^{N} \gamma(t_{i-1})(1 - u^2(t_i)) . \tag{12}$$

The first term in equation (12) will be referred to as the volatility error component VE and the second as the truncation error component TE. The rest of our analysis focuses on the role of $\delta(t_{i-1})$[4], as we want to isolate the impact of volatility misspecification on replication strategies. The variance of the expiry hedging error will then be equal to the variance of the volatility error component $VE(T)$:

$$\underset{j=1\ldots M}{Var}(HE(T)) = \underset{j=1\ldots M}{Var}\left[\sum_{i=1}^{N} (\delta(t_{i-1}) u(t_i))\right]. \tag{13}$$

Since $u \sim N(0,1)$, $u^2 \sim \chi_1^2$, $\mathrm{E}(u^2) = 1$ and $\mathrm{var}(u^2) = 2$, we can show that,

$$\underset{j=1\ldots M}{Var}(HE(T) \cong \underset{j=1\ldots M}{E}\left[\sum_{i=1}^{N} \delta^2(t_{i-1})\right] \tag{14}$$

Equation (14) is an approximation because of the statistical dependency among $u(t_i)$ and $\delta(t_i)$, used in calculating $H(t_i)$ and $H(t_{i+1})$ respectively. Nevertheless, in section III, we show that (14) provides a good approximation of the unconditional variance of the final hedging errors.

Using a Taylor series expansion, we express $\delta(t_{i-1})$ with respect to the 1st and 2nd partial derivatives of delta with respect to volatility. Let $\varepsilon(t_i)$ designate the percentage error in the volatility forecast, $\varepsilon(t_i) = (\sigma_F(t_i) - \sigma_R(t_i)) / \sigma_R(t_i)$. Ignoring any high-order terms, the discrepancy between the deltas evaluated at the ex-post and at the hedging volatility respectively can be written as follows,

[4] Table 4 shows that, for a reasonably frequent rebalancing of the hedge, the variance of the hedging error due to discrete hedging takes relatively small values.

$$c_S^F(t_{i-1}) - c_S^R(t_{i-1}) = -(\sigma_R(t_{i-1}) - \sigma_F(t_{i-1}))\frac{\partial c_S^F(t_{i-1})}{\partial \sigma} - \tfrac{1}{2}(\sigma_R(t_{i-1}) - \sigma_F(t_{i-1}))^2 \frac{\partial^2 c_S^F(t_{i-1})}{\partial \sigma^2}$$

Therefore,

$\delta(t_{i-1}) =$

$$(15) \quad \left[-(\sigma_R(t_{i-1}) - \sigma_F(t_{i-1}))\frac{\partial c_S^F(t_{i-1})}{\partial \sigma} - \tfrac{1}{2}(\sigma_R(t_{i-1}) - \sigma_F(t_{i-1}))^2 \frac{\partial^2 c_S^F(t_{i-1})}{\partial \sigma^2}\right]\sigma(t_{i-1})S(t_{i-1})\sqrt{\Delta t}$$

where the partial derivative of delta with respect to volatility denotes the 'delvar', or 'Ddeltadvol'[5], and the second partial of delta with respect to volatility designates the 'Ddelvardvol' (Taleb, 1997). Delvar and Ddelvardvol measure the sensitivity of delta and delvar respectively to the implied volatility.

Combining equations (14) and (15) and ignoring the second-order term of the series expansion, we obtain that

$$(16) \qquad \underset{j=1\ldots M}{Var}(HE(T)) \cong \underset{j=1\ldots M}{E}\left\{\sum_{i=1}^{N}\left[-(\sigma_R(t_{i-1}) - \sigma_F(t_{i-1}))\frac{\partial c_S^F(t_{i-1})}{\partial \sigma}\right]^2 \sigma^2(t_{i-1})S^2(t_{i-1})\Delta t\right\}.$$

The Ddelvardvol merely gives a second-order effect on the aggregate variance. We focus on the first-order effect of the delvar on the variance of the hedging errors. Equation (16) provides a general approximation for the unconditional variance of hedging errors. For many purposes, this approximation gives a sufficient indication of risk exposure. In fact, since $u(t_i)$ is not correlated with $u(t_{i+1})$, the resulting correlation between $H(t_i)$ and $H(t_{i+1})$ is sufficiently small, and our variance seems to be fairly robust. Nevertheless, in section IV we will develop alternative methods for deriving the exact distribution of both unconditional and conditional moments of the aggregate hedging errors.

[5] Pronounced "D-delta-d-vol".

In a continuous-time framework, formula (16) is transformed to the corresponding double integral. The partial derivative of this double integral with respect to volatility is a new option 'Greek,' which we name 'eta' and denote by the Greek letter η[6]. η measures the size of the change in the standard deviation of the replication error with respect to movements in volatility levels:

$$\eta = \frac{\partial SD(HE(T))}{\partial \sigma} = \sqrt{\underset{j=1\ldots M}{E}\left\{\sum_{i=1}^{N}\left[\frac{\partial c_S^F(t_{i-1})}{\partial \sigma}\right]^2 \sigma^2(t_{i-1})S^2(t_{i-1})\Delta t\right\}}. \tag{17}$$

η predicts the error in our delta hedge for every 1% error in our volatility estimate, $SD(HE(T)) = \eta\Delta\sigma$, where $\Delta\sigma = \sigma_F - \sigma_R$. Table 2 presents the values for the new option 'Greek' for three different contracts.

[Insert Table 2 here]

η provides a tool ranking these options based on their hedging difficulty. The barrier contract displays the greatest value and the geometric the lowest, as expected. Using η, we can investigate how much easier is to hedge a portfolio, compared to hedging its individual options. Although all other 'Greeks' are linear in terms of options portfolios, η is non-linear, but easily computable. If a portfolio P consists of an amount w_k of option k $(1 \le k \le K)$, the η of the portfolio, η_P, is given by,

$$\eta_P = \sqrt{\frac{1}{M}\sum_{j=1}^{M}\sum_{i=1}^{N}\left(\frac{\partial \Delta_P}{\partial \sigma}(S_{jt_i})\right)^2 \sigma^2(t_{i-1})S_{jt_i}^2\Delta t} \tag{18}$$

where

[6] The Greek letter 'eta' sounds the same with the Greek word 'etta' which means to lose (in a battle).

$$\frac{\partial \Delta_P}{\partial \sigma}(S_{jt_i}) = \sum_{k=1}^{K} w_k \frac{\partial \Delta_k}{\partial \sigma}(S_{jt_i}) .$$

S_{jt_i} denotes the value of the underlying asset at time t_i in sample path j. $\frac{\partial \Delta_P}{\partial \sigma}(S_{jt_i})$ and $\frac{\partial \Delta_k}{\partial \sigma}(S_{jt_i})$ designate the delvar of the portfolio P and of the k^{th} option respectively. This calculation is relatively easy to implement on most portfolio risk systems. Note in particular that the η of a portfolio is less than the sum of the η's of the individual options. Another property, which makes η distinct from other 'Greeks', is that a negative and a positive position can have the same η.

One of the most important aspects of our analysis is that it applies to any contingent claim that can be spanned under a general diffusion process. It therefore provides a theoretical tool for understanding the risk involved in misspecified replicating strategies of more exotic products with path-dependence features. Moreover, it provides a measure of hedging difficulty and an ordinal ranking of options. The following section includes applications of the tools presented on various option contracts.

III. Applications on Vanilla, Asian and Barrier Options

This section applies the tools described in section II to three distinct hedging situations. In order to demonstrate the generality of our approach and that some contracts are much easier to hedge than others, we have chosen three different call options: a European, an average rate, and an up-and-out with a continuous-time monitored barrier. We assume that the options mature after 1 year and are struck at $50. They are written on stock with an initial value of $50. The simulations are run for different numbers of rebalancing dates so that the effect of hedging more frequently to be investigated. Before presenting the three options in detail, we include Table 3.

[Insert Table 3 here]

Table 3 presents a summary statistics for the replication errors of all the three contracts. It underlines the distinctness of each option and therefore justifies the use of three distinct hedging situations to demonstrate our tools. The first, second and third columns present the results for the option payoff, hedging portfolio and replication error at expiry respectively. The Asian option exhibits the smallest mean and standard deviation of the expiry hedging errors, whereas the barrier option displays the greatest expected error and standard deviation. In the fourth column, we present a summary statistics for standardised hedging errors. We scale the errors for each contract by the standard deviation of the corresponding option payoff $C(T)$ at maturity. The standardised hedging errors then correspond to a short position of $1/SD(C(T))$ in each option. The payoff at expiry of such a short position shows unit standard deviation.

A. Vanilla Option

If the claim is a European call option, it is evaluated and hedged according to the Black and Scholes (1973) pricing formula. In the case of the plain vanilla, the delvar is given by,

(19) $$\frac{\partial c_S}{\partial \sigma} = -c_{SS} d_2 S\sqrt{T-t=} - n(d_1)\frac{d_2}{\sigma}.$$

Figure 1 presents the call payoff as well as the replication portfolio at expiry. The delta used in the actual hedging strategy corresponds to the trader's 'forecast' volatility, which deviates from the volatility still to be realised. Under perfect replication, the portfolio price should coincide with the option payoff. The distance between the line and the scatter plot in Figure 1 depicts the replication error at the maturity of the option. The impact of incorporating an incorrect volatility estimate is clear. Figure 1 also implies that the closer to the strike the option finishes, the greater the magnitude of the replication error. A similar pattern in the behaviour of the aggregate replication errors will also prevail when incorrectly delta hedging an average rate and a reverse knock-out option. This is mostly attributed to the volatility error component. Figure 2 gives an insight to the behaviour of this component. It depicts the error in delta, the difference between the actual delta used in hedging and the delta evaluated at the ex-post volatility, with respect to the underlying asset values.

[Insert Table 4 here]

Table 4 presents the unconditional variance, skewness and kurtosis for the replication error and its two components, for different numbers of rebalancing dates. It also displays the approximation of the hedging error variance. The very small difference observed between $\underset{j=1...M}{Var}(HE(T))$ and $\underset{j=1...M}{E}\sum_{i=1}^{N}\delta^2(t_{i-1})$ is due the statistical dependencies among $\delta(t_i)$ and $u(t_i)$, used in calculating $H(t_i)$ and $H(t_{i+1})$ respectively. At each date, $u(t_i)$ is uncorrelated to all previous deltas and normal disturbances, but it affects the subsequent $\delta(t_i)$ value when calculating $H(t_{i+1})$. If the assumption of independence were valid, the hedging error would be distributed as a mixture of normal distributions, with zero skewness. Nevertheless, (16) gives a good approximation of the unconditional variance of the final hedging errors. The expiry errors exhibit negative skewness for all the different numbers of time steps and a kurtosis less than 3.

As expected, Table 4 illustrates that, for a given number of paths, the variance of the truncation error component is inversely related to the frequency of rebalancing. Boyle and Emanuel (1980) reach similar conclusions. At this point, we should underline the fact that the variance of the truncation error component remains constant as volatility error changes. Finally, the variance of the volatility error component is slightly reduced when we increase the frequency of hedge rebalances.

To draw conclusions on the impact of the level of moneyness on the replication errors, we run the simulations for out-, at-, and in-the-money calls. According to Table 5, the variance of the expiry error increases the nearer the underlying asset is to the strike. Indeed, the at-the-money option displays the biggest variance.

[Insert Table 5 here]

It is also clear from Tables 4 and 5 that the volatility error component attributes mostly to the magnitude of the final hedging error. Moreover, we find that for standard out-, at-, and in-the-money vanilla options, the standard deviation of the final delta error component as well as of the final hedging error is remarkably linear over a wide range of volatility error.

B. Geometric Average Rate Call Option

The average rate option is chosen as an easier case to hedge than the vanilla. The option is struck on the geometric average of the underlying stock with initial value of $50. The option has 1-year maturity with a strike of $50. Starting from the beginning of the option contract, the averaging is taken over a period of 1 year and reset 101 times at equidistant dates. That is, the geometric average is updated every two hedge rebalances. The formulae used in evaluating and delta hedging consider a discrete averaging (Clewlow and Strickland, 1997).

We expect the average rate option to be easier to hedge than a vanilla for two main reasons: a geometric average rate option has a smaller standard deviation of payoff, and its gamma does not exhibit a spike at expiry. A geometric average on the underlying stock exhibits a variation smaller by roughly a factor of 3 compared to the underlying. Indeed, Table 3 shows that the standard deviation of the Asian option payoff is smaller by approximately $\sqrt{3}$ (Kemna and Vorst, 1990). We can also observe that the hedging errors arising from delta hedging such an option display a smaller variance when compared to a plain vanilla. In particular, the delta of a geometric average rate option converges to zero towards expiration, even when the option dies in-the-money, whereas the vanilla hedge ratio approaches either 0 or 1, and may get 'spiked' between them. Nevertheless, when the errors are scaled by the standard deviation of the options' payoffs, the replication errors display similar dispersion for both types of claim. This means that a short position of $1/SD(C(T))$ in the geometric average rate option is, apparently, no easier to hedge than the same position in the vanilla. η justifies our results. Whereas the η of a vanilla and a geometric average rate call option is 7.51 and 4.38 respectively, the η of a short position of $1/SD(C(T))$ in the vanilla and in the average rate option is 1.14 and 1.22 respectively[7]. Moreover, attention must be given to the skewness, kurtosis and η. The European option exhibits a greater kurtosis, smaller skewness and greater η when compared to the average rate option.

[7] If we hedge more frequently at the beginning and less frequently at the end so that the time steps provide equal increments in the quadratic variation of the forward average, then the average rate option will be 'more efficiently' hedged and display smaller hedging errors. Nevertheless, the 'eta' of the standardised replication errors is equal to 1.15, which is still very close to the 'eta' of the plain vanilla.

C. Up-and-Out Call Option with a Continuous-Time Monitored Barrier

As an example of an instrument which is more difficult to hedge than a vanilla option, we have chosen to examine an up-and-out call option. The payoff of a barrier option not only depends on the final price of the underlying but also on whether or not the price of the underlying asset has crossed a predetermined barrier level throughout the life of the contract. An up-and-out call option knocks out if it goes too far into the money. It is therefore more difficult to hedge than a down-and-out, which has no such singularity in its intrinsic value. The discontinuity in the delta, as the underlying asset triggers the barrier, makes the up-and-out call notoriously difficult to hedge. According to Table 3, the up-and-out call exhibits the largest variance of all the options. It is therefore expected to have a greater η than the vanilla and the Asian option. This is clearly shown in Table 2.

The first published analytical solution of 'out' barrier options, with constant barrier level over time, is found in Rubinstein and Reiner (1991). In order to restrict the effect of the singularity near the constant barrier, Rich (1994) presents an analytical solution for the case of a continuous-time monitored barrier, which grows exponentially with time. We have chosen to apply an exponential barrier in order to restrict the replication error stemming from the discontinuity near the level of the barrier. Let $C^{uo}(t_i)$ denote the value of an up-and-out call option at time t_i with strike price K and initial barrier $B(t_0)$, where $K < B(t_0)$. $B(t_i)$ denotes the value of the continuous monitored barrier at t_i, with $B(t_i) = B(t_0)e^{-\theta(T-t_i)}$, where $\theta \geq 0$. Closed-form solutions of the up-and-out option[8] can be derived as long as the exponential barrier does not intersect with the strike price.

We assume that a writer is short an up-and-out call option and then creates a replicating portfolio to eliminate his delta exposure. The initial barrier level $B(t_0)$ and θ are set to \$70 and 0.1 respectively. Figure 3 shows the final payoff of the up-and-out call as well as of the hedging portfolio. The misspecified replicating portfolio of the up-and-out call presents the same spike around the strike as the vanilla option. The volatility as well as the truncation error explains the shape of the final hedging errors around the strike and in the neighbourhood of the barrier.

To conclude this section we present how well η predicts the sensitivity of the expiry replication errors of the three options, with respect to the error in volatility. Figure 4 includes the actual as well as the forecast standard deviation of the hedging error, which is remarkably linear over a wide range of volatility error.

IV. Estimation Of Conditional and Unconditional Distribution of Replication Errors at expiry

For many purposes, our approximation to the unconditional hedging variance is sufficient. If necessary, we can compute both the conditional and unconditional distributions of hedging error much more precisely. In this section, we describe methods for doing so.

A. Hedging Errors in a Binomial Tree

The analysis of hedging errors within a binomial context provides a convenient numerical scheme, through which the distribution of hedging errors can be estimated. It allows the

[8]The formulae for the delta, gamma and delvar of an up-and-out call with exponential barrier are available from the authors.

computation of all the moments of the conditional distribution at each node of the tree and of the unconditional at expiry.

Each node of the tree is designated by $[j,i]$. Let j denote the space grid and i the time grid, with $i \in [0, N]$, where N is the number of time steps and $j \in [1, i+1]$. The price of the underlying stock of the option will evolve according to the binomial option pricing model in Cox *et al* (1979), $S[j,i] = S[1,0]u^{i-1}d^{\,i-j+1}$, $u = e^{\sigma\sqrt{\Delta t}}$, $d = e^{-\sigma\sqrt{\Delta t}}$, and $\Delta t = T/N$. $S[1,0]$ represents the initial stock price, and σ denote the volatility driving the evolution of the underlying asset. Table 6 summarises the values stored at each node.

[Insert Table 6 here]

The framework of our analysis is essentially the same as previously. A writer sells a call option and then embarks upon a hedging strategy to cover his short position. The writer uses the BS formula to delta hedge his risk exposure. The difference between the trader's delta and the 'true' delta, multiplied by the stock price, corresponds to the value misinvested in the hedge.

Under this binomial tree setting, the moments about zero of the cumulative hedging error can be estimated. The analysis is first illustrated for node [1,0] and then for nodes [1,1] and [2,1]. The expected sum and higher moments at node [1,0] are set to zero. With the use of a simple recursive formula[9], forward calculations are conducted and the misinvested value at each node of the tree is calculated. In the next time step, the initial stock price can either increase to $S[2,1]$ or decrease to $S[1,1]$. Suppose the stock price goes up. At node [2,1], the hedging error

[9] See Appendix A.

comes from two sources: the return (loss) from the value misinvested in the hedge at [1,0], and the expected sum of cumulative hedging errors at [1,0], which is set to zero. It should be pointed out that there is only one path from [1,0] to [2,1]. The same applies when the underlying price goes to $S[1,1]$. However, in the second time step, there are two possible paths leading to [2,2]. In this case, the calculation of the expected sum of the cumulative hedging errors involves considering the hedging errors coming from the two paths and weighting them respectively. This way, the expected sum of cumulative hedging errors can be calculated recursively at each node of the tree. It must be stressed that the binomial tree analysis involves forward calculations[10]. Once the expected sum is estimated, the second, third and fourth moment about zero of the cumulative hedging errors can also be computed. Then, the moments about the mean can be recovered from the moments about zero. This way, both conditional and unconditional moments as well as skewness and kurtosis can be retrieved at each node and each time step.

Consider a binomial tree with 200 time steps and a standard vanilla option with a strike and initial underlying price of $50.A writer sells the option and delta hedges at a volatility of 10%, while the underlying volatility is 20%. Comparison with the analysis of section II shows that the binomial tree is quite accurate. Table 7 summarises the results.

[Insert Table 7 here]

Finally, Figures 5 present the conditional mean, standard deviation, skewness and kurtosis of hedging errors as a function of the underlying asset at expiry. It is clear that the conditional standard deviation has increased values around the strike.

[10] One of our future objectives is to derive a forward Kolmogorov equation for the distribution of hedging errors.

B. Kernel Estimation of the Hedging Errors Distribution

An alternative way of estimating both conditional and unconditional distribution of the hedging errors is to apply a kernel estimation technique to the simulation results of section II. Having derived the conditional moments via the kernel estimators, our objective is to compare the kernel estimation technique to the binomial tree.

Let us consider a writer selling a 1-year standard vanilla option at the correct volatility and then embarking upon a misspecified delta hedging strategy. 5000 simulations have been carried out for 200 time steps, providing us with 5000 pairs of final stock prices S_j and final hedging errors ε_j, where j denotes the corresponding sample path[11]. We will now proceed to derive kernel estimators of the local mean and variance of ε_j close to any given stock value S.

In the kernel estimation technique, a weight function w_j must be constructed. The objective is to assign weights to each hedging error ε_j, according to how close it is to S. To succeed, we will first have to choose a bandwidth β. If β is very large, the weighting will be over larger neighbourhoods of S, and if β is very small, the neighbourhood will be smaller. It is clear that controlling the weight function comes to adjusting the bandwidth. Equations (21) and (22) gives the bandwidth as well as the weight function,

(20) $$\beta = \frac{(S_{\max} - S_{\min})}{k},$$

(21) $$w_j = e^{\left\{-\frac{1}{2}\left(\frac{S_j - S}{\beta}\right)^2\right\}}.$$

[11] For reasons of simplification, we use a different notation in this section.

Having constructed β and w_j, we can now carry through our analysis. Given any stock value, the local mean $\tilde{\mu}$ and local variance $\tilde{\sigma}$ of the hedging errors can be calculated according to the following equations:

$$E[\varepsilon_j] = \frac{\sum_{j=1}^{M} w_j \varepsilon_j}{\sum_{j=1}^{M} w_j}, \tag{22}$$

$$E[\varepsilon_i^2] = \frac{\sum_{j=1}^{M} w_j \varepsilon_j^2}{\sum_{j=1}^{j=M} w_j}. \tag{23}$$

The standard error of the local mean can now be computed:

$$SE(\tilde{\mu}) = \tilde{\sigma}\sqrt{\sum_{j=1}^{M} W_j^2} \tag{24}$$

where the W_J are the normalised weights.

By applying a gaussian kernel estimator to the expiry replication errors (Ullah, 1988), we also obtain the unconditional distribution[12]. Figure 6 shows the unconditional distribution of the expiry hedging error for the plain vanilla, the geometric average rate and the up-and-out call option.

[12] A third way of deriving the conditional distribution is to use a Brownian bridge. This way, we can construct sample paths ending up at a given price, and then calculate the conditional moments of the hedging error at that price. The Brownian Bridge is a convenient way of obtaining the conditional distribution for the plain vanilla and the barrier option, depending on the level of the barrier. However, it is difficult to apply it to the geometric average rate option.

V. Summary

This paper analyses the distribution of delta hedge errors and provides a general decomposition for them. For a large class of contingent claims as well as for portfolios of claims, it enables us to quantify the prospective hedging errors which stem from lack of information about future volatility.

The decomposition is based on the idea that if we knew in advance what the future sample quadratic variation was, the hedging error would be very small and its only source would be due to the discrete (vs. continuous) rebalancing of the hedge. Hedging errors are analysed in terms of three components. The first component involves the exposure from the error in delta, resulting from volatility misestimation. The second component measures the exposure from the error in gamma discretisation. Finally, the third component is a price bias resulting from the option having been sold at the incorrect level of volatility. Using this decomposition, we develop an approximate method for calculating the unconditional variance of the hedging errors at maturity. We demonstrate the use of this technique by applying it to three distinct options: a vanilla, a geometric average rate and an up-and-out call.

We draw the following conclusions from our analysis. The standard deviation of the final hedging error is remarkably linear over a wide range of volatility errors. This means that the sensitivity of the standard deviation, with respect to different levels of volatility, is a meaningful new 'Greek', which we name 'eta' and denote by the Greek letter η. η measures the sensitivity of hedging error to volatility error and provides us with the probable hedging error.

Although the options' 'Greeks' are all linear, η is non-linear, but easily computable for portfolios. For realistically-frequent rebalances of the hedge, the gamma component is small for the vanilla as well as the average rate option, and the delta component is likely to dwarf the gamma. However, for the up-and-out call, the exposure to the gamma error is greater owing to the singularity around the barrier. The above analysis provides an approximate method for deriving the unconditional variance due to the statistical dependencies among the variables involved in hedging errors. Nevertheless, the resulting correlation is small, and our variance estimate is fairly robust.

For many purposes, our approximation to the unconditional variance of hedging error is sufficient. Yet if necessary, we can compute both the conditional and unconditional distribution of the hedging errors arising from not knowing the volatility. We develop two different techniques for doing this: a binomial approach and kernel estimation. We also suggest a third one: a Brownian bridge construction. Binomial tree allows for estimating all the moments of the conditional and unconditional distributions of hedging errors. The calculations can be done quite efficiently. Moreover, we can derive as many moments of the distributions as we are interested in. Alternatively, using kernel estimation, we compute the unconditional distribution of the expiry hedging errors, as well as local estimates and confidence intervals for the hedging errors, conditional either on the payoff or on the underlying asset value at expiry. These techniques demonstrate that the mean and variance strongly depend on how close to the strike the underlying finishes. This confirms practitioners' beliefs about getting 'spiked' at expiry.

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Appendix: Binomial Tree Derivation

The amount misinvested $V[j.i]$ in the hedge at each node $[j,i]$ can be calculated recursively by,

$$V[j,i] = S[j,i](N(d_1^F) - N(d_1)) .$$

$N(d_1^F)$ represents the delta ratio as calculated by the writer's assumptions about volatility. Both deltas are calculated according to BS. $N(d_1^F)$ is calculated using the forecast volatility, while $N(d_1)$ is calculated using the volatility which evolves the stock price in the tree.

Let $a[j,i]$ be the expected sum of the cumulative hedging errors at $[j,i]$. At node $[1,0]$, $a[j,i]$ is zero, while for all other nodes, $a[j,i]$ is given by the following equation:

$$a[j,i] = W_{up}[j,i](a[j-1,i-1] + (u-1)V[j-1,i-1]) + W_{dn}[j,i](a[j,i-1] + (d-1)V[j,i-1]$$

where $W_{up}[j,i] = \frac{j-1}{i}$ and $W_{dn}[j,i] = \frac{i-j+1}{i}$. For simplicity, $(1-u)V[j,i]$ will be denoted by $r_{up}[j.i]$ and $(d-1)V[j,i]$ V[j, i](d-1) by $r_{dn}[j.i]$. The above formula then becomes,

$$a[j,i] = W_{up}[j,i](a[j-1,i-1] + r_{up}[j-1,i-1]) + W_{dn}[j,i](a[j,i-1] + r_{dn}[j,i-1] .$$

The second, third and fourth moments about zero of the cumulative hedging errors can be also calculated according to the following equations respectively:

$$b[j,i] = W_{up}[j,i](b[j-1,i-1] + 2r_{up}[j-1,i-1]a[j-1,i-1] + r_{up}^2[j-1,i-1]) +$$

$$W_{dn}[j,i](b[j,i-1] + 2r_{dn}[j,i-1]a[j,i-1] + r_{dn}^2[j,i-1]),$$

$$c[j,i] = W_{up}[j,i](c[j-1,i-1] + 3r_{up}[j-1,i-1]b[j-1,i-1] + 3r_{up}^2[j-1,i-1]a[j-1,i-1] + r_{up}^3) +$$

$$W_{dn}[j,i](c[j,i-1] + 3r_{dn}[j,i-1]b[j,i-1] + 3r_{dn}^3[j,i-1]a[j,i-1] + r_{dn}^3[j,i-1]),$$

$$d[j,i] = W_{up}[j,i](d[j-1,i-1] + 4r_{up}[j-1,i-1]c[j-1,i-1] + 6r_{up}^2[j-1,i-1]b[j-1,i-1] +$$

$$4r_{up}^3[j-1,i-1] + r_{up}^4[j-1,i-1]) + W_{dn}[j,i](d[j,i-1] + 4r_{dn}[j,i-1]c[j,i-1] +$$

$$6r_{up}^3[j,i-1]b[j,i-1] + 4r_{dn}^3[j,i-1]a[j,i-1] + r_{dn}^4[j,i-1]),$$

where $b[1,0]$, $c[1,0]$ and $d[1,0]$ are set to be zero. At each node, the conditional moments about the mean of the cumulative hedging error can be calculated from the moments about zero:

$$\mu_k^{'} = E[X^k], \ \ \mu_k = E[(X-\mu_1)^k].$$

The unconditional moments about zero at maturity can be calculated from the conditional moments about zero. Finally, the unconditional moments about the mean can be computed.

TABLE 1

Summary of Notation

σ	Underlying volatility
σ_R	Ex-post (realized) volatility
σ_F	Hedging (forecast) volatility
$H(t_i)$	Return on the hedged portfolio at time t_i
$HE(t_i)/HE(T)$	Intermediate Hedging error/Final (Expiry) Hedging Error
$\Delta = c_S = \frac{\partial c^{F/R}}{\partial \sigma}$	Delta of an option evaluated at σ_F / σ_R
$\Delta_\sigma = \frac{\partial c_S^{F/R}}{\partial \sigma}$	'Delvar' or 'Ddeltadvol' of an option evaluated at σ_F and σ_R respectively
$\Delta_{\sigma\sigma} = \frac{\partial^2 c_S^{F/R}}{\partial \sigma^2}$	'DdelvarDvol' of an option evaluated at σ_F / σ_R
$\Gamma = c_{SS} = \frac{\partial^2 c^{F/R}}{\partial S^2}$	'Gamma' of an option evaluated at σ_F / σ_R

TABLE 2

'Eta' for different option contracts

Geometric Average Rate Option	Vanilla	Up-and-Out with Exponential Barrier
4.3834	7.5073	23.683

Spot Price	\$50	Maturity	1 year
Strike	\$50	Number of Hedge Rebalances	200
Initial Barrier	\$70	Number of Simulations	2000
Averaging Resets	101	Underlying volatility $\sigma(t)$	20%
Averaging Period	1 year		

TABLE 3

Summary Statistics of Option Payoffs, Replication Portfolios and Errors at Expiry

		$C(T)$	$H(T)$	$HE(T)$	$HE(T)/SD(C(T))$
Vanilla	Mean	4.1684	4.1784	0.0100	0.0015
	S.D.	6.6046	6.9843	1.0046	0.1584
	Skewness	1.9550	1.8997	-0.4995	-0.4995
	Kurtosis	4.1237	3.8691	-0.4058	-0.4058
	Min	0	-3.5083	-3.8311	-0.5801
	Max	43.4935	44.934	1.7852	0.2703

TABLE 3 (continues)

		$C(T)$	$H(T)$	$HE(T)$	$HE(T)/SD(C(T))$
Geometric Average Rate	Mean	2.3443	2.3664	0.0221	0.0062
	S.D.	3.5846	3.8418	0.6042	0.1686
	Skewness	1.7310	1.6671	-0.4918	-0.4918
	Kurtosis	2.7386	2.5172	-0.4323	-0.4323
	Min	0	-1.9654	-1.9654	-0.5483
	Max	22.9598	23.9902	1.1295	0.3151
Up-and-out	Mean	2.3683	2.3410	-0.0273	-0.0065
	S.D.	4.1869	5.9597	2.9363	0.7013
	Skewness	1.8617	2.3124	3.2512	3.2512
	Kurtosis	2.6433	6.6598	16.2315	16.2315
	Min	0	-7.6754	-7.6754	-1.8332
	Max	19.6997	42.6782	25.7027	6.1389

Spot Price	\$50	Maturity	1 year
Strike	\$50	Number of Hedge Rebalances	200
Initial Barrier	\$70	Number of Simulations	2000
Averaging Resets	101	Underlying volatility	20%
Averaging Period	1 year	% Error in volatility forecast	-50%

TABLE 4

Unconditional Variances, Skewness & Kurtosis for Expiry Hedging Errors for a Vanilla Option under Misspecified Hedging Volatility

Number of Hedge Rebalances	$E\left[\sum_{i=1}^{N}\delta^2(t_{i-1})\right]$	$Var\left[\sum_{i=1}^{N}\delta(t_{i-1})u(t_i)\right]$	$Var\left[\sum_{i=1}^{N}\gamma(t_{i-1})(1-u^2(t_i))\right]$	$Var(HE(T))$
200	1.0096	1.0362	0.06176	1.0183
500	1.0099	1.0191	0.02348	1.0099
750	1.0219	0.9673	0.01684	0.9645
1000	1.0187	0.9930	0.01292	0.9884
2000	1.0184	0.9981	0.00648	0.9965

Number of Hedge Rebalances	$Skewness[HE(T)]$	$Kurtosis[HE(T)]$		
200	-0.3326	2.1901	Spot Price	\$50
500	-0.3644	2.2794	Strike	\$50
750	-0.2704	2.2026	Maturity	1 year
1000	-0.3438	2.2675	Number of simulations	2000
2000	-0.3137	2.2160	% Error in volatility forecast	-50%
			Underlying volatility	20%

TABLE 5
A Comparison of Unconditional Variance of Final Hedging Errors for Out-, At-, and In-the-Money Vanilla Options under Misspecified Hedging Volatility

$S(t_0)$	$E\left[\sum_{i=1}^{N}\beta^2(t_{i-1})\right]$	$Var\left[\sum_{i=1}^{N}\beta(t_{i-1})u(t_i)\right]$	$Var\left[\sum_{i=1}^{N}\gamma(t_{i-1})(1-u^2(t_i))\right]$	$Var[HE(T)]$
35	0.1788	0.1697	0.00710	0.1621
40	0.7147	0.7726	0.02846	0.7618
50	1.0096	1.0362	0.06176	1.0183
60	0.9864	1.0003	0.03769	0.9785
65	0.5911	0.5745	0.02305	0.5590

Strike	\$50
Maturity	1 year
Number of Hedge Rebalances	200
Number of Simulations	2000
Error in volatility forecast	-50%
Underlying volatility $\sigma(t)$	20

TABLE 6
Values stored at each node of the tree

$S[j,i]$	Price of the underlying stock
$V[j,i]$	Hedging error
$p[j,i]$	Probability of reaching $[j,i]$
$a[j,i]$	Expected sum of cumulative hedging errors
$b[j,i]$	Second moment about zero of cumulative hedging errors
$c[j,i]$	Third moment about zero of cumulative hedging errors
$d[j,i]$	Fourth moment about zero of cumulative hedging errors

TABLE 7
Comparison of Binomial Tree with Monte Carlo simulations

	Mean	Variance
Binomial Tree	0	1.0222
Monte Carlo Simulations		
Variance of the Volatility Error Component	0.00025	1.0329
Variance of Expiry Hedging Error	0.00465	1.0087

Initial Stock Price	\$50	Number of Simulations	10000
Strike	\$50	Error in volatility forecast	-50%
Maturity	1 year	Underlying volatility $\sigma(t)$	20%
Number of Hedge Rebalances	200		

FIGURE 1
Delta Hedging a European Call Option at the Incorrect Volatility
200 hedge rebalances, 2000 simulations, strike=$50,
1-year maturity, underlying volatility=20%,
& absolute volatility error=50%

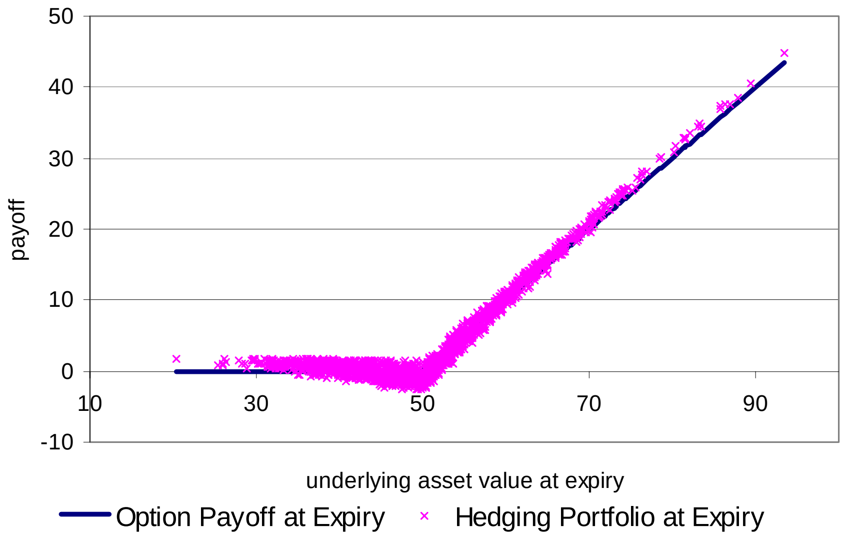

FIGURE 2
Error in Delta and Volatility
'correct' volatility=20%

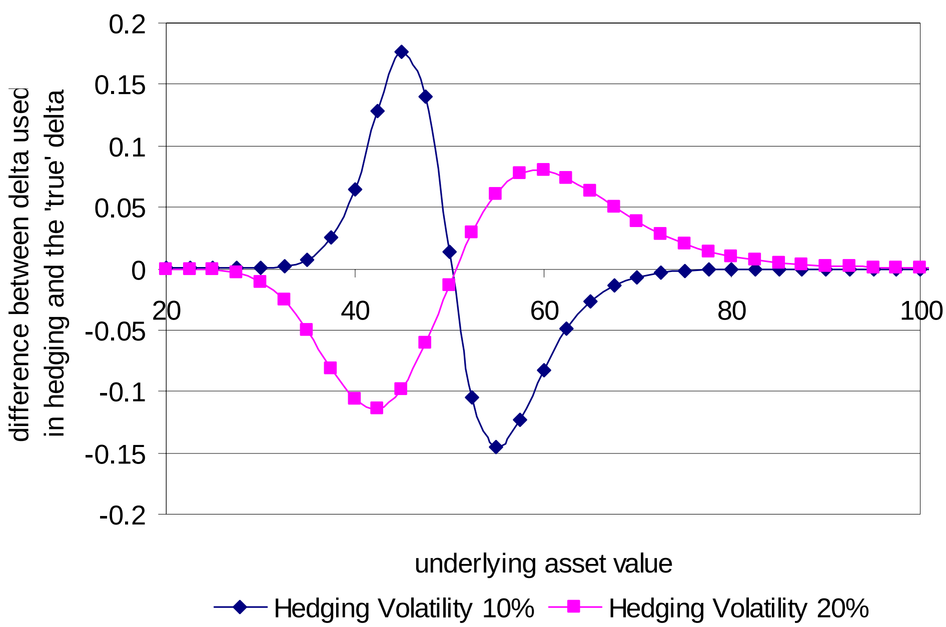

FIGURE 3
Delta Hedging an Up-and-Out Call Option at the Incorrect Volatility
200 hedge rebalances, 2000 simulations, strike=$50,
1-year maturity, underlying volatility =20%,
& absolute volatility error=50%

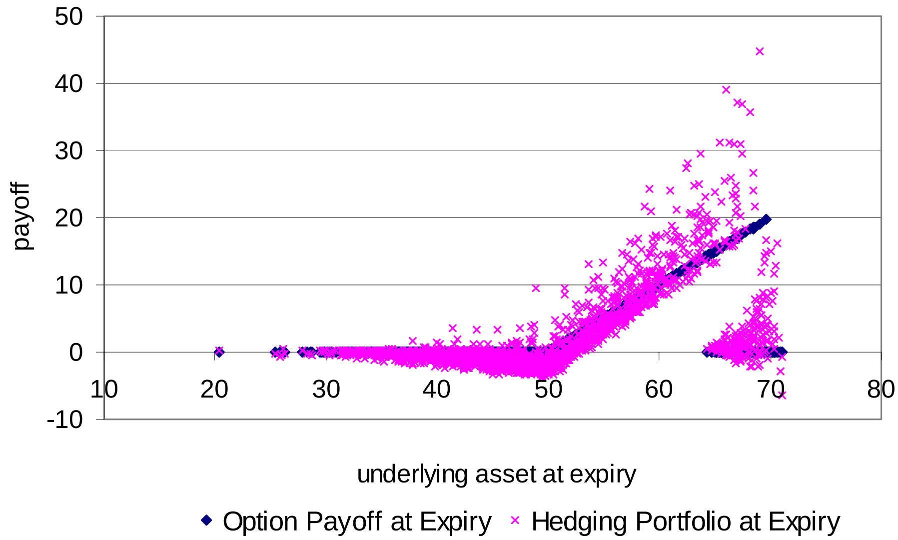

FIGURE 4
Actual and Forecast Standard Deviations of Expiry Replication
Errors due to Misspecified Deltas: the new Greek 'eta'

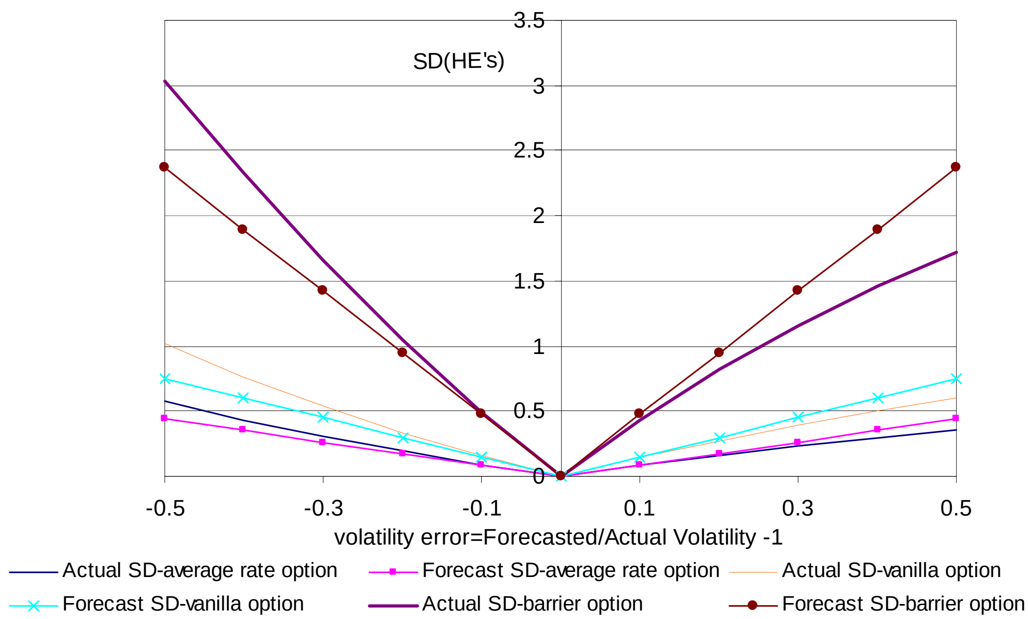

FIGURE 5
Conditional Moments of Hedging Errors
in a Binomial Tree with 200 timesteps

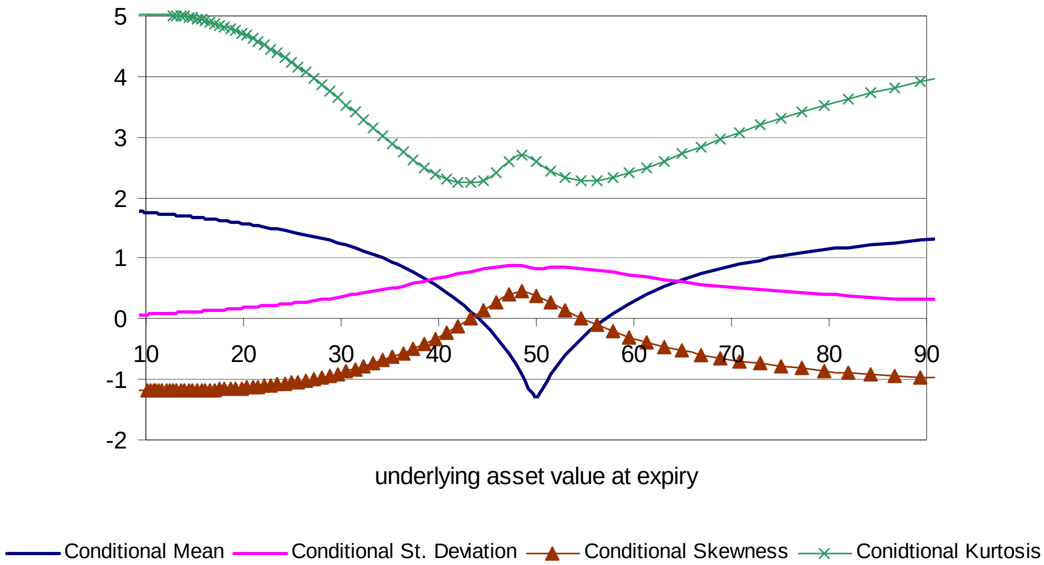

FIGURE 6
Kernel Estimation of the Unconditional Distribution
of Expiry Hedging Errors, for 200 hedge rebalances,
2000 paths, strike=$50, absolute volatility error=50%