Expanding the Merit of Utility Maximisation for Portfolio Choice

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Abstract

In the Full-Scale Optimization approach the complete empirical financial return probability distribution is considered, and the utility maximizing solution is found through numerical optimization. Earlier studies have shown that this approach is useful for investors following non-linear utility functions (such as *bilinear* and *S-shaped* utility) and choosing between highly non-normally distributed assets, such as hedge funds. We clarify the role of (mathematical) smoothness and differentiability of the utility function in the relative performance of FSO among a broad class of utility functions.

Using a portfolio choice setting of three common assets (FTSE 100, FTSE 250 and FTSE Emerging Market Index), we identify several utility functions under which Full-Scale Optimization is a substantially better approach than the mean variance approach is. Hence, the robustness of the technique is illustrated with regard to asset type as well as to utility function specification.

Keywords: Portfolio choice; Utility maximization; Full-Scale Optimization, S-shaped utility, bilinear utility.

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1 Introduction

Full-Scale Optimization (*FSO*) is increasingly attracting the interest of both portfolio choice researchers and financial industry. In this approach, the portfolio choice is based on direct maximization of a parametric utility function over the historical returns of the admissible assets (subject to adjustment to reflect investor expectations). Using numerical optimization to find the optimal portfolio, it allows for usage of non-linear utility functions. Cremers, Kritzman and Page (2005), and Adler and Kritzman (2007) apply the model to four different utility functions, showing that the difference to the mean variance (*MV*) model (Markowitz 1952) can be substantial in hedge fund selection problems when investors have non-linear utility functions. In this study we apply the FSO model to a wider selection of utility functions and show that it is useful for other asset classes than hedge funds.[1]

For half a century Markowitz's MV model (1952, 1959) has been the default model in financial engineering practice, and the benchmark model for new theories of portfolio choice. In this model, the choice of asset allocations is asserted to depend solely on the expected return (mean) and risk (variances and covariances) of the admissible assets. This makes the model simple to apply, but it is based on the assumption that either (1) the return distribution feature spherical symmetry[2], or that (2) investors are indifferent to higher moments and equally averse to downside and upside risk[3].

With respect to returns, ex post financial returns rarely are normal[4] - more

[1] The FSO approach is related to the Scenario-Based approach by Koskosidis and Duarte (1997) and Grinold (1999), and of course to numerous utility maximizing portfolio choice models, some of which are referred below.

[2] This was pointed out by Chamberlain (1983). Below, we refer to such distributions as *normal*, even though e.g. the Joint Normal, the Unified and the Binomial distributions also feature spherical symmetry.

[3] These limitations of the MV approach were established at an early point by e.g. Tobin, (1958, 1965), Markowitz (1959), Arrow (1965), Feldstein (1969), Hanoch and Levy, (1969, 1970), Chipman (1973) and Pratt (1976). Rothschild and Stiglitz (1970) showed from several perspectives that the usage of variance as a definition of risk is insufficient. More recent assessments of the mean-variance approach include e.g. Epps (1981), Michaud (1989), MacKinlay and Richardson (1991), Chopra and Ziemba (1993), and Clare, Smith and Thomas (1997).

[4] First recognized by Mandelbrot, (1963), there is now overwhelming evidence available on the non-normality of returns. This is referred extensively below.

often than not, skewness and kurtosis deviate from normality, making the MV approximation unlikely to select the optimal portfolio when the higher moments matter to the investor. With respect to utility, an investor *indifferent* to higher moments than variance follows a *quadratic utility function*. While perhaps acceptable as a local approximation, this function has well-known unattractive global properties. It has the unrealistic implication that the investor's absolute risk aversion is increasing in wealth, so that he at some point of wealth is worse off as he gets richer.

Markowitz (1987) shows that the difference between quadratic utility and power utility functions is very small in practice, and several other studies have come to the same conclusion (e.g. Levy and Markowitz 1979). This conclusion rests, however, on the existence of continuous higher-order derivatives of the utility function. In realistic portfolio management situations, investors often express preferences that do not admit continuous derivatives (Litterman 2003, Meucci 2005, Ch.2 and Ch.5 respectively). However, the relevance of higher moments for investment decisions, first pointed out by Levy (1969) and Samuelson (1970), has caused a plethora of higher moment modifications of the MV approach to emerge.[5]

As an alternative solution to the problems of MV analysis, the utility based approach is well motivated. The problem of non-normal return distribution is avoided by taking all individual returns into account instead of approximating the first two moments only, and the utility function can be chosen in accordance to true investor preferences. The idea of utility maximization as a methodology for portfolio optimization problems, based on the utility theory founded by Von Neumann and Morgenstern (1947), can be traced back at least to Tobin (1958), and also appeared in the early assessments of the mean-variance

[5]Lai (1991), Konno, Shirakawa and Yamazaki (1993), Markowitz, Todd, Xu and Yamane (1993), Konno and Suzuki (1995), Chunhachinda, Dandapani, Hamid and Prakash (1997), Leung, Daouk and Chen (2001), and Wang and Xia (2002) suggest models taking skewness into account, Lai, Yu and Wang (2006) consider both skewness and kurtosis, Athayde and Flôres Jr, (2003, 2004) produce higher moments efficient frontiers, and Britten-Jones (1998) and Harvey, Liechty, Liechty and Muller (2003) approach higher moments with Bayesian methodologies.

approach (e.g. Rothschild and Stiglitz 1971, Levy and Markowitz 1979). In addition to the common result that the difference to MV was negligible, it was frequently argued that that utility maximization was computationally too cumbersome for the improvement achieved.

Due to its theoretical appeal, however, utility maximization for portfolio choice has remained present in academia. A relatively recent development of the method is Full-Scale Optimization. Introduced by Cremers et al. (2005) (inspired by Grinold 1999), the name captures the fact that the entire historical return distribution is considered in the optimization problem, rather than summary statistics thereof. To do this, instead of pursuing an analytical solution, numerical optimization and dynamic programming is used. This has the second advantage that the utility function specification is not restricted to mathematical convenience. Indeed, the function need not exist as a closed form analytical equation, but rather may be only a convex mapping of wealth and risk into utility. The computational burden of FSO is even worse than the traditional utility maximization approaches, but immense technological development during the last decades has according to Cremers et al. (2005) and Gourieroux and Monfort (2005) rendered the computational argument obsolete.[6] The econometrics of the FSO approach have recently been examined by Gourieroux and Monfort (2005), who explore estimator properties and derive a framework for asset pricing, monitoring constraints and efficiency tests under such estimation. Cerny (2004) describes how a maximization routine can be programmed.

Assessments of the FSO approach have shown that as long as the utility function chosen display either continuous derivatives and/or risk aversion that is a monotone function of wealth, the utility maximizing asset allocations are close to solutions on the MV efficiency frontier (Grinold 1999, Cremers, Kritzman and Page 2003, Cremers et al. 2005, Adler and Kritzman 2007). Cremers et al. and

[6] This argument is often overstated however (Meucci 2005). notes that asset allocation via FSO is infeasible for more than a small number of assets (his largest example considers 8 assets), even when the objective function is convex and the feasible set is the intersection of a hyperplane and a convex set. He recommends conic programming, an approach not pursued here.

Adler and Kritzman show, however, that the difference between FSO and MV is substantial when non-linear utility functions, such as *bilinear* or *S-shaped*, are chosen. This is shown in a setting of hedge fund returns, which deviate substantially from normality[7].

We assess the performance of the FSO approach relative to the MV approach under a spectrum of utility function specifications, wider than previously considered. Our results illustrate the robustness of Cremers et al. (2005) and Adler and Kritzman (2007). While the previous studies have been performed on hedge fund selection problems only, we show that the FSO model is useful even when very general indices are considered.

This paper is organized as follows. Section 2 gives a detailed account on the characteristics and preferences of financial return distributions. Section 3 presents the methodology and Section 4 presents the data used for the empirical comparison of the FSO and MV approaches. The results are presented and analyzed in Section 5, and section 6 concludes.

2 The nature and preferences of return distributions

Understanding the deviations from normality in financial return distributions is key in forming an efficient portfolio selection model. We need to understand both what is causing the deviations, and what preferences investors have for different distribution characteristics. With that knowledge, appropriate utility functions can be formed and used for FSO portfolio selection.

2.1 Return distributions

The non-normality of financial asset returns is a problem for mean-variance analysis as well as for the *Capital asset pricing model* (Sharpe 1964, Lintner

[7]It is reasonable to believe that the degree of non-normality influences the difference between FSO and MV positively, but this has yet to be proven.

1965, Mossin 1966, Merton 1973a) and the *Option pricing model* (Black and Scholes 1973, Merton 1973b). It was discovered long ago that financial asset price changes formed distributions that are more peaked than samples drawn from Gaussian distributions. Mandelbrot (1963) credited Mitchell (1915) for this discovery, and Olivier (1926) and Mills (1927) for the proof of the phenomenon. After Mandelbrot's article, an extensive literature on the proper mathematical formulation of financial return distributions emerged, and eventually the problem was addressed with techniques such as *conditional heteroscedasticity models* (Engle 1982, Bollerslev 1986), *stochastic volatility models* (Clark 1973, Taylor 1982, Taylor 1986), and *regime switching models* using a mixture of normal distributions (Goldfeld and Quandt 1973, Hamilton 1989).[8]

In general, financial assets display leptokurtic return distributions (i.e. distributions with higher kurtosis than the normal distribution). The logic behind this fact, pointed out by Clark (1973) and illustrated in Figure 1, is that the information flows reach the market in a non-linear fashion, causing calm days when no significant news appear, and chaotic trading on days of path-breaking news. The calm days form the high peak of the return probability distribution, and the fat tails are the large price changes due to important news. Non-linear behavior of investors has also been pointed out as a reason for this pattern (Aparicio and Estrada 2001), which may be due to uncertainty of information, or insider trading (Clark). A high level of kurtosis means that the asset has a high probability of extreme events (Lai et al. 2006). This uncertainty typically make risk averse investors dislike kurtosis, and such preferences have been theoretically proven by Scott and Horvath (1980). Aparicio and Estrada point out that an investor that mistakenly assumes normality when a return distribution is leptokurtic, will substantially underestimate the risk of the asset. The non-normality decreases when long horizon returns are studied, as the news flow to markets cause less variation in returns when comparing longer periods.

[8] As the return distribution parameterization is side-stepped by the FSO model, it is not referred further here, but e.g. Aparicio and Estrada (2001) gives a brief summary of the literature.

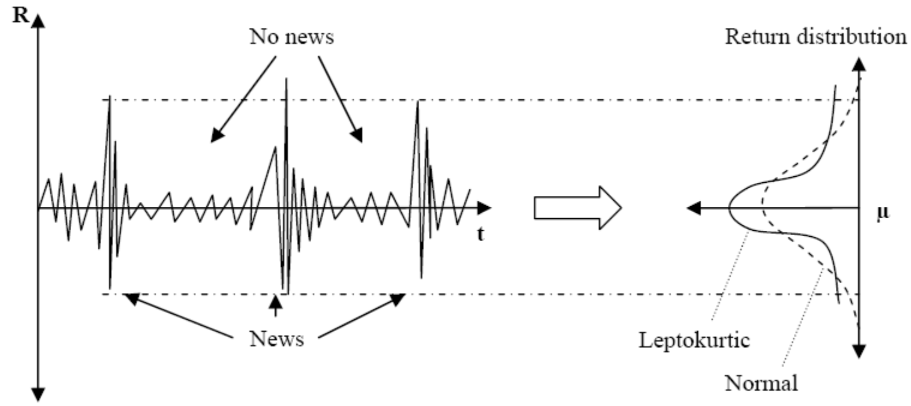

Figure 1: The formation of kurtosis

A second non-normality feature that is important for investors to follow is asymmetry of the return distribution, i.e. skewness. Skewness appear when the mean and the median do not coincide. A mean higher (lower) than the median characterize positive (negative) skewness, and implies a higher chance (risk) of extreme positive (negative) events. An example of when this is the case is a company whose value falls deeper when negative news arrive than it rises when good news arrive (Guidolin and Timmermann, 2005, shows in a regime switching model that large negative skewness appears in the switch from a "bull" to a "bear" state of the market). This is illustrated in Figure 2. Negative skewness means that the risk for a substantial loss is bigger than the chance of a substantial gain, and, typically, investors are hence averse to it. Asset allocations with positive skewness are preferred by investors to those with negative skewness, but such are less common in financial markets (appears when one go short in a negatively skewed stock). Preferences for positive skewness have been shown to hold theoretically by Arditti (1967) and Scott and Horvath (1980), and empirically by e.g. Sortino and Price (1994), Sortino and Forsey (1996), Levy and Sarnat (1984). This implies that an investor would be ready to trade off some average return for a lower risk of high negative returns (as argued by Harvey et al. 2003). This reasoning links to the literature on downside risk, including *Value at Risk models.*

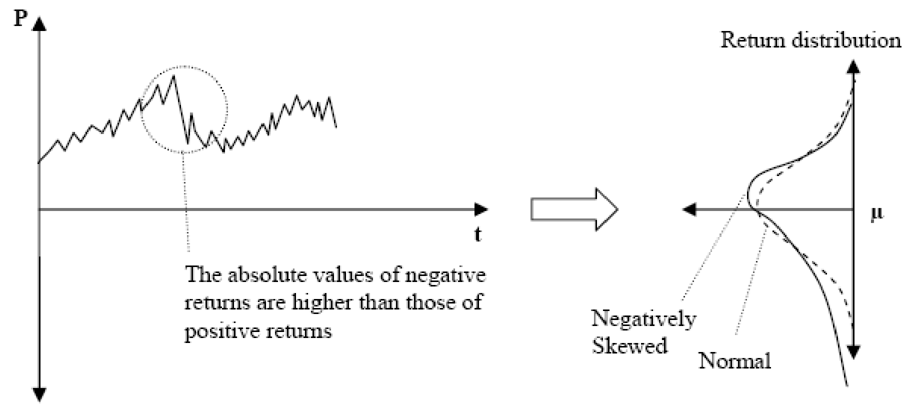

Figure 2: The formation of negative skewness

Recent financial innovation developments have aggravated the problems of non-normality of asset returns, as many financial derivatives feature larger deviations from normality than equity in general. Kat and Lu (2002) survey a large number of hedge funds and identify significant degrees of both kurtosis and skewness. This makes it increasingly important to consider these features in the investment decision.

Any single period portfolio selection problem can be described with an expression such as

$$\begin{gathered} \theta^* = \arg\max_\theta U(\theta_i' R_{i,t}) \\ \theta \in \Omega \end{gathered} \tag{1}$$

where $R_{i,t}$ is a matrix of returns of the admissible assets i over available return history t, and θ_i' is the allocation chosen for each asset. Utility is dependent on the expected portfolio return, $\theta_i' R_{i,t}$, which the vector θ_i is chosen to maximize, subject to the constraint matrix Ω. Typically, Ω includes a budget constraint such as $\theta_i'\iota = 1$ (where ι is a vector of ones), but it may also include a short selling constraint ($0 < \theta_i < 1$) or a investor loss aversion constraint.

The history of each asset combined in accordance of θ_i form a portfolio return probability distribution, R_p. The functional form of the utility function should mirror the investor's preferences of the shape of this distribution. Expanding the utility function in a Taylor series around the mean (μ_1) and take expectations on both sides, as done in Equation 2, yields measures of the investors' preferences in terms of the distribution's moments[9]. Let U^n denote the n^{th} derivative of the utility function and μ_i the i^{th} moment of R_p, then (set up in the same fashion as in Scott and Horvath 1980)

$$E(U) = U(\mu_1) + \frac{U^2(\mu_1)}{2}\mu_2 + \Sigma_{i=3}^{\infty}\frac{\mu_i}{i!}U^i(\mu_1). \tag{2}$$

The expression shows that the expected utility equals the utility of the expected returns, plus the impact on utility of deviations from the expected return. The influence of each moment on expected utility is weighted by the corresponding order derivative of the utility function. Typically, $U^2(R_p)$, for variance is negative; $U^3(\mu_1)$ for skewness is positive (as discussed above); and $U^4(\mu_1)$ for kurtosis is negative.[10]

The MV approach can be viewed in this framework as a utility function with $U^1(\mu_1) = 1$, $U^2(\mu_1) < 0$ (usually referred to as the risk aversion parameter), and $U^n(\mu_1) = 0$ for all $n > 2$.[11] This is the quadratic utility function referred to above, which is an explicit function of the first two moments ($U = \mu_1 - \lambda\mu_2$, where λ is the risk aversion parameter).[12]

The explicitness of the MV utility function is not necessary – in the utility functions to be presented below, the moment preferences are implicit. In principle there is no limit on the number of moments to take into account, but higher

[9]The term $(\theta_i' R_{i,t} - \mu_1)U'(\mu_1)$ disappears when taking expectations, as $E(\theta_i' R_{i,t} - \mu_1) = 0$

[10]The Taylor expansion has frequently been used in the study of investor preferences, see e.g. Arditti (1967) and Markowitz (1987). Hlawitschka (1994) analyzes the methodology.

[11]In the MV model the *covariances* of the assets play an important role. These appear in the utility function when the portfolio return is expressed by its constituents: the individual assets. Accordingly, utility functions with higher moments different from zero will implicitly take *co-skewness* and *co-kurtosis* into account. Co-skewness is a phenomenon extensively discussed by Harvey et al. (2003).

[12]As referred above, the MV approach is based on either assuming quadratic utility or a normal return distribution. If the latter is assumed, all odd moments ($n = 3, 5...$) will be zero, and all even moments will be functions of the variance (see Appendix to Chapter 1 in Cuthbertson and Nitzsche, 2004).

moments than kurtosis ($n > 4$) have not been considered in the finance literature, and will not be discussed here. However, according to Scott and Horvath (1980), most investors have utility functions where moments of odd order (i.e. $n = 1, 3, 5$ etc...) have positive signs on its respective derivative, and moments of even order have negative derivatives.

We note that recently popular utility functions such as linear splines, S-shaped functions and functions that include limits on decreases in wealth, often have discontinuous first or second derivatives. These functions are not in the range of MV analysis.

2.2 Utility functions

The choice of utility function for the estimation is of course crucial for a successful approximation of the optimal portfolio. The utility function must capture the investor's preferences as accurately as possible. In the strive for analytical solutions, mathematical feasibility has traditionally constrained the choice of utility function. This constraint does not apply for the FSO model, as no analytical solution is pursued, making the choice truly flexible.

The preferences we want to capture with the utility function have been described above: favor of mean and skewness and aversion to variance and kurtosis.[13] The parametric, closed form utility functions that are most common in the finance literature are the families of *exponential* and *power* utility functions. The former is characterized by constant *absolute* risk aversion ($CARA$), and the latter by constant *relative* risk aversion ($CRRA$), meaning that risk aversion varies with wealth level. To consider the investor preferences of skewness and kurtosis in particular, two types of utility functions that have been suggested are the *bilinear* and the *S-shaped* utility function families. Both of these are characterized by a critical point of investment return, under which returns are given disproportionally bad utility. The bilinear utility functions

[13] The process of interviewing investors to determine their risk tolerance is discussed by Litterman (2003,Ch.2) and Meucci (2005, Ch.5). The preferences of actual investors often are nonlinear due to particular biases; modeling these is a topic for future research.

Exponential utility (CARA)	$U = -exp(-A\theta_i' R_{i,t})$
Power utility (CRRA)	$U = \begin{cases} \frac{(\theta_i' R_{i,t}+w)^{1-\gamma}-1}{1-\gamma} & \text{for} \quad \gamma > 0 \\ ln(\theta_i' R_{i,t} + w) & \text{for} \quad \gamma = 1 \end{cases}$
Bilinear utility (BILI)	$U = \begin{cases} ln(1+\theta_i' R_{i,t}) & \text{for} \quad \theta_i' R_{i,t} \geq x \\ P(\theta_i' R_{i,t} - x) + ln(1+\theta_i' R_{i,t}) & \text{for} \quad \theta_i' R_{i,t} < x, P > 0 \end{cases}$
S-shaped utility (SSHA)	$U = \begin{cases} -A(\theta_i' R_{i,t} - x)^{\gamma_1} & \text{for} \quad \theta_i' R_{i,t} \leq x \\ +B(x - \theta_i' R_{i,t})^{\gamma_2} & \text{for} \quad \theta_i' R_{i,t} > x \end{cases}$ Subject to $A, B > 0, 0 < \gamma_1, \gamma_2 \leq 1$

Table 1: Utility function equations

(Cremers et al. 2005, Adler and Kritzman 2007) have a kink at the critical point and is formed by straight lines of different slope on each side of it (i.e. the functions are linear splines), which obviously yields discontinuity even in the first derivatives. The kink may constitute a level of wealth under which the investor has to lower his living standards. The S-shaped utility functions have an inflection point at the critical point. Introduced by Kahnemann and Tversky (1979), this utility function captures a investor behavior observed in behavioral finance studies (see further discussion below). In these functions, first derivatives are continuous, but second derivatives are not. The general mathematical form for the four utility functions discussed is presented in Table 1, and their graphical characteristics are displayed in Figure 3.

Gourieroux and Monfort (2005) apply the FSO model to the exponential and power utility functions, which allows them to derive the asymptotic properties of the FSO estimator. They establish in this context that the utility maximizing estimator yields greater robustness than the MV counterpart, as no information in the return distribution is ignored. However, they do not derive to what extent the FSO yields better approximations of the optimal portfolio than MV does. Levy and Markowitz (1979) and Cremers et al. (2003, 2005) also apply a power utility function, but conclude that the difference to the MV approach under such

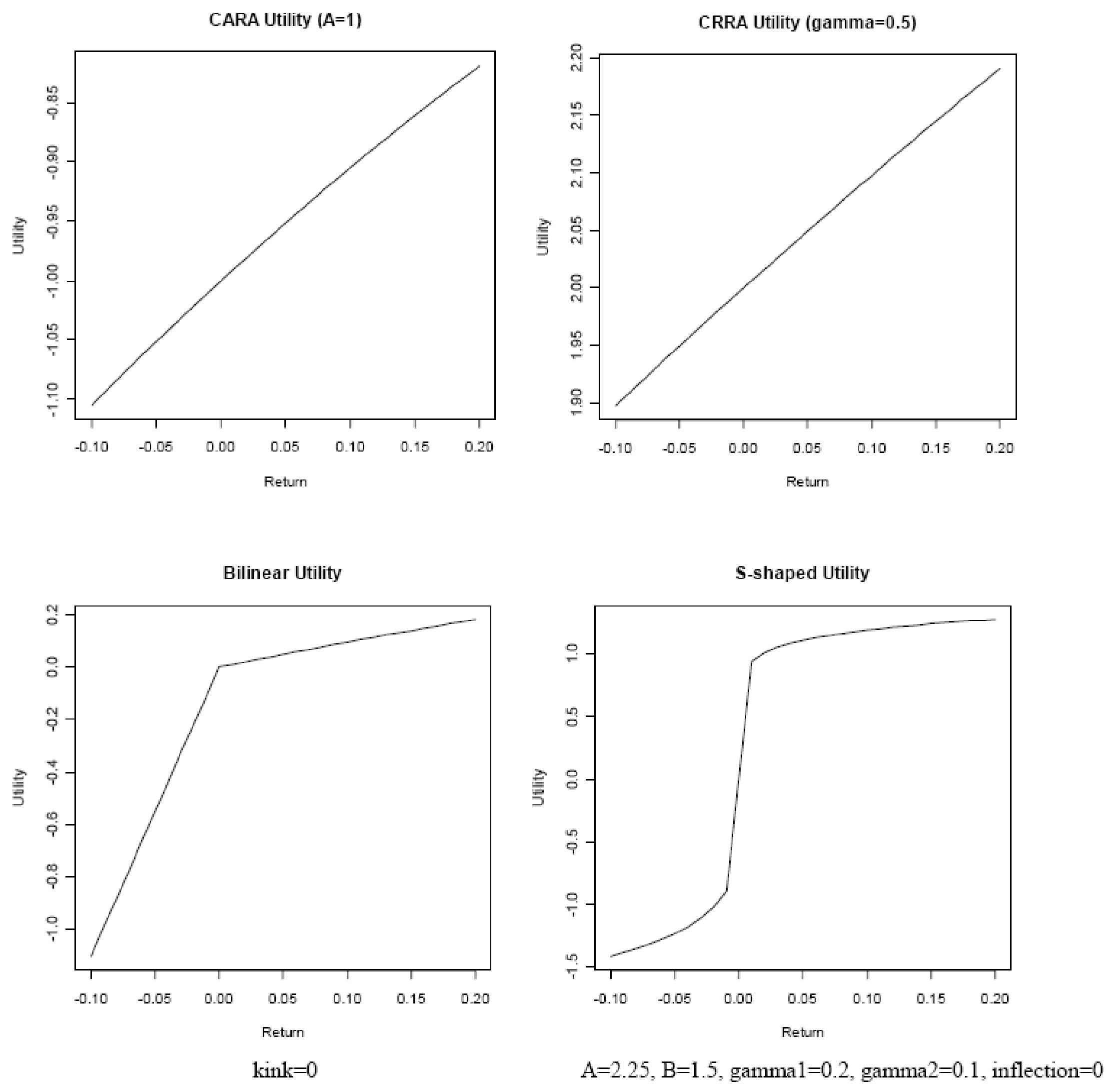

Figure 3: Utility functions considered for FSO models

preferences is negligible. In the applications of the bilinear and S-shaped utility functions, provided by Cremers et al. (2005) and Adler and Kritzman (2007), substantial differences in performance between FSO and MV approaches are identified. Such results must however be evaluated in light of how closely the analytic utility functions describe investor preferences.

The bilinear functions capture a phenomenon that is central in investment management today: loss aversion. The objective of limiting losses is motivated by monetary as well as legal purposes. The issue is traditionally treated with Value-at-Risk models, and can also be incorporated in FSO theory through a constraint on the maximization problem (as shown by Gourieroux and Monfort 2005). Cremers et al. (2005) show that under bilinear preferences, the resulting portfolio displays less kurtosis and less down-side risk than the portfolio resulting from MV analysis, confirming that if loss aversion is desirable to account for, this is a reasonable way to do it.

The S-shaped utility function is motivated by the fact that it has been shown in behavior studies that an investor prefers a certain *gain* to an uncertain gain with higher expected value, but he also prefers an uncertain *loss* to a certain loss with higher expected return. The inflection point is where these certainty preferences changes. The utility function implies high absolute values of marginal utility close to the inflection point, but low (absolute) marginal utility for higher (absolute) returns. Hence, this function may make sense in cases when investor behavior needs to be captured by the model, but it can be questioned whether it is a utility function investors are striving to fulfill. In the application by Cremers et al. (2005) it yields a portfolio, which, when compared to the MV outcome, has fewer negative returns but equal kurtosis.

Utility functions are not all about moments. Further possible extensions include liquidity preferences, preferences for firm-specific features such as ethical standards and geographical orientation, and habit formation functions that put a value to stable consumption patterns. In this study, we limit the investigation of differences between MV and FSO models to the four utility functions presented above. This decision, in spite of the plethora of other utility functions, is based on that they have been applied in former FSO performance studies, with which we intend to compare our results.

3 Methodology

The methodology for comparing the two approaches is to a large extent inspired by that applied by Cremers et al. (2005), where the performance of the different approaches is measured in utility.

In the FSO approach the optimal portfolio is found using numerical optimization, often beginning with a grid search. The problem may be thought of as constructing a matrix Θ containing each possible allocation combination. The allocation matrix dimension is $(n \times m)$, where n is the number of assets considered, and m is a function of n and a precision parameter p. Each column of Θ represents one allocation combination vector θ $(n \times 1)$. To find the optimal θ, the utility for each theta is evaluated for each asset returns vector R_t, which contains returns on each asset i at time t, $i = 1, 2, ..., n$ and $t = 1, 2, ..., T$. Each of the $T - 1$ R_t vectors will have dimension $(n \times 1)$ and elements $R_{i,t} = \frac{P_{i,t}}{P_{i,t-1}}$, where $P_{i,t}$ is the price of asset i at time t. The θ with highest average utility over time will be the optimal allocation combination, θ_{FSO}. This is shown formally in Equation 3,

$$\theta_{FSO} = \max_{\theta_a} \left(T^{-1} \sum_{t=1}^{T} U(\theta'_{a,i} R_{i,t}) \right), \tag{3}$$

where $a = 1, 2, ..., m$.

The study is performed in a one-period setting – no rebalancing of the portfolio is considered. As the allocation matrix grows quickly when more assets are added or the allocation precision p is increased, we use a three asset setting with $p = 1\%$, and we do not allow for short-selling. This yields an allocation matrix of dimension (3×5151), which we evaluate over 129 monthly observations. This rather limited amount of assets and precision is due to the computational burden of the technique. We analyze the full grid of possible allocations – no search algorithm is applied. As it turns out this level of detail is enough to illustrate the difference between the FSO and MV.[14] To illustrate the computational burden problem, a routine for deriving m is provided in Appendix A.

A common problem in dynamic programming is the choice of starting values. Litterman (2003) suggests starting the optimization at a vector of "equilibrium" returns determined via a CAPM analysis (he labels this the Black-Litterman approach). Unfortunately, CAPM has been widely rejected in empirical studies as a description of U.S. asset returns.

For the bilinear and S-shaped utility functions, we also calculate success rates of the same type as in Cremers et al. (2005)). These are the fraction of all points in time that yield portfolio returns superior to the investor's specified critical level (kink and inflection point respectively).

We apply the same portfolio choice problem to a wide range of utility functions, including the CARA functions, CRRA functions, bilinear functions, and S-shaped functions. The same utility functions have been investigated before, but only a few cases of each type. We perform the exercise under several different utility function parameter values, chosen with the intention to cover all reasonable levels. The range of utility parameters tested is given in Table 2.

Utility function	**Parameter values**			
Exponential utility	$0.5 \leq A \leq 6$			
Power utility	$0 \leq \gamma \leq 2$			
Bilinear utility	$-4\% \leq Kink \leq +0.5\%$		$0.1 \leq P \leq 10$	
S-shaped utility	$0 \leq \gamma_1 \leq 0.5$	$1 \leq \gamma_2 \leq 0.5$	$A = 1.5$	$B = 1.5$
	$\gamma_1 = 0.5$	$\gamma_2 = 0.5$	$1.5 \leq A \leq 0.1$	$1.5 \leq B \leq 2.9$

Table 2: Utility function parameters

[14]In Cremers et al. (2005) and Adler and Kritzman (2007), a search algorithm is applied to find the FSO optimum. Such algorithms are necessary when using larger number of assets. Cremers et al. consider 61 assets in their application, and use a precision of 0.1% and do not allow for short selling. This implies $m = 7.23 \times 10^{98}$, which is the number of vectors to be evaluated over their 10 annual observations.

Gourieroux and Monfort (2005), Cremers et al. (2003, 2005) and Adler and Kritzman (2007) all argue that the computational burden of the FSO technique has become obsolete with the ample computational power on hand nowadays. There is however, to our knowledge, no study verifying this. This is an important point for future research.

For the exponential utility function, the only parameter to vary is the level of risk aversion (A), which we vary between 0.5 and 6. The γ parameter in the power utility function determines level of risk aversion and how risk aversion decreases with wealth. As we let it vary between 0 and 2, we include the special case when the power utility function is logarithmic, which happens when γ is zero. The higher γ is the lower is the risk aversion. For the bilinear utility function, we vary the critical point (the kink, varied from -4% to +0.5%) under which returns are given a disproportionate bad utility. We also vary the magnitude, P, of this disproportion from 0.1 to 10. In the S-shaped utility function there are five parameters to vary. We choose to hold the inflection point fixed at zero, as the idea behind this utility type is loss aversion. The parameters γ_1 and A respectively determine the shape and magnitude of the upside of the function, whereas γ_2 and B determines the downside characteristics in the same way. The disproportion between gains and losses can be determined either by the γ parameters or the A and B parameters, or both. We perform one set of tests where the γ's vary ($\gamma_2 \geq \gamma_1$) and the magnitude parameters are hold constant and equal, and one set of tests where the gammas are constant and equal, but where A and B varies ($B \geq A$).

4 Data

For the empirical application we use three indices that are published by the Financial Times, downloaded from Datastream (2006):FTSE 100, FTSE 250, and FTSE All-World Emerging Market Index (*EMI*).[15] The FTSE 100 includes the 100 largest firms on the London Stock Exchange (LSE) and FTSE 250 include mid-sized firms, i.e. the 250 firms following the hundred largest. The EMI reflects the performance of mid- and large-sized stocks in emerging markets[16]. All series are denoted in British pounds (£). We calculate return series for 10

[15] The Datastream codes for the indices are FT100GR(PI),FT250GR(PI), and AWALEG£(PI).

[16] For exact definition, see http://www.ftse.com/Indices/FTSE_Emerging_Markets/Downloads/FTSE_Emerging_Market_Indices.pdf

years of monthly observations (25/11 1996 - 27/11 2006), yielding 129 observations. As shown in Figure 4, the period covers two expansionary periods and one recession on the UK stock market, whereas the pattern is less clear for the EMI (likely due to the Russian and Asian crises in the late 1990's).

The data properties are presented in Table 3. All three indices display positive means over the sample period. The least volatile choice of the three is the FTSE100, followed by FTSE250 and the FTSE EMI. All of them are characterized by negative skewness and excess kurtosis. It is shown with a Jarque-Bera test of normality (Jarque and Bera 1980) that normality can be rejected for all three indices[17].

Summary statistics	**Mean**	**Variance**	**Skewness**	**Kurtosis**	**J–B stat.**	**p**
FTSE 100	0.0043	0.0018	-0.043	4.99	19.67	0.00
FTSE 250	0.0064	0.0046	-0.244	4.19	8.15	0.02
FTSE EMI	0.0090	0.0060	-0.409	5.24	28.09	0.00

Table 3: Summary statistics

5 Results and analysis

The portfolio selection problem described was repeated 100 times using different utility function specifications. There were 12 tests with CARA type utility, 9 tests with CRRA type utility, 60 tests with bilinear utility and 19 tests with S-shaped utility. All results are presented in an appendix. Below the test results are presented and interpreted for each utility function type separately. The section is concluded with some general observations.

In order to compare the FSO optimum to the MV solution, the resulting total portfolio return is calculated ($R_p = \theta'_{FSO,i} R_{i,1T}$), where $R_{i,1T}$ is the return

[17] The Jarque-Bera test is appropriate for serially uncorrelated data (such as white-noise regression residuals) but inadequate for temporally dependent data such as certain financial returns. We do not pursue this further; the interested reader is referred to Bai and Ng (2005)

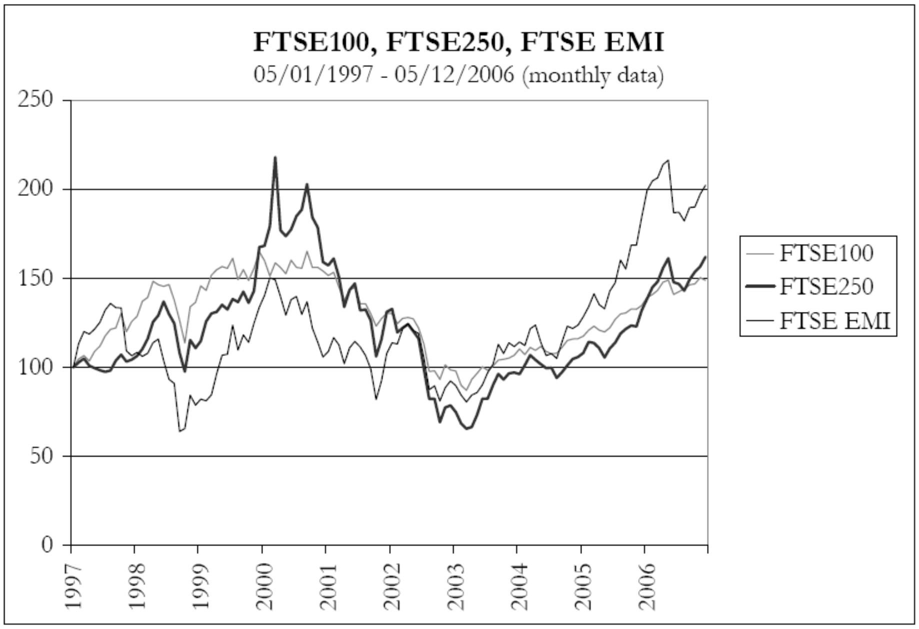

Figure 4: Development of the three indices over the time period considered

of asset i over the whole time period considered $\left(R_{i,1T} = \frac{P_{i,1}}{P_{i,T}}\right)$. The MV optimal portfolio, θ_{MV}, is the variance-minimizing allocation that yields the same portfolio return[18] . This will be a point on the MV efficiency frontier.[19] From each solution, a return distribution over time is calculated ($\theta' R_t$). By inserting each of these in the utility function applied for the FSO method, a measure of the MV approximation error, ϵ_{MV}, can be calculated as in Equation 4.

$$\epsilon_{MV} = U(\theta'_{FSO,i} R_{i,t}) - U(\theta'_{MV,i} R_{i,t}) \tag{4}$$

5.1 Exponential utility functions

The tests performed with CARA type utility and differing levels of risk aversion yielded, as expected, portfolios overweighted to risky assets when A was low and

[18]We allow the portfolio return to differ by 0.5% upwards or downwards. This interval is necessary in order to include alternatives to the FSO optimum.

[19]We use the Markowitz MV model from 1952 as benchmark in this study. In this way, it can be established whether the FSO model is superior to that model. The rich supply of MV extensions, however, is yet to be compared to the FSO model.

underweighted to less risky assets when A was high. Portfolio allocations chosen on the mean variance efficiency frontier yielded small differences in portfolio return - only in one case did the difference exceed 0.005%. These results conform well to those of Cremers et al. (2005) and Adler and Kritzman (2007), showing that portfolio allocations chosen by the utility maximizing approaches constitute very small improvements relative to the MV approach, when the investor's utility is well described by the CARA or CRRA utility functions.

5.2 Power utility functions

Portfolio allocations based on power utility functions were selected with 9 different levels of gamma, implying different levels of risk aversion and change of risk aversion as wealth grows. For low levels of gamma, the allocations are overweighted to the risky assets, gradually changing to the non-risky assets as gamma grows. The special case when gamma is 1 yields a non-diversified portfolio allocating all money to the EMI - the most risky asset. This is logical, because the size of the returns in this case is all that matters ($U = ln(R)$). The differences in utility to the corresponding MV portfolio allocations are small, reaching at most 0.02%. These small differences conform to the previous studies.

5.3 Bilinear utility functions

The tests on bilinear utility functions (i.e. linear splines) were performed with the kink (knot) at different levels and various penalties (P) on sub-kink returns. Each kink value was tested for 10 different penalty levels. Referring back to the utility function specification in Table 1, it is seen that the penalty is subtracted from a fixed level of utility on all returns. For the lowest penalty, 0.1, well-diversified portfolio allocations are chosen, but higher penalty cause less risky assets to be preferred. For penalty levels above one, the deviations from the safest portfolio possible (100% on the least risky asset) are small. This occurs when the incentive to avoid returns less than those associated with the kink dominates other investor incentives, such as maximizing returns or minimizing risk by diversification.

The differences to the MV choices are in many cases substantial. The differences range up to 16% and have an average of 0.89%. As shown in Figure 5, the utility difference under high penalty levels (> 0.5) tends to be larger for lower kink levels. For penalty levels at 0.1 and 0.5 that tendency is less clear.

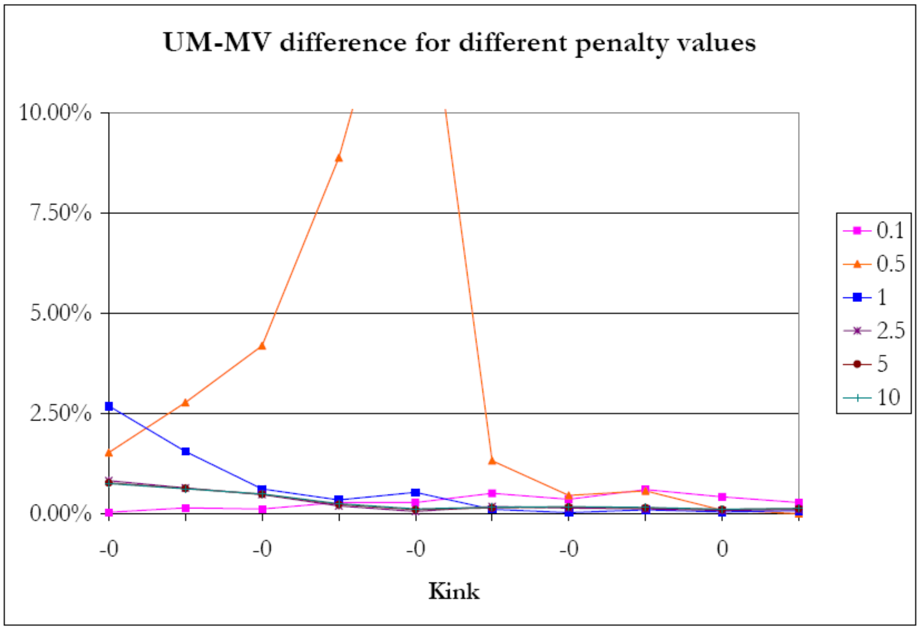

Figure 5: FSO-MV difference under bilinear utility functions

An alternative way of evaluating performance of the bilinear function, which is also applicable to S-shaped functions, is to investigate the frequency of returns exceeding the critical point under which a disproportionately bad utility is assigned. Results of such evaluation, called success rates, are shown in Table 4[20]. For the bilinear utility function, the success rate is not improved by using the FSO portfolio. However, as a majority of the bilinear utility functions led to close to non-diversified portfolios, not much scope for variation was given. The same result was shown by Cremers et al. (2005), who applied the same measure.

[20]These are average success rates for all utility function specifications. The individual success rates can be found in the appendix.

Success rates	**FSO**	**MV**
Bilinear utility	71.30%	71.50%
S-shaped utility	63.51%	58.34%

Table 4: Average success rates

5.4 S-shaped utility functions

The portfolio selections performed under S-shaped utility were all done with the inflection point at zero. Returns that fall short of that point are assigned a more or less disproportionate utility. The proportions between negative and positive returns' utility are regulated with the gamma values, γ_1 and γ_2, and with the magnitude values, A and B. In the first set of tests, the magnitude values were held equal and constant and the proportions of the gammas varied. In the second set of tests, the gammas were equal and constant, whereas the magnitude variables' proportions varied.

The utility from the FSO approach under S-shaped utility is considerably better than the utility obtained with S-shaped preferences when evaluating allocations chosen via the MV approach. The average difference is 21.84%, ranging up to 58% and down to zero. The success rates, presented in Table 5 (above), are also clearly in favor of the FSO approach in cases where S-shaped utility are believed to describe the investor's preferences well. This extends the findings in Cremers et al. (2005) to a wider range of utility specification values.

The variation of gamma proportions does not alter the allocations much. The gamma primarily determines the bends of the S-shape, and the influence on allocations is apparently marginal. The variation of A and B, on the other hand, has a big influence on allocations. The higher the ratio A/B gets, the less risk is chosen for the portfolio. The benefit of using the FSO approach is indicated to be higher when the ratio is closer to one than when it is higher (no case where it is lower than one was considered in this study).

5.5 General results

This empirical application illustrates the robustness of the finding of Cremers et al. (2005) and Adler and Kritzman (2007) that the FSO approach in general is useful when investor preferences are well described by non-mathematically smooth utility functions, including the bilinear and S-shaped utility functions. If investor preferences are well described by a power or exponential utility function, there is little benefit of using the FSO rather than the MV approach. This is shown in a more general framework than in preceding studies, investigating a wide scope of utility function parameter level.

The fact that these results appear in an application of such general indices as the FTSE 100 and FTSE 250 increases the scope of the FSO applicability. It has earlier only been shown that the FSO is useful in allocation problems involving hedge funds.

6 Conclusions

The empirical application of this study constitutes the widest FSO-MV comparison to date with respect to utility functions. The results extend earlier findings by Cremers et al. (2005) and Adler and Kritzman (2007), establishing the robustness of those studies' results. Our results also emphasize the important role of the mathematical smoothness of the utility function. The FSO methodology is useful when investor utility function features a threshold, such as in the bilinear and the S-shaped utility functions. For traditional utility functions (exponential and power utility) the MV approach yields similar results. The main empirical contribution of this study is that the scope of the FSO applicability is extended to more general stocks, as the sample used features the FTSE 100 and FTSE 250 with limited non-normality in the return distributions (in previous studies the methodology has been proven useful only in hedge fund applications).

Future research on the Full-Scale Optimization methodology needs to focus on the choice of utility function and its parameters. These are crucial for an adequate portfolio optimization. The scope of the usefulness of the FSO approach also deserves more attention. Furthermore, the computational burden of the numerical optimization needed for the FSO model needs to be clarified before the simplicity of the mean variance approach can be deemed an obsolete argument.

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Appendix A: The dimensions of the allocation matrix Θ

The asset allocation matrix contains all possible combinations of allocations of n assets ($n = 1, 2, $), with precision p ($0 < p < 100\%$ such that $1/p$ is an integer). The matrix dimension is $(n \times m)$. This appendix studies how m, the number of allocation combinations, is related to n and p when no short selling is allowed[21].

The simplest possible Θ matrix is the one asset case, when the only option is to invest 100% of the portfolio in that asset and $m = 1$. When going from one asset to the two asset case, the width growth factor, g_i will always be ($g_1 = m_2 = p^{-1} + 1$). When 10% precision is used, the two asset case has a matrix width $m_2 = 11$, as illustrated in Figure 6.

1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1	0.0
0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

Figure 6: Example of Θ matrix ($n = 2$; $p = 10\%$)

Adding another asset to this example yields $m_3 = 66$, implying a growth factor $g_2 = 6$. If precision is increased to $p = 1\%$ (note that increased precision implies that p declines) we will get matrices of the width $m_2 = 101$ and $m_3 = 5151$, implying growth rates of $g_1 = 101$ and $g_2 = 51$. At this stage the following tendencies can be noted about the function $m = f(n, p)$:

- The width is growing in both asset number and precision: $\frac{\delta m}{\delta n} > 1$, $\frac{\delta m}{\delta p} > -1$
- The growth rate decreases with the number of assets: $\frac{\delta g_n}{\delta n} < 0$
- The growth rate decreases with precision: $\frac{\delta g_n}{\delta p} > 0$

We have not been able to mathematically prove a formula for m, but by observation on how the Θ matrix grows, we have derived a three-step formula that holds exactly for all matrices we have generated. These steps are given in

[21] If no constraint on short selling is set, m will be infinite.

Equations 5–7, and are explained below.

$$\Delta_m g_n = \begin{cases} p^{-1}+1 & \text{for} \quad n=1 \\ (-pn(n+1))^{-1} & \text{for} \quad n>1 \end{cases} \tag{5}$$

$$g_n = \sum_{i=1}^{n} \Delta_m g_i \tag{6}$$

$$m_n = \prod_{i=1}^{n-1} g_i \quad \text{for} \quad n>1 \tag{7}$$

Equation 5 gives the change in growth factor of m as one asset is added to the n asset case, $\Delta_m g_n$. This is positive only when the second asset is added, which is given as a special case for $n=1$. At addition of subsequent assets, the growth change is the inverse product of the sum of the arithmetic sequence $(1, 2, ..., n)$ and the precision factor times two. The sum of the arithmetic sequence[22] is given by the expression $\sum_{i=1}^{n} = \frac{n(1+n)}{2}$ which multiplied by $2p$ and inversed yields the $n>1$ part of Equation 5. Now, at addition of the nth asset, m will grow with a factor g_n, which Equation 6 shows equals the sum of all growth rate changes. In this way the growth factor for each addition of assets can be calculated, and their product will be the matrix width m, as shown by Equation 7. Using this routine, the Θ matrix dimensions in Table 5 have been calculated as an illustration of the problem of computational burden.

n	p=10% m	p=5% m	p=1% m	p=0.1% m
1	1	1	1	1
2	11	21	101	1001
3	66	231	5151	501501
4	286	1771	176851	167668501
5	1001	10626	4598126	42084793751
10	92378	10015005	4.26E+12	2.88E+21
20	20030010	68923264410	4.91E+21	9.93E+39
50	62828356305	1.16E+17	6.71E+39	5.49E+84

Table 5: Examples of matrix dimensions

[22] Traced back to the work of Leonardo of Pisa, 1202.

Appendix B: Full results

In this appendix the complete set of results from the empirical application for each utility function is presented.

	FSO approach					MV approach					
A	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	ϵ_{MV}
0.5	0.00	0.00	1.00	1.02	-0.9963	0.00	0.08	0.92	1.02	-0.9963	0.01%
1	0.00	0.00	1.00	1.02	-0.9940	0.00	0.08	0.92	1.02	-0.9941	0.00%
1.5	0.11	0.13	0.76	0.98	-0.9932	0.10	0.17	0.73	0.97	-0.9932	0.00%
2	0.38	0.08	0.54	0.87	-0.9930	0.4	0.06	0.54	0.86	-0.9930	0.00%
2.5	0.54	0.06	0.40	0.78	-0.9932	0.54	0.07	0.39	0.78	-0.9932	0.00%
3	0.65	0.03	0.32	0.72	-0.9938	0.65	0.04	0.31	0.72	-0.9938	0.00%
3.5	0.73	0.02	0.25	0.68	-0.9949	0.73	0.03	0.24	0.67	-0.9949	0.00%
4	0.79	0.01	0.20	0.64	-0.9965	0.78	0.02	0.20	0.64	-0.9965	0.00%
4.5	0.83	0.00	0.17	0.61	-0.9985	0.82	0.01	0.17	0.62	-0.9985	0.00%
5	0.86	0.00	0.14	0.59	-1.0010	0.85	0.01	0.14	0.60	-1.0010	0.00%
5.5	0.89	0.00	0.11	0.57	-1.0039	0.88	0.01	0.11	0.57	-1.0039	0.00%
6	0.91	0.00	0.09	0.56	-1.0073	0.9	0.01	0.09	0.56	-1.0074	0.00%

Table 6: CARA utility results

	FSO approach					MV approach					
γ	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	ϵ_{MV}
0	0.00	0.00	1.00	1.02	1.01	0.00	0.08	0.92	1.02	1.01	0.02%
0.25	0.00	0.00	1.00	1.02	1.34	0.00	0.08	0.92	1.02	1.34	0.01%
0.5	0.00	0.00	1.00	1.02	2.01	0.00	0.08	0.92	1.02	2.01	0.01%
0.75	0.00	0.00	1.00	1.02	4.01	0.00	0.08	0.92	1.02	4.01	0.00%
1	0.00	0.00	1.00	1.02	1.01	0.00	0.08	0.92	1.02	1.01	0.02%
1.25	0.00	0.12	0.88	1.01	-3.99	0.00	0.15	0.85	1.01	-3.99	0.00%
1.5	0.14	0.13	0.73	0.96	-2.00	0.14	0.15	0.71	0.96	-2.00	0.00%
1.75	0.29	0.10	0.61	0.91	-1.33	0.30	0.10	0.60	0.90	-1.33	0.00%
2	0.40	0.08	0.52	0.86	-1.00	0.42	0.06	0.52	0.85	-1.00	0.00%

Table 7: CRRA utility results

Parameters		FSO approach					MV approach					Evaluation		
$Kink$	P	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	ϵ_{MV}	SR_{FSO}	SR_{MV}
-0.04	0.1	0.07	0.10	0.83	0.994	0.005	0.07	0.13	0.80	0.989	0.005	0.04%	78.99%	79.83%
-0.035	0.1	0.09	0.07	0.84	0.992	0.005	0.07	0.14	0.79	0.987	0.005	0.15%	75.63%	76.47%
-0.03	0.1	0.09	0.07	0.84	0.992	0.004	0.07	0.14	0.79	0.987	0.004	0.12%	75.63%	75.63%
-0.025	0.1	0.10	0.06	0.84	0.990	0.004	0.07	0.15	0.78	0.985	0.004	0.28%	73.11%	72.27%
-0.02	0.1	0.14	0.04	0.82	0.980	0.004	0.11	0.13	0.76	0.975	0.004	0.28%	70.59%	69.75%
-0.015	0.1	0.12	0.05	0.83	0.985	0.004	0.09	0.14	0.77	0.980	0.004	0.51%	68.07%	67.23%
-0.01	0.1	0.15	0.04	0.81	0.977	0.004	0.12	0.13	0.75	0.972	0.004	0.36%	63.87%	64.71%
-0.005	0.1	0.14	0.07	0.79	0.975	0.004	0.10	0.17	0.73	0.971	0.004	0.60%	63.87%	63.87%
0	0.1	0.12	0.09	0.79	0.979	0.003	0.09	0.17	0.74	0.974	0.003	0.42%	60.50%	60.50%
0.005	0.1	0.12	0.08	0.80	0.981	0.003	0.09	0.16	0.75	0.976	0.003	0.28%	57.14%	58.82%
-0.04	0.5	0.63	0.00	0.37	0.748	0.002	0.61	0.05	0.34	0.743	0.002	1.52%	84.03%	85.71%
-0.035	0.5	0.68	0.00	0.32	0.716	0.001	0.66	0.05	0.29	0.711	0.001	2.77%	82.35%	80.67%
-0.03	0.5	0.71	0.00	0.29	0.696	0.001	0.70	0.03	0.27	0.692	0.001	4.19%	78.99%	78.99%
-0.025	0.5	0.74	0.00	0.26	0.676	0.000	0.73	0.03	0.24	0.672	0.000	8.88%	77.31%	78.99%
-0.02	0.5	0.75	0.00	0.25	0.669	0.000	0.74	0.03	0.23	0.665	0.000	15.67%	73.95%	75.63%
-0.015	0.5	0.76	0.00	0.24	0.662	-0.001	0.75	0.03	0.22	0.658	-0.001	1.32%	71.43%	73.95%
-0.01	0.5	0.76	0.00	0.24	0.662	-0.002	0.75	0.03	0.22	0.658	-0.002	0.46%	66.39%	67.23%
-0.005	0.5	0.75	0.00	0.25	0.669	-0.002	0.74	0.03	0.23	0.665	-0.002	0.57%	63.03%	62.18%
0	0.5	0.79	0.00	0.21	0.641	-0.003	0.78	0.01	0.21	0.645	-0.003	0.09%	57.14%	57.14%
0.005	0.5	0.84	0.00	0.16	0.606	-0.005	0.83	0.01	0.16	0.609	-0.005	0.01%	52.10%	51.26%
-0.04	1	0.89	0.00	0.11	0.570	-0.001	0.88	0.01	0.11	0.573	-0.001	2.69%	85.71%	86.55%
-0.035	1	0.85	0.00	0.15	0.599	-0.001	0.84	0.01	0.15	0.602	-0.001	1.55%	83.19%	84.03%
-0.03	1	0.85	0.00	0.15	0.599	-0.002	0.84	0.01	0.15	0.602	-0.002	0.62%	81.51%	81.51%
-0.025	1	0.85	0.00	0.15	0.599	-0.003	0.84	0.01	0.15	0.602	-0.003	0.34%	78.99%	78.15%
-0.02	1	0.88	0.00	0.12	0.577	-0.004	0.87	0.01	0.12	0.581	-0.004	0.53%	74.79%	75.63%
-0.015	1	0.91	0.00	0.09	0.555	-0.006	0.90	0.01	0.09	0.559	-0.006	0.11%	72.27%	73.11%
-0.01	1	0.89	0.00	0.11	0.570	-0.007	0.88	0.01	0.11	0.573	-0.007	0.03%	69.75%	69.75%
-0.005	1	0.89	0.00	0.11	0.570	-0.009	0.88	0.01	0.11	0.573	-0.009	0.10%	61.34%	60.50%
0	1	0.94	0.00	0.06	0.533	-0.011	0.93	0.01	0.06	0.536	-0.011	0.05%	60.50%	59.66%
0.005	1	0.98	0.00	0.02	0.503	-0.013	0.97	0.01	0.02	0.507	-0.013	0.08%	52.94%	52.94%

Table 8: Bilinear utility results, first half

Parameters		FSO approach					MV approach					Evaluation		
$Kink$	P	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	ϵ_{MV}	SR_{FSO}	SR_{MV}
-0.04	2.5	1.00	0.00	0.00	0.488	-0.007	0.99	0.01	0.00	0.491	-0.007	0.82%	84.03%	84.03%
-0.035	2.5	1.00	0.00	0.00	0.488	-0.009	0.99	0.01	0.00	0.491	-0.009	0.64%	83.19%	83.19%
-0.03	2.5	1.00	0.00	0.00	0.488	-0.011	0.99	0.01	0.00	0.491	-0.011	0.48%	81.51%	81.51%
-0.025	2.5	1.00	0.00	0.00	0.488	-0.014	0.99	0.01	0.00	0.491	-0.014	0.19%	77.31%	78.15%
-0.02	2.5	1.00	0.00	0.00	0.488	-0.016	0.99	0.01	0.00	0.491	-0.017	0.06%	75.63%	76.47%
-0.015	2.5	0.98	0.00	0.02	0.503	-0.020	0.97	0.01	0.02	0.507	-0.020	0.18%	72.27%	71.43%
-0.01	2.5	0.97	0.00	0.03	0.511	-0.023	0.96	0.01	0.03	0.514	-0.023	0.14%	68.91%	68.91%
-0.005	2.5	1.00	0.00	0.00	0.488	-0.028	0.99	0.01	0.00	0.491	-0.028	0.12%	63.87%	63.87%
0	2.5	1.00	0.00	0.00	0.488	-0.032	0.99	0.01	0.00	0.491	-0.032	0.08%	60.50%	60.50%
0.005	2.5	1.00	0.00	0.00	0.488	-0.038	0.99	0.01	0.00	0.491	-0.038	0.11%	53.78%	53.78%
-0.04	5	1.00	0.00	0.00	0.488	-0.017	0.99	0.01	0.00	0.491	-0.017	0.77%	84.03%	84.03%
-0.035	5	1.00	0.00	0.00	0.488	-0.021	0.99	0.01	0.00	0.491	-0.021	0.63%	83.19%	83.19%
-0.03	5	1.00	0.00	0.00	0.488	-0.026	0.99	0.01	0.00	0.491	-0.026	0.49%	81.51%	81.51%
-0.025	5	1.00	0.00	0.00	0.488	-0.031	0.99	0.01	0.00	0.491	-0.031	0.23%	77.31%	78.15%
-0.02	5	1.00	0.00	0.00	0.488	-0.036	0.99	0.01	0.00	0.491	-0.036	0.10%	75.63%	76.47%
-0.015	5	1.00	0.00	0.00	0.488	-0.043	0.99	0.01	0.00	0.491	-0.043	0.15%	72.27%	72.27%
-0.01	5	1.00	0.00	0.00	0.488	-0.050	0.99	0.01	0.00	0.491	-0.050	0.17%	68.91%	68.91%
-0.005	5	1.00	0.00	0.00	0.488	-0.059	0.99	0.01	0.00	0.491	-0.059	0.15%	63.87%	63.87%
0	5	1.00	0.00	0.00	0.488	-0.068	0.99	0.01	0.00	0.491	-0.068	0.10%	60.50%	60.50%
0.005	5	1.00	0.00	0.00	0.488	-0.078	0.99	0.01	0.00	0.491	-0.079	0.13%	53.78%	53.78%
-0.04	10	1.00	0.00	0.00	0.488	-0.038	0.99	0.01	0.00	0.491	-0.038	0.75%	84.03%	84.03%
-0.035	10	1.00	0.00	0.00	0.488	-0.046	0.99	0.01	0.00	0.491	-0.046	0.62%	83.19%	83.19%
-0.03	10	1.00	0.00	0.00	0.488	-0.054	0.99	0.01	0.00	0.491	-0.055	0.49%	81.51%	81.51%
-0.025	10	1.00	0.00	0.00	0.488	-0.064	0.99	0.01	0.00	0.491	-0.065	0.25%	77.31%	78.15%
-0.02	10	1.00	0.00	0.00	0.488	-0.076	0.99	0.01	0.00	0.491	-0.076	0.12%	75.63%	76.47%
-0.015	10	1.00	0.00	0.00	0.488	-0.089	0.99	0.01	0.00	0.491	-0.089	0.16%	72.27%	72.27%
-0.01	10	1.00	0.00	0.00	0.488	-0.104	0.99	0.01	0.00	0.491	-0.104	0.18%	68.91%	68.91%
-0.005	10	1.00	0.00	0.00	0.488	-0.120	0.99	0.01	0.00	0.491	-0.121	0.16%	63.87%	63.87%
0	10	1.00	0.00	0.00	0.488	-0.139	0.99	0.01	0.00	0.491	-0.139	0.11%	60.50%	60.50%
0.005	10	1.00	0.00	0.00	0.488	-0.160	0.99	0.01	0.00	0.491	-0.160	0.13%	53.78%	53.78%

Table 9: Bilinear utility results, second half

Parameters				FSO approach					MV approach					Evaluation		
γ_1	γ_2	A	B	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	θ_1	θ_2	θ_3	R_p	$\bar{U}_p$	ϵ_{MV}	SR_{FSO}	SR_{MV}
0.0	1.0	1.5	1.5	0.10	0.71	0.19	0.746	-0.488	0.63	0.02	0.35	0.741	-0.634	29.95%	64.71%	55.46%
0.05	0.95	1.5	1.5	0.10	0.71	0.19	0.746	-0.404	0.63	0.02	0.35	0.741	-0.509	25.78%	64.71%	55.46%
0.1	0.9	1.5	1.5	0.24	0.61	0.15	0.718	-0.332	0.66	0.04	0.30	0.715	-0.404	21.86%	64.71%	56.30%
0.15	0.85	1.5	1.5	0.24	0.61	0.15	0.718	-0.267	0.66	0.04	0.30	0.715	-0.321	20.17%	64.71%	56.30%
0.2	0.8	1.5	1.5	0.24	0.61	0.15	0.718	-0.210	0.66	0.04	0.30	0.715	-0.251	19.22%	64.71%	56.30%
0.25	0.75	1.5	1.5	0.24	0.61	0.15	0.718	-0.159	0.66	0.04	0.30	0.715	-0.190	19.41%	64.71%	56.30%
0.3	0.7	1.5	1.5	0.24	0.61	0.15	0.718	-0.112	0.66	0.04	0.30	0.715	-0.136	21.78%	64.71%	56.30%
0.35	0.65	1.5	1.5	0.28	0.47	0.25	0.769	-0.067	0.57	0.06	0.37	0.765	-0.080	18.40%	64.71%	59.66%
0.4	0.6	1.5	1.5	0.28	0.48	0.24	0.764	-0.024	0.58	0.06	0.36	0.759	-0.036	50.29%	64.71%	59.66%
0.45	0.55	1.5	1.5	0.08	0.60	0.32	0.817	0.020	0.48	0.08	0.44	0.813	0.010	48.95%	63.87%	59.66%
0.5	0.5	1.5	1.5	0.00	0.38	0.62	0.950	0.065	0.18	0.14	0.68	0.946	0.061	6.14%	62.18%	60.50%
0.5	0.5	1.5	1.5	0.00	0.38	0.62	0.950	0.065	0.18	0.14	0.68	0.946	0.061	6.14%	62.18%	60.50%
0.5	0.5	1.7	1.3	0.28	0.48	0.24	0.764	0.024	0.58	0.06	0.36	0.759	0.013	45.29%	64.71%	59.66%
0.5	0.5	1.9	1.1	0.29	0.47	0.24	0.763	-0.015	0.58	0.06	0.36	0.759	-0.023	57.93%	64.71%	59.66%
0.5	0.5	2.1	0.9	0.54	0.46	0.00	0.601	-0.053	0.84	0.00	0.16	0.606	-0.061	15.80%	63.87%	57.14%
0.5	0.5	2.3	0.7	0.70	0.30	0.00	0.571	-0.088	0.88	0.01	0.11	0.573	-0.093	6.48%	62.18%	57.98%
0.5	0.5	2.5	0.5	0.89	0.11	0.00	0.523	-0.122	0.95	0.00	0.05	0.526	-0.123	1.28%	59.66%	60.50%
0.5	0.5	2.7	0.3	1.00	0.00	0.00	0.488	-0.155	0.99	0.01	0.00	0.491	-0.155	0.00%	60.50%	60.50%
0.5	0.5	2.9	0.1	1.00	0.00	0.00	0.488	-0.188	0.99	0.01	0.00	0.491	-0.188	0.01%	60.50%	60.50%

Table 10: S-Shaped utility results