A STOCHASTIC VOLATILITY SWAP MARKET MODEL

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Abstract

This paper derives a stochastic volatility extension of the Swap Market Model where a multiplicative stochastic factor equally affects all instantaneous forward swap rate volatilities. First, qualitative support for such extension is provided, and second, based on the fast fractional Fourier transform and a specific functional form of the instantaneous swap rate volatility a calibration methodology to European swaption prices is performed.

EFM Classification : 550, 410.
Keywords : Swap market model, stochastic volatility, fast fractional Fourier transform, European swaptions.

1 Introduction

Since the emergence of market models (Brace et al. (1997), Jamshidian (1997), Miltersen et al. (1997) and Musiela and Rutkowski (1997)) most of the academic focus has been put on studying Libor-based models. Many issues have then been investigated within this framework : pricing, hedging, calibration and extensions. The Libor market model is used as a ground to price both caps and swaptions. Even though the Libor market model

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assumes lognormal forward Libor rates it is also used to deal with swaptions where the underlying swap rates are also assumed to be lognormal. To overcome this inconsistency[1], academics have relied on approximations (see Hull and White (2000) and Jäckel and Rebonato (2003) for example[2].) to price European swaptions within the Libor market model. Another approach consists in using the Swap market model (hereafter SMM)(Jamshidian (1997)) to directly price European swaptions. However, very few research papers have investigated the use of the Swap market model when it comes to deal with swaptions and other related derivatives (see Galluccio and Hunter (2004) for the case of co-initial swap rates and Galluccio et al. (2004) for that of co-terminal swap rates). Obviously, the standard version of the Swap market model does not account for the smile observed in the swaption market. Contrary to the case of the Libor market model, insofar extension of the Swap market model has only been formulated as a jump-diffusion model (Glasserman and Kou (2003)). Nevertheless, since swaptions are mostly long maturity options we may expect that a jump-diffusion extension would not be very satisfactory.[3]

In this paper I derive a stochastic volatility extension of the Swap market model. Stochastic volatility models are well known to account for the smile for intermediate and long option maturities. This feature makes them very suitable for the European swaptions. In the model considered here, swap rate volatilities are subject to a multiplicative stochastic factor that is common to all of them. This stochastic factor follows a square-root diffusion process *à la* Heston (1993).This stochastic volatility extension has already been applied in the context of the Libor market model by Andersen and Brotherton-Ratcliffe (2001) and Wu and Zhang (2003), however, none of these papers has provided a justification of this choice. Based on market data, I provide a qualitative investigation in support of the extension. The model is then calibrated to a set of market data composed of European swaption prices of various option maturities and swap lengths. Using a specific parametric form of the instantaneous swap rate volatilities and relying on the fast fractional Fourier transform (FFrFT) a fast calibration is achieved. Actually, it has become standard in the

[1]Since a Swap rate can be a sum of Libor rates thus if the latter is lognormal then the former cannot be lognormal.

[2]See Brigo and Mercurio (2001) for an empirical comparison of these approximations.

[3]See for instance, Das and Sundaram (1999) and Jarrow et al. (2003).

academic literature to use the fast Fourier transform (FFT) to obtain option prices in various setting (see for instance Carr and Madan (1999), Dempster and Hong (2000) and Benhamou (2002)). This method offers a significant computational time gain without loss of accuracy which makes it very appealing for calibration. However, the method lacks flexibility with respect to its implementation : to achieve high accuracy, one has to care about the choice of the number of points and the upper integration bound. The log-strike grid cannot be chosen freely, though. This disadvantage is circumvented when applying the fast fractional Fourier transform (FFrFT) since this method permits an independent choice of both the integration grid as well as the log-strike grid. Besides, as shown by Chourdakis (2005), the FFrFT may be faster than the FFT.

The specific parametric form of the instantaneous forward swap volatility takes into account, contrary to existing literature (see for example De Jong et al. (2001) and Galluccio et al. (2004)) both the option maturity and the swap period length. This feature provides a further "ease" to the calibration process.

The outline of the paper is as follows. In the next section, the Swap market model is briefly reviewed. Section (3) derives a stochastic volatility extension of the SMM, motivates the extension choice, and tests the computational speed and pricing accuracy of the fast fractional Fourier transform with respect to the Monte Carlo method. A calibration methodology relying on the fast algorithm is discussed and presented in section (4). Section (5) concludes.

2 The Swap Market Model : a review

Jamshidian (1997) developed a Swap Market Model where swap rates are assumed to be lognormal. This assumption, as in the case of the Libor market model, meets market practice which uses Black's model to price European swaptions (Black (1976).).
Consider a tenor structure $T_1 < \cdots < T_n$ and n zero-coupon bonds maturing at time T_i, $i = 1, \ldots, n$. Let $S^{i,n}(t)$ the swap rate spanning the period $T_n - T_i$, and the accrual period $\delta_j = T_j - T_{j-1}$, $j = 1, \ldots, n$ with $T_0 = 0$. In the following, we drop the subscript from the accrual periods and set them all equal to a constant δ.

Let $B(t,T_i)$ the time t price of the zero-coupon bond maturing at time T_i. Its process satisfies, under the risk-neutral measure $\mathbf{Q}$ where $\beta(t)$ (the money market account) is its associated numeraire, the following dynamics :

$$dB(t,T_i) = B(t,T_i)\big(r(t)\,dt + \sigma(t,T_i)\,dW(t)\big) \tag{2.1}$$

The forward swap rate satisfies the following relation :

$$S^{i,n}(t) = \frac{B(t,T_i) - B(t,T_n)}{\sum_{j=i+1}^{n} \delta\, B(t,T_j)} \qquad \forall \quad t \in [0,T_i] \tag{2.2}$$

Denote $B_{i,n}(t)$ the fixed-leg process. $B_{i,n}(t) = \delta \sum_{j=i+1}^{n} B(t,T_j)$. If we associate the numeraire $B_{i,n}(t)$ to the probability measure $Q^{i,n}$ (called *the forward swap measure*) then the forward swap rate process $S^{i,n}(t)$ is a martingale under $Q^{i,n}$.
Its dynamics is :

$$dS^{i,n}(t) = S^{i,n}(t)\,\sigma^{i,n}(t)\,dW^{i,n}(t) \tag{2.3}$$

where $\sigma^{i,n}(t)$ is the volatility of the forward swap rate.
Define

$$\frac{B_{i,n}(t)}{B(t,T_n)} = \tau_i(t) = \delta \sum_{j=i}^{n-1} \prod_{k=i+1}^{j} \big(1 + \delta\, S^{k,n}(t)\big) \qquad \text{thus} \qquad \frac{B(t,T_i)}{B(t,T_n)} = 1 + \tau_i(t) S^{i,n}(t)$$

$$\text{and} \qquad \tau_i(t) = \delta + \tau_{i+1}(t)\Big(1 + \delta\, S^{i+1,n}(t)\Big) \tag{2.4}$$

The price at time t of a European payer swaption giving the right to enter at time T_i into a swap maturing at time T_n is given by :

$$\Pi(t) = B_{i,n}(t)\mathbf{E_t^{i,n}}\Big[\big(S^{i,n}(T_i) - K\big)^+\Big] \tag{2.5}$$

3 A stochastic volatility extension

The Swap market model can be extended to a stochastic volatility model by a means of a common multiplicative stochastic factor that affects uniformly all swap rate volatilities. In this stochastic volatility framework, the multiplicative factor process follows a square-root diffusion. This extension has been applied in the case of the Libor market model by

Wu and Zhang (2003) and Andersen and Brotherton-Ratcliffe (2001). However, none of these studies had motivated this model choice. The next sub-section fills this gap.

3.1 Motivation for the extension : a qualitative investigation

I propose in the following a qualitative examination of the swaption implied volatility matrix. The goal is to investigate whether there is a common factor (stochastic) that affects in similar way all the volatilities across option maturities and swap periods. The data[4] used for this task consist in time-series of daily at-the-money implied volatilities (IV) from may 14, 2001 to october 30, 2003. For each date, swaptions of option expiry of 1, 2, 3, 4, 5 and 10 years and swap period of 1, 2, 3, 4, 5, 6, 7, 8, and 9 years are considered. Figure (1) plots the time-series of various IV. We can notice that all the curves exhibit a similar behavior : volatilities react simultaneously to the same impact and move in the same direction. The amplitudes due to the shock are different, though.

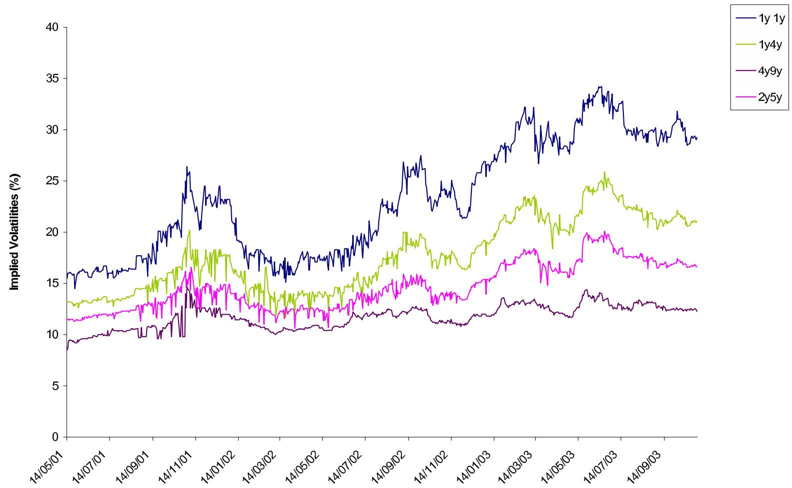
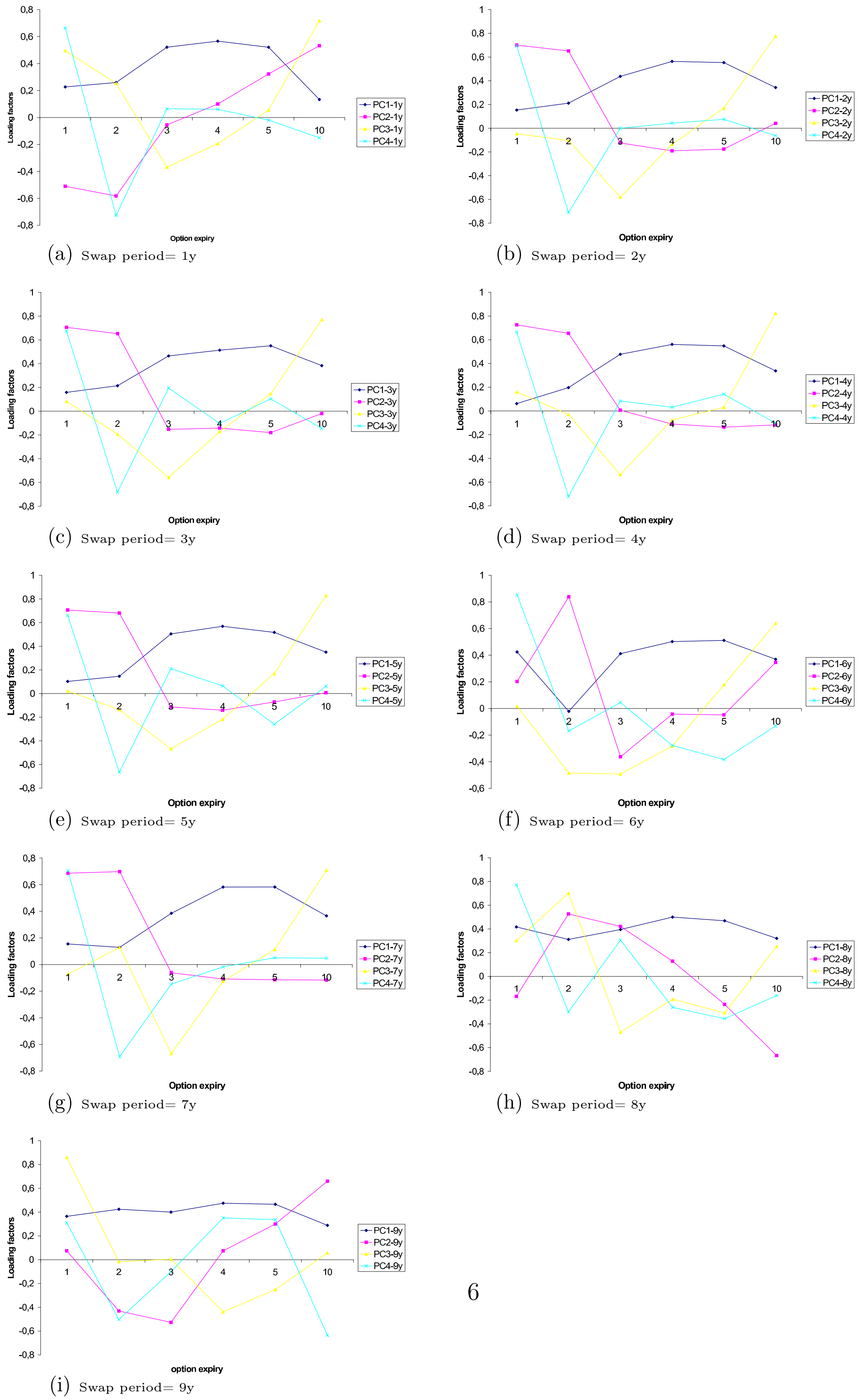


FIG. 1 – Implied Volatilities

In addition, to gain further insights, I construct correlation matrices with respect to each swap period (9 sub-matrices) of percentage changes in the IV and compute the eigenvalues and eigenvectors for each correlation sub-matrix. The results (figure (2)) show that the most principal components[5] display similar qualitative patterns across different swap

[4]The data are obtained from Bloomberg.

[5]The four most important principal components explain, in each case, more than 90% of innovations



(a) Swap period= 1y

(b) Swap period= 2y

(c) Swap period= 3y

(d) Swap period= 4y

(e) Swap period= 5y

(f) Swap period= 6y

(g) Swap period= 7y

(h) Swap period= 8y

(i) Swap period= 9y

The most significant principal components for swap options for different swap periods.

FIG. 2 – Principal components

periods and option expiries. This feature is another indication in favor of the suggested model.

3.2 Derivation of the extended model

Under the risk-neutral measure $\mathbf{Q}$ the dynamics of the Swap rate and the volatility factor are :

$$dS^{i,n}(t) = S^{i,n}(t)\,\sqrt{V(t)}\,\sigma^{i,n}(t)\left[dW(t) - \frac{\sqrt{V(t)}}{\tau_i(t)}\sum_{j=i+1}^{n}\delta\big(1+\tau_j(t)\,S^{j,n}(t)\big)\,\sigma(t,T_j)\,dt\right] \tag{3.1}$$

$$dV(t) = \kappa(\theta - V(t))\,dt + \eta\sqrt{V(t)}\,dZ(t) \tag{3.2}$$

respectively. W and Z are two independent Wiener processes. This zero-correlation assumption is supported by a recent empirical paper in which Chen and Scott (2004) didn't find evidence on the presence of significant correlation between changes in interest rates and changes in interest rate volatility.
To price the swaption, we need to use (2.5). Therefore, we must re-write the dynamics of both processes in equations (3.1) under a new probability measure $\mathbf{Q^{i,n}}$. This can be achieved using the Radon-Nikodym derivatives and the Girsanov theorem. Hence, we have :

$$\begin{aligned}\frac{d\mathbf{Q^{i,n}}}{d\mathbf{Q}} &= \frac{B_{i,n}(t)}{B_{i,n}(0)\,\beta(t)}\\ &= \mathcal{E}\left(\frac{\sqrt{V(t)}}{\tau_i(t)}\sum_{j=i+1}^{n}\delta\big(1+\tau_j(t)\,S^{j,n}(t)\big)\,\sigma(t,T_j)\right)\\ &= \zeta_t\end{aligned} \tag{3.3}$$

where $\mathcal{E}$ is the Doléans-Dade exponential.
Therefore,

$$\frac{d\zeta_t}{\zeta_t} = \sqrt{V(t)}\frac{\delta}{\tau_i(t)}\sum_{j=i+1}^{n-1}\big(1+S^{j,n}(t)\,\tau_j(t)\big)\sigma(t,T_j)\,dW(t)$$

in the implied volatility surface.

So equations (3.1) and (3.2) become under the forward swap measure :

$$dS^{i,n}(t) = S^{i,n}(t)\sqrt{V(t)}\sigma^{i,n}(t)\, dW^{i,n}(t) \tag{3.4}$$

$$dV(t) = \kappa(\theta - V(t))\, dt + \eta\sqrt{V(t)}\, dZ(t) \tag{3.5}$$

respectively.

Set $Y(t) \equiv \log(\frac{S^{i,n}(t)}{S^{i,n}(0)})$, and let $\Psi(y, v, T; u)$ the characteristic function defined by :

$$\Psi(y, v, t; u) = \mathbf{E}[e^{iuY(T)}|Y(t) = y, V(t) = v]$$

where $i^2 = -1$. Applying Ito's lemma to $\Psi(y, v, t; u)$, we obtain (given the martingale property) the following partial differential equation :

$$\frac{\partial \Psi}{\partial t} - \frac{1}{2}(\sigma^{i,n})^2\, v\frac{\partial \Psi}{\partial y} + \frac{\partial \Psi}{\partial v}\, \kappa\left(\theta - v\right) + \frac{1}{2}(\sigma^{i,n})^2 v\frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{2}\eta^2 v\frac{\partial^2 \Psi}{\partial v^2} = 0 \tag{3.6}$$

with terminal condition

$$\Psi(u) = \exp(iuy) \tag{3.7}$$

In order to compute the characteristic function we define the following exponential affine form of Ψ :

$$\Psi(y, v, \epsilon; u) = \exp(C(\epsilon; u) + vD(\epsilon; u) + iuy) \tag{3.8}$$

where $\epsilon = T - t$ or more precisely $T_i - t$. Substituting this functional form into Eq. (3.6) we obtain two ordinary differential equations for D and C :

$$\frac{\partial D}{\partial \epsilon} = \frac{1}{2}\eta^2\, D^2 - \kappa\, D - \frac{1}{2}(\sigma^{i,n})^2\, u(i + u) \tag{3.9}$$

$$\frac{\partial C}{\partial \epsilon} = \kappa\, \theta\, D \tag{3.10}$$

respectively, with initial conditions $D(0; u) = 0$ and $C(0; u) = 0$. To obtain the explicit expressions of D and C one has to first solve the Riccati equation (3.9) and then use its solution to determine C (Eq.(3.10)).

Proposition 1. *The explicit expressions of D and C are as follows :*

$$D(\epsilon; u) = \frac{\kappa + \Delta}{\eta^2} + \frac{\varrho\, \Delta\, \exp(\Delta\, \epsilon)}{0.5\eta^2\left[\varrho(1 - \exp(\Delta\, \epsilon)) + 2\, \Delta\right]} \tag{3.11}$$

$$C(\epsilon) = C(0) + \frac{\kappa\,\theta}{\eta^2}\left[(\Delta+\kappa)\,\epsilon - 2\,\log\left(\frac{\varrho\,(1-\exp(\Delta\,\epsilon))+2\,\Delta}{2\,\Delta}\right)\right] \tag{3.12}$$

with

$$\Delta = \sqrt{\kappa^2+\eta^2(\sigma^{i,n})^2 u(i+u)} \qquad \textit{and} \qquad \varrho = 2\,\eta^2 D(0) - \kappa - \Delta$$

3.3 Application of the fast fractional Fourier transform

Eq. (2.5) has to be evaluated numerically. It is now standard to rely on Monte Carlo simulations to obtain prices. This is however achieved, as will be shown later, at the cost of speed. To avoid this disadvantage, various numerical methods are considered in the literature as well as by practitioners. Among them, the fast Fourier transform (see Carr and Madan (1999)) is one of the most used. It offers the crucial advantage of a low computational time. Nonetheless, its implementation requires a careful choice of its parameters. The fast fractional Fourier transform[6] (hereafter FFrFT), introduced by Choudakis (2005) in option pricing, offers the advantages of speed and the freedom of choosing the parameters without any loss of accuracy[7].

I compute swaption prices through the FFrFT and compare them, w.r.t. speed and pricing differences, to those obtained by Monte Carlo.

Let's first write the integral version of eq. (2.5) and then apply the FFrFT. This is achieved in the Swap market model setting as follows :
Denote $k = \log(\frac{K}{S^{i,n}(0)})$ so eq. (2.5) becomes :

$$\Pi(k) = \int_k^\infty B_{i,n}(0)\,S^{i,n}(0)(e^y - e^k)f(y)dy \tag{3.13}$$

where $\Pi(k)$ denotes now the price of a payer swaption at time 0 for a strike $\exp(k)$; and $f(y)$ is the density function of y satisfying

$$\Psi(u) = \int_{-\infty}^\infty e^{iuy} f(y)dy \tag{3.14}$$

$\Psi(.)$ is the characteristic function.
Note that equation (3.13) is not square integrable over $(-\infty,\infty)$: when $k\to\infty$, $\Pi(k)=0$

[6]The fast Fourier transform used in Carr and Madan (1999) is considered as a specific case of the FFrFT.

[7]See Chourdakis (2005) for a comparison.

and when $k \to -\infty$, $\Pi(k) = S^{i,n}(0)$. The following modified swaption price circumvents the problem :

$$\widetilde{\Pi}(k) = \exp(\gamma k)\,\Pi(k) \tag{3.15}$$

for γ a positive constant satisfying $\mathbf{E}[(S_{T_i}^{i,n})^{\gamma+1}] < \infty$. Now we can consider a Fourier transform of $\widetilde{\Pi}(k)$:

$$\varphi(u) = \int_{-\infty}^{\infty} e^{iuk}\,\widetilde{\Pi}(k)\,dk \tag{3.16}$$

which yields,

$$\begin{aligned}
\varphi(u) &= \int_{-\infty}^{\infty} e^{iuk} e^{\gamma k} \int_{k}^{\infty} B_{i,n}(0)\,S^{i,n}(0)(e^{y} - e^{k}) f(y)\,dy\,dk \\
&= \int_{-\infty}^{\infty} B_{i,n}(0)\,S^{i,n}(0) f(y) \int_{-\infty}^{y} (e^{y+(iu+\gamma)k} - e^{(1+iu+\gamma)k})\,dk\,dy \\
&= B_{i,n}(0)\,S^{i,n}(0)\Big(\frac{1}{\gamma + iu} - \frac{1}{1+\gamma+iu}\Big)\int_{-\infty}^{\infty} f(y)\,e^{(1+iu+\gamma)y} dy \\
&= \frac{B_{i,n}(0)\,S^{i,n}(0)\Psi(u - (\gamma+1)i)}{(\gamma+iu)^2 + \gamma + iu}
\end{aligned} \tag{3.17}$$

Having obtained an analytic expression of $\varphi(u)$, I can recover the price of the swaption by applying the inverse transform to Eq.(3.16) :

$$\Pi(k) = \frac{e^{-\gamma k}}{\pi}\int_{0}^{\infty} e^{-iuk}\,\varphi(u)\,du \tag{3.18}$$

To apply the fast fractional Fourier transform[8], we need to approximate the integral in (3.18) so that a discrete fractional Fourier transform can be obtained. This is achieved by using a numerical integration scheme and then re-write the sum hence obtained in a manner that the FFrFT can be applied. Using the extended trapezoidal rule yields :

$$\int_{0}^{\infty} e^{-iuk}\,\varphi(u)\,du \approx \sum_{s=1}^{N} e^{-i(s-1)\delta_s k}\varphi(u_s)\,\delta_s\big(\frac{1}{2}\mathbf{1}_{s=1;N} + \mathbf{1}_{s=2,\ldots,N-1}\big) \tag{3.19}$$

where δ_s are evenly spaced points (pertained to the characteristic function) and $\mathbf{1}$ is the indicator function. Furthermore, k has also equidistant spacing grids ς ; $k_l = -\dfrac{N\varsigma}{2} + \varsigma(l-$

[8]One of the advantages of this method over the fast Fourier transform is that it works with less restrictions on the parameters.

1).
Hence (3.19) becomes :

$$\int_0^\infty e^{-iuk}\,\varphi(u)\,du \approx \sum_{s=1}^{N} e^{-i(s-1)\delta_s(-\frac{N\varsigma}{2}+\varsigma(l-1))}\varphi(u_s)\,\delta_s\big(\frac{1}{2}\mathbf{1}_{s=1;N}+\mathbf{1}_{s=2,\ldots,N-1}\big)$$

Defining $x_s = e^{i(s-1)\delta_s\frac{N\varsigma}{2}}\varphi(u_s)\,\delta_s\big(\frac{1}{2}\mathbf{1}_{s=1;N}+\mathbf{1}_{s=2,\ldots,N-1}\big)$, we obtain :

$$\Pi(k) \approx \frac{e^{-\gamma k_l}}{\pi}\sum_{s=1}^{N} e^{-i(s-1)\delta_s\varsigma(l-1)}x_s \tag{3.20}$$

The discrete fractional Fourier transform[9] (hereafter DFrFT) of order[10] α has the following form :

$$\mathcal{F}_l(x,\alpha) = \sum_{s=1}^{N} e^{-2\pi i(s-1)(l-1)\alpha}x_s$$

So it suffices that one chooses the values of δ_s and ς independently[11] and then recover the value of α through the relation $\alpha = \dfrac{\varsigma\delta_s}{2\pi}$ to transform the sum in (3.20) to a DFrFT. Therefore a fast algorithm can be used to compute the sum obtained.

Following Bailey and Swarztrauber (1991, 1993) and Chourdakis (2005), this is achieved as follows :

$$\begin{aligned}\mathcal{F}_l(x,\alpha) &= \sum_{s=1}^{N} e^{-2\pi i(s-1)(l-1)\alpha}x_s\\ &= e^{-\pi i(l-1)^2\alpha}\sum_{s=1}^{N} e^{-\pi i(s-1)^2\alpha}\,e^{\pi i(l-s)^2\alpha}\,x_s\\ &= e^{-\pi i(l-1)^2\alpha}\sum_{s=1}^{N} a_s\,b_{k-s}\\ &= e^{-\pi i(l-1)^2\alpha}\mathcal{F}_l^{-1}\big(\mathcal{F}_l(a)\mathcal{F}_l(b)\big)\end{aligned} \tag{3.21}$$

where $\big(\mathcal{F}_l(a)\mathcal{F}_l(b)\big)$ is an element-by-element multiplication.

[9]The DFrFT is a generalization of the discrete Fourier transform (See Bailey and Swarztrauber (1991))
[10]No restriction on the value of the parameter α i.e. it can be real or complex
[11]In the discrete Fourier transform case these two quantities are linked and hence cannot be chosen freely

3.4 Monte Carlo Simulation

The forward swap rates are simulated under the terminal measure. Each forward swap rate process then satisfies the following SDE :

$$dS^{i,n}(t) = S^{i,n}(t)\,\sqrt{V(t)}\,\sigma^{i,n}(t)\left[dW^n(t) - \sqrt{V(t)}\sum_{j=i+1}^{n-1}\frac{\delta\,S^{j,n}(t)\,\sigma^{j,n}(t)}{1+\delta\,S^{j,n}(t)}\,\frac{\tau_{il}(t)}{\tau_i(t)}\,dt\right] \tag{3.22}$$

with

$$\tau_{il}(t) = \delta\sum_{j=i}^{n-1}\prod_{k=l+1}^{j}\left(1+\delta\,S^{k,n}(t)\right) \qquad\qquad i < l \leq n-1$$

As one can notice from Eq.(3.22), discretizing the drift is very challenging. To overcome this feature, I discretize the swap rates as in Glasserman and Zhao (2000).
This is achieved as follows : Since τ_i is a $\mathbf{Q^n}$-martingale, the process Y_{i-1} defined by $\delta\,Y_{i-1} = \tau_{i-1} - \tau_i - \delta$ $(i = 1, \ldots, n-1)$ is also a martingale under the terminal measure.

The dynamics of Y_i is

$$\frac{dY_i(t)}{Y_i(t)} = \sqrt{V(t)}\,\sigma_i^Y\,dW^n(t) \tag{3.23}$$

where

$$\sigma_i^Y = \sigma^{i+1,n} + \frac{\delta}{\tau_{i+1}}\left[\sum_{j=i+2}^{n-1}\sigma^{j,n}\,Y_{j-1}\prod_{k=i+2}^{j-1}\frac{\tau_{k-1}-\delta}{\tau_k}\right]$$

Once[12] Y_i at time $(t+\Delta t)$ is determined, I can obtain the swap rate $S^{i,n}$ at $(t+\Delta t)$ using the relationship $S^{i,n} = \dfrac{Y_{i-1}}{\tau_i}$, with $\tau_i = 1 + \delta\,[n-1-i+\sum_{j=i}^{n-2} Y_j]$. Also from (2.4), $\tau_0 = \delta + (1+\delta\,S^{1,n})\,\tau_1$.

To implement the square-root process (Eq.(3.2)), a moment-matching discretization scheme[13] for the volatility is used.
Hence, paths for the volatility process are computed as follows :

$$V(T_{k+1}) = \left(\theta + \left(V(T_k) - \theta\right)e^{-\kappa\,(T_{k+1}-T_k)}\right)e^{-\frac{1}{2}\Gamma(T_k)^2 + \Gamma(T_k)\nu_k} \tag{3.24}$$

[12] $i = n-2, \ldots, 0$

[13] See Andersen and Brotherton-Ratcliffe (2001)

Where

$$\Gamma(T_k)^2 = \log\left[1 + \frac{\frac{\eta^2}{2\kappa}\left(2V(T_k)\left[e^{-\kappa\Delta t} - e^{-2\kappa\Delta t}\right] + \theta\left[1 - e^{-\kappa\Delta t}\right]^2\right)}{\left(\theta + e^{-\kappa\,\Delta t}(V(T_k) - \theta)\right)^2}\right]$$

and ν_k, $k = 1, 2, ..., n-1$ are independent standard normal random variables.
The simulation algorithm is built in the following way :

i. Generate P paths for the volatility process as in (3.24)

ii. For each path $p = 1, \ldots, P$ simulate M paths of the swap rates via the processes Y and compute an average price of the swaption.

iii. The price of the swaption is the average of over P prices generated in (ii.)

3.5 Numerical results

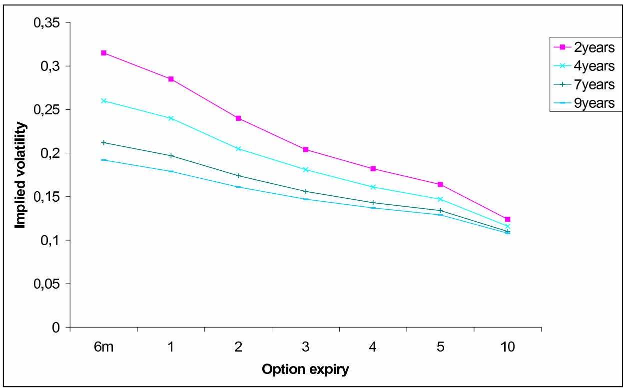
To price European swaptions I implement a one factor version of the stochastic volatility SMM. This low dimensional choice is motivated by the fact that there is empirical evidence (see Driessen et al. (2003) for example) that high pricing performance can be achieved with as few as a one-factor model.
In addition, an examination of the data introduced at the beginning of section (3) shows that the implied volatilities exhibit a decreasing pattern both in the option maturity and in the swap period (see figure (3).). The following volatility structure guarantees this feature : $\sigma^{i,n}(T_k) = 0.187e^{-0.083(i-k)}$. The discount factors used for the calculation are reported in table (1). I also set $\delta = 1$.

For the stochastic volatility dynamics, the parameters used are $V(0) = \theta = \kappa = 1$ and $\eta = 1.5$. Thousands of Monte Carlo simulations ($M = 100000$ and $P = 512$) with antithetic variates are used to obtain the prices of swaptions across strikes. Figure (4) plots the results which indicate that the prices depend on both the strike level as well as on the time-to-maturity (option expiry). Hence the model can confidently account for the smile (and/or skew) present in the swaption market. Applying the fast fractional Fourier transform has several advantages as will be shown below. Let's first say a word on the flexibility of this method over the fast Fourier transform. Both numerical methods aim at computing, in a fast way, the sum (and hence the integral) in Eq. (3.20). For the FFT, one has to decide on the choice of the parameters : the number of points N

Maturity	Discount factors
0	1
1	0.978883539
2	0.949155972
3	0.915053829
4	0.877723523
5	0.838785605
6	0.799090969
7	0.759352775
8	0.720508249
9	0.682892585
10	0.64643697

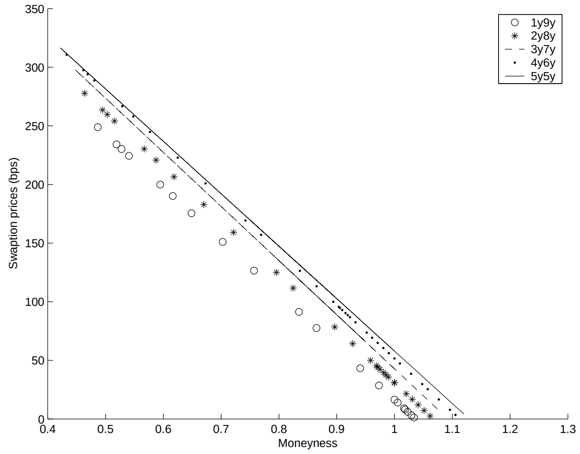
The discount factors used to perform the calculations for the simulation section.

TAB. 1 – **Discount factors**



This figure plots European swaptions implied volatilities across option expiries for different swap periods.

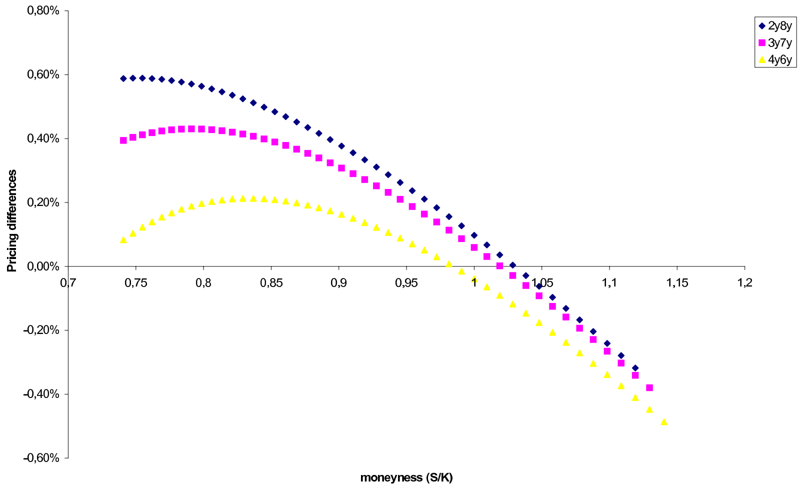
FIG. 3 – **Market implied volatility patterns.**



European swaption prices obtained by Monte Carlo simulation. Moneyness is computed as the ratio of the forward swap rate to the strike rate.

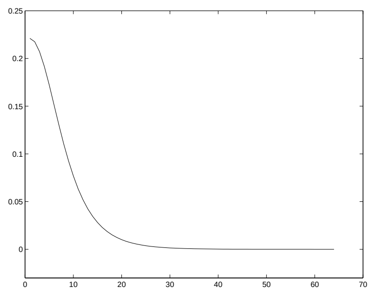
FIG. 4 – **Prices of European swaptions.**

and the integration grid to imply the log-strike grid. Therefore, getting a small log-strike grid hinges on the choice of a big integration grid which may attenuate the accuracy of the overall results. Thus, from a practical point of view this method turns out to be less appealing than the FFrFT since under the latter a free choice of the parameters is made possible. One can choose independently the values of N, δ_s and ς. Hence, the model is easily implemented. In this setting, extensive tests have been carried out : a $\delta_s = 0.2$ combined with a 64-point FFrFT yield very satisfactory results as shown in figure (5). One can notice that the difference between Monte Carlo prices and FFrFT prices is very small for at-the-money options. As we move away this difference increases but still within a reasonable and acceptable interval (less than 1%). Figure (6) shows that the choice of δ_s is appropriate since the real and imaginary parts of $\varphi(u)$ for a given swaption are well under 10^{-4}. Lee (2004) discusses various others conditions for the choice of δ_s. These results are obtained with $\gamma = 3$. The choice of the value of γ turns out to be very crucial since for values $\gamma \leq 2$ I have obtained poor results (more than 1% difference) especially for long maturity options. Finally, the computational speed is very high : On a Pentium4 3Ghz, the execution time for a single price obtained with Monte Carlo method is 126.90 seconds, whereas, using FFrFT to obtain 64 prices lasts much less than one second (0.23

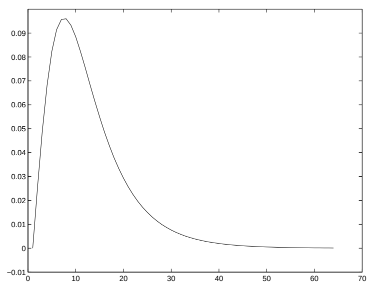


Pricing differences between the FFrFT and the Monte Carlo method.

FIG. 5 – FFrFT VS. MC



(a) Real part



(b) Imaginary part

The real and imaginary parts of 64-point FFrFT combined with an integration grid of 0.2.

FIG. 6 – **5y5y swaption**

second).

4 Calibration

Calibration is very important in financial modeling. However, when one uses a stochastic volatility model the calibration procedure becomes very time consuming if one resorts to Monte Carlo simulations. As I have shown in the previous section, the flexibility, speed and accuracy of the FFrFT makes it very appealing to be applied to the calibration phase. This section describes and discusses the calibration methodology to be employed in the stochastic volatility SMM setting. First I assume a functional form of the swap rates instantaneous volatility structure. The chosen form has to meet the following empirical evidence : the volatility decreases with long time-to-maturity option and with large swap periods[14]. This is ensured by taking a modified form of the structure used in the previous section, i.e. $g(T_i - t) = ae^{-b(T_i - t)} + d$. A perfect calibration of the volatility's parameters is achieved when scaling factors, $\beta_i(t)$, are introduced. Hence,

$$\sigma^{i,n}(t) = \beta_i(t)\, g(T_i - t) \tag{4.1}$$

The $\beta_i(t)$ have to be as close as possible to unity. As one can notice, the volatility structure does not depend on the length of the swap period $T_n - T_i$. Therefore, seperate calibration can be performed for each swap period $(1, 2, \ldots, 9$ years). This procedure is followed, for instance, in Galluccio et al. (2004) and De Jong et al. (2001). However, one can still calibrate the whole swaption volatility matrix by making the coefficient a in (4.1) decreasing in the swap period $(T_n - T_i)$:

Proposition 2. *A swap rate instantaneous volatility is decreasing with swaption expiry and swap period. The functional form below meets this feature and ensures perfect calibration to market data.*

$$\sigma^{i,n}(t) = \beta_i\, f(T_n - T_i, T_i - t) \tag{4.2}$$

[14]In the data considered in this paper as few as 2% of the volatility shapes exhibit a "hump". This is different form the cap market where a hump at around two years is much more frequent.

with

$$f(T_n - T_i, T_i - t) = a\,(T_n - T_i)^{-\frac{1}{2}}\, e^{-b(T_i - t)} + c \tag{4.3}$$

where a, b and c are positive constants.

This parametric form allows to recover the desired features of the market volatility, specifically, time-homogeneity and a decreasing structure both in option expiry and swap length, without using additional parameters with regard to swap period specific calibration procedure.

Calibration of the stochastic volatility SMM can be carried out in a two-step procedure. First the parametric instantaneous volatilities are calibrated as if the smile does not exist. And second, using the obtained instantaneous volatilities, the calibration for the multiplicative factor's parameters minimizes the sum of pricing errors between the model and market prices, namely

$$\min_{\vartheta} \sum \left[C(T_i, T_n, \sigma^{i,n}, K_i, \vartheta) - C^{market}(T_i, T_n, \sigma^{i,n}_{Black}, K_i) \right]^2 \tag{4.4}$$

with $\vartheta = (V0, \eta, \kappa, \theta)$. $C(T_i, T_n, \sigma^{i,n}, K_i, \vartheta)$ and $C^{market}(T_i, T_n, \sigma^{i,n}_{Black}, K_i)$ are the model and market prices of European swaptions, respectively. K is the strike rate.
The main advantage of this methodology, besides not using a constrained optimization procedure, is that it avoids over-parametrization which may cause an undesired over-fitting. Specifically, this two-stage calibration uses at each step as few as three or four parameters comparing to seven free parameters in a global minimization procedure.

I propose in the following to calibrate, using the FFrFT, the stochastic volatility SMM to a set of market data. The data used here consist of forward swap rates and at-the-money implied volatilities for swaptions which total maturities (option expiry + swap length) are equal to or less than 10 years. Table (2) shows the scaling factors obtained from the calibration of a whole swaption matrix. Apart from the short swap period (1 year) where the scaling factors are quite far from unity, although centered around a single value, all the remaining scaling factors satisfy the requirement. The calibrated parameters of the swap rate instantaneous volatilities are given in table (3).

Expiry	Swap period								
	1	2	3	4	5	6	7	8	9
1	0,80643	0,95017	0,99647	1,01676	1,02446	1,02783	1,01139	1,01352	1,00820
2	0,79691	0,93334	0,99083	1,01240	1,02560	0,99818	1,03042	1,03358	
3	0,7697	0,91613	0,99755	1,01532	1,01894	1,03367	1,04255		
4	0,74573	0,90730	0,98689	1,00301	1,00825	1,02900			
5	0,73226	0,89492	0,97558	0,99363	1,00412				

Scaling factors obtained from the calibration of a swaption matrix to market data on 05-21-2003.

TAB. 2 – **Scaling factors**

a	0,4546
b	0,5336
c	0,0514

TAB. 3 – **Fitted Swap rate instantaneous volatility parameters**

Table (4) gives the fitted parameters for the stochastic volatility SMM and figure (7) plots the pricing errors in basis points (bps) across swap periods for different option expiries.

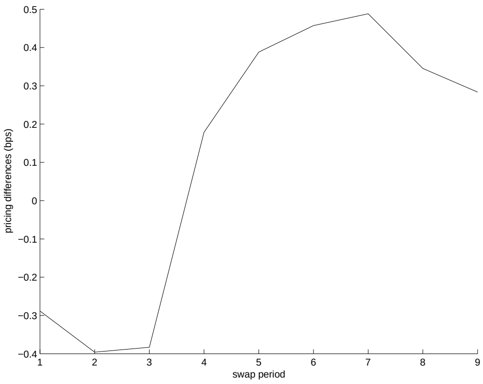
κ	0.028472
θ	2.9098
η	0.63347
$V(0)$	1.0084

Fitted parameters are obtained by minimizing mean squared swaption prices differences. All the options with total maturities $\leq$ 10 years are used for the calibration.

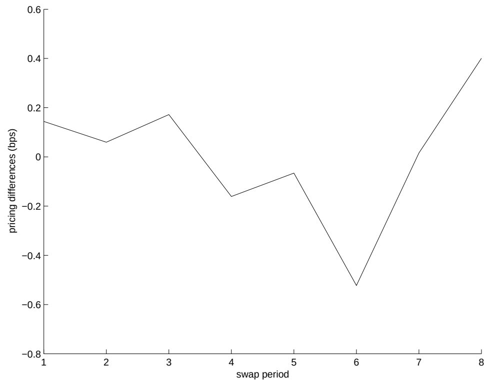
Tab. 4 – **The Stochastic Volatility SMM calibrated parameters on 05/21/2003**

5 Conclusion

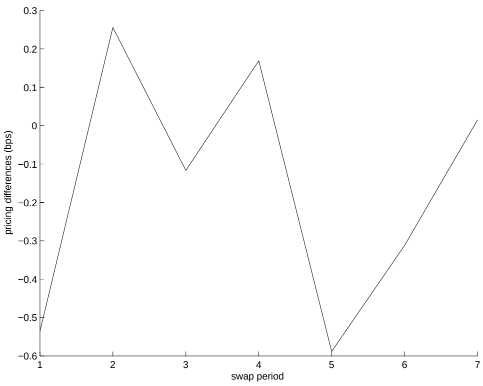
This paper develops a stochastic volatility extension of the Swap market model. In this setting all swap rates volatilities are subject to a common stochastic multiplicative factor that follows a square-root process. Empirical insight for such a model choice has been provided. Furthermore, since the accuracy of Monte Carlo simulations is achieved at the expense of speed I assess the performance of the fast fractional Fourier transform and employ it to calibrate the model. The calibration methodology is enhanced by means of a specific form of the instantaneous swap rate volatility that depends on both the option time-to-maturity and the swap length. A future line of research may assess the pricing and hedging performance of the model.



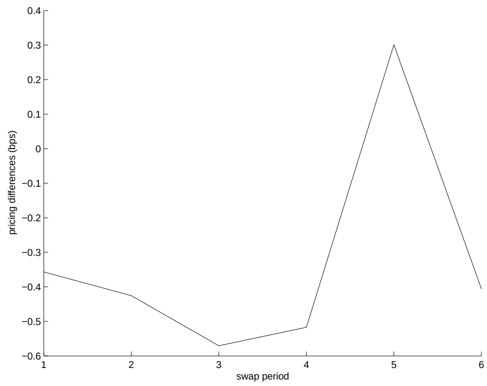
(a) 1 year expiry



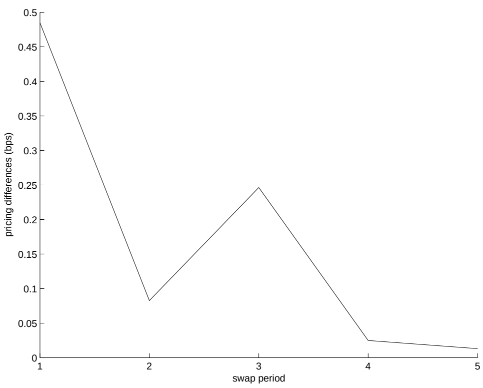
(b) 2 years expiry



(c) 3 years expiry



(d) 4 years expiry



(e) 5 years expiry

FIG. 7 – **Pricing differences**

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