View Bias towards Ambiguity, Expectile CAPM and the Anomalies[☆,☆☆]

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Abstract

Information ambiguity introduces view bias. By defining this view bias, we develop a novel reward-risk measurement framework, an extended CAPM a sequence of empirical test procedures to explain asset pricing anomalies. U.S. stock market data (1926-1999) implies a pessimistic view on average for people with rational risk preference; that explains the equity premium puzzle. The extended CAPM still admits a single beta representation. The amount of risk becomes the weighted average of systematic risk and latent risk. The price of risk, or the expected market excess return, is adjusted by view bias. The momentum effect has two alternative explanations within this framework. Either the winner has a low systematic risk but a high latent risk, and the adjusted price of risk is positive; or the winner has a low systematic risk and a low amount of risk (a weighted average of systematic risk and latent risk), but the adjusted price of risk is negative. Post-war U.S. data supports the latter explanation.

Keywords:

Behavioural Finance, Ambiguity asset pricing, Equity premium puzzle, Momentum, Reward-risk measure, View bias

EFM Classification code: 310, 320

[☆] We thank Siobhan Austen, Rong Chen, Miaoxin Chen, Adrian Cheung, Robert Durand, Patricia Fraser, Paskalis Glabadanidis, William Greene, Bin Li, Hai Lin, Subhrendu Rath, Jianfeng Shen, K.C. John Wei, and seminar participants at Xiamen University, Curtin University, and the University of Adelaide. We thank Ellen Young for minor editorial work.

[☆☆] We acknowledge support from the National Natural Science Foundation of China (Grant Nos. 70971114 and 71101121).

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1. Introduction

Theorists develop models with testable predictions; empirical researchers document puzzles stylised facts that fail to fit established theories and this stimulates the development of new theories. Such a process is part of the normal development of any science.

– Campbell (2000), Asset Pricing at the Millennium

Fundamental asset pricing theory established an asset pricing framework, that is, the asset price of any contingent claim is the expectation of its terminal payoff discounted by a stochastic discount factor (SDF). However, the actual data cannot reconcile with what the theory suggests, either from the mean and variance of stochastic discount factor perspective[1], e.g., the equity premium puzzle, risk-free rate puzzle, and correlation puzzle, etc., or from the factor structure of SDF perspective[2], e.g., size effect, value effect, and momentum. The main purpose of this paper is to generalise the above asset pricing framework to make the actual data reconcile with this novel theory. Specifically we first introduce a new coherent risk-reward measurement framework reflecting people's view bias towards information ambiguity, and then based on which, we derive a generalised CAPM theory. We also develop a corresponding econometrics methodology suitable for the new risk-reward measurement framework, and finally we provide the empirical evidence on the advantage on explaining the anomalies.

Since the anomalies are against the basic asset pricing equation, which is the first order condition of rational investor's portfolio optimization problem, hence the optimization

[1]Equity premium puzzle first documented by Mehra and Prescott (1985). In Cochrane (2001), over the last 50 years in the U.S. real stock returns have averaged 9% with a standard deviation of about 16%, while the real return on treasury bills has been about 1%. Thus, the historical annual market Sharpe ratio has been about 0.5. Aggregate consumption growth has been about 1%. Thus, we can only reconcile these facts with the theory, if investors have a risk aversion coefficient of 50! However the experimental data shows the risk aversion coefficient for normal person is between 1 and 4. Also see Campbell (2000) for the survey paper of asset pricing theoretical development and empirical anomalies. An equivalent anomaly, risk free rate puzzle, first documented by Weil (1989).

[2]Size effect in US stock returns was first documented by Banz (1981), debates on whether size premium is a compensation for systematic risk is ongoing. Momentum first documented by Jegadeesh and Titman (1993), which shows that past winners continue to outperform the past losers, while the beta estimate for the winner portfolio is even lower. Fama and French (1996) find that among several CAPM anomalies, momentum is the only one unexplained by the three-factor model. Also See Cochrane (2001) for the discussion of asset pricing anomalies.

problem itself should be the cradle land of the research. The optimization problem can be described as follows: By choosing the optimal consumption and the weight of each individual security within the portfolio, the investor is to maximize the expectation of the integration of the utility generated by the consumption of whole life as well as a utility generated by the lump sum terminal wealth, with budget equation and security price dynamics as the restrictions. However sometimes, e.g. in the post second world war, people only know the possible outcomes of an uncertainty, and they do not know the exact probabilities of each state, but just have a vague assessment[3]. Then for the investor, the expectation is not obtainable. Hence the pessimistic investor will take the maxmin strategy and optimistic investor will take the maxmax strategy. Table 1 shows us an example.

Table 1

Decision making under perfect and imperfect information

	Strategy X				Strategy Y			
Rate of return	1%	3%	50%	100%	-1000%	3%	50%	100%
Probability	5%	15%	70%	10%	0%	10%	80%	10%
Summary	Mean=45.5%, VAR=0.068				Mean=50.29%, VAR=0.048			

Assume a risk averse investor is facing the risk under perfect information of the probability of each state, he will choose Strategy Y, because Y is generating a higher expectation of the rate of return (50.29%) than Strategy X (45.5%), with a lower variance of the rate of return (0.048) than Strategy Y (0.068). However if the investor cannot observe the probability, by comparing the state payoff sets and following the maxmin priciple, a pessimistic investor will choose Strategy X, because he will first anticipates that the worst state of each strategy will happen (State S1 for both), then select a strategy with a better result. In reverse, the optimistic investor will anticipate the best state will always happen, then select a strategy with a better result. In contrast to the risk preference, we

[3] Knight (1921) distinguishes knowing the outcomes without aware of the corresponding possibility as uncertainty from risk.

name the pessimism and optimism[4] the view bias of a rational investor facing the imperfect information. Not necessarily a risk-averse investor is pessimistic or a risk taking investor is optimistic. A risk averse investor can be either pessimistic or optimistic due to their opinion on the circumstances when the probability is not observable. We find the following stylized fact in Section 5: if the market return and consumption growth move in the same direction, a risk-averse investor is more like a pessimistic investor; if the market return and consumption growth move in a counter-direction, a risk-averse investor is more like an optimistic investor, which is against Bossaerts et al. (2010)'s finding that those two are positively correlated.

From the above example, we can see the essential conceptual difference between the risk preference and the view bias. First, risk preference exists no matter information is perfect or not, however the view bias deviates from neutral when information is imperfect. Second, risk preference reflects people's character, which is nature born, and quite stable during the whole life, hence it does make sense to get an experimental estimate for risk-averse coefficient for normal person; while view bias reflects people's attitude toward the information ambiguity, and it is quite variable. It is highly possible that for the same person, his view bias changes dramatically from pessimistic to optimistic within one day. Dow Industrials declined 6.54 percent (the seventh largest in the postwar period) on Sep., 26, 1955 due to President Eisenhower's heart attack, and the decline lasted for a couple of months. People couldn't adjust their believes on the stock return distribution pattern in a timely manner, hence although he expected that the stock price will fall, he didn't realised that the stock price will fall that much. In other words, he relatively amplified the good state's probability and technically possessed an optimistic view bias, although in real life he is not optimistic at all[5].

Then what if the extreme states of each strategy are all the same? E.g., the worst stock

[4] Abel (2002) shows that pessimism and doubt in the subjective distribution of the growth rate of consumption reduce the equity premium puzzle. Also see Guidolin (2006).

[5] The monthly historical view bias calculated based on our model shows that from Jan. 1955 to Aug. 1955, the aggregated view bias are all less than 0.5, indicating a pessimistic view; the view bias in Sep. and Otc. are both above 0.5, indicating an optimistic view. See Siegel (2008) for the event.

price is always zero, the worst rate of return is always negative infinity, etc.? That's the reason why many researches have be done on using quantile to measure the reward of the rate of return instead of using the extreme worst outcome. The logic is that even though the worst outcomes for each strategy are all the same, their 5 percent quantile might be different enough for us to make a choice. However there are two reasons why we don't like the quantile. First, quantile is not a sufficient statistics, and making decision based on quantile, is a waste of information. Second, using quantile as a reward measurement, is not directing us to a nice concise SDF pricing framework, which is integrating all the asset pricing problems on a contingent claim level. Without that framework, the expectation-conceptual-driven martingale theory is not available any more, the elegant risk neutral valuation option pricing theory collapses.

The above concerns motive us to establish an axiomatic reward-risk measurement framework by introducing a novel concept of expectile which takes the merits of expectation and quantile both, without affording the drawbacks of either. Our goal is that the expectile we develop in this paper should be able to reflect the imperfect information, capture people's view bias towards the information ambiguity, without losing the concise integrated asset pricing framework, and eventually solves two categories of anomalies within that particular framework. Our paper makes the following methodological and empirical contributions to the existing literature.

First, we define the expectile as an argument minimizing the weighted (by view bias) mean squared deviation from that particular argument. The logic is as follows, the median of a random variable is an argument minimizing the mean absolute deviation from that particular argument. The quantile is an argument minimizing the weighted mean absolute deviation from that particular argument. The expectation is an argument minimizing the mean squared deviation from that particular argument. Hence in an induction manner, it is quite natural to define the expectile as above. By doing so, the measure of risk, named as variancile by us, is being defined simultaneously as the weighted mean squared deviation from the expectile. In order to make sure the expectile and variancile are suitable for measuring reward and risk, we needs to check if the coherency of reward (risk), namely the additivity (sub-additivity), homogeneity and risk-free condition are satisfied. Among those conditions, only the additivity of the reward measure is not satisfied. However that is not violating any pricing principle, actually that extends the traditional law of one price under perfect information as follows: Imperfect information may induce instantaneous profit by repackaging the portfolios. The premium is not to compensate for bearing information ambiguity risk, but being awarded for providing more information. We differentiate those two types premium as the first and second category information premium respectively.

Second, we revise the expectation utility maximization axiom into an expectile utility maximization axiom[6]. We redo Merton Problem under the expectile framework, and extend the CAPM theory. The main result is that the expectile of the rate of return, which is view bias adjusted, can still be described as a single beta representation, except that the beta is now the weighted average of systematic risk and latent risk[7].

$$\mu_i + \frac{\sigma_i}{\sqrt{dt}} - r^f = \frac{\Theta^2 \widetilde{\sigma}_{iM} + \Phi \sigma_{iM}}{\Theta^2 \widetilde{\sigma}_M^2 + \Phi \sigma_M^2} \left(\mu_M + \frac{\sigma_M}{\sqrt{dt}} - r^f \right). \tag{1}$$

Under perfect information, the investor will possess a neutral view bias, Θ will degenerate into zero, and Φ will degenerate into one. Therefore we obtain the traditional CAPM. We explain the equity premium puzzle by using this new theoretical framework, that people possess a pessimistic view bias where there is no perfect information in post-war U.S.; We explain the momentum as follows, the people's view bias will be pendulous to digest the past information and accommodate itself to the new information. Hence there is a cycle for people's view bias reciprocates from pessimism to optimism. Since beta is the weighted average of systematic risk and latent risk, and the weights Θ^2 and Φ are affected by movement of view bias, the beta will surge accordingly. The adjusted

[6] Krausa and Sagi (2006) claim that the agents' preferences with respect to unforeseen contingencies must be non-expected utility.

[7] Latent risk is closely related to the idiosyncratic risk, however their definitions are not exactly the same. We will brief it in section 3.3 in details.

price of the risk ($\mu_M + \frac{\sigma_M}{\sqrt{dt}} - r^f$) will be the same for all portfolios, hence the amount of risk will be the only determinant of portfolio ranking. If $\mu_M + \frac{\sigma_M}{\sqrt{dt}} - r^f > 0$, the momentum appears when the winner portfolio is having a low systematic risk and a high latent risk, while the loser portfolio is having a high systematic risk and a low latent risk. If $\mu_M + \frac{\sigma_M}{\sqrt{dt}} - r^f < 0$, the momentum appears when the winner portfolio is having a low systematic risk and a low amount of risk (weighted average of systematic risk and latent risk), while the loser portfolio is having a high systematic risk and a high amount of risk. If the view bias to reciprocate around neutral symmetrically (asymmetrically), the relative amount of risk, beta, will vary across time with a period being half of (the same long as) the period of view bias. Jacobs and Wang (2004) investigate the importance of idiosyncratic consumption risk for the cross sectional variation in asset returns. Since idiosyncratic risk and latent risk are closing related, our theoretical results support what their empirical finding suggests.

Third, we clarify the necessity of developing new econometrics methodology, an expectile regression to test the extended CAPM theory. We distinguish the expectile regression between ordinary least square regression (OLS), weighted least square regression (WLS), and quantile regression. We establish the expectile regression methodology by listing all the assumptions, finding new estimators, and proving the asymptotic consistency and normality in large sample analysis. We develop the hypothesis testing by the case of conditional homoskedadticity and heteroskedasticty. We estimate and test the expectile based unconditional CAPM theory through the conditional GMM being restricted by view bias distorted linear conditions.

Finally, we demonstrate the advantage of the expectile based asset pricing theory through empirical application, in which we reach dramatically different conclusions on CAPM hypothesis testing. Specifically, we obtain a statistically significant view bias based beta, and there is no evidence that we should reject the null hypothesis that moment conditions are satisfied.

Our paper is related to the ambiguity[8] asset pricing approach. The difference between risk and uncertainty was documented by Knight (1921) and Ellsberg (1961). Ambiguity aversion is a preference for risks over uncertainty. Historical contributions includes Gilboa and Schmeidler (1989)'s "maxmin" model, Ghirardato et al. (2004)'s "α-maxmin" model, Klibanoff et al. (2005)'s "smooth ambiguity" model. Bossaerts et al. (2010) argue that attitudes towards ambiguity are heterogeneous across the population. Ui (2011) study the relationship between limited market participation and the equity premium which is decomposed into the risk premium and the ambiguity premium. The paper establishes rational-expectations equilibrium with an assumption that the ambiguity averse investor evaluates his portfolio in terms of the minimum of expected utility. Araujo et al. (2012) consider the pricing rules of single-period securities markets with finitely many states, and develop an Arrow-Debreu ambiguous state price valuation. Their results, however, explain the anomalies to an experimental level, not to an empirical test level, due to the reason of model complexity and data availability. Their approach loses the integrated SDF pricing framework while making extensions; hence the modeling complexity will be compounded with the problem complexity, and soon becomes unmanageable.

The advantage of our approach is that not only does the expectile take the merits of quantile, which conceptually reflects people's behaviour of amplifying the bad state probability and shrink the good state one if he is pessimistic, and vice versa, when facing the information ambiguity, but also the expectile based asset pricing takes the merits of the expectation framework. Now the pricing equation of any contingent claim is the expectile of terminal payoff discounted by SDF, which can be rewritten into the expectation of the product of the view bias adjustment, SDF and the terminal payoff, moreover technically the view bias adjustment and SDF are separable. That is critical important from the model simplification's perspective, and because of that, the whole theoretical framework can be easily applied to solve more complicated problems, e.g.,

[8]See Hirshleifer (2001) for psychology theoretical support.

considering the heterogeneity, limited market participant, etc. Also the expectile takes the merits of expectation, which has mature and astronomical mathematics knowledge as support. Hence after some modification work, the expectile based asset pricing framework can be applied to the martingale theory to provide explanation to derivatives pricing puzzles. Moreover since the expectation is being widely used in all finance area, e.g., portfolio management, corporate finance, financial risk management, fixed income, etc., it is worth trying to apply the expectile concept to any of them to solve the puzzles in those different areas. For the same reason, expectile approach is superior to many other reward-risk measure approaches, e.g., Bassett et al. (2004)'s Choquet expected utility approach, being formulated as a problem of linear quantile regression, Gourieroux and Liu (2006)'s VaR, Tail-VaR and proportional hazard-distortion risk measure, and Giorgi and Post (2011)'s second-order stochastic dominance measure.

Our paper is also different from the utility or consumption based approaches, the endeavours made on which includes: 1) Loss aversion [See Benartzi and Thaler (1995), Fielding and stracca (2007), Giorgi and Post (2011)]. 2) Disappointment ex post [See Athanasoulis and Sussmann (2007) and Gollier and Muermann (2010)]. 3) State dependent utility [See Danthine et al. (2004), Falato (2009)]. 4) Habit formation [See Meyer and Meyer (2005), Du (2011), and Otrok et al. (2002)]. 5) Interrelation between time preference and risk aversion coefficient [see Kang and Kim (2012)]. 6) Consumption based CAPM [See Parker (2003)]. 7) Markov regime switching of consumption, etc. Although the modification on utility and consumption will enter into the SDF, which might solve both two categories of the anomalies, either from the mean and variance of SDF perspective, or the factor structure of SDF perspective, the utility or consumption based approaches can only solve the equilibrium asset pricing anomalies. However our approach is replacing the key factor expectation into an expectile, and that will help to solve the equilibrium asset pricing anomalies as well as the risk-free arbitrage pricing puzzles. One way to distinguish our paper from all the remaining literature is that our approach has a by-product (expectile regression) to empirically exploit the advantage of the expectile based asset pricing theory, while the econometrics contributions are not necessary for the other approaches.

The rest of the paper is organized as follows. Section 2 establish the axiomatic expectile based reward-risk measurement framework, and check the coherency of the measures. Section 3 revise the expectation utility maximization axiom into the expectile utility maximization axiom, redo Merton problem, extend CAPM theory, and finally explain two categories of asset pricing anomalies. Section 4 brief the necessity of introducing new econometric methodology to test the expectile based theories, list the hypotheses, and discuss the empirical methodologies. Section 5 provide empirical evidence on the advantages of the expectile based model for explaining the asset pricing anomalies. Section 6 conclude. Proofs are in Appendix.

2. Expectile based reward-risk measurement framework

In this section, we first introduce a novel concept of expectile based reward-risk measure framework. We then discuss the seven advantages of using the expectile as a reward measure to reflect people's view bias towards the information ambiguity. We also check the coherency of both the reward measure and risk measure. Finally, we compare the view bias approach and the risk preference approach, and summarise the results in Table **2**.

As what we've discussed in Section 1, we want the new reward measure taking the merits of expectation and quantile both, without affording the drawbacks of either. The probabilistic setting will be as follows: the Brownian motion W will be defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$, and we shall denote by $\mathcal{F}_t$ the $\mathcal{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s); 0 \leqslant s \leqslant t)$. For simplicity, at this stage, we drop time t, and denote $W(t)$ as W, We give the following definition.

Definition 1. (Unconditional Expectile) *Suppose the above probability setting holds, and obviously W is absolute-integrable, i.e., $\int |W| f_W(w)dw < \infty$, the unconditional expectile* *of W is*

$$\mathbb{E}_\theta(W) \triangleq argmin_q \left[(1-\theta)\int_{W<q}(W-q)^2 f_W(w)dw + \theta\int_{W\geqslant q}(W-q)^2 f_W(w)dw\right] \quad (2)$$

where $f_W(\cdot)$ is the probability density function of W, and θ is the view bias coefficient.

Under perfect information, namely if $f_W(\cdot)$ is clearly known to the investor, θ is 50%, which implies view bias neutral. Under imperfect information, namely the investor doubting about $f_W(\cdot)$, if he is possessing a pessimistic view bias, he will add more weight on the bad state probability (i.e., make $(1-\theta) \geqslant 50\%$), and put less weight on the good state probability (i.e., make $\theta < 50\%$); if he is possessing an optimistic view bias, he will do in a reverse way. Hence θ captures the tendency of how people response to the information ambiguity. Optimizing Eq. (2) by taking the first order derivative with respect to (henceforth w.r.t.) q, we equivalently define the unconditional expectile as follows,

$$\mathbb{E}_\theta(W) \triangleq q^* = \int \pi_W(\theta) w f_W(w)dw \quad (3)$$

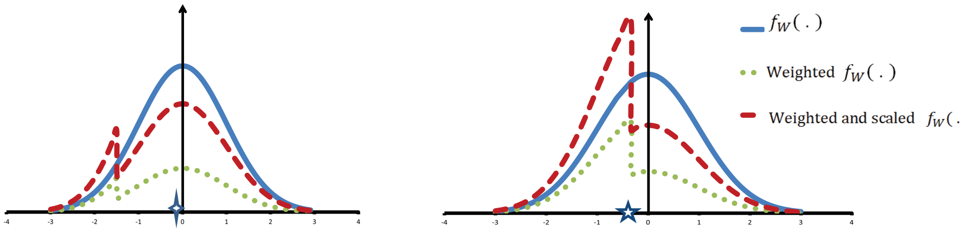
where

$$\pi_W(\theta) = \frac{(1-\theta)\mathbf{1}_{W<q^*} + \theta\mathbf{1}_{W\geqslant q^*}}{\int\left[(1-\theta)\mathbf{1}_{W<q^*} + \theta\mathbf{1}_{W\geqslant q^*}\right] f_W(w)dw} \quad (4)$$

The unconditional expectile $\mathbb{E}_\theta(W)$, (for simplicity, denoted as q^* in Eq. (3) and Eq. (4), appears on both sides of Eq. (3). Hence Eq. (3) is an implicit function w.r.t. $\mathbb{E}_\theta(W)$. The disadvantage is that there is no closed form solution, and we need to numerically solve the equation, but it is for good. However the advantage is of far more meanings.

First, Implicit function format makes the expectile a self-triming measure[9], and that makes the expectile superior to the weighted average. Fig. **1.** provides a graphical illustration of how expectile is obtained, and of the differences between the expectile and the weighted average. The horizontal axis represents the value of W. The curve [— $f_W(\cdot)$] represents the probability density of a standard normal distributed random

[9]We define self-trimming within this context as the weighted average being equal to the weighting division point.



(a) Weightd average

(b) Expectile

Fig. 1.

Comparison of non-selftrimming and selftrimming reward measure. This graph illustrates the conceptual differences between weighted average [see Fig. 1(a)] and expectile [see Fig. 1(b)]. The blue solid curve [— $f_W(\cdot)$] represents the original probability measure of standard normal distribution. The light green dot-dash curve [... weighted $f_W(\cdot)$] represents the view bias weighed measure. The red dash curve [- - weighted and scaled $f_W(\cdot)$] represetns the weighted and scaled probability measure. Cross shaped star in Fig. 1(a) represents the weighted average of standard normal distribution under [- - weighted and scaled $f_W(\cdot)$], where the weight is arbitrarily chosen. Pentacle in Fig. 1(b) represents the weighted average of standard normal distribution under [- - weighted and scaled $f_W(\cdot)$], where the weight is chosen to be to equal to the weighted average itself (pentacle) itself.

variable. We arbitrarily choose a q, and then assign a weight of $(1-\theta)$ to the left hand side of q, and θ to the right hand side of q. We obtain the weighted density $[(1-\theta)\mathbf{1}_{W<q^*}+\theta\mathbf{1}_{W\geqslant q^*}]\,f_W(w)$, see curve [... weighted $f_W(\cdot)$] of Fig. 1(a). In order to make it a probability density, we scaled it by dividing it by the area under curve [... weighted $f_W(\cdot)$], then we have a probability density, $\frac{(1-\theta)\mathbf{1}_{W<q}+\theta\mathbf{1}_{W\geqslant q}}{\int\left[(1-\theta)\mathbf{1}_{W<q}+\theta\mathbf{1}_{W\geqslant q}\right]f_W(w)dw}$, see curve [- - weighted and scaled $f_W(\cdot)$] of Fig. 1(a). With the reshaped probability, we calculate the expectation, which is different from q and marked as a cross-shaped star. We search for a q^*, which will make the expectation under the weighted and scaled probability density the same as q^*, and then define q^* as expectile and mark it as the pentacle in Fig. 1(b).

By comparing Fig. 1(a) and Fig. 1(b), we understand the reason why the implicit-function format makes the expectile a self-triming measure. When we add the weighted average operator onto a random variable, we can capture people's view bias towards the ambiguity, however when we solving more complicated problem, we might have to add the weighted average operator onto another weighted average of a random variable, since the weighting division point is different from the weighted average, the results will become more and more fussy. However, the expectile is defined as the solution of an implicit function, which guarantee the convergence of weighting division point is the expectile itself, hence no matter how complicated the research problem is, and how many layers that the expectile operator needs to be added onto another expectile operator, it will always reach a nice and concise expression, which can fully exploit the essential truth of the problem without showing too much technical bits and pieces being with very limited financial intuitions.

The second advantage of the expectile is that as a tool being used to discover the mystery of asset pricing, it is able to describe the information ambiguity. We use relative entropy (Kukkback-leibler Distance), which is a measure of how different the distorted distribution is from the original one, to quantify the information lost. We find that the expectile models the process where the information is being lost continuously w.r.t. the view bias adjustment. However using quantile to measure the reward of return, the probability information will be lost out of the control of the view bias adjustment. Moreover expectile is a sufficient statistics, which means that calculating the sample-based estimator of expectile will maintain all the information contained in the sample set. However quantile is not a sufficient statistics, and the calculation of sample-based estimator is a waste of the information contained in the sample set. We formalise it as Lemma 1.

Lemma 1. (Relative entropy comparison [10]) $0 \leqslant D(f \| h_{expextile}) < D(f \| h_{quantile}) = +\infty$, *with the first equality if and if* $\theta = 50\%$, *where* $\mathbb{Q}_\theta(x)$ *is the* $\theta-$*quantile of* X, $D(f \| h_{expectile}) = \int f_X(x) ln \left(\frac{f_X(x)}{\pi_X(\theta) f_X(x)} \right) dx$ *and* $D(f \| h_{quantile}) = -\int f_X(x) ln(\frac{1}{\theta} \times \boldsymbol{1}_{\{X < \mathbb{Q}_\theta(x)\}}) dx$.

Proof. See the Appendix.

The third advantage of the expectile framework is that the reward measure and the

[10] $D(f \| h)$ is the relative entropy of distribution h w.r.t f, if they are the same, $D(f \| h)$ is zero, as h deviates from f, $D(f \| h)$ deviates from zero.

risk measure can be defined simultaneously, similar to the expectation and the variance. However the weighted average, with an arbitrarily chosen weighting division point, loses this important character. We defined the new risk measure as follows.

Definition 2. (Unconditional variancile) *Suppose the above probability setting holds, and obviously W is absolute-integrable, i.e., $\int |W| f_W(w)dw < \infty$, the unconditional variancile of W is*

$$VAR_\theta(W) \triangleq \int (W - q^*)^2 \pi_W(\theta) f_W(w) dw \tag{5}$$

where $f_W(.)$ is the probability density function of W, and θ is the view bias coefficient[11].

Proposition 1. (Monotonicity of expectile) *As people goes from extreme pessimism to extreme optimism, his reward measure, expectile, goes from negative infinity to positive infinity monotonously.* (Invariance of variancile) *As people goes from extreme pessimism to extreme optimism, his risk measure, variancile, remains to be unchanged.*

Proof. See the Appendix.

The fourth advantage of expectile framework is formalised as Proposition 1. The Girsanov theory describes how the dynamics of stochastic change when the original measure is changed to an equivalent measure. The Doléans exponential shifts the original distribution density rightward (leftward) when it is changing from the risk averse (risk taking) world into a risk neutral world. The risk preference will not change people's measurement on the amount of risk (i.e., the variance remains to be constant), but will affect the price of the risk, or how much excess rate of return on average that people will ask for being compensated for taking per unit of the market risk. Proposition 1 also describes a change of measure. The $\pi_W(\theta)$ reshapes the original distribution density by amplifying the left-hand side and shrinking the right-hand side of the expectile. Similarly, (*Invariance of variancile*) guarantees that the view bias will not change people's measure on the amount of risk. (*Monotonicity of expectile*) w.r.t. view bias is compatible with the actual behaviour of how people adjust their anticipation on the Brownian motion by

[11]It is easy to prove $VAR_\theta(W) = \mathbb{E}_\theta(W^2) - [\mathbb{E}_\theta(W)]^2$.

taking the risk of information ambiguity into account. The adjustment is being called the first category information premium.

However, based on the stock price Geometric Brownian motion assumption, the rate of return of the stock is a linear function of Brownian motion,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{6}$$

Moreover, the portfolio rate of return is the weighted rate of return of each individual security. Hence in order to do further analysis, we need to investigate the coherency i.e., additivity, homogeneity, and risk-free condition of expectile as a reward measure. We also need to check the coherency, namely sub-additivity, homogeneity, and risk-free condition of variancile as a risk measure. The result is that all of them are satisfied, except the additivity of expectile. See Proposition 2 to 4 as below.

Proposition 2. (Homogeneity of expectile) $\mathbb{E}_\theta(\sigma W) = \sigma\mathbb{E}_\theta(W)$, *where* σ *is a constant;* (Risk-free condition of expectile) $\mathbb{E}_\theta(r + W) = r + \mathbb{E}_\theta(W)$, *where* r *is a constant;* (Additivity of expectile with full information)*If there is no risk-source dimensional receding, i.e., if people knows the joint distribution of individual stock returns within the portfolio, his expectile of the rate of return of the portfolio is equal to the sum of the expectile of each individual security, namely* $\mathbb{E}_\theta(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathbb{E}_\theta(X_i)$, *where* $X_i = r_i + \sigma_i W_i$, *is normal distributed.* (Non-additivity of expectile with information dimensional receding) *If people only knows the distribution density of the rate of return of the portfolio, and he doesn't know the joint distribution of individual stock returns, his expectile of the rate of return of the portfolio is not equal to the sum of the expectile of each individual security.* $\mathbb{E}_\theta(\overline{\sum_{i=1}^n X_i}) = \sum_{i=1}^n \mathbb{E}_\theta(X_i)$, *where* $\overline{\sum_{i=1}^n X_i}$ *represents the rate of return of the portfolio as a one dimensional random variable, rather than the sum of n dimensional random variables.*

Proof. Trivial and omitted.

The fifth advantage of expectile reward-risk measure framework is fully exploited in Proposition 3, based on which we provide policy recommendation to complete the market.

Proposition 3. (Extended law of one price) *Under imperfect information, when portfolios are repackaged, the return remains to be the same, if and only if people are view neutral. If pessimism exists, people can assemble them into a portfolio to do riskless arbitrage; if optimism exists, they can split the package to earn non-zero excess return.*

Proof. See the Appendix.

There is no conflict between this result and the traditional no arbitrage theory, since under this general reward-risk measurement framework, the amount of information being contained in the asset is another factor that influences asset pricing besides time, market risk, and information ambiguity risk. People who earn non-zero excess rate of return through repackaging must know more about the probabilities of each individual security. The premium is not to compensate for bearing the market risk or the information ambiguity risk. It is a reward for providing more information. We name it the second category information premium. Within the general framework of reward-risk measurement, we re-examine the market completeness in contingent claim market, Arrow-Debreu security market and ordinary security market. The market completeness expands itself from the security level to a portfolio level. The policy recommendation is that to ensure each elementary adopted consumption process obtainable, there should be no portfolio repackaging constraints. Getting the second category information premium through repackaging the portfolio can improve the welfare of both two parties, and it is a Pareto equilibrium allocation process.

The sixth advantage of expectile reward-risk measure framework can be briefly described as follows. We've checked the traditional coherency conditions of reward measure, however under imperfect information, the above conditions are not sufficient. As a reward measure, it should reflect the "exparte effect". The story is that a two sons' father worries about his elder son, a fisherman, since tomorrow might be a rainy day, and worries about his younger son, an umbrella seller, since tomorrow might be a sunny day. Under expectation based framework, where risk averse takes effects only, if the expected rate of return of asset X is negative, and there is a perfect negative correlation between asset X and Y, the expected rate of return of asset Y is positive. However, in the above example, the father's reward measure for X and Y are both negative. Hence it is not appropriate to describe the father's behaviour as risk averse, and that effect actually reflects more about his pessimism. As an appropriate reward measure, expectile should be able to capture that meaningful difference. We formalised the "exparte effect" as another coherency condition in Lemma 2.

Lemma 2. (Exparte effect[12]) *Assume X and Y are the rate of returns of two assets. Assume $Y = bX$, then $\mathbb{E}_\theta(Y) = b\mathbb{E}_\theta(X)$, iff $b \geqslant 0$, and $\mathbb{E}_\theta(Y) = b\mathbb{E}_{(1-\theta)}(X)$, iff $b \leqslant 0$.*

Proof. See the Appendix.

Lemma 2 discusses the case of perfect correlation between two assets. Naturally, we are interested in the case where the correlation is not tight. To answer this question, we need to develop an approach to construct two standard normal distributed random variables X and Y with correlation being ρ. We know that under perfect information, Y can always be constructed as $Y_1 = \rho X + \sqrt{1-\rho^2}Z$ or $Y_2 = \rho X - \sqrt{1-\rho^2}Z$, where X,Z are two independent standard normal distributions $N(0,1)$. However under imperfect information, $\mathbb{E}_\theta(\rho X + \sqrt{1-\rho^2}Z) \neq \mathbb{E}_\theta(\rho X - \sqrt{1-\rho^2}Z)$. According to Proposition 2 *(Non-additivity of expectile with information dimensional receding)*, $\mathbb{E}_\theta(\rho X + \sqrt{1-\rho^2}Z)$ and $\mathbb{E}_\theta(\rho X - \sqrt{1-\rho^2}Z)$, the expectiles based on the joint distribution of (X, Z) are both different from $\mathbb{E}_\theta(Y)$, the one dimensional distribution of a single Y. However we should choose the construction of Y_1 or Y_2 to make the expectile based on the joint distribution of (X, Z) and the expectile based on the one dimensional distribution of Y distort the anticipation in the same direction.

Lemma 3. (Construct correlated standard normal distributions[13]) *Assume X,Z are independent $N(0,1)$, and $Y = \rho X + sign(\rho) \times \sqrt{1-\rho^2}Z$, then $Y \sim N(0,1)$, $\rho_{XY} = \rho$,*

[12] Due to the "Exparte effect", it is not always correct to modify the existing expectation based theory to an expectile based theory simply by replacing the expectation into expectile.

[13] $\mathbb{E}_\theta(\rho X + sign(\rho) \times \sqrt{1-{}^2}Z)$ means that the expectile is calculated based on the one dimensional distribution of random variable,Y, where Y is constructed as $\rho X + sign(\rho) \times \sqrt{1-\rho^2}Z$.

where ρ_{XY} is the correlation between X and Y, and $\mathbb{E}_\theta(Y) = \mathbb{E}_\theta(\overline{\rho X + sign(\rho)\sqrt{1-\rho^2}Z})$ and $\mathbb{E}_\theta(\rho X + sign(\rho)\sqrt{1-\rho^2}Z)$ will have the same sign under the same view bias, θ.

Proof. Trivial and omitted.

So far, we've checked the coherency of the reward measure, in order to check the risk measure and to understand how the risk can be diversified through portfolio management under the new measurement framework, we still need to define the following concepts.

Definition 3. (Conditional expectile (linear[14])) *Suppose the probability setting holds, the unconditional expectile of Y w.r.t. X is defined as*

$$\mathbb{E}_\theta(Y|X) \triangleq \underset{g\in A}{argmin}\mathbb{E}_\theta\left[Y - g(X)\right]^2 = \underset{g\in A}{argmin}\mathbb{E}\left[\pi_X(\theta)\pi_{Y|X}(\theta)(Y-g(X))^2\right] \tag{7}$$

where X and Y are random variables (r.v.) with normal distribution, set $A = \{g : \mathcal{R}^2 \mapsto \mathcal{R}|g(x) = \beta_0 + \beta_1 X\}$, and

$$\pi_X(\theta) \triangleq \frac{(1-\theta)\boldsymbol{1}_{\{X<\mathbb{E}_\theta(X)\}} + \theta\boldsymbol{1}_{\{X\geqslant\mathbb{E}_\theta(X)\}}}{\int\left[(1-\theta)\boldsymbol{1}_{\{X<\mathbb{E}_\theta(X)\}} + \theta\boldsymbol{1}_{\{X\geqslant\mathbb{E}_\theta(X)\}}\right]f_X(x)dy} \tag{8}$$

$$\pi_{Y|X}(\theta) \triangleq \frac{(1-\theta)\boldsymbol{1}_{\{sign(\beta_1)(Y-g(X)<0)\}} + \theta\boldsymbol{1}_{\{sign(\beta_1)(Y-g(X)\geqslant 0)\}}}{\int\left[(1-\theta)\boldsymbol{1}_{\{sign(\beta_1)(Y-g(X)<0)\}} + \theta\boldsymbol{1}_{\{sign(\beta_1)(Y-g(X)\geqslant 0)\}}\right]f_{Y|X}(x)dy} \tag{9}$$

Definition 4. (Two dimensional unconditional Expectile) *Suppose the probability setting holds, the conditional expectile of measurable function $h(X,Y)$ is well defined, if $\mathbb{E}\left[\pi_X(\theta)\pi_{Y|X}(\theta)h(X,Y)\right] = \mathbb{E}\left[\pi_Y(\theta)\pi_{X|Y}(\theta)h(X,Y)\right]$, and the conditional expectile is defined as $\mathbb{E}_\theta\left[h(X,Y)\right] = \mathbb{E}\left[\pi_X(\theta)\pi_{Y|X}(\theta)h(X,Y)\right]$.*

Definition 5. (Two dimensional unconditional Covariancile) *Suppose the probability setting hold and then the conditional expectile of $h(X,Y) = XY$ is well defined, where X and Y are random variables with joint normal distribution, then the Covariancile[15] of X and Y is defined as $COV(X,Y) \triangleq \mathbb{E}\left[\pi_Y(\theta)\pi_{X|Y}(\theta)(X-\mathbb{E}_\theta(X))(Y-\mathbb{E}_\theta(Y))\right]$.*

[14] Why is linear relationship sufficient for asset pricing? When we use n-dimensional Geometric Brownian Motion to model the stock price dynamics, the rate of return of the stock is a linear function of the independent increments. Why do we study the expectile on normal distribution or the function of normal distribution? It ensure the extended law of one price holds. Why is the conditional expectile defined as $\mathbb{E}_\theta(Y|X) = b_0 + b_1 X$ good enough for pricing purpose? When we are solving portfolio optimization, expectile on n-dimentional rate of return can be converted into the summation of the expectile of the product of two standard normal random variables.

[15] It is equivalent to $COV_\theta(X,Y) \triangleq \mathbb{E}_\theta(X,Y) - \mathbb{E}_\theta(X)\mathbb{E}_\theta(Y)$.

Proposition 4. (Homogeneity of Variancile) $VAR_\theta(\sigma W) = \sigma^2 \ VAR_\theta(W)$, *where* σ *is a constant;* (Risk-free condition of expectile) $VAR_\theta(r + W) = VAR_\theta(W)$, *where* r *is a constant;* (Sub-additivity of variancile) $VAR_\theta(X + Y) = VAR_\theta(X) + VAR_\theta(Y) + 2 \times COV_\theta(X, Y)$.

Proof. See the Appendix.

The seventh advantage of the expectile framework is that the view bias adjustment, π_θ and the risk averse adjustment, SDF are product separable, which is very important for simplifying the problem. When we apply the above coherent reward-risk measure to the portfolio optimization problem, we get the first order condition (FOC), which is an extended SDF based asset pricing formula,

$$p(g(X)) = \mathbb{E}_\theta(m(X)g(X)) = \int_\Omega \pi_X(\theta)m(X)g(x)f_X(x)dx \tag{10}$$

where X is the rate of return, a normal distributed random variable, $g(X)$ is the payoff of the contingent claim, represented as a measurable function. Assuming the probability setting holds, $m(X)$ is the stochastic discount factor, and $f_X(x)$ is the probability distribution function of X. To conclude Section 2, we compare the view bias approach with the traditional risk preference approach, and summarise the results in Table 2.

3. Expectile based asset pricing model and anomalies

In this section, we first revise the expectation utility maximization axiom into an expectile utility maximization axiom, and then we redo Merton Problem under the expectile framework, and extend the CAPM theory. Finally, we explain the first category asset pricing anomaly, equity premium puzzle, and the second category asset pricing anomaly, momentum within the expectile based reward-risk measurement framework.

3.1. Expectile CAPM with view bias adjustment

The technical difficulty we face in revising the expectation utility maximization axiom into an expectile utility maximization axiom is that so far we just defined the expectile

Table 2

Comparison of risk preference approach and View bias approach

Risk preference approach	View bias approach
Panel A: Similarity	
Change the price of market risk, keep the quantity of market risk unchanged	
Self-trimming measure	
Sufficient statistics	
Define reward measure and risk measure simultaneously	
Risk diversification	
Panel B: Dissimilarity	
Perfect and imperfect information	Imperfect information
Shifts	Reshape
Character	Attitude
Stable	Variable
No exparte effect	Exparte effect
Market risk premium	First & Second category information premium
No instant payoff from repackaging	Instant payoff from repackaging
Panel C: Relationship	
Product separable, $p(g(X)) = \mathbb{E}_\theta(m(X)g(X)) = \int_\Omega \pi_X(\theta)m(X)g(x)f_X(x)dx$	

Table 2 compares the traditional Risk preference approach and newly developed View bias approach. The similarity includes: (1) Both risk preference and view bias adjustments are a change of measure from real probability to subjective probability. (2) As risk preference (view bias) deviates from risk neutral (view neutral), only the reward measure, expectation (expectile) will be adjusted, the value of the risk measure, variance (variancile) remain to unchanged. In other words, both two approaches change the price of the market risk, and either of the approach change the quantity of the market risk. (3) Both expectation and expectile are self-trimming, and that provides the possibility for us to get a nice and concise expression of results. (4) Both expectation and expectile are sufficient statistics, the empirical analysis will never waste the information of sample sets. (5) Both two approaches define the reward measure and risk measure simultaneously. The logic of the framework is internally consistent. (6) The risk can be diversified through portfolio management under both two measurement frameworks. The dissimilarity includes: (1) Risk preference is a concept under both perfect and imperfect information; View bias is a concept under imperfect information only. (2) Risk preference shifts the probability distribution curve; View bias reshapes the probability distribution curve. (3) Risk preference describes people's character. View bias describes people's attitude. (4) Risk preference is stable. It can be obtained through experiments; View bias is variable, it is affected by the status quo. (5) Expectation does not capture the "exparte effect"; Expectile captures the "exparte effect". (6) Risk preference will compensate the investor for taking market risk, and the premium is market risk premium; View bias will compensate the investor for taking information ambiguity risk, and the premiums include the first and second category information premium. (7) Law of one price holds, and repackaging portfolio does not induce non-zero excess return; Extended law of one price holds, and repackaging portfolio induces non-zero excess return. (8) Market completeness is on security level; Market completeness is on portfolio level. The relation between two approaches is that the expectile based asset pricing formula implies that the view bias adjustment and SDF are product-separable.

of a measurable function of maximum two random variables. The optimization problem can be described as follows: By choosing the optimal consumption and weight of each individual security within the portfolio, the investor is to maximize the expectation of the integration of the utility generated by the consumption of the whole life as well as the utility generated by the lump sum terminal wealth, with budget equation and security price dynamics as the restrictions. However the utility of intermediate consumption or the utility of terminal wealth are a function of high dimensional random variables. Fortunately, the essential part of the problem can be eventually converted into an expectile of pairs of the product of two standard normal distributions.

Theorem 1. (Expectile CAPM) *Suppose Assumption A.1 to A. 6 in the Appendix holds. Then the following expectile CAPM representation holds for $\mathbb{E}_\theta(\cdot)$:*

$$\mathbb{E}_\theta(r_i - r_f) = \beta^\theta \mathbb{E}_\theta(r_M - r_f), i = 1, 2, \cdots, n \tag{11}$$

where $\beta^\theta \triangleq \dfrac{\Theta^2 \widetilde{\sigma}_{iM} + \Phi \sigma_{iM}}{\Theta^2 \widetilde{\sigma}^2_M + \Phi \sigma^2_M}$, $\sigma_{iM} \triangleq \sum \varpi_j \sigma_i \sigma_j \rho_{ij}$, $\sigma^2_M \triangleq \sum_{j=1}^n \varpi_j \sigma_M \sigma_j \rho_{ij}$, $\widetilde{\sigma}_{iM} \triangleq \sum_{j=1}^n \varpi_j \sigma_i \sigma_j sign(\rho_{ij}) \sqrt{1-\rho^2_{ij}}$, $\widetilde{\sigma}^2_M \triangleq \sum_{j=1}^n \varpi_j \sigma_M \sigma_j sign(\rho_{jM}) \sqrt{1-\rho^2_{jM}}$, $\Theta \triangleq \mathbb{E}_\theta(w)$, $\Phi \triangleq \mathbb{E}_\theta(w^2)$, ϖ_i *is the weight of security i within the portfolio, and $w \sim N(0, 1)$.*

Proof. See the Appendix.

We convert Eq. (11) into an equivalent expectation based formula as below,

$$\mathbb{E}(r_i) + \frac{\sigma_i}{\sqrt{dt}} - r_f = \beta^\theta (\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}} - r_f), i = 1, 2, \cdots, n \tag{12}$$

The left hand side of the equation is the expected excess rate of return being adjusted by the view bias coefficient to compensate for taking information ambiguity risk. σ_{iM} is the absolute amount of the systematic risk of individual security, σ^2_M is a benchmark, the amount of systematic risk inherent in the market portfolio, $\widetilde{\sigma}_{iM}$ is the absolute amount of latent risk of individual security, $\widetilde{\sigma}^2_M$ is the absolute amount of latent risk inherent in the market portfolio. β^θ is the relative amount of risk, being different from traditional β, it is not the relative amount of systematic risk, but of the weighted average of systematic risk and latent risk, with the weight being $\frac{\Theta^2}{((\Theta^2+\Phi)}$ and $\frac{\Phi}{((\Theta^2+\Phi)}$ respectively. Under perfect information, the investor will possess a neutral view bias, then Θ becomes zero, and Φ becomes one. The extended expectile based CAPM will degenerate into a tradition CAPM. We are able to explain the equity premium puzzle using this new theoretical framework.

3.2. Theoretical explanation for equity premium puzzle

Equity premium puzzle first documented by Mehra and Prescott (1985), also see Cochrane (2001). Over the last 50 years in the U.S. real stock returns have averaged 9% with a standard deviation of about 16%, while the real return on treasury bills has been about 1%. Thus, the historical annual market Sharpe ratio has been about 0.5. Aggregate consumption growth has been about 1%. Thus, we can only reconcile these facts with the theory, if investors have a risk aversion coefficient of 50! If considering the correlation between consumption growth rate and market return, the risk aversion coefficient, risk aversion coefficient has to be 250. However the experimental data shows that the risk aversion coefficient for normal person is between 1 and 4.

Theorem 2. (Equity premium puzzle) *Assume the correlation between consumption growth rate and the market returns is ρ_{CM}, then $\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}}\Theta - r_f = \alpha\sigma_C\sigma_M(\Phi\rho_{CM} + \Theta^2 sign(\rho_{CM})\sqrt{1-\rho_{CM}^2})$, where σ_C is the volatility of the aggregate consumption growth rate and α is the risk averse coefficient.*

Proof. See the Appendix.

Under view bias adjusted expectile framework, with a normal constant risk aversion coefficient 3, we solve a monthly average (1945-1999) of implied view bias coefficient, $\theta = 0.473$, which is just a slight deviation from view neutral. However if we neglect this important factor, the implied risk aversion coefficient will be highly irrational.

3.3. Theoretical explanation for momentum

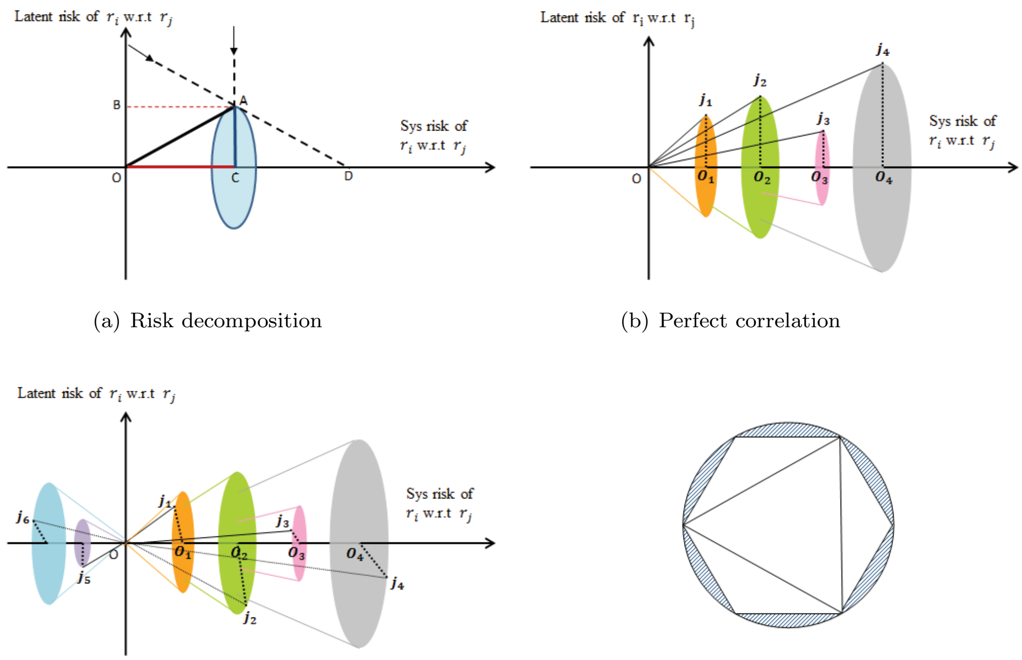
In this subsection, we will explain how the momentum effect is generated. We first discuss the risk decomposition between each pair of the security, σ_{ij} and $\widetilde{\sigma}_{ij}$. Second, we discuss the risk decomposition of security i w.r.t. the market portfolio, and then brief the difference between idiosyncratic risk and latent risk. Third, we develop an approximate estimation of the latent risk $\widetilde{\sigma}_{iM}$ and the benchmark $\widetilde{\sigma}_M^2$ without using the joint distribution of each pair of securities within the portfolio[16]. Fourth we exploit how the view bias towards ambiguity affects β^θ, we then make the hypothesis that view bias vacillates with a periodicity, and prove that if it is the case, the period of β^θ is half of or the same as the period of θ. Finally, we explain the momentum, size effect and value effect based on the above analysis.

The difficulty of estimating β_θ is that while we redo the Merton problem, we solve the optimal weight for each individual security, and then we have a market portfolio, which consists of n individual securities, but according to β_θ's definition, we can only figure out the absolute amount of the systematic risk σ_{iM} (and the benchmark σ_M^2) by studying the relationship between security i and the market portfolio (by studying the rate of return of the market portfolio), however in terms of the latent risk $\widetilde{\sigma}_{iM}$ and benchmark $\widetilde{\sigma}_M^2$, we need to study each pair of the relationship between security i and j, where j varies from 1 to n, and then sum them over.

First, we discuss the risk decomposition between each pair of security. The risk of security i can be decomposed into a risk perfectly correlated with the risk of security j (See Fig. 2(a): OC), i.e. $\sigma_{ij} \triangleq \sigma_i \sigma_j \rho_{ij}$, and a risk being independent to the risk of security j (See OB), i.e. $\widetilde{\sigma}_{ij} \triangleq \sigma_i \sigma_j sign(\rho_{ij})\sqrt{1-\rho_{ij}^2}$. Surprisingly under the view bias based expectile framework, the risk being compensated is the weighted sum of both (See OD), i.e. $\Theta^2 \sigma_i \sigma_j \rho_{ij} + \Phi \sigma_i \sigma_j sign(\rho_{ij})\sqrt{1-\rho_{ij}^2}$. That is different from the traditional risk averse based expectation framework, under which only σ_{ij} will be compensated.

Second, we consider the risk decomposition of security i w.r.t. the market portfolio. As discussed in the last paragraph, the risk of security i can be decomposed into σ_{ij} and $\widetilde{\sigma}_{ij}$, where $j = 1, 2, n$. In Fig. 2(b), σ_{ij} are the vectors from the origin to the center of

[16]According to the definition of the latent risk of security i and market portfolio $\widetilde{\sigma}_{iM}$ and $\widetilde{\sigma}_M^2$, the sample estimation of latent require the joint distribution of each pair of individual securities, which makes the accurate estimation practically impossible. We will discuss it in details later in this section.



(a) Risk decomposition

(b) Perfect correlation

(c) Imperfect correlation

(d) Latent risk by industry

Fig. 2.

Risk decomposition. This graph intuitively explains why we need each pair of the security returns to estimate the latent risk of security i w.r.t the market portfolio, and illustrate why an approximate estimation by industry works. Fig. 2(a) shows not only the projection of on return i, but also the independent component perpendicular to return i should be taken into account, which is considered as the latent risk. The total risk of r_i w.r.t r_j can be decomposed into OB and OC, where OC is the systematic risk, OB is the latent risk). Fig. 2(b) assumes that the securities in the market index are all pefectly correlated. Then the latent risk equals the idyosyncratic risk ($\widetilde{\sigma}_{iM} = \sigma_i \sigma_M \sqrt{1-\rho_{iM}^2}$). Systematic risk ($\sigma_{iM}$) is the sum of distances from origin O to O_l. Latent risk ($\widetilde{\sigma}_{iM}$) is the sum of radius $O_l j_l$ of the circles, where l represents the number of securities in the market index. Fig. 2(c) describes the real world situation that the securities in the market index are not pefectly correlated. Then latent risk does not equal the idyosyncratic risk. Again systematic risk (σ_{iM}) is the sum of distances from origin O to O_l. Latent risk ($\widetilde{\sigma}_{iM}$) is the sum of radius $O_l j_l$. Since the rate of return of each individual security might be negatively correlated within the market index, if we consider the market portfolio as a one dimensional security j^*, and calculate the latent risk of security i w.r.t j^*, by doing so, some latent risks will be canceled out with each other, hence it will wrongly estimate the latent risk of security i w.r.t the market portfolio. See Fig. 2(d), the securities in the same industry tend to have strong positive correlations; hence it makes sense to approximate the latent risk of security i w.r.t the market portfolio by dividing the market portfolio into several sub-portfolios by industry. The accuracy improves as the section becomes finer. The marginal improvement of latent risk estimation accuracy decrease rapidly.

the circles, and $\widetilde{\sigma}_{ij}$ are the radiuses of the circles. Suppose for each pair of securities, the rate of return are all perfectly correlated, then all vectors $(j_1, j_2, \cdots, j_n)$ points at 12 o'clock if $\rho_{ij} \geqslant 0$ (or 6 o'clock if $\rho_{i,j} < 0$) as shown in Fig. 2(b), otherwise the vectors points at disordered directions as in Fig. 2(c). We claim that the following equation holds,

$$\widetilde{\sigma}_{iM} \triangleq \sum_{j=1}^{n} \varpi_j \sigma_i(t)\sigma_j(t) sign(\rho_{ij}(t))\sqrt{1-\rho_{ij}^2(t)} = \sigma_i(t)\sigma_M(t) sign(\rho_{iM}(t))\sqrt{1-\rho_{iM}^2(t)} \tag{13}$$

The left hand side is defined as the latent risk of security i w.r.t. the market portfolio, and right hand side is defined as idiosyncratic risk of security i w.r.t. the market portfolio. The latent risk equals to the idiosyncratic risk, if and only if the individual security in the market portfolio are all perfectly correlated as shown in Fig. 2(b), otherwise the negatively correlated securities tends to offset each other, and the latent risk will distinguish itself from idiosyncratic risk. Hence we cannot estimate $\widetilde{\sigma}_{iM}$ and $\widetilde{\sigma}_M^2$ simply using the aggregated rate of return of the market portfolio, just as what we did in estimate σ_{iM} and σ_M^2. We need the joint distribution of each pair of the securities in the market portfolio. Therefore an accurate estimate of β^θ is practically infeasible.

Third, we develop an approximate estimation of the latent risk $\widetilde{a}_{iM}$ and the benchmark $\widetilde{\sigma}_M^2$. We first divide the market portfolio into several sub-portfolios by category of industry, and then estimate the latent risk $\widetilde{\sigma}_{iM}$ using $\sum_{l=1}^{N} \varpi_l \sigma_i \sigma_l sign(\sigma_{il})\sqrt{1-\rho_{il}^2}$ where N represents the number of industry, σ_l is the volatility of the rate of return of industry l portfolio. The securities in the same industry tend to have strong positive correlations; on the graph, they are falling in the same section. We figure out the latent risk of security i w.r.t. each industry portfolio, and then calculate the weighted average. Through this way, we approximate the latent risk of security i w.r.t. the market portfolio. The accuracy improves as the section becomes finer. (See Fig. 2(d), if the number of industry goes from 3 to 6, the marginal improvement of latent risk estimation accuracy is significant, however if the number of industry goes from 6 to 12, the improvement is no more than the shaded area. We summarize the above fact in Lemma 4.

Lemma 4. (Estimating latent risk using industries' rate of return)

$$\lim_{N\to n}\sum_{m=1}^{N}\varpi_l\sigma_i\sigma_l sign(\sigma_{il})\sqrt{1-\rho_{il}^2}=\widetilde{\sigma}_{iM} \tag{14}$$

Proof. Obvious and omitted.

Fourth, we exploit how the view bias towards ambiguity affects β^θ. If the view bias is neutral i.e., $\theta = 50\%$. The $\Theta^2 = 0$ and $\Phi = 1, \beta^\theta = \frac{\sigma_{iM}}{\sigma_M^2}$, which is the projection of total risk on the market portfolio risk. See Fig. 2(a), intuitively it is the shadow OC of OA on the horizontal axis at 12 o'clock under the sun on top of head. If the view bias is pessimism or optimism, i.e., $\theta \neq 50\%$, then $\Theta^2 > 0$ and $0 < \Phi < 1$, then β^θ is the relative weighted average of systematic risk and latent risk of security i w.r.t. the benchmark, market portfolio. Again, See Fig. 2(a), with a biased view, β^θ is the shadow OD of OA on the vertical axis under the sun early in the morning or at dawn. Under imperfect information, it takes time for people to digest the current and past information, to adjust his view bias, and then to accommodate his view bias to the new information. Hence usually people will overreact to the ambiguity. For example, he starts at taking a too pessimistic view bias and carrying a very conservative trading strategy, but later he realizes that the real situation is much better than what he thought, and he starts to be optimistic to the new coming information. Overreaction will be eliminated if he is reacting to the same information, because he will learn from the past lessons, however since the information he is facing to is progressing, and not stationary, the overreaction can never be eliminated. Therefore it is reasonable to make the hypothesis that view bias vacillates with a periodicity, under which we are able to explain the momentum effect. See Corollary 1 and Corollary 2.

Corollary 1. (Symmetry of beta) $\forall\theta, \beta^\theta = \beta^{1-\theta}$.

Proof. Obvious from the definition of β^θ, Θ, and Φ.

Corollary 2. (Match between periods) *Under* $\mathbb{H}_0$*:* THE VIEW BIAS COEFFICIENT

θ VACILLATES WITH A PERIODICITY, *if the dynamics of θ is symmetric periodic waves around neutral (like a sine wave, a square wave), the period of β^θ is half of the period of θ, otherwise the period of β^θ is equal to the period of θ.*

Proof. Obvious from Corollary 1.

Fifth, we explain the momentum, size effect and value effect based on the above analysis. Momentum first documented by Jegadeesh and Titman (1993), shows that past winners continue to outperform the past losers, while the beta estimated for the winner portfolio is even lower. Fama and French (1996) find that among several CAPM anomalies, momentum is the only one unexplained by the three-factor model. We claim that the real factor causing the momentum effect is the fluctuation of people's view bias, which adjusts the price of the risk (the market excess rate of return) by $\frac{\sigma_M}{\sqrt{dt}}\Theta$, also makes the relative risk amount β^θ fluctuate accordingly either with a period half of or the same length as the period of the view bias, depending on whether or not the dynamic of θ when it is pessimism is a phase lagged mirror reflection to the dynamic of θ when it is optimism. There are two alternative explanations.

Explanation 1. (View bias adjusted beta approach) *After being adjusted by view bias, a high relative amount of risk and a positive market price of risk make the winner portfolio outperform the loser portfolio with a lower beta. Assume that after the view bias adjustment, the price of the risk, $\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Theta - r^f$, is positive. See Fig. 3.(d), the view bias at time t_1 is the extreme optimism, according to the expectile CAPM, $\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Theta - r^f = \beta^\theta(\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Theta - r^f)$, the optimism will adjust the expected rate of return for different stocks in the same direction, by $\frac{\sigma_i}{\sqrt{dt}}\Theta$. We scale the portfolios to make them on a same level of volatility σ, the optimism view bias will shift the expected rate of return by the same amount, and it will not change the rank of the security. However as view bias changes, it will also change the amount of risk, beta, and that will change the ranking of the stock accordingly. At time t_1, see Fig. 3.(a) to (c), their risk amount which determines the excess rate of return is $\frac{\Theta^2\widetilde{\sigma}_{iM}+\Phi\sigma_{iM}}{\Theta^2\widetilde{\sigma}_M^2+\Phi\sigma_M^2}$. If the winner portfolio is more composed of the securities with low systematic risk σ_{iM} and greater amount of latent risk $\widetilde{\sigma}_{iM}$ and the loser portfolio is more composed of the securities with high systematic risk*

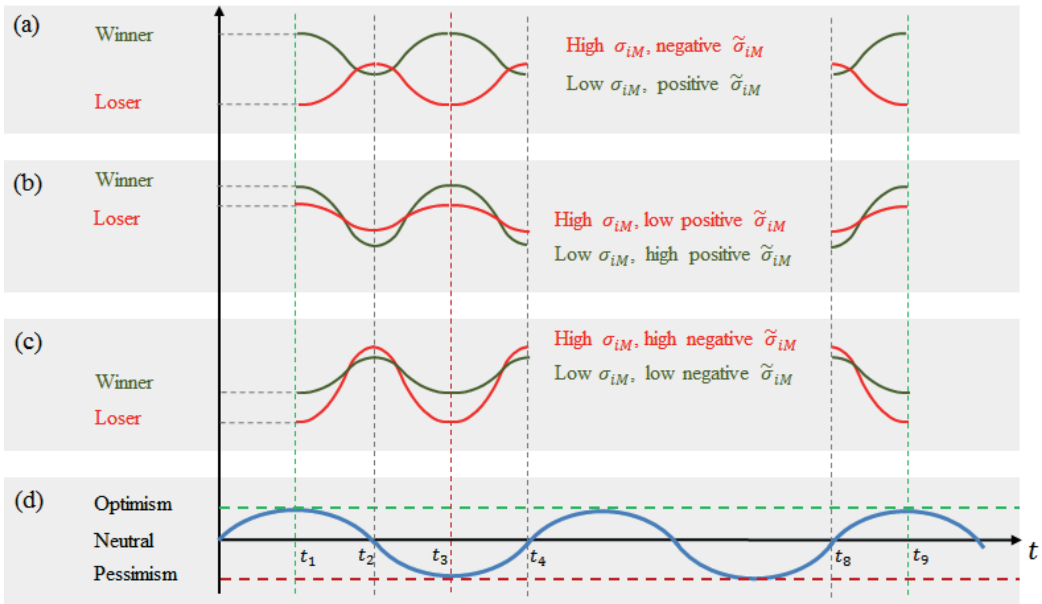


Fig. 3.

Momentum-View bias adjusted beta approach. This graph illustrates how the view bias reciprocation explains the momentum effect. Fig. 3.(d) describes a symmetric view bias fluctuation w.r.t view neutral. Fig. 3.(a) to (c) depict the behaviour of three types of winners and three types of losers under the corresponding view bias at different times. If the winner portfolio is more composed of the securities with low σ_{iM} and high $\widetilde{\sigma}_{iM}$, and the loser portfolio is more composed of the securities with high σ_{iM}, low $\widetilde{\sigma}_{iM}$, then momentum effect appears, namely the past winners continue to outperform the past losers, while the beta $\frac{\sigma_{iM}}{\sigma_M^2}$ estimate for the winner portfolio is even lower, under the conditiona that the view bias adjusted price of risk is positive.

σ_{iM} and less amount of latent risk $\widetilde{\sigma}_{iM}$, then it will explain the momentum effect that past winners continue to outperform the past losers, while the beta $\frac{\sigma_{iM}}{\sigma_M^2}$ estimate for the winner portfolio is even lower. Hence in order to accept the explanation to the momentum effects, we shouldn't find any empirical evidence to against the following five facts. First, the view bias adjusted market execess rate of return is positive. Second, σ_{iM} of the winner portfolio is low, and σ_{iM} of the loser portfolio is high. Third, $\widetilde{\sigma}_{iM}$ of the winner portfolio is greater than $\widetilde{\sigma}_{iM}$ of the loser portfolio. Fourth, the periods of both winner portfolio and loser portfolio are the same, either to be half of or the same length as the period of view bias. Fifth, the time that momentum exists is the time that view bias deviates from neutral most; The time that momentum is being relieved or eliminated is the time that view bias is close to neutral.

Explanation 2. (View bias adjusted market price of risk approach) *After being adjusted by view bias, a low relative amount of risk and a negative market price of risk will make the winner portfolio outperform the loser portfolio with a lower beta. Assume that after the view bias adjustment, the price of the risk,* $\mu_M + \frac{\sigma_M}{\sqrt{dt}}\Theta - r^f$ *is negative, then the greater the relative amount of the risk is, the lower the compensation for taking the risk is, the lower the ranking of the portfolio is. E.g. the market excess rate of return is 5% on average, the adjustment,* $\frac{\sigma_M}{\sqrt{dt}}\Theta$*, is -8% under pessimism view bias. Hence the new price of risk is -3%. Assume view bias adjusted beta for winner portfolio is 0.8, for loser portfolio is 2. then the compensation for taking risk for winner is -2.4%, the compensation for loser is -6%. Assume the view bias adjustment* $\frac{\sigma_i}{\sqrt{dt}}\Theta$ *are the same for both, being 8%, then the excess rate of return for winner portfolio is 5.6%, and 2% for loser portfolio. That explains the momentum effect. Hence in order to accept this alternative explanation to the momentum effects, we shouldn't find any empirical evidence to against the following two facts. First, the view bias adjusted market execess rate of return is negative. Second, the view bias adjusted beta for the winner portfolio is even lower.*

The size effect in US stock returns first documented by Banz (1981), debates on whether size premium is a compensation for systematic risk. Recent studies consider the possibility that liquidity is a priced state variable, and the returns on small stocks are sensitive to this state variable [See van Dijk (2011)]. However Amihud (2002) finds that liquidity risk can only absorb part of the size effect. Another anomaly is the value effect, first documented by Sanjoy (1983). The returns are predicted by the ratios of the market value to the accounting measures such as the earnings or the book value of equity. Our paper explain the size effect and value effect that under imperfect information, the size and the value of a firm is a signal, based on which the investor will extrapolate the past performance and adjust their view bias towards the ambiguity, and then the view bias is the driving force of the distorted pricing.

4. Expectile based econometrics model

In this section, we first brief the necessity of introducing new econometric methodology to test the expectile based theories and hypotheses, and then we find the estimator and study the expectile based law of large number (LLN), and central limit theory (CLT). The small sample properties does not exists, however the large sample properties including consistency and asymptotic normality can be proved. Finally, we employ the conditional GMM restricted by expectile regression condition to empirically test view bias based unconditional expectile CAPM theory.

4.1. Necessity of developing expectile regression

First, there is a one to one correspondence between mathematical model and econometrics model. If a theory is developed under the expectation and variance reward-risk measurement framework, the corresponding econometrics method to test that theory has to be the OLS regression. If a theory is developed under the quantile and absolute deviation reward-risk measurement framework, the corresponding econometrics method to test that particular theory has to be the quantile regression. Hence in an induction manner, intuitively we need a new expectile regression to test the view bias based expectile CAPM.

Second, in particular the model correct specification of expectile based model is $\mathbb{E}_\theta(\epsilon|X) = 0$, and $E_\theta(\epsilon|X) = 0 \Leftrightarrow \mathbb{E}(\epsilon|X) = 0$, iff $\theta = 50\%$, the model is misspecified in an OLS regression point of view. We usually use the instrumental variable regression to solve the problem. However, for a given θ, being different from 50%, $\mathbb{E}_\theta(\epsilon|X) = 0$ implies $\mathbb{E}(\epsilon|X) \neq 0$, but not the reverse. It simply because $\mathbb{E}(\epsilon|X) \neq 0$ contains no information about θ, and obviously we cannot decide whether or not $\mathbb{E}_\theta(\epsilon|X) = 0$ is true. In other words, using instrumental variable method, we neglect the information of θ, the 2SLS estimator would not be efficient.

Third, do we still need an expectile regression, if we can translate the expectile CAPM (Eq. (11)) into an expectation based linear relationship between the rate of return of security i, and the market portfolio (See Eq. (12) as below)?

$$\mathbb{E}(r_i) + \frac{\sigma_i}{\sqrt{dt}} - r_f = \beta^{\theta}(\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}} - r_f)$$

The answer is yes. The reason is that by reviewing OLS regression, the estimator of β for $r_i^e = \alpha + \beta r_M^e$, e.g., (where superscript e represents the excess rate of return), is $\hat{\beta} = \frac{\sigma_{iM}}{\sigma_M^2}$,. Hence calculating portfolio i's beta is no difference from estimating the parameter β of a linear regression. Theoretically we can estimate the parameter of regression by calculating the sample beta of securtity i w.r.t. the market portfolio. However as what we discussed in Section 3.3, in practice it is not feasible, because we need the joint distribution of each pair of the securities within the market portfolio to get the β^{θ}. Hence Eq. (12) is useful to compare itself with the ordinary CAPM to explain the asset pricing anomalies, because both of them are taking the expectation format, but Eq. (12) is not appropriate for parameter estimation purpose.

4.2. Expectile regression model

In this subsection, we develop an expectile regression, including rewriting regression identity, resetting the assumptions, looking for new estimators, developing the expectile based asymptotic tools (WLLN and CLT), proving the consistency and asymptotic normality of expectile regression with i.i.d Observations, We develop asymptotic χ^2 test for the case of Conditional Homoskedasticity and Conditional Heteroskedasticity respectively. Finally, we use randomly generated sample sets to simulate the expectile regression.

We start from a finite sample linear regression with one independent variable only, namely $Y = g(X) + \epsilon$, where $g(X) = \beta_0 + \beta_1 X$, however we cannot find an unbiased estimator. The sampling distribution of the estimated parameter $\hat{\beta}$ is not a normal distribution. Therefore, we focus on large sample properties of expectile regression.

Theorem 3. (Regression identity) *Given $\mathbb{E}_{\theta}(Y|X)$, we can always write*

$$Y = \mathbb{E}_{\theta}(Y|X) + \epsilon \tag{15}$$

where ϵ is called the regression distribution and has the property that

$$\mathbb{E}_\theta(\epsilon|X) = \mathbb{E}(\pi_X \pi_{Y|X} \epsilon|X) \tag{16}$$

The random variable ϵ represents the part of Y that is not captured by $\mathbb{E}_\theta(Y|X)$. It is usually called a noise, while $\mathbb{E}_\theta(Y|X)$ is called the signal. We denote the estimator as $\hat{\beta} = (\hat{\beta}_0, \hat{\beta}_1)'$, the estimate error as $\epsilon_t = Y_t - \hat{g}(X_t)$, where $\hat{g}(X_t) = \hat{\beta}_0 - \hat{\beta}_1 X_t$.

Proof. Obvious and omitted.

Suppose Z_t are i.i.d *r.v.*, with $\mathbb{E}|Z_t| < \infty$, and define

$$\bar{Z}_n = \frac{\sum_{t=1}^n (\theta \mathbf{1}_{\{Z_t \geqslant \bar{Z}_n\}} Z_t + (1-\theta)\mathbf{1}_{\{Z_t < \bar{Z}_n\}} Z_t)}{\sum_{t=1}^n (\theta \mathbf{1}_{\{Z_t \geqslant \bar{Z}_n\}} + (1-\theta)\mathbf{1}_{\{Z_t < \bar{Z}_n\}})}, where\ t = 1, 2, \ldots, n, \tag{17}$$

Lemma 5. (SLLN of Expectile for i.i.d. samples)

$$\bar{Z}_n \xrightarrow{a.s.} \mathbb{E}_\theta(Z_t), as\ n \to \infty \tag{18}$$

Proof. See the appendix.

Theorem 4. (Consistency of expectile regression with i.i.d observations) *Suppose Assumption A.7 to A.10 in the Appendix holds, $\hat{\beta} \xrightarrow{p} \beta$ as $n \to \infty$, where*

$$\hat{\beta} = (X'\hat{\Pi}_X X)^{-1} X' \hat{\Pi}_{Y|X} Y \tag{19}$$

and

$$\hat{\Pi}_X = \begin{pmatrix} \hat{\pi}_{X_1} & & 0 \\ & \ddots & \\ 0 & & \hat{\pi}_{X_n} \end{pmatrix}_{n \times n}, \hat{\Pi}_{Y|X} = \begin{pmatrix} \hat{\pi}_{Y_1|X_1} & & 0 \\ & \ddots & \\ 0 & & \hat{\pi}_{Y_n|X_n} \end{pmatrix}_{n \times n} \tag{20}$$

and

$$\hat{\pi}_{X_t} = \frac{\theta \boldsymbol{1}_{\{X_t \geqslant \bar{X}_n\}} + (1-\theta)\boldsymbol{1}_{\{X_t < \bar{X}_n\}}}{\frac{1}{n}\sum_{t=1}^n (\theta \boldsymbol{1}_{\{X_t \geqslant \bar{X}_n\}} + (1-\theta)\boldsymbol{1}_{\{X_t < \bar{X}_n\}})}, \tag{21}$$

and

$$\hat{\pi}_{Y_t|X_t} = \frac{\theta \mathbf{1}_{\{sign(\hat{\beta}_1)(Y_t-\hat{g}(X_t))\geqslant 0\}} + (1-\theta)\mathbf{1}_{\{sign(\hat{\beta}_1)(Y_t-\hat{g}(X_t))<0\}}}{\left(\frac{\sum_{t=1}^{n}\left(\theta \mathbf{1}_{\{X_t\geqslant \bar{X}_n\}}+(1-\theta)\mathbf{1}_{\{X_t<\bar{X}_n\}}\right)\left(\theta \mathbf{1}_{\{sign(\hat{\beta}_1)(Y_t-\hat{g}(X_t))\geqslant 0\}}+(1-\theta)\mathbf{1}_{\{sign(\hat{\beta}_1)(Y_t-\hat{g}(X_t))<0\}}\right)}{\sum_{t=1}^{n}\left(\theta \mathbf{1}_{\{X_t\geqslant \bar{X}_n\}}+(1-\theta)\mathbf{1}_{\{X_t<\bar{X}_n\}}\right)}\right)} \tag{22}$$

Proof. See online proof.

Theorem 5. (Asymptotic normality of expectile regression) *Under assumptions A.7-A.12, we have*

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \xrightarrow{d} N\left(0, Q^{-1}VQ_{-1}\right), \text{ as } n \to \infty$$

where

$$Q = \mathbb{E}(X\pi_X X'), \text{ and } V = \mathbb{E}\left(X(\pi_X)^2 X'(\pi_{Y|X}\epsilon)^2\right).$$

Proof. See online proof.

So far, we have the estimator $\hat{\beta}$, and we prove the consistency and asymptotic normality of expectile regression with i.i.d observations. Next we discuss how to construct a test statistic for the null hypothesis $\mathbb{H}_0 : R\beta = r$, where R is a $J \times 2$ constant matrix, and r is a $J \times 1$ constant vector, J represents number of equalities.

Theorem 6. (Asymptotic χ^2 test with conditional homoskedasticity) *Suppose assumption A.7- A.13 hold, then under $\mathbb{H}_0 : R\beta = r$,*

$$\mathcal{W} = (R\hat{\beta}-r)'\left[R\left(X'\hat{\Pi}_X'X\right)^{-1}X'\hat{\Pi}_X'D(e)D(e)'\hat{\Pi}_X X\left(X'\hat{\Pi}_X'X\right)^{-1}R'\right]^{-1}(R\hat{\beta}-r) \sim X_J^2 \tag{23}$$

where

$$D(e) = diag\left(\hat{\Pi}_{Y_1|X_1}e_1, \hat{\Pi}_{Y_2|X_2}e_2, \cdots, \hat{\pi}_{Y_n|X_n}e_n\right). \tag{24}$$

Proof. See online proof.

We use randomly generated sample sets to simulate the expectile regression. The main conclusions are as follows: First, for a given positively correlated (X, Y), we run expectile regression of Y on X, as θ goes from 0 to 1, the estimated intercept becomes greater, while the slope becomes smaller. For a given negatively correlated (X, Y), as theta goes from 0 to 1, the estimated intercept becomes smaller, while the slope becomes greater. Second, the estimation is robust when view bias coefficient θ varies within the range of $[30\%, 70\%]$. As θ deviates from neutral (50%), the contour of $\hat{\beta}$ turns bumpy and becomes more twists. Hence although we can always find the estimation $\hat{\beta}$, it contains more noises rather than the true information as θ goes to 1 or goes to 0. See Fig. 4..

4.3. Expectile CAPM empirical test methodology

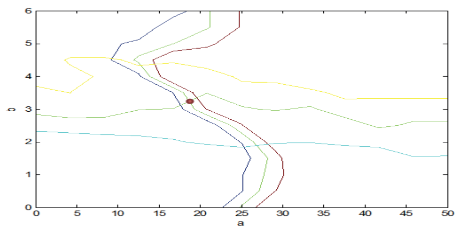
In this subsection, we employ conditional GMM restricted by expectile regression condition to empirically test view bias based expectile CAPM theory. We test the unconditional expectile CAPM,

$$\mathbb{E}_\theta(r_i - r_f) = \beta^\theta \mathbb{E}_\theta(r_M - r_f),$$

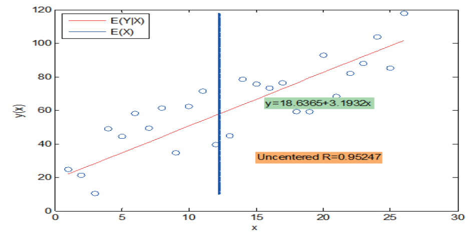
We rewrite it into

$$\mathbb{E}_\theta \left(r_i^e - \beta^\theta r_M^e \right) = 0,$$

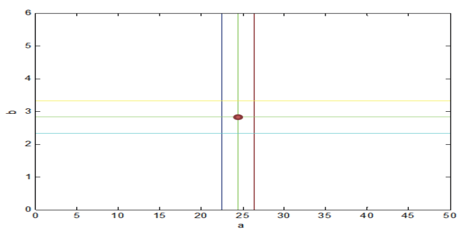
where r_i^e is the excess rate of return of security i, and r_M^e is the excess rate of return of market portfolio. Hence testing CAPM is testing a moment condition. The difficulty is that there is only one equation, but with two parameters (zero intercept and statistically significant non-zero slope) to be tested. We know that only when the number of moment conditions is greater than the dimension of the parameter vector, the model is said to be over-identified. Over-identification allows us to check whether the model's moment conditions match the data well or not. Fortunately, the unconditional expectile CAPM is just a special case of conditional expectile CAPM. Therefore, we solve the problem by estimating an unconditional expectile CAPM model using conditional GMM, which is



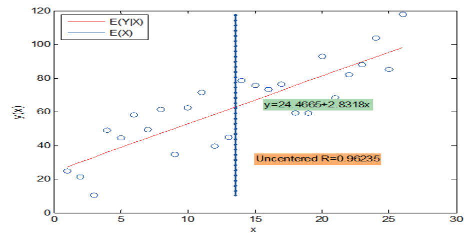
(a) Contour of solving Eq. (19) with $\theta = 40\%$



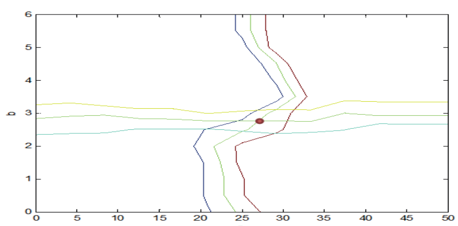
(b) $Y = 18.64 + 3.19X$; Uncentered $R^2 = 0.95$



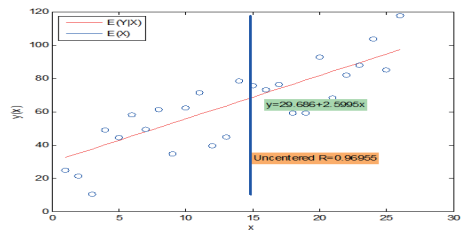
(c) Contour of solving Eq. (19) with $\theta = 50\%$



(d) $Y = 24.47 + 2.83X$; Uncentered $R^2 = 0.96$



(e) Contour of solving Eq. (19) with $\theta = 60\%$



(f) $Y = 29.69 + 3.60X$; Uncentered $R^2 = 0.97$

Fig. 4.

Expectile regression. This graph illustrates how the view bias affects the expectile estimation results. See e.g. Fig. 4(a), the horizontal axis represents the estimated intercept, and horizontal axis represents the estimated slope of the expectile regression $Y = \mathbb{E}_\theta(Y|X) + \epsilon$, where $\mathbb{E}_\theta(Y|X) = a + bX$, with view bias θ being 40% (indicating pessimism). The sample sets are generated by random seeds. Fig. 4(a) is the contour of solving Eq. (19). There are two twisting bands, the intersection of two midlines of the bands are the estimated intercept (18.6365) and slope (3.1932). The Uncentered R^2 is 0.95247. See Panel A. (b), the scattered diagram with best fitting line. The estimating process is that we search for a sample expectile of X (see Fig. 4(b), the vertical line perpendicular to the X axis), and assign a weight of 70% to the samples less than the sample expectile and a weight of 30% to the samples greater than the sample expectile. The sample expectile needs to found satisfying the condition that itself is equal to the calculated weighted average. We then search for the best fitting line based on the rule that if the slope is upwarding, we assign a weight of 70% to the samples above the fitting line, and 30% to the samples below, otherwise we assign the weight in a reverse way, namely 70% to the samples below the fitting line, and 30% to the samples above, and then we calculate the OLS estimators based on the weighted sample sets. The parameters of the best fitting line is the one to make itself equal to the OLS estimators weighted based on that particular fitting line. The estimation is robust when view bias coefficient θ varies within the range of $[30\%, 70\%]$. By comparing Fig. 4(a), 4(c), and 4(e), we observe that as θ deviates from neutral (50%), the contour of $\hat{\beta}$ turns bumpy and becomes more and more twists. Hence although we can always find the estimation β, it contains more noises rather than the true information as view bias goes to extreme pessimism (0%) or goes to extreme optimism (100%). By comparing Fig. 4(b), 4(d), and 4(f), as view bias goes from 40% to 60%, the estimated intercept becomes smaller, while the slope becomes greater. Hence it is possible that we find evidence to reject the null hypothesis under view neutral, but we cannot find evidence to reject and have to accept the null hypothesis under pessimistic view or optimistic view.

restricted by the expectile regression condition. The econometrics model is as follows,

$$\begin{cases} \mathbb{E}\left(\pi_{r^e_{i,t}} - \beta_0 - \beta_1 \pi_{r^e_{M,t}} r^e_{M,t}\right) = 0 \\ \mathbb{E}\left(\pi_{r^e_{i,t-1}} \pi_{r^e_{i,t}|r^e_{i,t-1}} r^e_{i,t} r^e_{i,t-1} - \beta_0 \pi_{r^e_{i,t-1}} r^e_{i,t-1} - \beta_1 \pi^e_{r_{i,t-1}} \pi_{r^e_{M,t}|r^e_{i,t-1}} r^e_{M,t} r^e_{i,t-1}\right) = 0 \\ \mathbb{E}\left(\pi_{r^e_{M,t-1}} \pi_{r^e_{i,t}|r^e_{M,t-1}} r^e_{i,t} r^e_{M,t-1} - \beta_0 \pi_{r^e_{M,t-1}} r^e_{M,t-1} - \beta_1 \pi^e_{r_{M,t-1}} \pi_{r^e_{M,t}|r^e_{M,t-1}} r^e_{M,t} r^e_{M,t-1}\right) = 0 \end{cases} \tag{25}$$

St: $\pi_{Y|X}$ *are implied by* $\mathbb{E}_\theta(Y|X) = \alpha_0 + \alpha_1 X$*, where* $Y \in \{r^e_{i,t} r^e_{M,t}\}$*,* $X \in \{r^e_{i,t} r^e_{M,t}\}$

where $r^e_{i,t}$ is the excess rate of return of security i at time t, $r^e_{M,t}$ is the excess rate of return of market portfolio at time t. The first equation of GMM is an unconditional moment condition, the second equation of GMM is a moment condition conditioning on the information of previous period's excess rate of return of security i, and the last equation is a moment condition conditioning on the information of previous period's excess rate of return of market portfolio. We will test the null hypothesis $\mathbb{H}_0 : \beta_0 = 0, \beta_1 = 0$, and the moment condition by the case of conditional homoskedasticity and heteroskedasticity according to the following steps.

- Step 1: assume view bias θ is given, search for the sample expectile to get $\pi_{r^e_{i,t}}$ and $\pi_{r^e_{M,t}}$; using the same sample set with one period lag to get $\pi_{r^e_{i,t-1}}$, and $\pi_{r^e_{M,t-1}}$.
- Step 2: run expectile regression of $r^e_{i,t}$ on $r^e_{i,t-1}$, obtain the estimated parameters, based on which estimate $\pi_{r^e_{i,t}|r^e_{i,t-1}}$. Repeat the same procedure to estimate $\pi_{r^e_{i,t}|r^e_{M,t-1}}$, $\pi_{r^e_{M,t}|r^e_{i,t-1}}$, $\pi_{r^e_{M,t}|r^e_{M,t-1}}$.
- Step 3: test statistical significance of $\pi_{r^e_{i,t}|r^e_{i,t-1}}$.
- Step 4: if the slope is insignificant, then replace $\pi_{r^e_{i,t}|r^e_{i,t-1}}$ using $\pi_{r^e_{i,t}}$.
- Step 5: repeat step 3 and step 4 for $\pi_{r^e_{i,t}|r^e_{M,t-1}}$, $\pi_{r^e_{M,t}|r^e_{i,t-1}}$, $\pi_{r^e_{M,t}|r^e_{M,t-1}}$.
- Step 6: run GMM to estimate β_0 and β_1.
- Step 7: test null hypothesis $\mathbb{H}_0$: $\beta_0 = 0$, $\beta_1 = 0$.

- Step 8: test the moment condition $\mathbb{H}_0$: $\mathbb{E}_\theta(r_i^e - \beta_0 - \beta_1 r_M^e) = 0$.

5. Empirical results

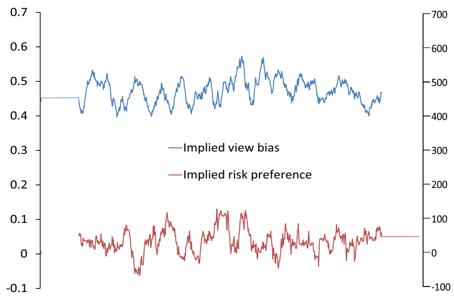
In this section, we provide empirical evidence on the advantages of the expectile based model for explaining the asset pricing anomalies. In particular, our empirical analysis is conducted in several steps. We first check the data availability. Second, by assuming a reasonable risk aversion coefficient, we obtain an aggregated implied view bias time series. Third, we construct momentum portfolio. We use the ranking time series of 10 momentum portfolios' return as proxies for β_t^θ. we do spectral analysis on β_t^θ and the excess rate of return of all momentum portfolio. The period of β_t^θ and excess rate of return are both compatible with the period of view bias coefficient, indicating that view bias towards ambiguity is being priced for that particular momentum portfolio. Therefore, fourth, we test the expectile based unconditional CAPM on 10 momentum portfolios under the average aggregated implied view bias coefficient over the observation period. We reach dramatically different conclusions on testing the moment condition. GMM reject the Null hypothesis suggested by traditional CAPM, $\mathbb{E}(r_i^e - \beta_0 - \beta_1 r_i^e) = 0$. View bias adjusted GMM accept the Null hypothesis suggested by the expectile based $\mathbb{E}_\theta(r_i^e - \beta_0 - \beta_1 r_i^e) = 0$. Finally, we analyse the systematic risk and latent risk of winner portfolio and loser portfolio, as well as the view bias adjusted excess rate of return to see if our statement hypothesis explains the momentum successfully.

First, we check the data availability. Daily stock returns data including common shares of all NYSE-, AMEX- and NASDAQ-listed firms are available in CRSP. We select the stocks with stock share code being 10 or 11 to exclude closed end funds, real estate investment trust, American depository receipts and foreign stocks. We get the monthly average of daily common share return and market return including dividends and the volatility excluding dividends. The annualized volatility is obtained by multiplying $\sqrt{252}$. Daily risk-free rate is proxied as 30 T-bill return from CRSP divided by 30.4. The U.S. quarterly aggregate nondurable goods and service consumption per capita is obtained from John Compbell's website. We interpolate it into a monthly time series. We use Gauss-Laguerre quadrature to numerically approximate Θ and Φ with respect to θ.

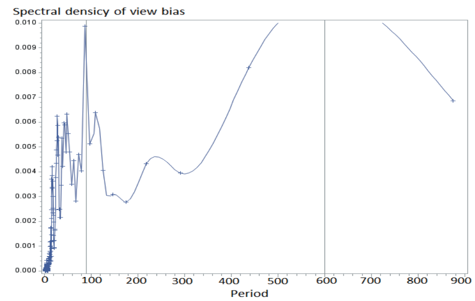
Second, we solve equation $\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}}\Theta - r_f = \alpha\sigma_C\sigma_M(\Theta^2 sign(\rho_{CM})\sqrt{1-\rho_{CM}^2} + \Phi\rho_{CM})$ to get an implied USA aggregated view bias coefficient time series from May 1926 to Sep 1999, by assuming a reasonable constant risk aversion coefficient being 3. The averaged view bias is 0.473, a slight deviation from view neutral, the maximum and minimum are 0.871 and 0.219 respectively. See Fig. 5(a), view bias when it is pessimism is not a phase lagged mirror reflection to the dynamic when it is optimism. We compare the equity premium puzzle implied view bias coefficient under a constant risk-averse coefficient being 3 and the equity premium puzzle implied risk averse coefficient under view neutral. The conclusion is that if we neglect the view bias factor, which is just lightly deviating from neutral, then we need a huge risk preference to reconcile $\mathbb{E}(r_M) - r_f = \alpha\sigma_C\sigma_M\rho_{CM}$ with the actual data, which is a stylized fact documented as equity premium puzzle. See Fig. 5(b), we draw the view bias periodogram within the time span from May. 1926 to Dec. 1999. The fist significant period is around 50 years, the second significant period is around 90 months. We group the data set of implied view bias and implied risk preference by the sign of correlations ρ_{CM}. We run regression by each group for the data pairs, whose risk aversion is within the range of $(-300, 300)$. The result indicates that they are strongly correlated for both two groups. See Fig. 5(c) and 5(d), we claim that the implied risk preference under view neutral and implied view bias under the constant risk preference being 3 are positively (negatively) correlated when aggregate consumption growth rate and market portfolio return are positively (negatively) correlated. In other words, if the market return and consumption growth move in the same direction, a risk-averse investor is more like a pessimistic investor. If the market return and consumption growth move in a counter-direction, a risk-averse investor is more like an optimistic investor.

Third, we get the rank of each selected stock in deciles. The sample range[17] is from

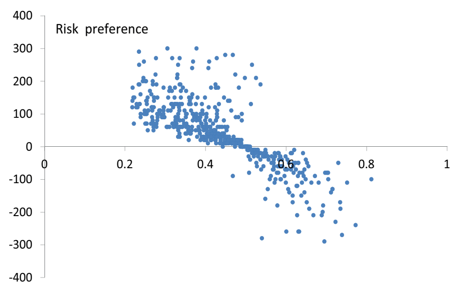
[17]The reason why we select this sample range is because we need the volatility of each momentum



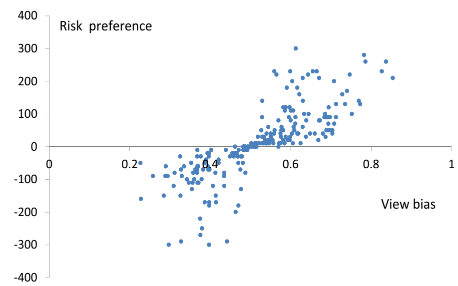
(a) Comparison of implied view bias and risk preference



(b) Spectral density of implied view bias



(c) Positive correlation between risk preference and view bias when $\rho_{CM} > 0$



(d) Negative correlation between risk preference and view bias when $\rho_{CM} < 0$

Fig. 5.

Implied view bias and implied risk preference. Fig. 5(a) compares the equity premium puzzle implied view bias under a constant risk preference being 3 and the equity premium puzzle implied risk preference under view neutral within the time span from Jun. 1963 to Dec. 1970. For the implied view bias, the maximum is 0.871, the minimum is 0.219, the average is 0.473. For the implied risk averse coefficient, we truncate it to [-300, 300] to exclude the outliers, then the average is 30. Fig. 5(b) is the view bias periodogram within the time span from May. 1926 to Sep. 1999. The first significant period is around 50 years months. The second significant period is 90 months. Fig. 5(c) indicates that the implied risk preference is positively correlated with the implied view bias when the aggregate consumption is positively correlated with market portfolio return. The intercept (-361) and the slope (712) are both statistically significant under 99% confidence, the adjusted R^2 is 0.58. Fig. 5(d) indicates that the implied risk preference is negatively correlated with the implied view bias when the aggregate consumption is negatively correlated with market portfolio return. The intercept (341) and the slope (-689) are both statistically significant under 99% confidence, the adjusted R^2 is 0.60.

Jun. 1963 to Dec. 1970. Following many studies, the rank periods have the length of six months. For any given month, the rank of a certain stock is determined based on the past 6-month returns. We construct momentum portfolios in July 1967, the middle of the sample range. We group the stocks by rank to get 10 portfolios, and get the value-weighted monthly return of each portfolio. We scale the portfolio return to make each of them having exactly the same volatility to control $\frac{\sigma_i}{\sqrt{dt}}\Theta$ adjustment in the expectile CAPM. We then do the spectral analysis to get the period of each portfolio return, period of β_t^θ, as well as the implied view bias coefficient with sample range (Jun. 1963 to Dec 1970). The first and second significant period for view bias are 62 and 19 months respectively. Actual data indicates that view bias when it is pessimism is not a phase lagged mirror reflection to the dynamic when it is optimism, then according to Corrolary 2, we are expecting the period of β^θ is equal to the period of view bias. And based on Theorem 1, the period of the excess rate of return of the portfolio should be equal to the period of view bias as well. This is a unique perspective, from which we hypothesize and prove that view bias impact the market. Since we are interested in the period instead of the wave swing of the β_t^θ dynamic, it makes sense to use the rank time series of return as a proxy for its β_t^θ. We make comparisons between the periods of view bias and the rank period of each momentum portfolio β_t^θ. See Table 3, quite many of them are consistent with what the expectile CAPM model suggests. So we account that the view bias towards ambiguity is being priced for that particular security. The rank periods don't have a period with the length being around 62, and the p-value for white noise test is large. That is because we are using the rank as a proxy for beta β_t^θ, and the range of the rank is restricted to [1, 10], hence it is not able to reflect the long term trend or cycle.

Fourth, we give an example of empirical evidence on the advantages of the expectile based model for testing CAPM. We take average of the implied view bias into expec-

portfolio and market index to be stable in order to carry out the following analysis, hence the 50 years view bias cycle is too long. The sample range (Jun. 1963 to Dec. 1970) covers 93 months, roughly the second statistical significant period, and it is the golden age for the united states' economy after the second world war.

Table 3

Value weighted momentum portfolio spectral analysis

	Rank spectral analysis		Return spectral analysis		
Portfolio	Period	p-value	1st period	2st period	p-value
Loser	14	0.3187	57	21	<0.0001
Loser+	19	0.6497	57	18	<0.0001
Loser++	19	0.2384	56	18	<0.0001
Loser+++	24	0.9419	57	19	<0.0001
Loser++++	24	0.7953	60	23	<0.0001
Winner_{----}	20	0.5192	58	22	<0.0001
Winner_{---}	20	0.5519	63	24	<0.0001
Winner_{--}	23	0.0019	68	23	<0.0001
Winner_{-}	19	0.1282	70	24	<0.0001
Winner	26	0.0618	—	25	<0.0001
Average	20.6		60.7	21.7	

Table 3 compares the period of rank time series and the period of return time series of 10 momentum portfoios, and both of them are compatible with view bias period.

tile CAPM to estimate the view bias based beta, and to test if the moment conditions are satisfied. We use GMM method with restriction to the estimated view bias adjustments,$\pi_{r^e_{i,t}|r^e_{i,t-1}}$, $\pi_{r^e_{i,t}|r^e_{M,t-1}}$, $\pi_{r^e_{M,t}|r^e_{i,t-1}}$, $\pi_{r^e_{M,t}|r^e_{M,t-1}}$. The sample range again is from Jun. 1963 to Dec. 1970, overall 93 months. We run GMM and classical OLS regression to estimate CAPM model. We also run view bias adjusted GMM for each momentum portfolio. We first group sample data by view bias. Then we run expectile regression to get the distortion factor for each data pair. The slope of the expectile regression are not statistical significant, hence we replace the conditional $\pi_{r^e_{j,t}|r^e_{i,t-1}}$ using unconditional $\pi_{r^e_{j,t}}$. We finally combine two groups of data, to run GMM (Eq. (25)). See Table 4. The view bias adjusted betas, estimated through GMM$_\theta$, GMM, OLS are of the same pattern for 10 momentum portfolios. The beta for the biggest winner estimated through GMM$_\theta$ is 0.7687, which is the lowest among 10 momentum portfolios. Hence the real data does not support Explanation 1.

Finally, we claim the actual data supports Explanation 2. The amount of risk of winner portfolio is lower, the adjusted market price of risk is negative. We first study the risk composition of each portfolio. See Table 5, we decompose the total risk of each portfolio into systematic risk (σ_{iM}) and idyosyncratic risk ($\sigma_i\sigma_M\sqrt{1-\rho^2_{iM}}$). According to the Theorem 1 and Eq. (13) the real priced risk is the weighted average of systematic risk and latent risk ($\widetilde{\sigma}_{iM}$), where the weights are Φ and Θ^2 respectively. Second, we arbitrarily choose a view bias. For any particular view bias, there exists a pair of Φ and Θ^2, using which we are able to get the view bias adjusted beta (β^θ). Third, we calculate the sample average of market excess rate, which is postive. However the market excess return adjusted by first category of information premium ($\frac{\sigma_M}{\sqrt{dt}}\Theta$) is negative (-2.991 annually), multiplying which by β^θ, we obtain that the adjusted exptected portfolio excess return ($\mathbb{E}(r_i) + \frac{\sigma_i}{\sqrt{dt}}\Theta - r_f$) are all negative. By substracting the item ($\frac{\sigma_i}{\sqrt{dt}}\Theta$), we get the exptected portfolio excess return ($\mathbb{E}(r_i) - r_f$) being all postive. Fourth, we search for a view bias for each portfolio to make the expected portfolio excess return implied by expectile CAPM being equal to the sample average of excess return, we then get the momentum effect implied view bias for each momentum portfolio. The corresponding view bias adjusted beta for each portfolio are reconcile with β^θ estimated by GMM$_\theta$ in Table **4**. Fifth, we find that the difference between β and β^θ are quite tiny. β^θ for the biggest winner portfolio is the lowest among all 10 portfolios, which is consistent with what the GMM$_\theta$'s results indicate. Finally, we find the advantage of expectile CAPM and the advantage of using view bias distorted GMM for testing expectile CAPM. See Table **4**, J-statistic for GMM$_\theta$ are all greater than 5%, which implies the moment conditions in Eq. (25) are all satisfied, and model is correctly specified. 6 out of 10 J-statistic for GMM are all less than 5%, which implies their moment conditions in Eq. (25) with all distortion multiplier $\pi \equiv 1$ are rejected, and traditional CAPM model are mis-specified. Those dramatically different results shows an empirical evidence on the advantages of the expectile based theoreticl and empirical model. All the above analysis supports Explanation 2.

Table 4

Summary of View bias adjusted GMM under expectile CAPM, and GMM, OLS under alternative traditional CAPM

		L	L+	L++	L+++	L++++	W____	W___	W__	W_	W
	Alpha (e-3)	−0.110	0.553	2.243**	2.302**	2.878**	2.281**	2.731**	5.753***	8.424***	7.063***
	t-statistic	[-0.12]	[0.56]	[2.85]	[2.74]	[2.72]	[2.70]	[3.62]	[4.08]	[8.22]	[4.69]
	Beta (e-1)	8.975***	9.529***	9.056***	9.095***	9.181***	9.325***	9.383***	9.058***	9.143***	7.687***
GMM_θ	t-statistic	[42.67]	[38.15]	[47.58]	[30.85]	[29.62]	[36.25]	[43.52]	[24.84]	[41.65]	[30.12]
	J-statistic	[27.47]	[29.10]	[25.65]	[24.80]	[30.57]	[33.81]	[31.81]	[32.13]	[31.74]	[29.14]
	P-value (%)	87.29	81.97	92.01	93.74	76.35	61.96	71.09	69.65	71.37	81.84
	Alpha (e-3)	0.281	-0.460	-1.071	2.449	3.437**	2.221	2.376*	5.790***	7.495***	5.588**
	t-statistic	[0.21]	[-0.38]	[0.86]	[1.90]	[2.95]	[1.81]	[2.56]	[3.65]	[5.56]	[2.81]
GMM	Beta (e-1)	8.924***	9.361***	9.282***	9.286***	9.050***	9.313***	9.569***	9.302***	9.340***	8.172***
	t-statistic	[29.17]	[29.05]	[26.18]	[27.20]	[26.86]	[26.74]	[35.86]	[20.28]	[32.30]	[18.19]
	J-statistic	[21.16]	[22.98]	[21.36]	[20.18]	[23.61]	[29.22]	[28.94]	[26.46]	[24.24]	[16.81]
	P-value (%)	6.98	4.20	6.62	9.08	3.49	0.61	0.67	1.48	2.90	20.80
	Alpha (e-3)	0.526	1.377	2.334	2.297	2.706	2.731*	2.818*	8.849***	7.339***	8.622**
	t-statistic	[0.32]	[0.97]	[1.52]	[1.48]	[1.66]	[2.00]	[2.39]	[3.55]	[4.72]	[3.34]
OLS	Beta (e-1)	9.111***	9.371***	9.263***	9.220***	9.160***	9.412***	9.580***	8.718***	9.212***	7.689***
	t-statistic	[20.67]	[24.99]	[22.81]	[22.56]	[21.30]	[26.17]	[30.83]	[17.13]	[22.44]	[11.29]
	R-square (%)	82.93	87.65	85.54	85.25	83.75	88.61	91.52	76.93	85.13	59.15

Table 4 compares view bias adjusted GMM regression results of expectile CAPM with GMM and OLS regression results of traditional CAPM using 10 momentum portfolios' monthly data of U.S. stock market from Jun. 1963 to Dec. 1970. L is the bottom loser portfolio, L+ is the bottum second loser, L++ is the bottum third loser... , W is the top winner, W- is top second winner, W– is top third winner... All regression coefficients marked by $*(**,***)$ are significant at the 5% (1%, <0.0001)level.

Table 5

Value weighted momentum portfolio risk composition

	Market	L	L+	L++	L+++	L++++	W----	W---	W--	W-	W
Sys risk (e-3)	1.417	1.291	1.327	1.309	1.310	1.297	1.335	1.356	1.242	1.308	1.091
Idyosyn (e-4)	0.000	5.847	4.978	5.435	5.399	5.709	4.773	4.136	6.827	5.458	9.041
Total risk (e-3)	1.417	1.876	1.825	1.852	1.850	1.868	1.812	1.769	1.925	1.854	1.996
Latent risk (e-3)	2.339	2.389	2.392	2.396	2.397	2.419	2.392	0.544	2.413	2.399	2.506
Priced risk (e-3)	1.441	1.315	1.351	1.333	1.334	1.321	1.358	1.368	1.265	1.332	1.115
View bias (e-1)	4.500	4.526	4.483	4.463	4.453	4.447	4.436	4.426	4.336	4.290	4.319
Θ^2 (e-3)	6.400	5.762	6.868	7.413	7.666	7.841	8.170	8.463	11.356	12.997	11.941
Φ	1.0064	1.0058	1.0069	1.0074	1.0077	1.0078	1.0082	1.0085	1.0114	1.0130	1.0119
Beta	1.000	0.911	0.936	0.924	0.925	0.915	0.942	0.956	0.876	0.923	0.770
Adjusted beta	1.000	0.912	0.937	0.925	0.926	0.917	0.943	0.947	0.879	0.925	0.776
Mkt r^e	0.653	0.653	0.653	0.653	0.653	0.653	0.653	0.653	0.653	0.653	0.653
Adjusted mkt r^e	-2.991	-2.991	-2.991	-2.991	-2.991	-2.991	-2.991	-2.991	-2.991	-2.991	-2.991
Adjusted portf r^e	-2.991	-2.728	-2.803	-2.766	-2.769	-2.742	-2.819	-2.831	-2.630	2.767	-2.321
Expected portf r^e	0.653	0.730	0.971	1.156	1.219	1.291	1.297	1.359	2.224	2.426	2.656
Averaged portf r^e	0.653	0.730	0.971	1.156	1.219	1.291	1.297	1.359	2.224	2.426	2.656

Table 5 decomposes the risk for 10 momentum portfolios respectively from Jun. 1963 to Dec 1999. Sys risk is systematic risk ($\sigma_{iM} = \rho_{iM}\sigma_i\sigma_M$). Idyosyn is idyosyncratic risk ($\sigma_\epsilon = \sigma_i\sigma_M\sqrt{1-\rho_{iM}^2}$). Total risk is the vector sum of systematic risk (σ_{iM}) and idyosyncratic risk (σ_ϵ). According to expectile CAPM, Priced risk , namely the risk being priced $\left(\Theta^2\widetilde{\sigma}_{iM} + \Phi\sigma_{iM}\right)$, is the weighted average of systematic risk (σ_{iM}) and Latent risk ($\widetilde{\sigma}_{iM}$). All numbers marked by (e-n) is expressed in scientific notation to a level of 10^{-n}. Assume view bias θ for each momentum portfolio are different. For any set of view bias, there is a corresponding pair of Θ and Φ . Beta ($\beta = \frac{\sigma_{iM}}{\sigma_M^2}$) is the relative amount of risk in traditional CAPM. Adjusted beta ($\beta^\theta = \frac{\Theta^2\widetilde{\sigma}_{iM}+\Phi\sigma_{iM}}{\Theta^2\widetilde{\sigma}_M^2+\Phi\sigma_M^2}$) is the relative amount of risk in expectile CAPM. Mkt r^e ($\mu_M^e = \mathbb{E}(r_M - r_f)$), representing the expectation of market excess rate of return, is price of risk in traditional CAPM. Adjusted mkt r^e ($\widetilde{\mu}_M^e = \mu_M^e + \frac{\sigma_M}{\sqrt{dt}}\Theta)$), is the price of risk in expectile CAPM. Adjusted portf r^e ($\widetilde{\mu}_i^e = \beta_i^\theta \times \widetilde{\mu}_M^e$), is portfolio i's reward for taking market risk in exectile CAPM. Expected portf r^e ($\mu_i^e = \widetilde{\mu}_i^e - \frac{\sigma_i}{\sqrt{dt}}\Theta$), is the price of risk in traditional CAPM. Averaged portf r^e is the sample average of excess rate of return of each portfolio. View bias is chosen to make the model forcasted Expected portf r^e equal to Average portf r^e.

6. Conclusion

In this paper, we develop an expectile based extended asset pricing theoretical framework. We also develop a novel econometrics method, expectile regression, for testing the extended CAPM and finding the empirical evidence to support our explanations to the anomalies. Comparing with the existing methods, our approach is more flexible. First, the traditional methods normally list many factors. The risk factor will be compensated by the risk premium directly. Some other factors, such as information asymmetry, restrictions, will change the model structure. Since they all exist at the same time, as fully incorporating those factors, the model is becoming more and more complicated. However the view bias is like a transducer, many risk factors affect the asset pricing indirectly through affecting the view bias factor. By modeling the relationship between asset pricing and view bias, we are able to explain both two categories of anomalies either from mean and variance of SDF perspective, e.g., equity premium puzzle, or from the factor structure of SDF perspective, e.g., momentum effect. Second, our approach is extending expectation into expectile. As a reward measure, expectile appears in both the equilibrium asset pricing and riskfree arbitrage pricing, however many traditional approach can solve equilibirum asset pricing anomalies only. Simulation studies and empirical example show that the non-zero intercept and statistical insignificant beta of traditional CAPM obtained by running OLS regression might become a zero intercept and statistical significant CAPM obtained by running expectile regression, the momentum condition which is not satisfied under expectation operator can be satisfied under expectile operator. Empirical studies indicate that the equity premium puzzle is consistent with what our view bias approach suggests, and people are pre-occupied with a aggregated pessimistic view bias on average during post war in U.S.; Momentum effect has two alternative explanations: either the winner portfolio has a low relative amount of risk (weighted average of systematic risk and latent risk) while the market price of risk is postive; or the winner portfolio is of a high relative amount of risk while the market price of risk is positive. Empirical analysis on post war U.S. stock market data supports the second explanantion.

Appendix

In this appendix, we provide the assumptions and detailed mathematical proofs of some of the results in the paper, for the others, please refer to online proof.

Assumption A. 1. Time interval between each decision is infinitesimal.

Assumption A. 2. Prices follow diffusion processes.

Assumption A. 3. Only consumption and portfolio process are controllable.

Assumption A. 4. There is no exogenous endowment.

Assumption A. 5. Investors are homogenous.

Assumption A. 6. Information is imperfect, and pessimism or optimism view bias exists.

Assumption A. 7. [I.I.D] $\{Y_t, X_t'\}_(t=1)^n$ is an $i.i.d$ random sample.

Assumption A. 8. [Linearity] $Y_t = X_t'\beta^0 + \epsilon_t, t = 1, , n$ for some unknown 2×1 parameter β^0 and some unobservable random variable ϵ_t.

Assumption A. 9. [Correct model specification] $\mathbb{E}_\theta(\epsilon_t|X_t) = 0 a.s.$ with $\mathbb{E}(\epsilon_t^2) = \sigma^2$.

Assumption A. 10. [Non-sigularity] The $K \times K$ matrix $Q = \mathbb{E}(X_t X_t')$ is nonsingular and finite.

Assumption A. 11. The $K \times K$ matrix $\mathbb{E}(X_t X_t' Y_t^2)$ is finite and positive definite.

Assumption A. 12. The $K \times K$ matrix $\mathbb{E}(X_t X_t' \epsilon_t^2)$ is finite and positive definite.

Assumption A. 13. $\mathbb{E}(\pi_{sign(\beta_1)(Y-\beta_0-\beta_1 X)}\sigma_t^2|X_t) = \sigma_\theta^2$ *a.s.*

Assumption A. 14. $\mathbb{E}(X_{jt}^4) < \infty$ for all $0 \leqslant j \leqslant k$; and $\mathbb{E}(\epsilon_t^4) < \infty$.

Lemma 1. (Relative entropy comparison) $0 \leqslant D(f\|h_{expextile}) < D(f\|h_{quantile}) = +\infty$, *with the first equality if and if* $\theta = 50\%$, *where* $\mathbb{Q}_\theta(x)$ *is the* $\theta-$*quantile of* X, $D(f\|h_{expectile}) = \int f_X(x) ln\left(\frac{f_X(x)}{\pi_X(\theta) f_X(x)}\right) dx$ *and* $D(f\|h_{quantile}) = -\int f_X(x) ln(\frac{1}{\theta} \times \boldsymbol{1}_{\{X<\mathbb{Q}_\theta(x)\}}) dx$.

Proof. According to the existing information theorem that defining $Supp(f) = \{x : f_X(x) > 0\}$, if the support $Supp(f) \nsubseteq Supp(h)$, then $D(f\|h) = +\infty$. By solving $Quantile_\theta(X) = argmin_q[(1-\theta)\int_{X<q}|x-q|f_X(x)dx + \theta\int_{X>q}|x-q|f_X(x)dx]$, we get the FOC: $1 = \int \frac{1}{\theta}1_{\{X<q^*\}}f_X(x)dx$. Hence, $Supp(f) \nsubseteq Supp(1_{\{X<q^*\}}f)$ are satisfied, and $D(f\|h_{quantile}) = +\infty$. In likely manner, $Supp(f) = Supp(\pi_X(\theta)f)$, therefore $0 \leqslant D(f\|h_{expectile}) < D(f\|h_{quantile}) = +\infty$ is proved. □

Proposition 1. (Monotonicity of expectile) *As people goes from extreme pessimism to extreme optimism, his expectile (reward measure) goes from negative infinity to positive infinity monotonously. (Invariance of variancile). As people goes from extreme pessimism to extreme optimism, his variancile (risk measure) remains to be unchanged.*

Proof. We use the orthogonal polynomials to approximate the integrand, then the integral is evaluated by Gaussian quadrature. Although Eq. (A.1) is still an implicit function with respect to q^*, it is of a much simpler form, and easy to get the result using root finding technique.

$$\mathbb{E}_\theta(W) \triangleq q^* = \frac{\int\left[(1-\theta)\mathbf{1}_{\{W<q^*\}} + \theta\mathbf{1}_{\{W>q^*\}}\right]f_W(w)wdw}{\int\left[(1-\theta)\mathbf{1}_{\{W<q^*\}} + \theta\mathbf{1}_{\{W>q^*\}}\right]f_W(w)dw} \tag{A.1}$$

where $f_W(w) = \frac{1}{2\pi}e^{-\frac{w^2}{2}}$. Equivalently, we rewrite it into

$$b^* = \frac{\int\left[(1-\theta)\mathbf{1}_{\{W<b^*\}} + \theta\mathbf{1}_{\{W>b^*\}}\right]e^{-w^2}wdw}{\int\left[(1-\theta)\mathbf{1}_{\{W<b^*\}} + \theta\mathbf{1}_{\{W>b^*\}}\right]e^{-w^2}dw},$$

where $b^* = \frac{q^*}{\sqrt{2}}$. Then we have

$$(1-\theta)\int_{-\infty}^{0} e^{-(w-b)^2}wdw + \theta\int_{-\infty}^{0} e^{-(w-b)^2}wdw = (\theta-1)\int_{0}^{-\infty} e^{-(w-b)^2}wdw + \theta\int_{-\infty}^{0} e^{-(w-b)^2}wdw$$

Namely,

$$(\theta-1)\int_{0}^{-\infty} e^{-w}e^{-(w^2-2bw+b^2-w)}wdw + \theta\int_{-\infty}^{0} e^{-w}e^{-(w^2+2bw+b^2-w)}wdw$$

We name e^{-w} the weight function, then $e^{-w^2-2bw+b^2-w}w$ is the integrand. By setting $q^* = \sqrt{2}b = \frac{(lnB)}{\sqrt{2}}$, and using Guass-Laguerre formula, we obtain the results as follows.

$$\sum_{k=1}^{n} \alpha_k \left[(\theta-1)B^{w_k} + \theta B^{-w_k}\right] = 0, \tag{A.2}$$

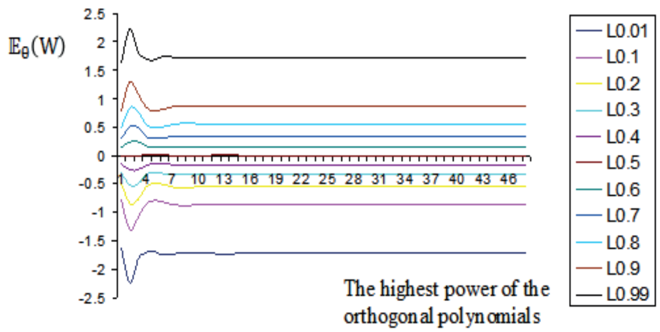
where $\alpha_k = A_k e^{-w_k^2+w_k}$, $A_k = \frac{w_k}{(n+1)^2[L_{n+1}(w_k)]^2}$, $L_{n+1}(w_k) = \sum_{k=0}^{n}(-1)^k \binom{n+a}{n-k} \frac{w^k}{k!} = 0$. That is, solving equation $L_{n+1}(w_k) = 0$, we get the w_k, we get the x_k, $k = 1, \cdots n$, and all the corresponding a_k, $k = 1, n$. After taking them into equation (A.2), we get the B using Newton root finding method, and q^* is obtained. Calculation of w_k and a_k is done for good. Given different , calculating q^* is to solve equation with respect to different , and the set of w_k and a_k, $k = 1, \cdots n$ are never changed. See Fig. A. 1(a), the expectile is converging quickly as the highest power of the orthogonal polynomials increases. Fig. A. 1(b) indicates that $E_\theta(W)$ is a monotonously increasing function of θ. We substitute the value of expectile into the definition of variancile, and get the constant 1, no matter what value the view bias is. □

Proposition 3. (Extended law of one price) *Under imperfect information, when portfolios are repackaged, the return remains to be the same, if and only if people are view neutral. If pessimism exists, people can assemble them into a portfolio to do riskless arbitrage; if optimism exists, they can split the package to earn non-zero excess return.*

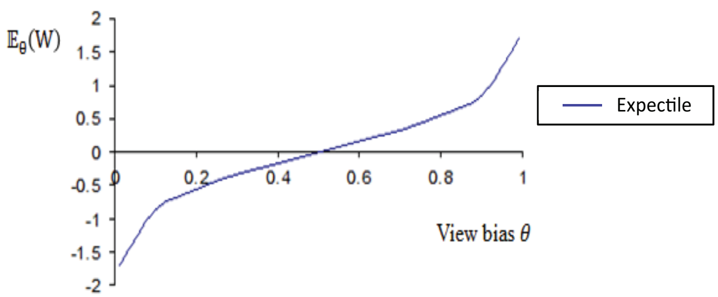
Proof. Equivalently, we only need to prove the following statement. If $X_i \sim N(\mu_i, \sigma_i)$, the correlation between X_i and X_j is $\rho_i j$, and $\exists \rho_{ij} \neq 1$, then

$$\mathbb{E}_\theta(\textstyle\sum_{i=1}^{n} X_i) = \mathbb{E}_\theta(\overline{\textstyle\sum_{i=1}^{n} X_i}) = \begin{cases} > 0\theta > 50\% \\ > 0\theta > 50\% \\ < 0\theta < 50\% \end{cases}.$$

Case 1: if $\theta = 50\%$, then $\mathbb{E}_\theta(\sum_{i=1}^{n} X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^{n} X_i}) = \mathbb{E}_\theta(\sum_{i=1}^{n} X_i) - \mathbb{E}_\theta(\sum_{i=1}^{n} X_i) = 0$.
Case 2: if $\theta > 50\%$ and $\exists \rho_{ij} \neq 1$,



(a) Numerically approximating expectile



(b) Monotonicity of expectile

Fig. A.1

Numerical analysis of expectile. Fig. A.1(a) indicates that the expectile $\mathbb{E}_\theta(W)$ (where W is a standard normal distributed random varaibel) is converging quickly as the highest power of the orthogonal polynomials [the n in Eq (A.2)] increases for all levels of view bias (from L0.01 to L0.99). Fig. A. 1(b) indicates that $\mathbb{E}_\theta(W)$ is a monotonously increasing function of θ.

- Step 1: if $n=2$, $X_i \sim N(\mu_i, \sigma_i)$, we have $X_1+X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2\rho_{12})$, then $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) = (\sigma_1+\sigma_2)\mathbb{E}_\theta(W) - \mathbb{E}_\theta(W)\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2\rho_{12}}$. If $\rho_{ij} \neq 1$, and from Proposition 1 that $\mathbb{E}_\theta(W)$ is a monotonously increasing function with respect to θ, $\mathbb{E}_\theta(W) > 0$, hence $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) > 0$, otherwise if $\rho_{ij} = 1$, $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) = \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i})$.

- Step 2: we make assumption that, if $n = k$, then $\mathbb{E}_\theta(\sum_{i=1}^2 X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) \geqslant 0$.

- Step 3: if $n = k+1$, then $\mathbb{E}_\theta(\sum_{i=1}^n X_i) = \mathbb{E}_\theta(\sum_{i=1}^{k+1} X_i) = \mathbb{E}_\theta(\sum_{i=1}^k X_i) + \mathbb{E}_\theta(X_k) \geqslant \mathbb{E}_\theta(\overline{\sum_{i=1}^k X_i}) + \mathbb{E}_\theta(X_k) \geqslant \mathbb{E}_\theta(\overline{\sum_{i=1}^{k+1} X_i})$.

Case 3: if $< 50\%$ and $\exists \rho_{ij} \neq 1$, since $\mathbb{E}_\theta(W) < 0$, ceteris paribus, we get $\mathbb{E}_\theta(\sum_{i=1}^n X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^2 X_i}) \leqslant 0$.

Case 4: if $\forall \rho_{ij} = 1$, then $\mathbb{E}_\theta(\sum_{i=1}^n X_i) - \mathbb{E}_\theta(\overline{\sum_{i=1}^n X_i}) = \mathbb{E}(\sum_{i=1}^n X_i) - \mathbb{E}(\sum_{i=1}^n X_i) = 0$.

□

Lemma 2. (Exparte effect) *Assume X and Y are the rate of returns of two assets. Assume $Y = bX$, then $\mathbb{E}_\theta(Y) = b\mathbb{E}_\theta(X)$, iff $b \geqslant 0$, and $\mathbb{E}_\theta(Y) = b\mathbb{E}_{(1-\theta)}(X)$, iff $b \leqslant 0$.*

Proof. 1) If $b > 0$, we have

$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(bx-\mu_Y)^2}{2\sigma_Y^2}} = \frac{1}{b}\frac{1}{\left(\frac{\sigma_Y}{b}\right)\sqrt{2\pi}} e^{-\frac{\left(x-\frac{\mu_Y}{b}\right)^2}{2\left(\frac{\sigma_Y}{b}\right)^2}},$$

denote $\mu_X = \frac{\mu_Y}{b}$ and $\sigma_X = \frac{\sigma_Y}{b}$, $f_Y(y) = bf_X(x)$, then

$$\mathbb{E}_\theta(Y) \triangleq q^* = b\frac{\int \left[(1-\theta)\mathbf{1}_{\{bX<q^*\}} + \theta\mathbf{1}_{\{bX>q^*\}}\right] f_X(x)xdx}{\int \left[(1-\theta)\mathbf{1}_{\{bX<q^*\}} + \theta\mathbf{1}_{\{bX>q^*\}}\right] f_X(x)dx}$$

Since $b > 0$, we have

$$\frac{q^*}{b} = \frac{\int \left[(1-\theta)\mathbf{1}_{\{X<\frac{q^*}{b}\}} + \theta\mathbf{1}_{\{X>\frac{q^*}{b}\}}\right] f_X(x)xdx}{\int \left[(1-\theta)\mathbf{1}_{\{X<\frac{q^*}{b}\}} + \theta\mathbf{1}_{\{X>\frac{q^*}{b}\}}\right] f_X(x)dx},$$

hence $\mathbb{E}_\theta(X) = \frac{q^*}{b}$, namely$\mathbb{E}_\theta(Y) = b\mathbb{E}_\theta(X)$.

2) If $b < 0$, we have

$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}}e^{-\frac{(bx-\mu_Y)^2}{2\sigma_Y^2}} = \frac{1}{b}\frac{1}{\left(\frac{\sigma_Y}{b}\right)\sqrt{2\pi}}e^{-\frac{\left(x-\frac{\mu_Y}{b}\right)^2}{2\left(\frac{\sigma_Y}{b}\right)^2}},$$

denote $\mu_X = -\frac{\mu_Y}{b}$ and $\sigma_X = -\frac{\sigma_Y}{b}$, $f_Y(y) = -bf_X(x)$, then

$$\mathbb{E}_\theta(Y) \triangleq q^* = b\frac{\int\left[(1-\theta)\mathbf{1}_{\{bX<q^*\}} + \theta\mathbf{1}_{\{bX>q^*\}}\right]f_X(x)xdx}{\int\left[(1-\theta)\mathbf{1}_{\{bX<q^*\}} + \theta\mathbf{1}_{\{bX>q^*\}}\right]f_X(x)dx},$$

hence $\mathbb{E}_{1-\theta}(X) = \frac{q^*}{b}$, namely $\mathbb{E}_{1-\theta}(Y) = b\mathbb{E}_{1-\theta}(X)$.

3) If $b = 0$, we have $\mathbb{E}_\theta(Y) = b\mathbb{E}_\theta(X) = b\mathbb{E}_{1-\theta}(X)$. $\square$

Proposition 4. (Homogeneity of Variancile) $VAR_\theta(\sigma W) = \sigma^2 VAR_\theta(W)$, *where* σ *is a constant;* (Risk-free condition of expectile) $VAR_\theta(r+W) = VAR_\theta(W)$, *where* r *is a constant;* (Sub-additivity of variancile) $VAR_\theta(X+Y) = VAR_\theta(X) + VAR_\theta(Y) + 2\times COV_\theta(X,Y)$.

Proof. 1) $VAR_\theta(\sigma W) \triangleq \int(\sigma W - \sigma q^*)^2\pi_W(\theta)f_W(w)dw = \sigma^2 VAR_\theta(W)$.
2) $VAR_\theta(r+W) \triangleq \int(r+W-(r+q^*))^2\pi_W(\theta)f_W(w)dw = VAR_\theta(W)$.
3) $VAR_\theta(X+Y) \triangleq \int(X+Y-(q_X^*+q_Y^*))^2\pi_Y(\theta)\pi_{X|Y}(\theta)f_{X,Y}(x,y)dxdy = \int[(X-q_X^*)^2 + 2(X-q_X^*)(Y-q_Y^*) + (Y-q_Y^*)^2]\pi_Y(\theta)\pi_{X|Y}(\theta)f_{X,Y}(x,y)dxdy = \int(X-q_X^*)^2\pi_X(\theta)\pi_{X|Y}(\theta)f_{X,Y}(x,y)dxdy + \int(Y-q_Y^*)^2\pi_Y(\theta)\pi_{X|Y}(\theta)f_{X,Y}(x,y)dxdy + \int[2(X-q_X^*)(Y-q_Y^*)]\pi_Y(\theta)\pi_{X|Y}(\theta)f_{X,Y}(x,y)dxdy = VAR_\theta(X) + VAR_\theta(Y) + 2\times COV_\theta(X,Y)$.
$\square$

Theorem 1. (Expectile CAPM) *Suppose Assumption A.1 to A. 6 in the Appendix holds. Then the following expectile CAPM representation holds for* $\mathbb{E}_\theta(.)$:

$$\mathbb{E}_\theta(r_i - r_f) = \beta^\theta\mathbb{E}_\theta(r_M - r_f), i = 1, 2, \cdots, n \tag{11}$$

where $\beta^\theta \triangleq \dfrac{\Theta^2\widetilde{\sigma}_{iM} + \Phi\sigma_{iM}}{\Theta^2\widetilde{\sigma}_M^2 + \Phi\sigma_M^2}$, $\sigma_{iM} \triangleq \sum\varpi_j\sigma_i\sigma_j\rho_{ij}$, $\sigma_M^2 \triangleq \sum_{j=1}^n\varpi_j\sigma_M\sigma_j\rho_{ij}$, $\widetilde{\sigma}_{iM} \triangleq \sum_{j=1}^n\varpi_j\sigma_i\sigma_j sign(\rho_{ij})\sqrt{1-\rho_{ij}^2}$, $\widetilde{\sigma}_M^2 \triangleq \sum_{j=1}^n\varpi_j\sigma_M\sigma_j sign(\rho_{jM})\sqrt{1-\rho_{jM}^2}$, $\Theta \triangleq \mathbb{E}_\theta(w)$,

$\Phi \triangleq \mathbb{E}_\theta(w^2)$, *$\varpi_i$ is the weight of security i within the portfolio, and $w \sim N(0,1)$.*

Proof. We first define the following variables. $W(t)$ $\triangleq$Total wealth at time t. $P_i(t)$ $\triangleq$Price of the ith asset at time $t(i = 1, \cdots, n)$. $S_j(t) \triangleq$ Value of the jth state variable at time $t(j = 1, \cdots, m)$. $C(t) \triangleq$ Consumption per unit time at time t. $w_i(t) \triangleq$ Proportion of total wealth in the ith asset at time $t(i = 1, \cdots, n)$. Note $\sum_{i=1}^{n} w_i(t) \equiv 1$ We model the consumption and portfolio choosing process as follows,

$$J[W(t), S(t), t] \equiv \max_{C(s),\varpi(s)} \mathbb{E}_{\theta,t}\{\int_t^T U_1\left[C(s), s\right] ds + U_2\left[W(T), T\right]\} \tag{A.3}$$

St: boundary condition:

$$J\left[W(T), S(T), T\right] = U_2\left[W(T), T\right].$$

budget equation:

$$W(t) = \sum_{i=1}^{n} w_i(t_0)\frac{P_i(t)}{P_i(t_0)}\left[W(t_0) - C(t_0)h\right]$$

assumption1: $t \equiv t + h$, $h \to 0$ assumption2:

$$\frac{dP_i(t)}{P_i(t)} = \mu_i(S,t)dt + \sigma_i(S,t)\sqrt{dt}w_i, i = 1, 2, \cdots, n$$

$$\frac{dP_i(t)}{P_i(t)} = \mu_i(S,t)dt + \sigma_i(S,t)\sqrt{dt}w_i, i = 1, 2, \cdots, n$$

$$V = \left[\sigma_i l\right], \sigma_{il} = \sigma_i\sigma_l\rho_{il}, i, l = 1, 2, \cdots, n$$

$$dS_j(t) = f_j(S,t)dt + g_i(S,t)\sqrt{dt}q_j, j = 1, 2, \cdots, m$$

By Taylor's theorem and the mean value theorem for integrals, Eq. (A.3) can be rewritten as,

$$
\begin{aligned}
J\left[W(t_0), S(t_0), t_0\right] = \underset{C(s),\varpi(s)}{max} & \mathbb{E}_{\theta,t_0}\{U_1\left[C(\bar{t}), \bar{t}\right] h + J\left[W(t_0), S(t_0), t_0\right] \\
& + \frac{\partial J\left[W(t_0), S(t_0), t_0\right]}{\partial dt} h + \frac{\partial J\left[W(t_0), S(t_0), t_0\right]}{\partial W}\left[W(t) - W(t_0)\right] \\
& + \sum_{j=1}^{m} \frac{\partial J\left[W(t_0), S(t_0), t_0\right]}{\partial S_j}\left[S_j(t) - S_j(t_0)\right] \\
& + \frac{1}{2}\frac{\partial^2 J\left[W(t_0), S(t_0), t_0\right])}{\partial W^2}\left[W(t) - W(t_0)\right]^2 \\
& + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 J\left[W(t_0), S(t_0), t_0\right]}{\partial S_k \partial S_j}\left[S_k(t) - S_k(t_0)\right]\left[S_j(t) - S_j(t_0)\right] \\
& + \sum_{j=1}^{m} \frac{\partial^2 J\left[W(t_0), S(t_0), t_0\right]}{\partial W \partial S_j}\left[W(t) - W(t_0)\right]\left[S_j(t) - S_j(t_0)\right] + O(h^2)\}
\end{aligned}
$$

where $\bar{t} \in [t_0, t]$, take limit as $h \to 0$, take the θ-adjusted expectation operators onto each term, and subtracting, $J[W(t_0), S(t_0), t_0]$ of both sides, we have

$$
\begin{aligned}
0 = & \underset{C(s),\varpi(s)}{max}\{U_1\left[C(t), t\right] dt + \frac{\partial J\left[W(t), S(t), t\right]}{\partial dt} dt \\
& + \frac{\partial J\left[W(t), S(t), t\right]}{\partial W}\mathbb{E}_{\theta,t}\left[dW(t)\right] \\
& + \sum_{j=1}^{m} \frac{\partial J\left[W(t), S(t), t\right]}{\partial S_j}\mathbb{E}_{\theta,t}\left[dS_j(t)(t)\right] \\
& + \frac{1}{2}\frac{\partial^2 J\left[W(t), S(t), t\right])}{\partial W^2}\mathbb{E}_{\theta,t}\left[dW(t)\right]^2 \\
& + \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \frac{\partial^2 J\left[W(t), S(t), t\right]}{\partial S_k \partial S_j}\left[S_k(t) - S_k(t)\right]\mathbb{E}_{\theta,t}\left[dS_j(t)\right] \\
& + \sum_{j=1}^{m} \frac{\partial^2 J\left[W(t), S(t), t\right]}{\partial W \partial S_j}\mathbb{E}_{\theta,t}\left[dW(t) dS_j(t)\right] + O(dt^2)\}
\end{aligned}
\tag{A.4}
$$

By subtracting $W(t_0)$ on both sides, the budget equation is rewritten as,

$$
W(t) - W(t_0) = \left[\sum_{i=1}^{n} \varpi_i(t_0)\frac{P_i(t) - P_0(t)}{P_i(t_0)}\right]\left[W(t_0) - C(t_0)h\right] - C(t_0)h
$$

The expectile of the limit process as $h \to 0$ is

$$\begin{aligned}\mathbb{E}_{\theta,t}[dW(t)] &= \{\sum_{k=1}^{n} \varpi_i(t)W(t)\mathbb{E}_{\theta,t}\left(\frac{dP_i(t)}{P_i(t)}\right) - C(t)dt\} + O(dt^2) \\ &= \{\sum_{k=1}^{n}\left(\varpi_i(t)W(t)\left[\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w_i)\right] - C(t)\right)\}dt\end{aligned} \tag{A.5}$$

Applying the same limit process to other terms,

$$\mathbb{E}_{\theta,t}[dW(t)]^2 = \sum_{l=1}^{n}\sum_{i=1}^{n} \varpi_i(t)\varpi_l(t)W(t)2\sigma_i(S,t)\sigma_l(S,t)\mathbb{E}_(\theta,t)[w_i w_l]dt \tag{A.6}$$

$$\mathbb{E}_{\theta,t}[dS_j(t)] = f_j(S,t)dt + g_j(S,t)\sqrt{dt}\mathbb{E}_{\theta,t}(q_i) \tag{A.7}$$

$$\mathbb{E}_{\theta,t}[dS_k(t)dS_j(t)] = g_k(S,t)g_j(S,t)\mathbb{E}_{\theta,t}[q_k(t)q_j(t)]dt + O(dt^2) \tag{A.8}$$

$$\mathbb{E}_(\theta,t)[dW(t)dS_j(t)] = \left[\sum_{i=1}^{n} \varpi_i(t)W(t)\sigma_i(S,t)g_j(S,t)\mathbb{E}_{\theta,t}[w_i q_j(t)]\right]dt \tag{A.9}$$

Take Eq. (A.5) to Eq. (A.9) into Eq. (A.4), and assume the nth asset is risk free asset, we get the following HJB function:

$$\begin{aligned}0 = \max_{C(s),\varpi(s)} &\{U_1\left[C(t),t\right]dt + \frac{\partial J\left[W(t),S(t),t\right]}{\partial dt}dt + \frac{\partial J\left[W(t),S(t),t\right]}{\partial W} \\ &\times \{\left[\left(\sum_{i=1}^{n-1}\left(\varpi_i(t)\left(\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w_i)\right) - r_f\right) + r_f\right)W(t)\right] - C(t)\} \\ &+ \sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial S_j}\left[f_j(S,t) + \frac{g_j(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(q_i)\right] \\ &+ \frac{1}{2}\partial^2 J[W(t),S(t),t]/W^2\sum_{l=1}^{n-1}\sum_{i=1}^{n-1}\varpi_i(t)\varpi_l(t)W(t)^2 \times \sigma_i(S,t)\sigma_l(S,t)\mathbb{E}_(\theta,t)(w_i w_l) \\ &+ \frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 J[W(t),S(t),t]}{\partial S_k \partial S_j}g_k(S,t)g_j(S,t)\mathbb{E}_(\theta,t)(q_k q_j) \\ &+ \sum_{j=1}^{m}\partial^2 J[W(t_0),S(t_0),t_0]\partial W\partial S_j[\sum_{i=1}^{n-1}\varpi_i(t)W(t)\sigma_i(S,t) \times g_j(S,t)\mathbb{E}_(\theta,t)(w_i q_j)]\}\end{aligned}$$

We let the derivatives of HJB w.r.t consumption $C(t)$ and weight assigned to each risky asset, $\varpi_1(t)$ to $\varpi_{n-1}(t)$, equal to zero, the first order conditions are,

$$U_{1,C}\left[C^*(t),t\right]-\frac{\partial J\left[W(t),S(t),t\right]}{\partial W}=0$$

$$\begin{aligned}
&\frac{\partial J[W(t),S(t),t]}{\partial W}\{\mu_i(S,t)+\frac{\sigma_i(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w_i)-r^f\}\\
&+\frac{\partial^2 J[W(t),S(t),t]}{\partial W^2}\sum_{l=1}^{n-1}\varpi_l^* W(t)\sigma_i(S,t)\sigma_l(S,t)\mathbb{E}_{\theta,t}(w_i w_l)\\
&+\sum_{j=1}^{m}\frac{\partial J[W(t),S(t),t]}{\partial W\partial S_j}\{\sigma_i(S,t)g_j(S,t)\mathbb{E}_{\theta,t}(w_i w_j)\}\\
&=0, i=1,2,\cdots,n-1
\end{aligned}$$

Define a more compact expression in the following way,

$$\begin{aligned}
&V=[\sigma_i\sigma_l\mathbb{E}_{\theta,t}(w_i w_l)], i,l=1,2,\cdots,n-1\\
&\Gamma=[\sigma_i g_j\mathbb{E}_{\theta,t}(w_i q_j)], i,l=1,2,\cdots,n-1; j=1,2,\cdots,m
\end{aligned}$$

Then, we can get the optimized portfolio process,

$$\begin{aligned}
\varpi^*=&-\frac{J_W[W(t),S(t),t]}{W(t)J_{WW}[W(t),S(t),t]}V^{-1}\{\mu_i(S,t)+\frac{\sigma_i(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w_i)\\
&-r^f\}-V^{-1}\Gamma\frac{J_{SW}[W(t),S(t),t]}{W(t)J_{WW}[W(t),S(t),t]}
\end{aligned}$$

We write the above formula in form of vectors, and sum K homogeneity investors' portfolio weight, we get the aggregated market portfolio weight.

$$\begin{aligned}
\varpi_M&=\frac{\sum_{k=1}^{K}\varpi^K W^K}{\sum_{k=1}^{K}W^K}\\
&=\frac{A}{M}V^{-1}\{\mu(S,t)+\frac{\sigma(S,t)}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w)-\gamma^f\}+V^{-1}\Gamma\frac{B}{M}
\end{aligned}$$

where

$$A = \sum_{k=1}^{K} \left(- \frac{J_W^k[W(t), S(t), t]}{J_{WW}^k[W(t), S(t), t]} \right); B = \sum_{k=1}^{K} \left(- \frac{J_{SW}^k[W(t), S(t), t]}{J_{WW}^k[W(t), S(t), t]} \right); M = \sum_{k=1}^{K} W^k.$$

The expectile excess return vector satisfies the following equation,

$$\mu(S,t) + \frac{\sigma(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(w) - \gamma^f = \varpi_M' V \frac{M}{A} - \Gamma \frac{B}{A}$$

We denote each scalar of the vector $\varpi_M' V$ as σ_{iM}^{θ}, and assuming that state variables are constants,

$$\mu_i(S,t) + \frac{\sigma_i(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(w) - r^f = \sigma_{iM}^{\theta} \frac{M}{A} \tag{A.10}$$

We specify the term of σ_{iM}^{θ}, and get

$$\begin{aligned} \sigma_{iM}^{\theta} &= \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \mathbb{E}\varpi\varpi_{\theta,t}(w_i w_j) \\ &= \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \{ [\mathbb{E}_{\theta,t}(w)]^2 \sqrt{1 - \rho_{ij}^2} \times sign(\rho_{ij} + \mathbb{E}_{\theta,t}(w^2)\rho_{ij} \} \\ &= \mathbb{E}_{\theta,t}(w)]^2 \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \sqrt{(1 - \rho_{ij}^2} \times sign(\rho_{ij}) + \mathbb{E}_{\theta,t}(w^2) \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j \rho_{ij} \end{aligned} \tag{A.11}$$

We consider a market portfolio as a whole, and then it is a one-dimension random variable. We use symbol $\bar{M}$ to distinguish it from n dimensional market portfolio,

$$\mu_{\bar{M}}(S,t) + \frac{\sigma_{\bar{M}}(S,t)}{\sqrt{dt}} \mathbb{E}_{\theta,t}(\varpi) - \gamma^f = \sigma_{\bar{M}M}^{\theta} \frac{M}{A} \tag{A.12}$$

where

$$\begin{aligned}
\sigma^{\theta}_{\bar{M}M} &= \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \mathbb{E}_{\theta,t}(w_{\bar{M}} w_j) \\
&= \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \{[\mathbb{E}_{\theta,t}(w)]^2 \sqrt{1-\rho^2_{j\bar{M}}} \times sign(\rho_{j\bar{M}}) + \mathbb{E}_{\theta,t}(w^2)\rho_{j\bar{M}}\} \\
&= [\mathbb{E}_{\theta,t}(w)]^2 \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \sqrt{(1-\rho^2_{j\bar{M}}} \times sign(\rho_{j\bar{M}}) + \mathbb{E}_{\theta,t}(w^2) \sum_{j=1}^{n} \varpi_j \sigma_{\bar{M}} \sigma_j \rho_{j\bar{M}}
\end{aligned}$$

The reason why we use symbol $\bar{M}$ to distinguish the market portfolio being taken as one security and as an n dimensional portfolio is that the expectile does not satisfy the additivity when there is a risk dimensional-receding. After the coefficients being taken out of the expectile operator, there is no difference between $\bar{M}$ and M, therefore we can rewrite the above expression as follows,

$$\sigma^{\theta}_{\bar{M}M} = [\mathbb{E}_{\theta,t}(w)]^2 \sum_{j=1}^{n} \varpi_j \sigma_M \sigma_j \sqrt{1-\rho^2_{jM}} \times sign(\rho_{jM}) + \mathbb{E}_{\theta,t}(w^2)\sigma^2_M \tag{A.13}$$

Take equations Eq. (A.11) and Eq. (A.13) into Eq. (A.10) and Eq. (A.12) respectively, and denote $\mu_i \triangleq \mu_i(S,t), \sigma_i \triangleq \sigma_i(S,t)$

$$\frac{\mu_i + \frac{\sigma_i}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - \gamma^f}{\mu_M + \frac{\sigma_M}{\sqrt{dt}}\mathbb{E}_{\theta,t}(w) - \gamma^f} = \frac{\sigma^{\theta}_{iM}}{\sigma^{\theta}_{\bar{M}M}}$$

We denote $\Theta \triangleq (w)$, $\Phi \triangleq \mathbb{E}_{\theta}(w^2)$, $\widetilde{\sigma}_{iM} \triangleq \sum_{j=1}^{n} \varpi_j \sigma_i \sigma_j sign(\rho_{ij})\sqrt{1-\rho^2_{ij}}$, $\widetilde{\sigma}^2_M \triangleq \sum_{j=1}^{n} \varpi_j \sigma_M \sigma_j sign(\rho_{jM})\sqrt{1-\rho^2_{jM}}$, ϖ_i is the weight of security i within the portfolio, and $w \sim (0,1)$. Hence we have two equivalent expressions of expectile CAPM.

$$\mathbb{E}_{\theta}(r_i - r_f) = \beta^{\theta}\mathbb{E}_{\theta}(r_M - r_f), i = 1, 2, \cdots, n.$$

or

$$\mu_i + \frac{\sigma_i}{\sqrt{dt}}\Theta - r^f = \beta^{\theta}(\mu_M + \frac{\sigma_i}{\sqrt{dt}}\Theta - r^f), i = 1, 2, \cdots, n.$$

$\square$

Theorem 2. (Equity premium puzzle) *Assume the correlation between consumption growth rate and the market returns is* ρ_{CM}*, then* $\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}}\Theta - r_f = \alpha\sigma_C\sigma_M(\Phi\rho_{CM} + \Theta^2 sign(\rho_{CM})\sqrt{1-\rho_{CM}^2})$*, where* σ_C *is the volatility of the aggregate consumption growth rate.*

Proof. We assume the investor is solving the following portfolio optimization problem.

$$
\begin{aligned}
&\underset{\varpi(t)}{max}\mathbb{E}_{\theta,t}\{\int_t^\infty e^{-\delta s}U\left(C(s)\right)ds\} \\
&= \mathbb{E}_{\theta,t}\{\int_t^{t+h} e^{-\delta s}U\left(C(s)\right)ds\} + \mathbb{E}_{\theta,t}\{\int_{t+h}^\infty e^{-\delta s}U\left(C(s)\right)ds\} \\
&= \mathbb{E}_{\theta,t}\{\int_t^{t+h} e^{-\delta s}U\left(C(s)\right)ds\} + \mathbb{E}_{\theta,t}\{\int_0^\infty e^{-\delta(t+h+s)}U\left(C(t+h+s)\right)ds\}
\end{aligned}
$$

$$
St: C(t) = e(t) - \varpi(t)P_M(t);\; C(t+s+h) = e(t+s+h) + \varpi(t)D(t+s+h)h;\; h \to 0
$$

where $\varpi(t)$ is number of shares of market portfolio, δ is time preference, $e(t)$ is endowment at time t, $D(t)$ is dividend at time t. We take the same approach shown in Cochrane (2001) and get the following similar result, which is expressed under expectile operator.

$$
\mathbb{E}_{\theta,t}\left(\frac{dP_M(t)}{P_M(t)}\right) + \frac{D(t)}{P_M(t)}dt - r^f dt = \alpha\mathbb{E}_{\theta,t}\left[\frac{dC(t)}{C(t)}\frac{dP_M(t)}{P_M(t)}\right] \tag{A.14}
$$

Taking the diffusion processes into the right hand side of the above equation, we have

$$
\mathbb{E}_{\theta,t}\left[\frac{dC(t)}{C(t)}\frac{dP_M(t)}{P_M(t)}\right] = \mathbb{E}_{\theta,t}\left[(\mu_C dt + \sigma_C\sqrt{dt}w_C)(\mu_M dt + \sigma_M\sqrt{dt}w_M)\right]
$$

where w_C, w_M are standard normal distributed, their correlation is ρ_{CM}. Omitting the high order derivatives, we obtain

$$\begin{aligned}\mathbb{E}_{\theta,t}\left[\frac{dC(t)}{C(t)}\frac{dP_M(t)}{P_M(t)}\right] &= \sigma_C\sigma_M dt\mathbb{E}_{\theta,t}[w_C w_M] \\ &= \sigma_C\sigma_M dt\left([\mathbb{E}_{\theta,t}(w)]^2\sqrt{1-\rho_{CM}^2}\times sign(\rho_{CM}) + [\mathbb{E}_{\theta,t}(w)]^2\rho_{CM}\right)\end{aligned} \tag{A.15}$$

Taking Eq. (A.15) into Eq. (A.14), and cancel dt on both sides, we have

$$\begin{aligned}&\mu_M + \frac{D(t)}{P_M(t)} + \sigma_M\sqrt{dt}\mathbb{E}_{\theta,t}(w) - r^f \\ &= \alpha\sigma_C\sigma_M\left([\mathbb{E}_(\theta,t)(w)]^2\sqrt{1-\rho_{CM}^2}\times sign(\rho_{CM}) + [\mathbb{E}_{\theta,t}(w)]^2\rho_{CM}\right)\end{aligned}$$

We drop t, assume $D(t) = 0$, and denote $\Theta \triangleq (w)$, $\Phi \triangleq \mathbb{E}_\theta(w^2)$, we get

$$\mathbb{E}(r_M) + \frac{\sigma_M}{\sqrt{dt}}\Theta - r_f = \alpha\sigma_C\sigma_M\left(\Theta^2 sign(\rho_{CM})\sqrt{1-\rho_{CM}^2} + \Phi\rho_{CM}\right)$$

If the volatility of aggregation consumption growth rate and the volatility of rate of return of market portfolio are completely correlated, that is, $\rho_{CM} = 1$.

$$\mathbb{E}(r_M) + \frac{\sigma_M}{dt}\Theta - r_f = \alpha\sigma_C\sigma_M(\Phi\rho_{CM}).$$

□

Reference

Abel, A.B., 2002. An exploration of the effects of pessimism and doubt on asset returns. Journal of Economic Dynamics and Control 26, 1075–1092.

Amihud, Y., 2002. Illiquidity and stock returns: Cross-section and time-series effects. Journal of Financial Markets 5, 31–56.

Araujo, A., Chateauneuf, A., Faro, J., 2012. Pricing rules and arrow-debreu ambiguous valuation. Economic Theory 49, 1–35.

Athanasoulis, S.G., Sussmann, O., 2007. Habit formation and the equity-premium puzzle: A sceptical view. Annals of Finance 3, 193–212.

Banz, R.W., 1981. The relationship between return and market value of common stock. Journal of Financial Economics 9, 3–18.

Bassett, G.W., Koenker, R., Kordas, G., 2004. Pessimistic portfolio allocation and choquet expected untility. Journal of Financial Econometrics 2, 477–492.

Benartzi, S., Thaler, R.H., 1995. Myopic loss aversion and the equity premium puzzle. The Quarterly Journal of Economics 110, 73–92.

Bossaerts, P., Ghirardato, P., Guarnaschelli, S., Zame, W.R., 2010. Ambiguity in asset markets: theory and experiment. The Review of Financial Studies 23, 1325–1359.

Campbell, J.Y., 2000. Asset pricing at the millennium. The Journal of Finance 4, 1515–1567.

Cochrane, J.H., 2001. Asset Pricing. Revised ed., Princeton University Press.

Danthine, J.P., Donaldson, J.B., Giannikos, C., Guirguis, H., 2004. On the consequences of state dependent preferences for the pricing of financial assets. Finance Research Letters 1, 143–153.

van Dijk, M.A., 2011. Is size dead? a review of the size effect in equity returns. Journal of Banking and Finance 35, 3263–3274.

Du, D., 2011. General equilibrium pricing of options with habit formation and event risks. Journal of Financial Economics 99, 400–426.

Ellsberg, D., 1961. Risk, ambiguity and the savage axioms. Quarterly Journal of Economics 75, 643–669.

Falato, A., 2009. Happiness maintenance and asset prices. Journal of Economic Dynamics and Control 33, 1247–1262.

Fama, E.F., French, K.R., 1996. Multifactor explanations of aseet pricing anomalies. Journal of Finance 51, 55–84.

Fielding, D., stracca, L., 2007. Myopic loss aversion, disappointment aversion, and the equity premium puzzle. Journal of Economic Behavior and Organization 64, 250–268.

Ghirardato, P., Maccheroni, F., Marinacci, M., 2004. Differentiating ambiguity and ambiguity attitude. Journal of Economic Theory 118(2), 133–173.

Gilboa, I., Schmeidler, D., 1989. Maxmin expected utility with non-unique prior. Journal of Mathematical Economics 18, 141–153.

Giorgi, E.G.D., Post, T., 2011. Loss aversion with a state dependent reference point. Management Science 57, 1094–1110.

Gollier, C., Muermann, A., 2010. Optimal choice and beliefs with ex ante savoring and ex post disappointment. Management Science 56, 1272–1284.

Gourieroux, C., Liu, W., 2006. Efficient Portfolio Analysis Using Distortion Risk Measures. Cahiers du CREF, HEC Montréal, Centre de recherche en e-finance.

Guidolin, M., 2006. Pessimistic beliefs under rational learning: Quantitative implications for the equity premium puzzle. Journal of Economics and Business 58, 85–118.

Hirshleifer, D., 2001. Investor psychology and asset pricing. Journal of Finance 56, 1533–1597.

Jacobs, K., Wang, K.Q., 2004. Idiosyncratic consumption risk and the cross-section of asset returns. Journal of Finance 59, 2211–2252.

Jegadeesh, N., Titman, S., 1993. Returns to buying winners and selling losers: Implications for stock market efficiency. Journal of Finance 48, 65–91.

Kang, J., Kim, H.S., 2012. An interrelation of time preference and risk attitude: an application to the equity premium puzzle. Applied Economics Letters 19, 483–486.

Klibanoff, P., Marinacci, M., Mukerji, S., 2005. A smooth model of decision making under ambiguity. Econometrica 73, 1849–1892.

Knight, F.H., 1921. Risk, Uncertainty and Profit. New York: Harper.

Krausa, A., Sagi, J.S., 2006. Asset pricing with unforeseen contingencies. Journal of Financial Economics 82, 417–453.

Mehra, R., Prescott, E.C., 1985. The equity premium a puzzle. Journal of Monetary Economics 15, 145–161.

Meyer, D.J., Meyer, J., 2005. Risk preferences in multi-period consumption models, the equity premium puzzle, and habit formation utility. Journal of Monetary Economics 52, 1497–1515.

Otrok, C., Ravikumar, B., Whiteman, C.H., 2002. Habit formation: a resolution of the equity premium puzzle? Journal of Monetary Economics 49, 1261–1288.

Parker, J.A., 2003. Consumption risk and expected stock returns. American Economic Review 93, 376–383.

Sanjoy, B., 1983. The relationship between earnings yield, market value and return for nyse common stocks: Further evidence. Journal of Financial Economics 12, 129–156.

Siegel, J.J., 2008. Stocks for the long run. 4th ed., Mc Graw Hill.

Ui, T., 2011. The ambiguity premium vs. the risk premium under limited market participation. Review of Finance 15, 245–275.

Weil, P., 1989. The equity premium puzzle and the risk-free rate puzzle. Journal of Monetary Economics 24, 401–421.