Fast trees for options with discrete dividends*

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Abstract

The valuation of options using a binomial non-recombining tree with discrete dividends can be intricate. This paper proposes three different enhancements that can be used alone or combined to value American options with discrete dividends using a non-recombining binomial tree. These methods are compared in terms of both speed and accuracy with a large sample of options with one to four discrete dividends. This comparison shows that the best results can be achieved by the simultaneous use of the three enhancements. These enhancements when used together result in significant speed/accuracy gains in the order of up to 200 times for call options and 50 times for put options. These techniques allow the use of a non-recombining binomial tree with very good accuracy for valuing options with up to four discrete dividends in a timely manner.

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American options valuation using binomial lattices can be cumbersome. Several authors have suggested different approaches to speed the computation or to increase the speed of convergence. Among others, some authors suggested ways to reduce the number of the nodes in the tree, thus speeding up the computation time (e.g.: Baule and Wilkens [2004]). One example of the latter approach is a paper by Curran [1995] that has been largely ignored in the literature. One advantage of Curran's [1995] method is that it is not an approximation to the value of a binomial tree, since it produces exactly the same result as Cox, Ross, and Rubinstein's [1979] (CRR) tree with the same number of steps at a fraction of the computation time.

The gains of speeding the computation of a binomial tree are more relevant for valuing options with underling assets that pay discrete dividends. In such cases the binomial trees do not recombine and the number of nodes rapidly explode even for a small number of time steps. There are several approximations that allow the use of recombining binomial trees, but all of them can in some occasions produce large valuation errors (usually they occur when a dividend is paid at the very beginning of the option life), or require a very large number of steps to avoid such valuation errors.[1]

The advantages of using a non-recombining binomial tree are twofold: first it considers the true stochastic process for the underlying asset which excludes arbitrage opportunities; and secondly it eliminates large valuation errors irrespective of when the dividends occur. This, in turn, results in a accurate valuation of such options. The problem of these trees is that the number of nodes in the tree grows exponentially with the number of dividends. This explains why usually a recombining approximation is used instead of the non-recombining tree.

Unfortunately the method proposed by Curran [1995] is not directly applied to options on assets which pay discrete dividends. In light of the techniques proposed by Curran [1995], which in turn are based on the work of Kim and Byun [1994], this paper adjusts their acceleration techniques to the valuation of American put options and also proposes a different accelerated binomial method to the valuation of American call options. We also suggest two other improvements with very good results. One, called here Adapted Binomial, consists in making a time step to coincide with the ex-dividend dates. The other is to apply the Black and Scholes [1973] formula to obtain the continuation value in the last steps of the binomial tree.

These adaptations along with the improvements here suggested result in significant speed/accuracy gains in the order of up to 200 times for call options and 50 times for put options. These techniques allow the use of a non-recombining binomial tree with very good

[1] Examples of such methods are: Schroder [1988], Hull and White [1988], Harvey and Whaley [1992], Wilmott, Dewynne, and Howison [1998], Vellekoop and Nieuwenhuis [2006].

accuracy for valuing options with up to four discrete dividends in a timely manner. Such improvements are particularly relevant since most stock options traded are for maturities under one year, which usually entails up to four dividends during the option lifetime.

These papers unfolds as follows: the next section presents the adaptations and improvements to the valuation of American options with an underlying asset that pays discrete dividends; section compares the performance of these methods in terms of speed and accuracy; section concludes.

Enhancements to the non-recombining binomial tree

There are a few approaches that deal with discrete dividends. The model sometimes designated as the *spot* model is the most realistic but it is not easy to implement in a lattice framework because it will make the trees non recombining. The model treats the discrete dividends by considering a price process where the asset price (S) jumps down by the amount of the dividend d_i paid at time t_i, and between dividends follows a geometric Brownian motion (GBM).

For this model the underlying asset price S does not follow a GBM over the time interval $[0, T]$, because of the jumps at times t_i. It is then a discontinuous process with deterministic jumps at deterministic moments in time. The risk-neutral process follows a GBM between the jumps:

$$dS_t = rS_t dt + \sigma S_t dz_t \quad t_i < t < t_{i+1} \tag{1}$$

$$S_{t_i}^+ = S_{t_i}^- - d_i \tag{2}$$

where $S_{t_i}^-$ is the asset price immediately before the payment of the discrete dividend and $S_{t_i}^+$ is the asset price immediately after such payment.

This model is the one which reflects reality best, but unfortunately there is no known closed form solution even for European options, and for American options the adaptation of a numerical method, such as a binomial/trinomial lattice, to value these options in a timely manner is not straightforward. The difficulty in applying lattices to value options on an asset which follows such a model is that the tree becomes non-recombining after each dividend. Therefore the number of nodes explodes as the number of discrete dividends increases.

Consider an option, with time to maturity T, on an asset which pays only one dividend, D, over its life, at time t_d which is between $k\Delta t$ and $(k+1)\Delta t$ ($\Delta t = \frac{T}{N}$). When $i \leq k$, the nodes in the binomial tree at time $i\Delta t$ correspond to stock prices:[2]

$$Su^j d^{i-j} \quad j = 0, 1, \ldots, i \tag{3}$$

[2]This description of the non recombining tree closely follows the one presented by Hull [1997].

When $i = k + 1$, the nodes on the tree correspond to stock prices:

$$Su^j d^{i-j} - D \quad j = 0, 1, \ldots, i \tag{4}$$

When $i = k + 2$, the nodes on the tree correspond to stock prices:

$$(Su^j d^{i-1-j} - D)u \ \text{ and } \ (Su^j d^{i-1-j} - D)d \quad \text{for } j = 0, 1, \ldots, i-1 \tag{5}$$

Thus there are $2i$ rather than $i + 1$ nodes. At time $(k + m)\Delta t$ there are $m(k + 1)$ instead of $k + m + 1$ nodes. Exhibit 1 illustrates this problem.

Insert Exhibit 1 about here.

Next, we describe the enhancements to non-recombining binomial tree for valuing options on underlying assets that pay discrete dividends, that result in a significant speed gain without sacrificing accuracy.

We first present what we here designate as the Accelerated Binomial method which consists in an adjustment of Curran's [1995] diagonal method to accelerate the valuation of American put options; and also a different accelerated binomial method to the valuation of American call options.

This is followed by a description of the improvement which adapts the binomial tree in order to make a time-step meet ex-dividends dates, here designated as Adapted Binomial. Finally, we describe another improvement designated as Binomial Black-Scholes, were the Black and Scholes [1973] formula is used as the continuation value in the last steps of the binomial tree.

These enhancements can be combined with each other allowing the use of a binomial non-recombining tree for valuing American options with up to four discrete dividends in a timely manner. The speed-accuracy improvements that are achieved are quite significant in an order of up to 200 times for call options and 50 times for put options.

Accelerated Binomial

American call options on assets paying discrete dividends are only optimally exercised at maturity or at the ex-dividend dates (Merton [1973]). A great deal of acceleration can be gained by checking the optimal exercise of the option only at ex-dividend dates, rather than at each step.

Another way to accelerate the computation time is to skip all the subtrees that have zero values on the final nodes. This is done for an entire subtree, originated at an ex-dividend date, when all final nodes are zero, or for the lower part of a subtree whose final nodes are zero.[3]

These two acceleration techniques are illustrated by Exhibit 2. For the hypothetical example depicted, the number of nodes that are computed as usual is only 12 out of 46 (26%). For the same number of time steps, the number of nodes that are not computed is higher the more out-the-money the option is.

Insert Exhibit 2 about here.

For the valuation of American put options we make use and adjust the acceleration techniques suggested by Kim and Byun [1994] and Curran [1995] to the problem of valuing options with discrete dividends. Kim and Byun [1994] suggest two propositions that are only valid for CRR binomial trees. Proposition I states that if a node generates two nodes that are early exercise nodes, that node is also an early exercise node. This proposition was extended to assets with a dividend yield by Curran [1995]. Proposition II states that if for a given underlying asset price the optimal decision is to continue, for that same price at an early date the decision is the same, which is based on the fact that an option with more time to expiration has more value. Curran [1995] makes use of these two propositions to suggest an accelerated method, designated by Diagonal Method (DM), that under certain conditions can decrease substantially the computation time of the tree in the case of American options with a continuous dividend yield.[4] The method computes the last no-early exercise diagonal of the tree, and then jumps to node 0, using first passage probabilities.[5] The method also skips the nodes on the subtrees with zero values.

These techniques cannot be applied directly to the case of American options with an underlying asset that pays a discrete dividend. For such options some adjustments have to be made.[6]

The use of DM method is only possible for subtrees originated at the last ex-dividend date. Before an ex-dividend date, Proposition II does not hold. The consequence is that when a diagonal of no-early exercise nodes is identified, it may not be the last no-early exercise diagonal, and one has to proceed up to the upper diagonal. Nevertheless, Proposition I can still be used to skip the nodes in the lower part of the subtree below the exercise price.

When a subtree has only zeros at the final nodes, the value at the origin is zero. When it has only exercise values at the final nodes, the value at the origin is also the exercise value, by following Proposition I.

[3] To locate the first zero at the last step of the tree, one just has to locate the exercise price K, by finding the smallest node a such that $Su^a d^{steps-a} < K$. For more details please refer to Kim and Byun [1994].

[4] Please refer to Curran [1995] for a detailed description of the method.

[5] Our experiments have shown that it is faster to simply compute the present value along the tree than to compute and make use of the first passage probabilities.

[6] Although the adjustments here suggested to the DM method could be applied to the case of American call options, it would not be as efficient as the afore mentioned Accelerated method.

Exhibit 3 illustrates the accelerated method for the valuation of put options. The number of nodes that are computed as usual is only 15 (33%) and the number of nodes that are not computed at all is 28 out of 46 (61%).

Insert Exhibit 3 about here.

Again, please note that the accelerated method produces exactly the same results as the traditional binomial non-recombining trees.

Adapted binomial

In this subsection we propose an adaptation to the binomial method to value American options with discrete dividends, in the same spirit of Areal and Rodrigues [2010] that suggested an adaptation to the simulation method to improve the valuation of European put and call options as well as American call options.

The adaptation consists in forcing a time-step to coincide with the ex-dividend dates. This can increase the accuracy of the method, particularly in the case of call options. Note that American call options on assets paying discrete dividends are only optimally exercised at maturity or at the ex-dividend dates (Merton [1973]).

This adaptation is done by changing the parameters of the tree at the ex-dividend dates. For the case of a single dividend, the lengths of the time steps that ensure that the time step k is at the ex-dividend date are:

$$\Delta t_1 = \frac{t_d}{k} \tag{6}$$

$$\Delta t_2 = \frac{T - t_d}{N - k} \tag{7}$$

The remaining parameters are adjusted accordingly:

$$p_j = \frac{e^{(r-\delta)\Delta t_j} - d_j}{u_j - d_j} \tag{8}$$

$$u_j = e^{\sigma\sqrt{\Delta t_j}} \tag{9}$$

$$d_j = 1/u_j \tag{10}$$

with $j \in \{1, 2\}$, $j = 1$ denoting the period until the ex-dividend date, and $j = 2$ denoting the remaining time to maturity.

For a call option on an asset paying one dividend, the adapted binomial method is illustrated by Exhibit 4.

Insert Exhibit 4 about here.

The only difference between call and put options binomial trees are the nodes at the ex-dividend dates. Put options are never optimally exercised immediately before the payment of any dividend. Call options, on the contrary, are never optimally exercised immediately after the payment of any dividend. As a result, put options valuation can be improved if the optimal exercise is checked at step k using ex-dividend values of the underlying asset rather than using prices before going ex-dividend.[7]

The expected improvement from the use of this Adapted Binomial method on the valuation of call options is greater than on the valuation of put options, since call options are only optimally exercised at the ex-dividend dates. The analysis performed later in this paper confirms this hypothesis. Remarkably it also shows that the Adapted Binomial method improves the valuation of put options.

Binomial Black-Scholes

A third enhancement is here designated as Binomial Black-Scholes. This was first suggest for binomial recombining trees by Broadie and Detemple [1996]. It consists on the use of the Black and Scholes [1973] formula as the continuation value in the last steps of the binomial tree.

At the step just before the maturity of the option the continuation value of an American option is equal to the value of a European option with similar parameters. The continuation value at that step is replaced with the value given by the Black and Scholes [1973] formula. Such approach increases the computation time since it requires the computation value of N (number of time steps) European options, but this extra time will be compensated by both a smoother and faster convergence to the real option price. This improvement is achieved by using the smooth continuation value given by the Black and Scholes [1973] formula which avoids the discretization problem. Note that this can only be done if the last dividend does not occur at the last step.

The Black and Scholes [1973] formula can be added also to the afore proposed Adapted an Accelerated Binomial trees. For call options valued using the Accelerated Binomial method, the Black and Scholes [1973] formula can be used at the last step after the ex-dividend date, replacing the computation of the hole subtree by the Black and Scholes [1973] formula. For the put options, since we use the DM acceleration technique, the Black and Scholes [1973] formula is only used scarcely. The number of nodes is small and often only one, since the number of nodes at the step before maturity that are computed and need the continuation value is minimal.

Comparison of the enhancements

In order to compare the numerical efficiency of the enhancements to the binomial non-recombining trees presented so far, both in terms of computation time and accuracy, we are

[7]Please note that by making the ex-dividend date coincide with a time-step of the tree, there is no time between the underlying asset value before going ex-dividend and the ex-dividend value.

going to compute option prices for a large sample of random parameters, obtain the relative mean squared error of each approach and compare it with the required computation time, for different step refinements.

Since the gains from the use of the proposed enhancements can differ depending on the number of the dividends, the time when the dividend is paid, the option moneyness and also the number of time steps of the tree, large samples of options were created. This will present a more realistic setting to measure the gains from the enhancements. A sample with one annual dividend, two dividends (one per semester), three quarterly discrete dividends, and four quarterly discrete dividends. For each of these samples a set of 2000 options (1000 calls and 1000 puts) is generated with random parameters. Sample volatility (σ) is distributed uniformly between 0.1 and 0.6. The asset price (S) follows a uniform distribution between 70 and 130 and the exercise price is fixed to 100. The riskless interest rate (r) is uniform between 0.0 and 0.1. For options with one annual and two semiannual discrete dividends, time-to-maturity (T) is uniform between 1 and 360 days.[8] For options with three quarterly discrete dividends time-to-maturity is uniform between 1 and 270 days, and for options with four quarterly discrete dividends time-to-maturity is uniform between 1 and 360 days. The first dividend date is uniform between 1 and 360 days, 1 and 180 days, 1 and 90 days, for one, two, and three or four discrete dividends, respectively. The annual dividend (D) is uniform between 1 and 10 and the sub-annual dividends are proportional.

The option valuation error is measured by the mean square relative error (MSRE) and by the root mean square relative error (RMSRE) which are given by:

$$\text{MSRE} = \frac{1}{M}\sum_{i=1}^{M} e_i^2 \tag{11}$$

$$\text{RMSRE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M} e_i^2} \tag{12}$$

where M is the number of sample options. The relative error is given by $e_i = \frac{\hat{\mathcal{O}}_i - \mathcal{O}_i}{\mathcal{O}_i}$, where $\mathcal{O}_i$ is the option benchmark price and $\hat{\mathcal{O}}_i$ is the estimated option price for a given method.

The benchmark used is a binomial non-recombining tree with Cox, Ross, and Rubinstein [1979] parameters and 1500 steps for the samples up to three dividends. Since it is not possible to compute the option value using a non-recombining tree with the same number of time steps in a timely manner for a number of dividends greater than three, the benchmark for the four dividends sample is computed by simulation using the Longstaff and Schwartz [2001] least squares Monte Carlo (LSMC) valuation method with the Conditional Estimation (CONT-CE) procedure proposed by Areal, Rodrigues, and Armada [2008] with powers polynomials with 5 basis functions and Halton [1960] low discrepancy sequences with Brownian bridges, 50 time steps and 200,000 paths.

[8]The year is assumed to have 360 days.

Insert Exhibit 5 about here.

Insert Exhibit 6 about here.

Insert Exhibit 7 about here.

Insert Exhibit 8 about here.

Exhibits 5 to 8 present the results for each sample.

The first thing to note is that all methods produce very accurate results (with a RMSRE of 0.1% or less), even using a small number of steps. However there are significant differences both in terms of speed and accuracy among the methods.

For the call options samples, the Binomial Black-Scholes non-recombining tree only improves efficiency after a certain number of steps. That number of steps increases as the number of dividends increases, making the method less competitive. This is valid for the traditional non-recombining trees as well as for the Adapted Binomial and Accelerated Binomial trees. The extra time to compute the formula penalizes the method and the smoother continuation value used is not worthy in terms of accuracy.

However, the simultaneous use of the three enhancements here suggested, Adapted Accelerated Binomial with Black-Scholes, can increase substantially the performance of the tree, particularly for a small number of dividends. For one dividend (Exhibit 5(a)), the same level of accuracy of the binomial non-recombining tree without any enhancement, can be achieved approximately 200 times faster with an Adapted Accelerated Binomial with Black-Scholes non-recombining tree. This method is consistently the best method for the all the samples with one to four dividends, although its advantage decreases with the number of dividends.

Most of the efficiency improvement comes from the Adapted Binomial tree. A similar accuracy to a traditional non-recombining tree can be obtained around 20 times faster for one dividend, and up to around 80 times faster in the case of four dividends. The Adapted Binomial tree is able to produce results that are more than twice as precise as the traditional tree, for the same computation time. The Accelerated Binomial method can increase the computation speed around two times for the exact same level of accuracy.

For the put options sample, the best method remains the Adapted Accelerated Binomial with Black-Scholes, except for the sample with four dividends, where a similar accuracy can be obtained with the Accelerated Binomial with Black-Scholes. The Adapted Accelerated Binomial with Black-Scholes has a similar accuracy of the traditional non-recombining binomial tree with a computation time that is more than 50 times faster, or it is able to produce results 5 times more accurate, for the same computation time.

The contribution of each enhancement is different from the call options samples and varies across samples with different number of dividends.

When comparing with the results for the American call samples, the Accelerated Binomial method produces a greater improvement, that decreases with the number of dividends and increases with the number of steps. Please note that the Accelerated Binomial method uses the diagonal method after the last dividend date, increasing substantially the speed of computation of the subtrees originated from the nodes at the last ex-dividend date. This acceleration is higher the higher the number of steps used, as is shown by Curran [1995]. On the other hand, the higher the number of dividends the lower the number of subtrees for which the diagonal method is used, since it can only be used after the last dividend date. For one dividend, the Accelerated Binomial tree is more than 10 times faster, and this superior performance decreases with the number of dividends, being 4 times faster for the sample with four dividends.

Another difference relates to the use of the Binomial Black-Scholes method, that now produces a significant improvement that decreases with the number of dividends, again comparing with the American call samples. The improvement of the Binomial Black-Scholes is potentiated when used along with the other enhancements here proposed.

Finally, the Adapted Binomial method is also able to improve the efficiency, particularly for options with more than one dividend. But this improvement is less pronounced for the put option samples than for the call options samples.

Conclusion

The valuation of options with discrete dividends can be intricate. The use of a traditional non-recombining binomial tree has been so far difficult to use in practice, since the number of nodes in the tree explodes as the number of dividends increase, even for a small number of steps. Nevertheless, this non-recombining tree is the one that can produce good results if a sufficient number of steps is used. Accelerating this method will allow the use of a non-recombining tree to value options that are commonly traded.

This paper contributes to the literature by proposing three different enhancements that can be used alone or combined to value American options with discrete dividends using a non-recombining binomial tree. The Accelerated Binomial method incorporates some techniques proposed by Curran [1995] and Kim and Byun [1994] for the valuation of American put options and also proposes a different accelerated binomial method to the valuation of American call options. Two other enhancements are suggested: the Adapted Binomial, which consists of making a time step coincide with the ex-dividend dates; and the Binomial Black-Scholes.

These methods are compared in terms of both speed and accuracy with a large sample of options with one to four discrete dividends. The number of dividends here considered are the ones that reflect the stock options most actively traded.

This comparison shows that the best results can be achieved by the simultaneous use of the three enhancements. These enhancements when used together result in significant speed/accuracy gains in the order of up to 200 times for call options and 50 times for put options. These techniques allow the use of a non-recombining binomial tree with very good accuracy for valuing options with up to four discrete dividends in a timely manner.

These improvements are particularly relevant since most stock options traded are for maturities under one year, which usually means up to four dividends during the option lifetime. Our results are therefore of interest for both academics and practitioners alike.

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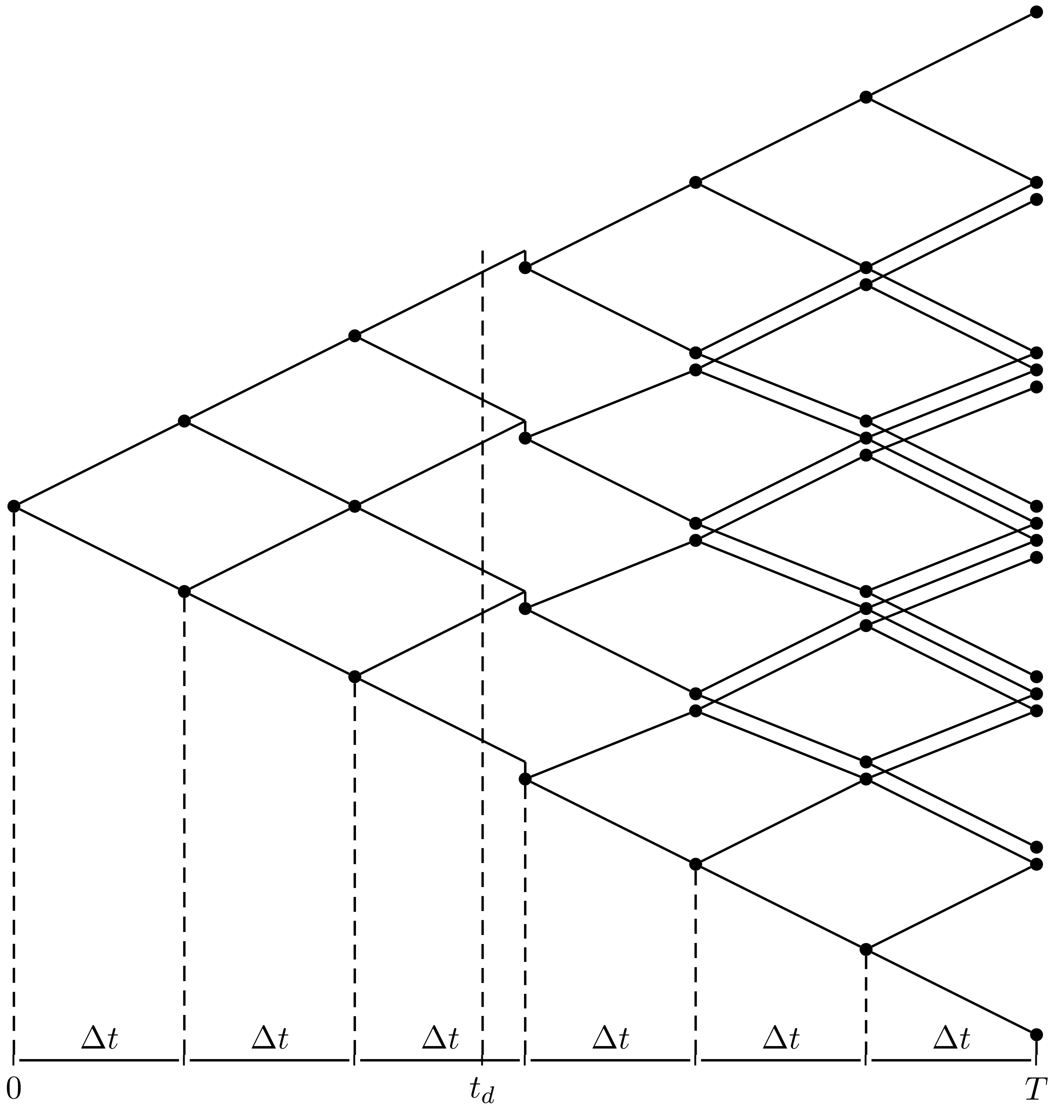
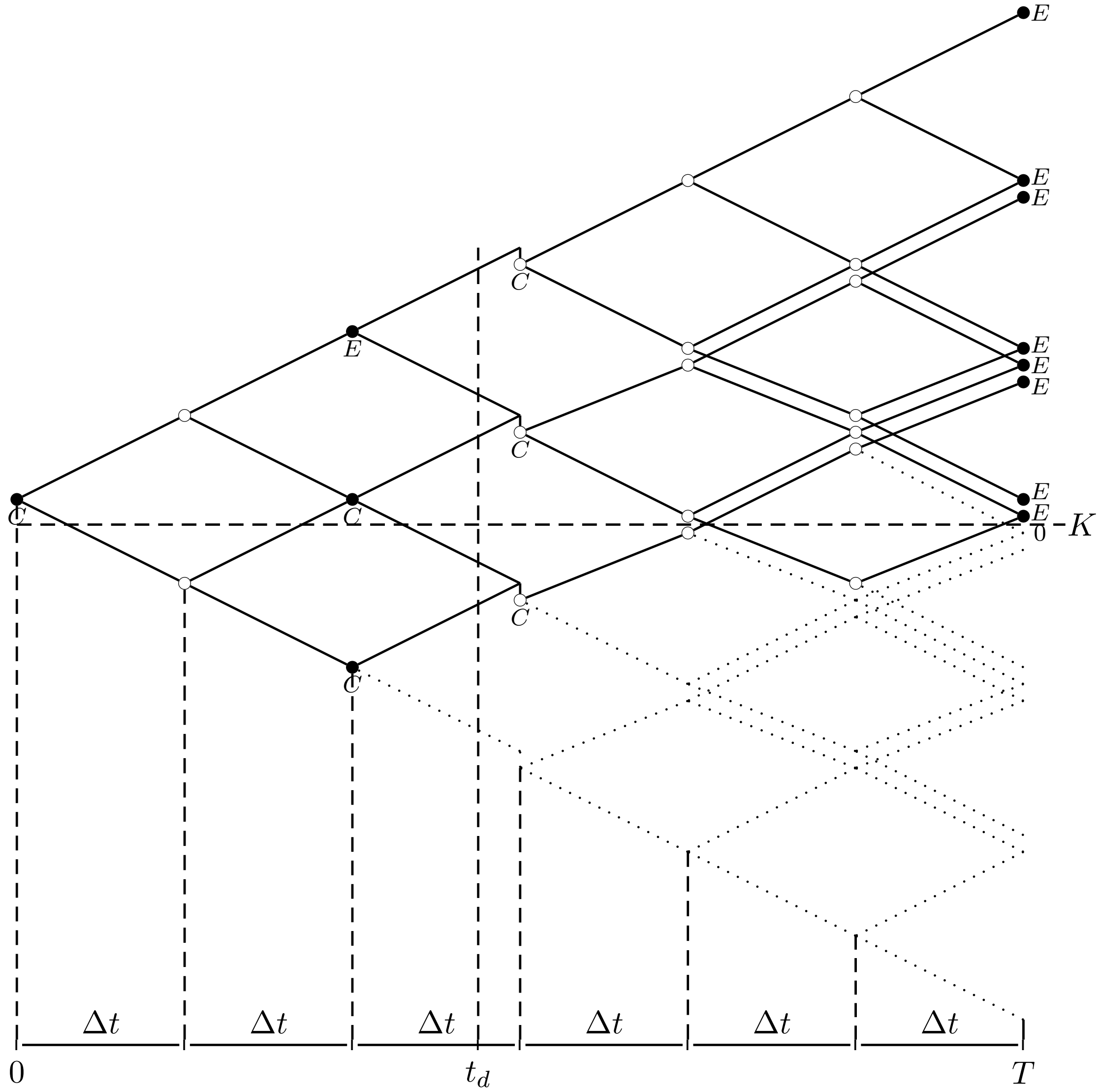
Exhibit 1: Binomial tree with one discrete dividend



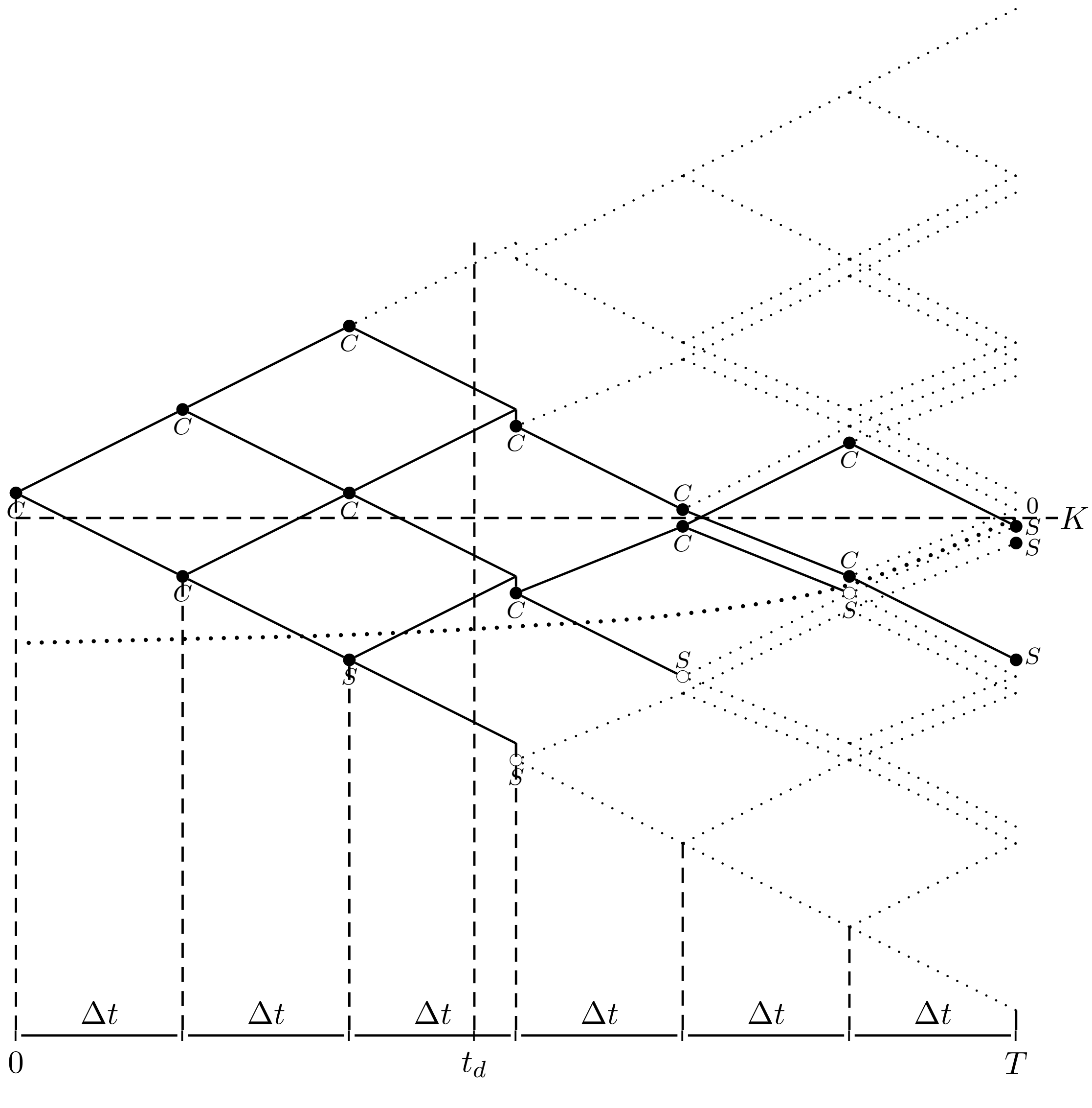
t_d is the ex-dividend date, T the time to maturity, $\Delta t = T/N$, N is the number of time steps, and K is the exercise price.

Exhibit 2: Accelerated Binomial tree - Call option on an asset paying one dividend



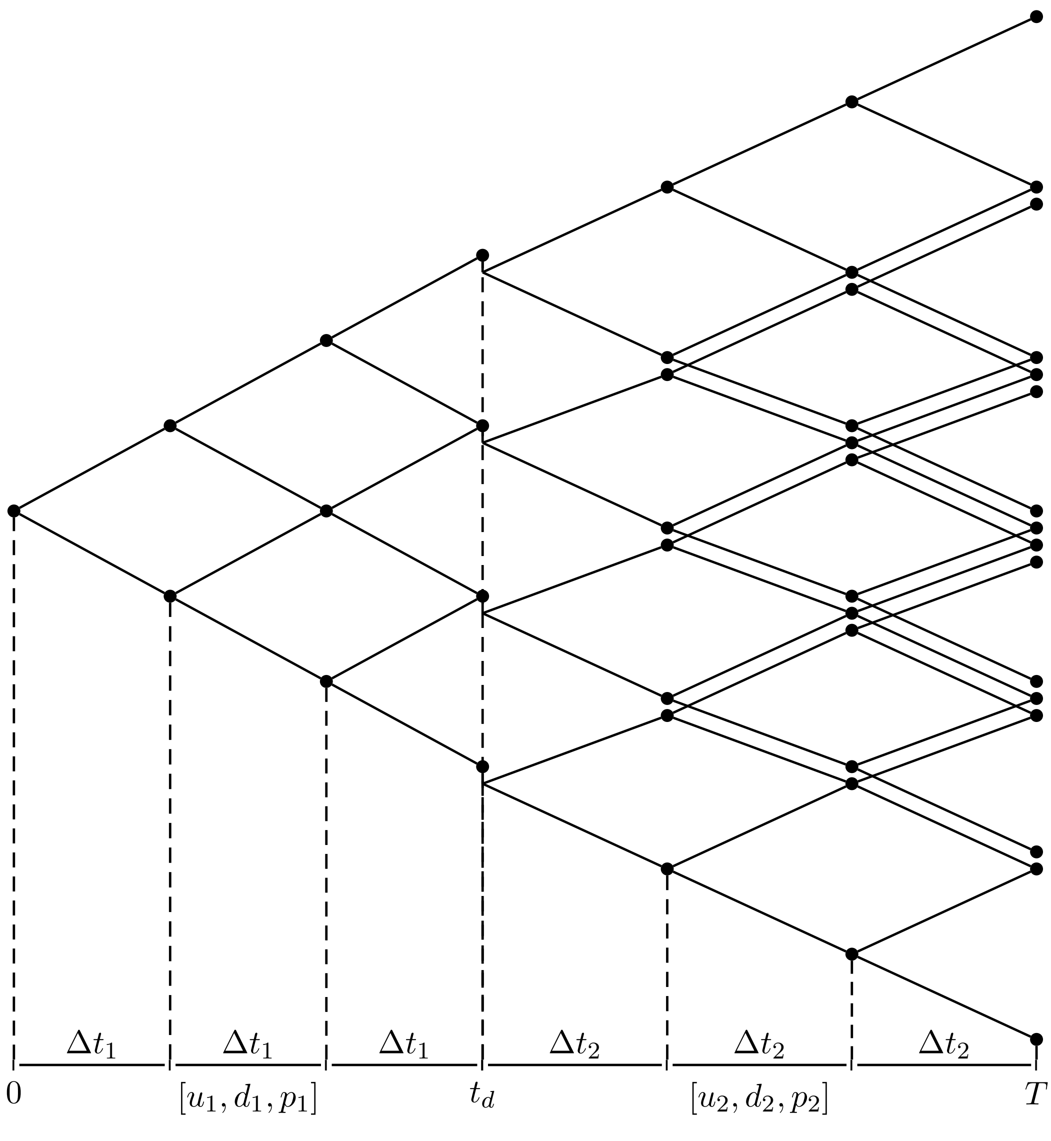
t_d is the ex-dividend date, T the time to maturity, $\Delta t = T/N$, N is the number of time steps, and K is the exercise price. E denotes an early exercise node, for which the exercise price is greater than the continuation value, and C denotes a continuation node when the opposite occurs. Empty bullets correspond to nodes for which the optimal exercise is not checked, whereas filled bullets are the nodes where early exercise is checked as is usual. Dotted lines are subtrees or parts of subtrees that are not computed. For this example the total number of nodes is 46. The number of nodes that are not computed is 17 and the number of nodes for which the optimal stopping is not checked is 17. The remaining 12 nodes are computed as usual.

Exhibit 3: Accelerated Binomial tree - Put option on an asset paying one dividend



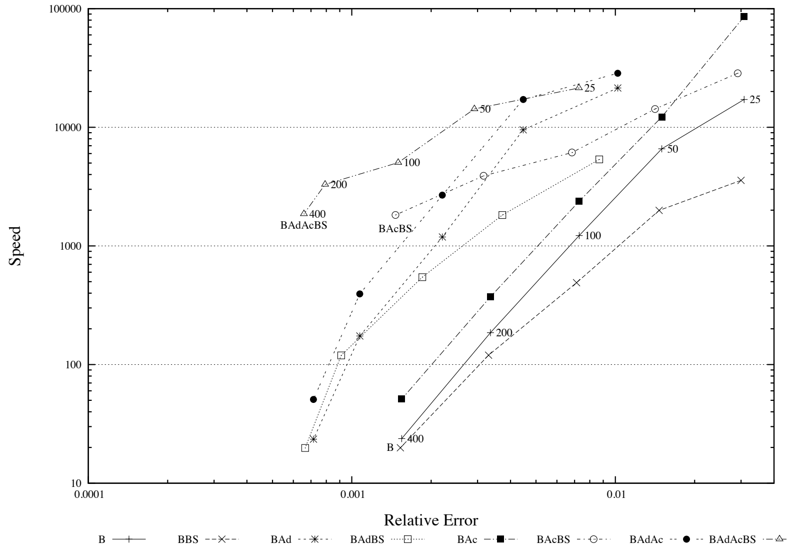
t_d is the ex-dividend date, T the time to maturity, $\Delta t = T/N$, N is the number of time steps, and K is the exercise price. E denotes an early exercise node, for which the exercise price is greater than the continuation value, and C denotes a continuation node when the opposite occurs. The dotted curved line depicts the hypothetical exercise boundary. Empty bullets correspond to nodes for which the optimal exercise is not checked, whereas filled bullets are the nodes where early exercise is checked as is usual. Dotted lines are subtrees or parts of subtrees that are not computed. For this example the total number of nodes is 46. The number of nodes that are not computed is 28 and the number of nodes for which the optimal stopping is not checked is 3. The remaining 15 nodes are computed as usual.

Exhibit 4: Adapted Binomial tree - Call option on an asset paying one dividend

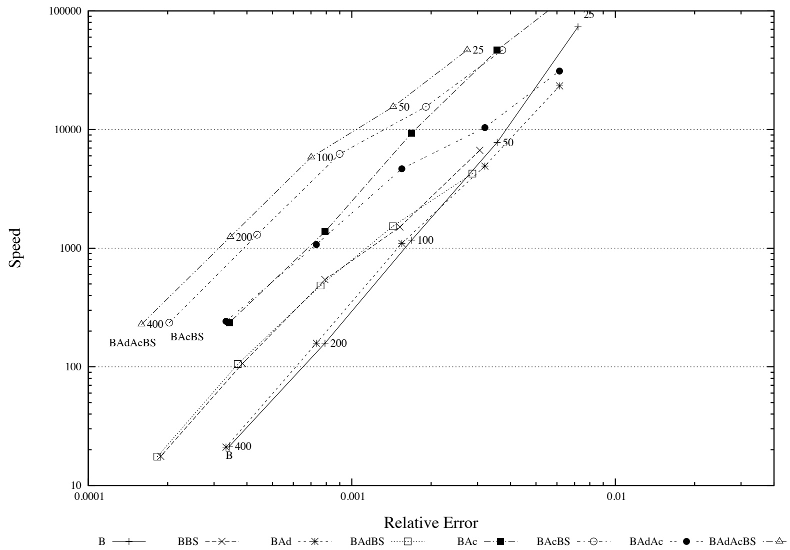


t_d is the ex-dividend date; T the time to maturity; N is the number of time steps; Δt_j are given by equations (6) and (7); u_j, d_j, p_j are given by equations (8) to (10); and, K is the exercise price.

Exhibit 5: Accuracy comparison - 1 dividend



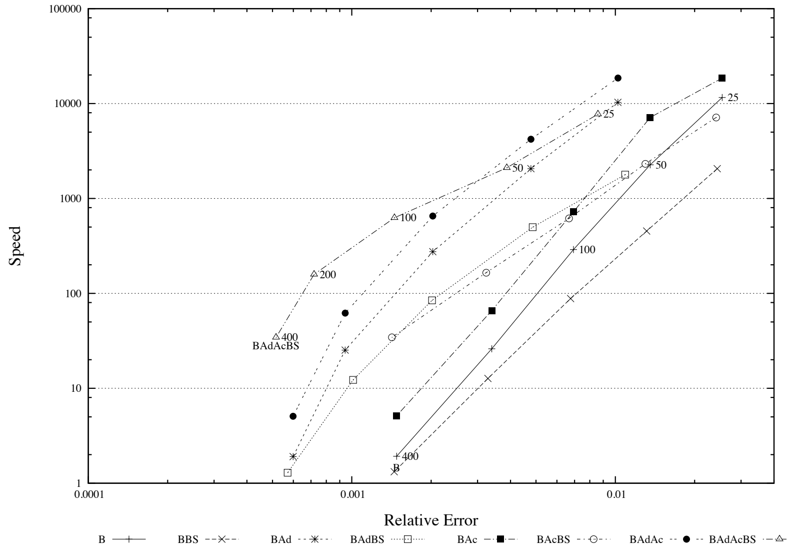
(a) Calls



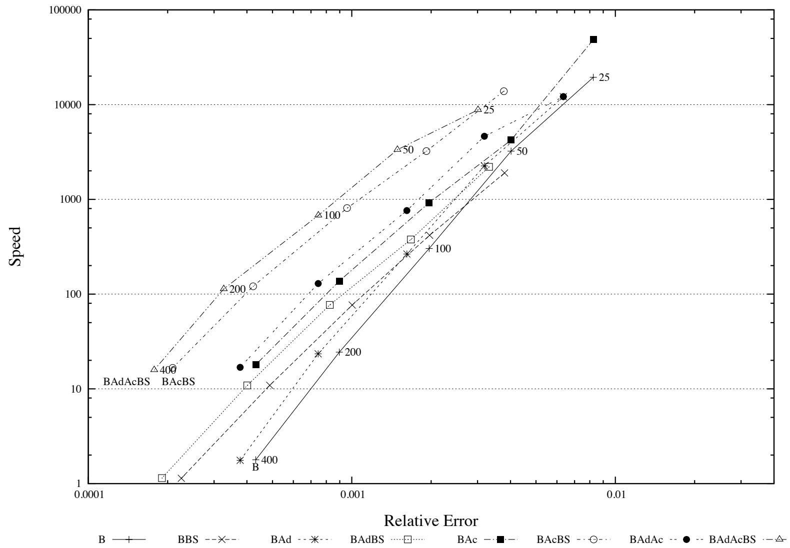
(b) Puts

The graph is in a log-log scale. Relative error is the RMSRE. Speed is measured by options values calculated per second. Only options with value above 0.5 were considered. The benchmark value of the American option is obtained using a non recombining binomial tree with Cox, Ross, and Rubinstein [1979] parameters and 1,500 steps. The binomial non recombining binomial tree methods are as presented in this paper: B stands for Binomial; BBS stands for Binomial with Black-Scholes; BAd stands for Adapted Binomial; BAdBS stands for Adapted Binomial with Black-Scholes; BAc stands for Accelerated Binomial; BAcBS stands for Accelerated Binomial with Black-Scholes; BAdAc stands for Accelerated Adapted Binomial; BAcBS stands for Accelerated Adapted Binomial with Black-Scholes. Numbers beside some methods stand for the number of time steps considered.

Exhibit 6: Accuracy comparison - 2 dividends



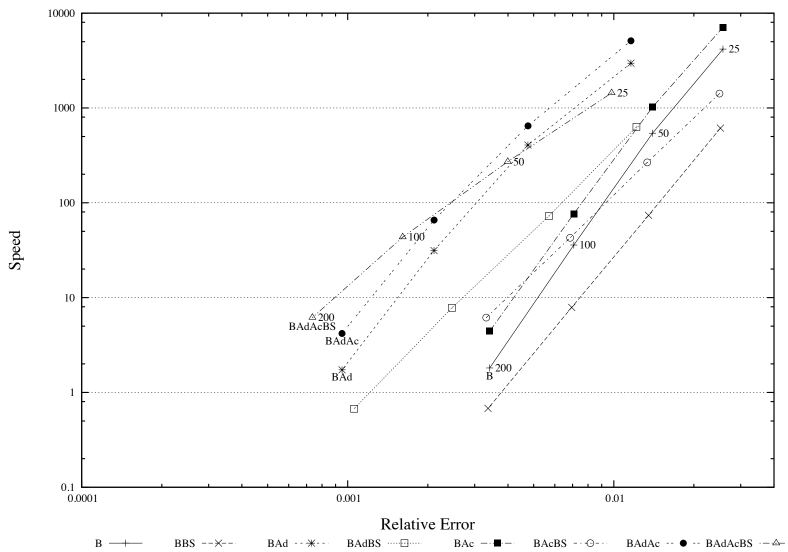
(a) Calls



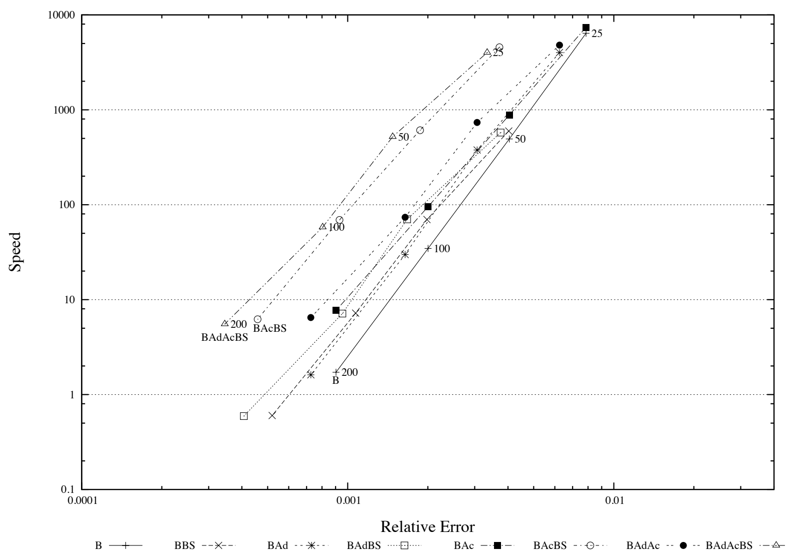
(b) Puts

The graph is in a log-log scale. Relative error is the RMSRE. Speed is measured by options values calculated per second. Only options with value above 0.5 were considered. The benchmark value of the American option is obtained using a non recombining binomial tree with Cox, Ross, and Rubinstein [1979] parameters and 1,500 steps. The binomial non recombining binomial tree methods are as presented in this paper: B stands for Binomial; BBS stands for Binomial with Black-Scholes; BAd stands for Adapted Binomial; BAdBS stands for Adapted Binomial with Black-Scholes; BAc stands for Accelerated Binomial; BAcBS stands for Accelerated Binomial with Black-Scholes; BAdAc stands for Accelerated Adapted Binomial; BAcBS stands for Accelerated Adapted Binomial with Black-Scholes. Numbers beside some methods stand for the number of time steps considered.

Exhibit 7: Accuracy comparison - 3 dividends



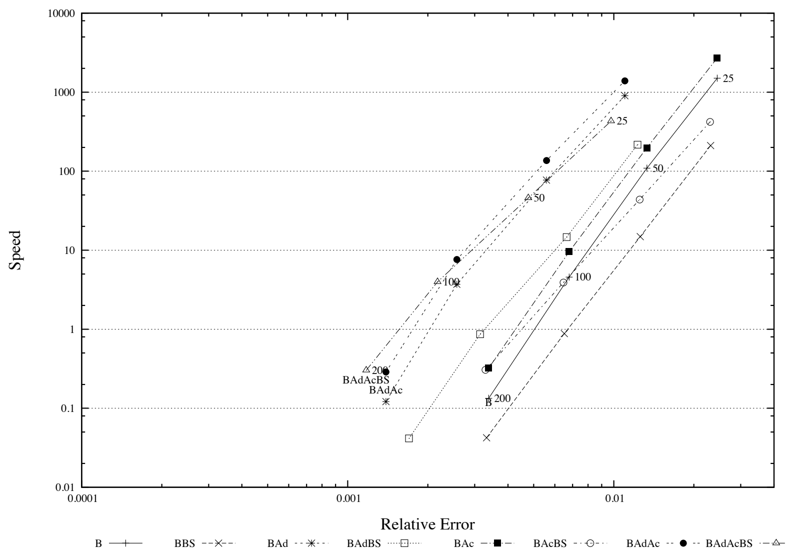
(a) Calls



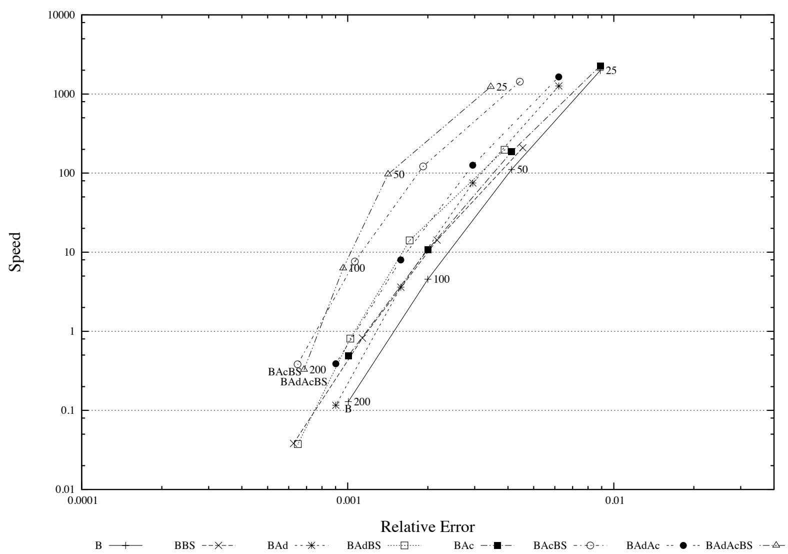
(b) Puts

The graph is in a log-log scale. Relative error is the RMSRE. Speed is measured by options values calculated per second. Only options with value above 0.5 were considered. The benchmark value of the American option is obtained using a non recombining binomial tree with Cox, Ross, and Rubinstein [1979] parameters and 1,500 steps. The binomial non recombining binomial tree methods are as presented in this paper: B stands for Binomial; BBS stands for Binomial with Black-Scholes; BAd stands for Adapted Binomial; BAdBS stands for Adapted Binomial with Black-Scholes; BAc stands for Accelerated Binomial; BAcBS stands for Accelerated Binomial with Black-Scholes; BAdAc stands for Accelerated Adapted Binomial; BAcBS stands for Accelerated Adapted Binomial with Black-Scholes. Numbers beside some methods stand for the number of time steps considered.

Exhibit 8: Accuracy comparison - 4 dividends



(a) Calls



(b) Puts

The graph is in a log-log scale. Relative error is the RMSRE. Speed is measured by options values calculated per second. Only options with value above 0.5 were considered. The benchmark value of the American option is obtained using a the Longstaff and Schwartz [2001] LSMC valuation method with CONT-CE procedure proposed by Areal, Rodrigues, and Armada [2008], and powers polynomials with 5 basis functions along with Halton [1960] sequences using Brownian bridges and 50 time steps and 200000 paths. The binomial non recombining binomial tree methods are as presented in this paper: B stands for Binomial; BBS stands for Binomial with Black-Scholes; BAd stands for Adapted Binomial; BAdBS stands for Adapted Binomial with Black-Scholes; BAc stands for Accelerated Binomial; BAcBS stands for Accelerated Binomial with Black-Scholes; BAdAc stands for Accelerated Adapted Binomial; BAcBS stands for Accelerated Adapted Binomial with Black-Scholes. Numbers beside some methods stand for the number of time steps considered.