Portfolio selection with commodities under conditional asymmetric dependence and skew preferences*

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Abstract

This article solves the portfolio selection problem for an investor with three-moment preferences, when returns follow a conditional asymmetric t copula with skewed and fat-tailed marginal distributions. The model can capture the impact on optimal portfolios of: time-varying investment opportunities, state dependence in the returns' correlations, and tail dependence. In the empirical test with oil and gold futures and equity from 1990 to 2010, the portfolios achieve better performance measures than they would under conventional alternatives. The factors explaining the significant differences among methods are the univariate process specification, the dynamic dependence, and the presence of tail and asymmetric dependence.

Keywords: Portfolio selection, commodity futures, conditional copulas, and skew preferences.

JEL classification: C46, G11, and G13.

I. Introduction

In the past ten years, commodities have attracted the attention of many financial investors, who perceive them as a new asset class. Financial institutions mainly take positions in commodity futures contracts as a natural way to gain exposure to commodity risk without owning the physical asset. This recent boom of commodity-related instruments as investment vehicles, and its economic impact, has been analyzed by Büyüksahin and Robe (2010), Etula (2010), Hong and Yogo (2012), Singleton (2011), Tang and Xiong (2010), among others.

Previous works, such as Erb and Harvey (2006) and Gorton and Rouwenhorst (2006), instead studied the properties and investment performance of commodity futures, individually and as an asset class. These works considered fully collateralized commodity futures and found that historically, such contracts exhibited little co-movements, zero or even negative correlations with stock returns, and Sharpe ratios fairly close to those of equities. Therefore, according to traditional portfolio theory, commodities should increase diversification when included in equity portfolios and may help enhance the portfolio's risk-return profile. Possibly boosted by the potential for such diversification benefits, investments in commodity indexes and related instruments grew quickly after the early 2000s (see Tang and Xiong (2010)).[1] Despite growing interest in commodities as investment vehicles, few recent works have obtained optimal portfolio allocation by considering the stylized features of commodities. To fill this gap, we address the portfolio selection problem of an equity investor when cash-collateralized commodity futures are part of the investment opportunity set.

Most works that analyze portfolios that contain commodities rely on standard mean-variance frameworks, but such frameworks might not be appropriate for commodity futures with their returns' specific distributional characteristics, such as the presence of serial correlation, heavy tails, and skewness (Gorton and Rouwenhorst (2006); Kat and Oomen (2007)). Instead, we propose a more general and realistic model to be used in the optimal port-

[1] Investments in commodity index funds increased from around \$50 billion, at the end of 2004 to a peak of \$200 billion in 2008; after a drop during the recession, they increased again to a second peak of around \$300 billion at the end of the third quarter of 2010. See Irwin and Sanders (2011).

folio selection process. Our approach combines a three-moment preferences specification with time-varying density models that describe the statistical properties of commodities and equity returns, as well as their interactions.

With respect to the investor's preferences, we solve an allocation problem in which the utility function is determined by the mean, variance, and skewness of portfolio returns. With fairly general assumptions, investors show a preference for positive skewness in return distributions and aversion to downside risk (negative skewness). In our proposed three-moment preferences specification, the investor is eager to decrease the chance of large negative deviations, which could reduce the final value of the portfolio. Expanding the standard mean-variance set-up by including a third moment of the portfolio returns similarly has been investigated for traditional assets, such as stocks, and suggested empirically that asset holdings can be quite different under third-moment preferences compared with their appearance in the standard mean-variance case.[2]

As far as we know, the effect of time-varying skew preferences on portfolio selection has not yet been analyzed. Nonetheless, due to the specific features of commodity assets, skewness seems likely to play a role in the investor's portfolio decisions. In particular, commodity prices depend to great extent on supply, demand, and inventories (Routledge, Seppi and Spatt (2000)). For example, the possibility of shortages in production or stockouts may produce spikes in commodity prices, leading to skewness in the returns of futures contracts.

Our three-moment preferences assumption can be interpreted in terms of the Taylor series expansion of a given underlying utility function, as shown by Guidolin and Timmermann (2008), Harvey, Liechty, Liechty and Müller (2010), and Jondeau and Rockinger (2012). Finally, as another reason to focus on downside risk, we consider the well-known loss aversion argument that is pervasive in behavioral finance literature.[3]

[2]Some early works on how skewness affects portfolio selection include Samuelson (1970) and Kraus and Litzenberger (1976). Harvey and Siddique (2000) build on these ideas to provide an empirical test of the effect of co-skewness on asset prices. Barberis and Huang (2008) and Mitton and Vorkink (2007) also suggest, from different perspectives, that the skewness of individual assets may also influence investors' portfolio decisions.

[3]For empirical support of these theories for traditional financial assets, see Shefrin (2008).

Regarding the multivariate density model, we offer a flexible approach to specify the joint distribution of returns using conditional copula models. Copula functions help disentangle the particular characteristics of the univariate distributions of equity and commodity returns from their dependence structure. We combine conditional copula theory, as presented in Patton (2006a,b), with the implicit copula functions of multivariate normal mixtures defined by Demarta and McNeil (2005) and Embrechts, Lindskog and McNeil (2003). In the most general case, we consider a conditional skewed t copula with generalized Student's t marginal distributions. Thus we can capture time-varying investment opportunities through time-varying moments and changes in the dependence structure. Furthermore, this copula model allows for tail dependence and asymmetry (i.e., differential dependence during bear and bull markets).

We empirically test the model using weekly data from the crude oil and gold futures, as well as the S&P 500 index, for the period from June 1990 to September 2010. We examine four primary issues:

(1) Is there asymmetric and tail dependence among commodities and equity returns?

(2) Are there discrepancies in the optimal portfolio allocations between our conditional copula approach and other more traditional benchmarks, such as the equally weighted or the Markowitz strategies?

(3) Do these discrepancies translate into economically relevant performance differences among methods?

(4) Is there a single key factor explaining these discrepancies?

First, we find strong evidence of skewness and tail dependence among equity and commodity futures. Second, we also uncover substantial discrepancies between our method's portfolio optimal weights and the portfolio weights provided by conventional alternatives. Third, in most cases, the differences in portfolio weights translate into economically more profitable investment ratios and relative performance measures with respect to the alternative procedures. Fourth, no single factor offers a sufficient explanation. Rather, we find many explanatory elements, including, in order, the proper specification of time-varying univariate behavior in terms of volatility, skewness, and fat tails; the dynamics in the dependence among marginal functions; and both asymmetric and extreme dependence.

The remainder of this article is organized as follows: Section II formulates the investor's preferences and portfolio choice problem. In Section III, we present the multivariate conditional copula model and the estimation methodology. Section IV describes the empirical application, before we conclude in Section V. In the online Appendix, we also provide technical details and additional tables and figures.

II. Portfolio choice with commodity futures and skew preferences

In this section, we present the portfolio selection problem when commodity futures are part of the investment opportunity set. Our methodological approach to solve this portfolio choice problem relies mainly on two points: the inclusion of the third moment of portfolio returns in the investor's preferences, and the effect of considering the collateral of commodity futures.

As is well known, returns on financial assets generally deviate from the Gaussian distribution, displaying heavy tails and skewness. This departure from normality is even greater in the case of commodities, magnified by the well-documented presence of spikes (positive and negative) in the data-generating process of commodity returns (Casassus and Collin-Dufresne (2005); Hilliard and Reis (1998)). The fundamentals underlying commodity price formation are key determinants of these statistical properties. Accordingly, the presence of jumps can be explained by the convex relations among commodity prices and supply, inventories, and demand (Routledge et al. (2000)). Shocks in supply, demand, or both can have great impacts on prices, especially when the balance in the market's supply and demand is particularly tight. Adding commodity assets to traditional portfolios constitutes a significant source of skewness for the portfolio's returns, increasing the importance of including the third moment in the portfolio selection problem.

With respect to to the second point, the collateral of commodity futures in the portfolio decision problem reflects the way futures exchanges operate. No money changes hands when futures are sold or bought; just a margin is posted to settle gains and losses. Without any upfront payment, it is not clear how to define the rate of return (Dusak (1973)). Taking collateral in futures contracts into account would affect the computation of their rates of return and the budget constraint of the investor's decision process. Following a common approach to analyze commodity futures as an asset class (Gorton and Rouwenhorst (2006), Hong and Yogo (2012)), we assume that long and short positions are fully collateralized; that is, the initial margin deposit corresponds with the overall notional value of the futures contract.[4]

With both these considerations, our specification of the investor's problem with commodity futures extends previous models of portfolio selection with skewness (e.g., Harvey et al. (2010)), such that the rates of return and budget constraints are determined by the particular characteristics of commodity futures contracts. Formally, the portfolio choice problem can be formulated in terms of an investor who maximizes expected utility at period $t+1$ by building at time t a portfolio that includes two group of assets: a group with n commodity futures contracts, and another group with $N-n$ spot contracts, such as stocks. We assume that the initial margin deposit of a fully collateralized futures position indicates the initial capital investment related to that position (long or short). Therefore, the gross return of a position in the commodity futures contract i at time $t+1$ is given by

$$(1) \qquad (1+R_{i,t+1}) = \frac{S_{i,t+1}}{S_{i,t}}(1+R^f_{t+1})\,, \quad i = N-n+1,\ldots,N\,,$$

where $S_{i,t}$ and $S_{i,t+h}$ are the futures settlement prices at times t and $t+1$, respectively, and $(1+R^f_{t+1})$ is the gross return on cash over the period, or the interest earned on the initial margin deposit. Finally, for this set of N investment opportunities, wealth at time $t+1$

[4]We thus control for the leverage involved in futures positions, and we can make fair comparisons with spot contracts. This assumption also can be relaxed, and smaller fractions of the nominal value can be considered, in the problem set-up.

equals the gross return of the portfolio over the period, $1 + R_{t+1}(\boldsymbol{\omega}_t)$, defined as

$$1 + R_{t+1}(\boldsymbol{\omega}_t) = 1 + \sum_{j=1}^{N} \omega_t^j (\exp(r_{j,t+1}) - 1)\,, \tag{2}$$

where $\boldsymbol{\omega}_t = (\omega_t^1, \ldots, \omega_t^{N-n}, \omega_t^{N-n+1}, \ldots, \omega_t^N)'$ is the vector of portfolio weights (for spot and futures contracts), chosen at time t, and $r_{j,t+1} = \log(1 + R_{j,t+1})$ is the continuously compounded return of asset j over the period.

In our approach, the investor's preferences are determined by the first three moments of the portfolio returns. Thus, the investor's objective consists of choosing a portfolio allocation $(\omega_t^1, \ldots, \omega_t^N)$ that maximizes the expected portfolio return, in Equation (2), penalized for risk (variance of portfolio returns) and negative skewness. That is, for each time t,

$$\max_{\boldsymbol{\omega}_t} \Big(\mathbb{E}_t[R_{t+1}(\boldsymbol{\omega}_t)] - \varphi_{\mathrm{V}} \operatorname{Var}_t[R_{t+1}(\boldsymbol{\omega}_t)] + \varphi_{\mathrm{S}} \operatorname{Skew}_t[R_{t+1}(\boldsymbol{\omega}_t)] \Big)\,, \tag{3}$$

where $\mathbb{E}_t(\cdot)$, $\operatorname{Var}_t(\cdot)$, and $\operatorname{Skew}_t(\cdot)$ are the first three moments of the portfolio returns conditioned on the information set $\mathcal{F}_t$ available at time t. The parameters $\varphi_{\mathrm{V}} \geqslant 0$ and $\varphi_{\mathrm{S}} \geqslant 0$ determine the impact of variance (risk aversion) and skewness (loss aversion) on the investor's utility. Aversion to risk, in the form of return variability, is a common, well studied feature of the investor's preferences. By adding aversion to negative skewness, we acknowledge the possibility that an investor might accept a lower expected return if there is a chance of high positive skewness, such as in the form of a large probability of positive jumps.

The maximization problem in Equation (3) is subject to the non linear constraint that the sum of portfolio weights in spot contracts plus the sum of the absolute value of futures contracts' weights must equal 1:

$$\sum_{j=1}^{N-n} \omega_t^j + \sum_{i=N-n+1}^{N} \left|\omega_t^i\right| = 1. \tag{4}$$

Because both long and short positions in commodity futures contracts require the same initial collateral, we take the absolute value of the futures weights $(\omega_t^{N-n+1}, \ldots, \omega_t^N)$, such that short positions in futures contracts cannot be used to leverage holdings of other assets.

The linear utility function in Equation (3) summarizes the investor's preferences in the first three moments of the distribution of portfolio returns. Alternatively, the three-moment preferences assumption could be interpreted as the expected value of a Taylor series expansion up to third order of an underlying utility function, as shown by Guidolin and Timmermann (2008) and Jondeau and Rockinger (2012), for example. In these works, the underlying utility function is the power utility with a coefficient of relative risk aversion $\mathcal{A}$. Thus, the impact of variance and skewness in the investor's decision rule, expressed in our specification by the parameters φ_{V} and φ_{S}, would depend on the coefficient of risk aversion $\mathcal{A}$ through the second and third derivatives of the power utility function.

III. Multivariate conditional copula model with asymmetry

In this section, we describe the model for the multivariate distribution of assets' log-returns $\boldsymbol{r}_{t+1} = (r_{1,t+1}, \ldots, r_{d,t+1})$, where $d \leqslant N$ is the number of risky assets, whether spot or futures contracts. We employ multivariate conditional copulas to obtain a flexible model for the joint distribution of asset returns. Every multivariate distribution model consists of marginal distribution functions that describe each univariate variable, as well as a joint dependence function that defines the relations among individual processes. Unlike traditional multivariate distributions, such as the Gaussian and Student's t distributions, copula models support the construction of multivariate distributions with arbitrary marginal processes and dependence.

Formally, a d-variate copula is a d-dimensional distribution function on the unit interval $[0,1]^d$, that is, a joint distribution with d uniform marginal distributions. Consider a multivariate conditional distribution $F_t(r_{1,t+1}, \ldots, r_{d,t+1})$ formed by d univariate conditional distributions $F_{i,t}(r_{i,t+1})$, where the subscript t denotes that joint and marginal distributions are conditioned on the information set $\mathcal{F}_t$ available at time t. Following Patton (2006b), there must exist a function C_t that maps the domain $[0,1]^d$ toward the interval $[0,1]$, called the *conditional copula*, such that

$$F_t(r_{1,t+1}, \ldots, r_{d,t+1}) = C_t\left(F_{1,t}(r_{1,t+1}), \ldots, F_{d,t}(r_{d,t+1})\right) . \tag{5}$$

This expression constitutes a d-dimensional extension of Sklar's (1959) theorem for conditional copulas.[5]

Using the expression in Equation (5), any copula C_t can be employed to define a joint distribution $F_t(\boldsymbol{r}_{t+1})$ with the arbitrary marginal distributions $F_{1,t}, \ldots, F_{d,t}$. Thus, using a bottom-up approach, we model the marginal distributions of asset returns, followed by the conditional copula function that describes their dependence structure.

A. Modeling univariate processes

We first specify the univariate distribution functions of the asset returns $\boldsymbol{r}_{t+1}$. Our multivariate copula model supports the use of various marginal distributions. Thus we can attend to the particular characteristics of each asset return, which is an useful feature when different types of assets appear in the portfolio, such as commodities and stocks. We present a marginal distribution model that captures individual skewness and heavy tails, as well as time-varying moments. We build on the autoregressive conditional density models of Hansen (1994), Harvey and Siddique (1999), and Jondeau and Rockinger (2003), and we propose, for modeling the conditional univariate distribution, a generalized Student's t distribution with possibly time-varying parameters. Thus, the univariate process for each asset returns $r_{i,t+1}$ $(i = 1, \ldots, d)$ can be expressed as follows:

$$r_{i,t+1} = \mu_{i,t+1} + \sqrt{\sigma^2_{i,t+1}}\, z_{i,t+1} \tag{6}$$

$$\mu_{i,t+1} \equiv \mathbb{E}_t\left(r_{i,t+1}\right) = \mu_{0,i} + \boldsymbol{\beta}_i' \boldsymbol{X}_t + \sum_{j=1}^{p} \Phi_{i,j} r_{i,t+1-j} \tag{7}$$

$$\sigma^2_{i,t+1} \equiv \mathrm{Var}_t\left(r_{i,t+1}\right) = \alpha_{0,i} + \alpha^+_{1,i}\, \sigma^2_{i,t} z^2_{i,t}\, \mathbb{1}_{\{z_{i,t} \geqslant 0\}} + \alpha^-_{1,i}\, \sigma^2_{i,t} z^2_{i,t}\, \mathbb{1}_{\{z_{i,t} < 0\}} + \alpha_{2,i}\, \sigma^2_{i,t}, \tag{8}$$

$$z_{i,t+1} \sim g_{i,t}(z_{i,t+1}\,;\, \nu_{i,t+1}\,, \lambda_{i,t+1})\,, \tag{9}$$

[5]This generalization of Sklar's theorem is a direct application of the concept of a conditional copula (Patton (2006b), Theorem 1) to a multivariate case (Nelsen (2006), Theorem 2.10.9), and requires simply that conditioning variables be the same for all marginal distributions and the copula. If margins are continuous, this copula is unique.

where $\mu_{i,t+1}$ and $\sigma^2_{i,t+1}$ are the mean and variance conditioned on $\mathcal{F}_t$ for the i-th asset returns, and $z_{i,t+1}$ is the corresponding residual.

The conditional mean, defined in Equation (7), is a linear function of p lagged returns, $r_{i,t+1-j}$ ($j = 1, \ldots, p$), and m further explanatory variables $\boldsymbol{X}_t$, with the coefficients $\Phi_{i,j}$ and $\boldsymbol{\beta}_i$, respectively, and the drift parameter $\mu_{0,i}$. This specification can capture the possible presence of autocorrelation and predictability in asset returns. As the exogenous regressors $\boldsymbol{X}_t$, we consider explanatory variables employed in previous literature to predict variation in stocks and commodity futures returns, including the short rate, default spread, momentum, basis, and growth in open interest (see Hong and Yogo (2012)).

As we describe in Equation (8), we employ an asymmetric or leveraged GARCH dynamic for the conditional variance. This specification is designed to account for volatility clustering and leverage effects, such as possible asymmetric responses to positive and negative shocks that have occurred in the previous period (Campbell and Hentschel (1992)).[6]Equation (9) then denotes that the univariate innovations $z_{i,t+1}$ are drawn from a generalized Student's t distribution, $g_{i,t}$, which can capture heavy tails and individual skewness through the degrees of freedom ν_i and asymmetry parameter λ_i (Hansen (1994)). The details of the functional form of this univariate distribution are summarized in the online Appendix A.

Finally, our specification of the marginal distributions addresses the possibility of time-varying higher moments as follows:

$$\nu_{i,t+1} = \Lambda_{(2,\infty)}\left(\delta_{0,i} + \delta^+_{1,i}\, z_{i,t}\, \mathbb{1}_{\{z_{i,t}\geqslant 0\}} + \delta^-_{1,i}\, z_{i,t}\, \mathbb{1}_{\{z_{i,t}<0\}} + \delta_{2,i}\Lambda^{-1}_{(2,\infty)}(\nu_{i,t})\right), \tag{10}$$

$$\lambda_{i,t+1} = \Lambda_{(-1,1)}\left(\zeta_{0,i} + \zeta^+_{1,i}\, z_{i,t}\, \mathbb{1}_{\{z_{i,t}\geqslant 0\}} + \zeta^-_{1,i}\, z_{i,t}\, \mathbb{1}_{\{z_{i,t}<0\}} + \zeta_{2,i}\Lambda^{-1}_{(-1,1)}(\lambda_{i,t})\right), \tag{11}$$

where $\delta^i_0, \delta^i_1, \delta^i_2, \zeta^i_0, \zeta^i_1$ and ζ^i_2 are constant parameters, and $y \equiv \Lambda_{(l,u)}(x) = (u + l\mathrm{e}^x)/(1 + \mathrm{e}^x)$ denotes the modified logistic map designed to keep the transformed variable y in the domain (l, u) for all $x \in \mathbb{R}$. Thus, shape parameters $\nu_{i,t+1}$ and $\lambda_{i,t+1}$ may depend on their lagged values and react differently to positive and negative shocks. This general specification also includes some well-known univariate distributions as particular cases. For instance, if the

[6]To guarantee positive and stationary volatility, the parameters of the variance dynamics in Equation (8) must satisfy the following constraints: $\alpha_{0,i} > 0$; $\alpha^+_{1,i}, \alpha^-_{1,i}, \alpha_{2,i} \geqslant 0$; and $\alpha_{2,i} + (\alpha^+_{1,i} + \alpha^-_{1,i})/2 < 1$.

asymmetry parameter goes to 0, we obtain the symmetric Student's t distribution; as degrees of freedom tend to infinity, we would converge to a Gaussian distribution.

B. Modeling copula functions

In this section, we present the copula functions that determine the dependence structure of our model. Following Sklar's theorem in Equation (5), the copula function acts like a joint distribution of the probability transformed vector $(F_{1,t}(r_{1,t+1}), \ldots, F_{d,t}(r_{d,t+1})'$, where $F_{i,t}(r_{i,t+1})$ are the marginal distribution functions of asset returns $r_{i,t+1}$, as described in Equations (6)-(9). In particular, we employ three multivariate copula functions: two well-known elliptical copulas, the Gaussian and the t copula (Embrechts et al. (2003)), and an asymmetric copula, the so-called skewed t copula (Demarta and McNeil (2005)). They are all implicit dependence functions of various multivariate normal mixtures. More specifically, they are the parametric copula functions contained in the multivariate Gaussian, Student's t, and generalized hyperbolic skewed t distributions, respectively.

Through a direct application of Sklar's theorem, we can obtain these implicit copulas by evaluating a given multivariate distribution (e.g., generalized hyperbolic skewed t) at the quantile functions of its corresponding marginal distributions. For example, the skewed t copula is given by:

$$C^{\mathrm{SK}}(\, u_1, \ldots, u_d\, ; \boldsymbol{P}, \nu, \boldsymbol{\gamma}\,) = H\Big(\, H_1^{-1}(u_1\, ; \nu, \gamma_1), \; \ldots, \; H_d^{-1}(u_d\, ; \nu, \gamma_d)\, ; \boldsymbol{P}, \nu, \boldsymbol{\gamma}\,\Big)\,, \tag{12}$$

where $H(\cdot\, ; \boldsymbol{P}, \nu, \boldsymbol{\gamma})$ is the generalized hyperbolic skewed t distribution with $d \times d$ correlation matrix $\boldsymbol{P}$, degrees of freedom ν, and d-dimensional asymmetry parameter vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$. The $H_i(\cdot\, ; \nu, \gamma_i)$ are the d univariate skewed t distributions, the H_i^{-1} are the corresponding quantile functions, and $(u_1, \ldots, u_d)'$ is the probability-transformed vector. Similarly, we can extract the Gaussian and t copulas, $C^{\mathrm{G}}(u_1, \ldots u_d\, ; \boldsymbol{P})$ and $C^{\mathrm{T}}(u_1, \ldots u_d\, ; \boldsymbol{P}, \nu)$, from their respective multivariate distributions.

In the online Appendix B, we provide more details about the functional forms of these three copulas, including their density functions. For illustrative purposes, in Figure 1, we present the contour plots and probability density functions of these copulas for a two-dimensional case. Although the examples in Figure 1 are for a bivariate case, a useful property of all three copulas considered is that they can be employed directly to specify the dependence structure of an arbitrary number of risky assets.

<Insert Figure 1 about here>

As Figure 1 reveals, using these three copulas, we can model three different types of dependence. The Gaussian copula, $C^{\mathrm{G}}(\cdot\,;\boldsymbol{P})$, defines linear, symmetric dependence, completely determined by the correlation matrix $\boldsymbol{P}$. Thus it is unable to capture tail dependence or asymmetries. The t copula, $C^{\mathrm{T}}(\cdot\,;\boldsymbol{P},\nu)$, is also elliptically symmetric but allows for tail dependence through the degrees-of-freedom parameter, ν. The plots in Figure 1 show that the t copula assigns more probability to the extremes than does the Gaussian copula. The greater the degrees of freedom, the smaller the level of tail dependence, converging in the limit $\nu \to \infty$ to the Gaussian copula. Finally, the skewed t copula, $C^{\mathrm{SK}}(\cdot\,;\boldsymbol{P},\nu,\boldsymbol{\gamma})$, can capture extreme and asymmetric dependence of the asset returns. Through the d-dimensional vector of asymmetry parameters $\boldsymbol{\gamma}$, the skewed t copula can assign more weight to one tail than the other. For example, in Figure 1, all elements of the asymmetry vector are negative, and therefore, the density contour is clustered in the negative-negative quadrant. Eventually, if $\boldsymbol{\gamma} \to \mathbf{0}$, asymmetric dependence goes to 0, and we recover the symmetric t copula.

Once we have defined the functional forms of the three implicit copulas, we can build the multivariate distribution model for our vector of asset returns. This multivariate distribution forms from the marginal distributions of the previous section and one of the implicit copulas we described previously. In addition, following pioneering works by Patton (2006a,b), we can parametrize time variation in the conditional copula function of our multivariate model. For that purpose, and in the spirit of Engle's (2002) dynamic conditional correlation model, we extend the notion to other types of dependence beyond the Gaussian one and allow that the dependence matrix $\boldsymbol{P}_t$ of our conditional copula may evolve over time, according to some GARCH-type process. In the most general case, the vector of return innovations,

$\boldsymbol{z}_{t+1} = (z_{1,t+1}, \ldots, z_{d,t+1})'$, follows the distribution specification:

$$(13) \qquad \boldsymbol{z}_{t+1} \sim C_t^{\mathrm{SK}}\Big(g_{1,t}(z_{1,t+1}\,; \nu_{1,t+1}, \lambda_{1,t+1}), \ \ldots, \ g_{d,t}(z_{d,t+1}\,; \nu_{d,t+1}, \lambda_{d,t+1})\,; \ \boldsymbol{P}_{t+1}, \nu, \boldsymbol{\gamma} \Big)\,,$$

where the $g_{i,t}(z_{i,t+1}\,; \nu_{i,t+1}, \lambda_{i,t+1})$ are the conditional univariate distributions in Equation (9), and the evolution equation for $\boldsymbol{P}_{t+1}$ is given by:

$$(14) \qquad \boldsymbol{P}_{t+1} = \Lambda_{(-1,1)}\Big(\omega_0\, \boldsymbol{P}_{\mathrm{c}} + \omega_1 \frac{1}{M} \sum_{m=1}^{M} \boldsymbol{x}_{t+1-m}\, \boldsymbol{x}'_{t+1-m} + \omega_2\, \boldsymbol{P}_t \Big).$$

In this case, $\boldsymbol{x}_t$ is the vector of transformed variables, $(H^{-1}(u_{1,t}; \nu, \gamma_1), \ldots, H^{-1}(u_{d,t}; \nu, \gamma_d))'$; $\boldsymbol{P}_{\mathrm{c}}$ is the constant correlation matrix; ω_0, ω_1 and ω_2 are constant parameters; and M is the number of lags we consider. The modified logistic function $\Lambda_{(-1,1)}(\cdot)$ ensures that the elements of $\boldsymbol{P}_{t+1}$ remain in the domain $(-1, 1)$.

C. Estimation

Our model structure, formed by the marginal distributions and the copula, allows for a two-step estimation procedure, similar to the conditional setups of Jondeau and Rockinger (2006) and Patton (2006a). In the first step, we obtain the maximum likelihood (ML) estimates of the individual processes; then, we determine the parameter estimates of the copula function. From this ML approach, we can compute the asymptotic and robust standard errors for the estimates.[7]

Formally, this procedure can be expressed as follows: Let $\bar{\boldsymbol{r}}_T = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T\}$ be the sample of returns of length T, where $\boldsymbol{r}_t = (r_{1,t}, \ldots, r_{d,t})$ for $i = 1, \ldots, T$. We want to find the set of parameter estimates $\widehat{\boldsymbol{\theta}}$ that maximizes the log-likelihood function $\mathcal{L}$, that is,

$$(15) \qquad \widehat{\boldsymbol{\theta}} \equiv \arg\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \mathcal{L}(\boldsymbol{\theta}\,; \bar{\boldsymbol{r}}_T) = \arg\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \sum_{t=1}^{T} \log f_t(\boldsymbol{r}_{t+1}; \boldsymbol{\theta})\,,$$

where $f_t(\boldsymbol{r}_{t+1}; \boldsymbol{\theta})$ is the probability density function of the multivariate model conditioned by the information set $\mathcal{F}_t$ and parameterized by $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

[7] This procedure is also known as the inference functions for margins method. The two-stage approach can estimate some multivariate GARCH models, such as constant (CCC) and dynamic (DCC) conditional correlation models (see Engle and Sheppard (2001)).

From the assumptions of Sklar's theorem in Equation (5), we can decompose the log-likelihood function $\mathcal{L}$ in Equation (15) into two parts, the margins and the copula (see the details in the online Appendix C):

$$
\begin{aligned}
(16) \qquad \mathcal{L}(\boldsymbol{\theta}_M, \boldsymbol{\theta}_C\,; \bar{\boldsymbol{r}}_T) &= \sum_{i=1}^{d} \mathcal{L}_i(\boldsymbol{\theta}_{i,M}; \bar{\boldsymbol{r}}_T) + \mathcal{L}_C(\boldsymbol{\theta}_C; \boldsymbol{\theta}_M, \bar{\boldsymbol{r}}_T) \\
&= \sum_{i=1}^{d} \sum_{t=1}^{T} \log f_{i,t}(r_{i,t+1}\, \boldsymbol{\theta}_{i,M}) + \sum_{t=1}^{T} \log c_t(u_{1,t+1}, \ldots, u_{d,t+1}\,; \boldsymbol{\theta}_C).
\end{aligned}
$$

where $\mathcal{L}_i$ and $\mathcal{L}_C$ are the log-likelihood functions for the i-th marginal process and the copula. In addition, $\boldsymbol{\theta}_M$ denotes the set of parameters corresponding to the d marginal distributions, $(\boldsymbol{\theta}_{1,M}, \ldots, \boldsymbol{\theta}_{d,M})'$, and $\boldsymbol{\theta}_C$ denotes the parameters of the copula function. The $u_{i,t+1}$ are the marginal distributions, $F_{i,t}(r_{i,t+1}\,; \boldsymbol{\theta}_{i,M})$, with corresponding marginal density functions $f_{i,t}(r_{i,t+1}\,; \boldsymbol{\theta}_{i,M})$; $c_t(\cdot\,; \boldsymbol{\theta}_C)$ is the copula density function. The online Appendix B provides the explicit expressions of the copula densities we consider. Therefore, according to Equation (16), we can estimate sequentially the d marginal distributions models, then, the copula function, whose ML estimators are given by $\widehat{\boldsymbol{\theta}}_{i,M} = \arg\max\ \mathcal{L}_i$ and $\widehat{\boldsymbol{\theta}}_C = \arg\max\ \mathcal{L}_C$, respectively.[8]

Some remarks should be considered though. First, with regard to the univariate processes, the quality of the copula estimation depends strongly on the goodness of fit of the parametric functions we use for the marginal distributions. Second, for the symmetric and skewed t copulas, we need extra parameters, apart from the correlation matrix $\boldsymbol{P}$; in these cases, because the objective function often falls in local maximums, convergence difficulties may arise when maximizing the log-likelihood function directly. To overcome this problem, we perform the ML estimations of these copula functions in two nested steps. The inner step maximizes the likelihood with respect to the correlation matrix, given fixed values for the shape parameters. This conditional optimization is placed within a maximization with respect to the shape parameters; thus maximizing the likelihood over all copula parameters.[9]

[8]Patton (2006a) shows that one-step maximum likelihood estimators and two-stage estimators are equally asymptotically efficient.

[9]Furthermore, we employ a global optimization approach, consisting of simulated annealing, to check the robustness of the local optimization results (Goffe, Ferrier and Rogers (1994)).

IV. Empirical application

In this section, we first present the data and their main univariate and multivariate statistical properties. Then, we estimate the conditional copula models and analyze their in-sample fitting performance. Finally, we solve the portfolio problem for the copula models numerically and obtain the optimal weights, investment ratios, and relative performance measures over the out-of-sample period.

A. Data and preliminary analysis

Our empirical application relies on three risky assets: two commodity futures, oil and gold, and the S&P 500 equity index. The oil futures correspond to West Texas Intermediate (WTI) crude oil from the New York Mercantile Exchange (NYMEX). The gold futures correspond to the gold bar, with a minimum of 0.995 fineness, from the New York Commodities Exchange (COMEX). These futures are two of the most actively traded commodity contracts in the world, and they do not have tight restrictions on the size of daily price movements.[10]In both cases, we employ the most liquid futures contracts, measured by daily trading volume, of all maturities available. The risk-free rate is computed from the three-month U.S. Treasury bills provided by the Federal Reserve System. All data are in U.S. dollars and came from Thomson-Reuters Datastream. The sample period considered ranges from June 20, 1990 to September 8, 2010, for a total of 1056 weekly observations. We divided the sample in two subperiods, such that the period from June 20, 1990 to June 20, 2006 supported the in-sample estimation analyses of the models, and the remaining observations from June 20, 2006 to September 8, 2010 were reserved for the out-of-sample portfolio performance exercise.

The online Appendix F contains the tables and figures related to the univariate and multivariate preliminary analyses.

[10]At the end of 2011, gold and crude oil futures represented 30% of the Dow Jones-UBS Commodity Index and 38% of the S&P-Goldman Sachs Commodity Index.

A.1. Univariate preliminary analysis

We first analyze the univariate behavior of the three asset returns. Appendix Table F.1 reports summary statistics for the weekly returns of the gold and oil futures and the equity index for the sample periods. With Exhibit 1 of Appendix Figure F.1, we display the relative price moves of each asset over the full-sample period.

We observe substantial changes in the sample moments of returns over time. The mean returns are all positive, except for equity during the out-of-sample period (Jun. 2006 – Sep. 2010). Returns volatilities per week for oil, gold, and equity increased from 4.4%, 1.9%, and 2.1% in the in-sample period to 5.4%, 3.2%, and 3.0% during the out-of-sample period. Looking at the ratio of the mean over the volatility (Sharpe's ratio), we find that for the in-sample observations, equity (0.07) performs better than oil (0.04) and gold (0.03). This pattern changed during the 2006-2010 period, during which ratios of oil (0.01) and equity (-0.02) were below their historical average, whereas gold's ratio (0.11) moved significantly above its historical average. According to the Ljung-Box (LB) and Lagrange multiplier (LM) statistics, reported in Table F.1, there is evidence of serial correlation in the returns and squared returns for all time series (except for oil returns over the in-sample period).

Assets returns are non-normal, skewed, and heavy tailed. According to the Jarque-Bera (JB) and Kolmogorov-Smirnov (KS) tests, normality in the returns' unconditional distribution is strongly rejected for all samples. Skewness and kurtosis of returns differs across assets and sample periods. From the in-sample to the out-of-sample period, the equity returns' skewness grew much more negative, while gold returns changed from positive to negative skewness, and oil returns from negative to positive. During 2006-2010, the oil and gold returns' kurtosis decreased with respect to the previous period, but equity returns' kurtosis strongly increased, as expected.

A.2. Multivariate preliminary analysis

It is also interesting to describe the interactions observed in the sample among the oil, gold, and equity index returns. In the online Appendix (Table F.2), we report some multivariate statistics and preliminary tests for the three-dimensional vector of asset returns. We first focus on the characteristics of the linear dependence; then we turn to analyzing non linear features observed in the vector of returns.

Appendix Table F.2 reports unconditional correlation coefficients. We find a large increase in linear dependence for the 2006-2010 period with respect to historical values. The sample correlation between oil and gold returns rises from 0.21 to 0.39. Furthermore, equity index returns, which were negatively correlated with oil (-0.06) and gold (-0.08) in 1990-2006, became positively correlated with both of these commodity returns over the 2006-2010 period, with coefficients equal to 0.39 and 0.17, respectively. These findings suggest that dependence between commodities and equity is no longer constant and evolves with time. To check this assumption, we carry out Engle and Sheppard's (2001) test for constant correlation. The probabilities of constant correlation (test p-values reported in Panel B of Table F.2) are less than 0.05 in all cases; therefore, we reject the hypothesis of constant dependence.

Panel C of Table F.2 reports the tri-variate measures of skewness and kurtosis proposed by Mardia (1970) to test multivariate normality. The corresponding statistics suggest that the hypothesis of multivariate normality should be rejected for the three sample periods considered. Following McNeil, Frey and Embrechts (2005) we test whether the standardized vector of returns is consistent with a spherical distribution. The corresponding KS test statistics, reported in Panel C of Table F.2, reject the ellipticity hypothesis for all samples. Visually, their associated quantile-quantile plots reveal that multivariate normality and elliptical symmetry are strongly rejected for our sample (see Exhibits 2 and 3 of Appendix Figure F.1).

Finally, to check for the presence of asymmetric dependence between asset returns, we analyzed the exceedance correlation and tail dependence in our sample. For each pair of asset returns, Appendix Figure F.2 plots the exceedance correlation function proposed in Ang and Chen (2002), Longin and Solnik (2001), and Patton (2004), which depicts the correlation between returns above or below a given quantile. In the case of symmetric dependence, the correlation for both extremes should be similar and equal to zero for Gaussian dependence.

According to these plots, any assumptions of normality or symmetry seem unrealistic for our sample. Oil and gold do not display the same level of diversification for bear and bull markets, and correlation between oil and equity is highly positive for large negative returns but smaller for large positive returns. The correlation between gold and equity is close to 0 for large negative returns and significantly positive for very large positive returns. Although oil and gold are very positively correlated for large negative returns, are not or even are negatively correlated for large positive returns.

If we estimate the tail dependence of each pair of returns in our sample, we observe an asymmetric pattern. Panel D of Table F.2 reports the fitted upper and lower tail dependence parameters, τ^U and τ^D, corresponding to the symmetrized Joe-Clayton (SJC) copula, defined by Patton (2006b). Tail dependence increases over the 2006-2010 period, and lower tail dependence estimates are generally larger than the upper ones, especially in the last sample period.

Both univariate and multivariate analyses suggest that the assumptions of normality and symmetry for the individual processes and dependence functions are very restrictive and probably should be rejected. A flexible model that captures all the features analyzed in the data thus is required. In the next section, we estimate the conditional copula model proposed in Section III for our vector of oil, gold, and equity returns. Subsequently, we investigate whether capturing these features (e.g., non-normality of the individual processes, time-varying moments, asymmetric dependence) using the more flexible model leads to economically better portfolio decisions.

B. Estimation of the conditional copula model

In this section, we estimate the conditional copula model using the multistage maximum likelihood procedure explained in Section III.C. We first present the in-sample estimation and goodness-of-fit test results for the marginal distribution models. In a second stage, we analyze the results for the copula model.

In the online Appendix F, we provide the tables and figures related to the in-sample estimation analyses of the proposed multivariate copula models.

B.1. Models for the marginal distributions

Appendix Table F.3 presents maximum likelihood estimates of the parameters of the marginal distributions for oil, gold, and equity index returns. We compute robust standard errors of these estimates and report their corresponding p-values in parentheses. These estimates correspond to the generalized t marginal distribution function with time-varying moments, described in Equations (6)-(11) of Section III.A.

In the mean equation, we find that the basis (oil returns) and momentum and risk-free rate (gold returns) are significant explanatory factors. The results for the variance equation further show that volatility is strongly persistent for all returns. For equity, only negative returns have an effect on subsequent variance. This result is consistent with the leverage effect studied by Campbell and Hentschel (1992), among others. Yet for both commodities, especially gold, we observe an *inverse* leverage effect; that is, positive shocks have a stronger effect on variance than do negative ones of the same size.

Regarding the dynamics of the degrees-of-freedom and asymmetry parameters, we find that both higher moments are rather persistent for all asset returns over the in-sample period. Large moves in oil returns, especially negative ones, diminish the posterior degrees of freedom ($\delta_1^- = 5.51 > 0$, $\delta_1^+ < 0$), increasing the likelihood of posterior large shocks. For equity returns, large moves, especially positive ones, increase the subsequent degrees of freedom ($\delta_1^- < 0$, $\delta_1^+ = 21.14 > 0$), so large returns are less likely. For gold returns, extreme events are generally more likely to cluster in periods of large positive moves: Positive shocks are followed by a decrease in posterior degrees of freedom ($\delta_1^+ = -3.63 < 0$), whereas negative shocks generally are followed by an increase ($\delta_1^- = -3.26 < 0$).

In general, lagged values of the asymmetry parameter are more significant for subsequent parameter values than is the effect of the previous returns shock. Over our study's in-sample period, only positive shocks in gold returns ($\zeta_1^+ = -0.90 < 0$) and negative shocks in equity returns ($\zeta_1^- = -2.55 < 0$) seem to have effects of opposite signs on the posterior asymmetry parameters. Therefore, for the three assets returns, we find significant time variation in the moments of the univariate processes. As a benchmark, Table F.3 also reports the degrees-of-freedom and asymmetry parameters, η_c and λ_c, of the conditional distribution with constant shape parameters. We find that for the in-sample period, the left tail of the conditional distribution of oil and equity returns is fatter than the right tail, with parameters η_c and λ_c equal to 11.24 and -0.09 for oil returns, and 12.25 and -0.23 for equity returns. In contrast, gold returns have positive (though not significant) asymmetry parameters and heavier tails than oil and equity returns (η_c =4.79).

A reliable estimation of copula models requires an appropriate specification of the univariate density functions (see Patton (2006a,b) and Jondeau and Rockinger (2006)). To avoid misspecified copula models, we conduct the in-sample goodness-of-fit test suggested by Diebold, Gunther and Tay (1998) for our estimated marginal distribution models. If the marginal model is correctly specified, the probability integral transform should be i.i.d. Uniform(0,1). According to the the LM statistics in Panel A of Appendix Table F.4, we must reject serial dependence in the first four moments of the probability integral transform (all p-values > 0.15). In addition, the KS statistics suggest that the shape of the conditional distribution model is correctly specified for the three returns (p-values > 0.90). Visually, Figure F.3 of Appendix F also supports these results. The asymmetric marginal model performs substantially better than the Gaussian and symmetric models, even for constant higher moments.

Finally, we compare our more general skewed t marginal model against different constrained alternative models using likelihood ratio (LR) tests. In Panel B of Table F.4, we report the LR test statistics for the next four alternative marginal models: the generalized t distribution with constant parameters, the standard Student's t distribution with time-varying and constant degrees of freedom, and the standard Gaussian distribution. In all cases, we reject the restricted specification in favor of a more general model, at least at a 5% significance level (p-values < 0.05).

B.2. Models for the copulas

In this second stage, we estimate the dependence function that links the three marginal distribution models. We analyze the three time-varying or conditional copula functions described in Section III. B: the Gaussian, t, and skewed t copulas. Appendix Table F.5 presents the in-sample ML estimates of the three conditional copula models, estimated on the transformed residuals of the generalized t univariate model. This table also reports the estimates p-values, computed from the asymptotic covariance matrix, and the likelihood values at the optimum for the conditional and unconditional copulas.[11]

According to the ML estimates, for all copula models, the dependence coefficient between oil and gold returns, $\rho_{\text{oil,gold}}$, is positive, whereas the dependence coefficient between gold and equity index returns, $\rho_{\text{gold,equity}}$, is negative, and that between oil and equity, $\rho_{\text{oil,equity}}$, is insignificantly different from 0. Therefore, the estimated dependence coefficients are consistent with the unconditional in-sample correlations reported in Table F.2.

Estimates of the degrees of freedom ν are strongly significant for symmetric and skewed t copulas, indicating the presence of a significant level of dependence in the extremes. Regarding the estimation of the skewed t copula, we find that all elements of the asymmetry parameters vector $(\gamma_{\text{oil}}, \gamma_{\text{gold}}, \gamma_{\text{equity}})$ are negative, especially for oil and equity, suggesting more extreme dependence among returns during extreme depreciations of these assets compared with during bullish markets.

The parameters ω_0, ω_1, and ω_2, which parametrized the dynamic equation of dependence, are significant for all conditional copulas, showing strong evidence of time variation and persistence in the conditional dependence. These results regarding the estimates of the dependence functions are consistent with the preliminary multivariate analysis of Section IV.A.2.

According to the LR test statistics reported in Appendix Table F.5, we observe, first, that conditional copulas are preferred over their corresponding unconditional versions (p-

[11] The conditional dependence follows the dynamics in Equation (14) for $M = 4$, which are the number of lags consistent with the autoregressive lags considered in the univariate models.

values $\leqslant 0.05$ for the three cases). Second, the presence of tail dependence in the in-sample data is not negligible. The p-values of the LR test of the conditional and unconditional t copulas with respect to the more restrictive Gaussian copulas are always less than 0.10. Third, there is evidence of asymmetry dependence over the in-sample period, captured by the skewed t copula, but the gains from modeling this asymmetry may not make up for the penalty associated with the inclusion of more parameters in the model. These gains seem less significant than those obtained from modeling time-varying and tail dependence.

B.3. Out-of-sample parameters forecasts

We proceed to obtain the optimal portfolio decisions over the out-of-sample period. These portfolio strategies are based on various multivariate copula models and require the forecasts of the different parameters at play over the 2006-2010 period. For that purpose, we recursively reestimate the marginal and copula models throughout the out-of-sample period (220 weekly observations) using a rolling window scheme that drops distant observations as more recent ones are added and therefore keeps the size of the estimation window fixed at 836 observations. Once we reestimated the model for each point in the out-of-sample period, we constructed the time-series of one-period-ahead parameter forecasts needed for the allocation stage.

Figure F.4 of the online Appendix shows the output of the forecasts of the conditional mean, volatility, and skewness of each return process throughout the out-of-sample period. The volatility forecasts of all asset returns are relatively high, especially around October 2008 (see Exhibit 2). Conditional skewness is negative for equity and oil returns during the 2006-2010 period, but it is positive for gold returns during that period (see Exhibit 3).

Appendix Figure F.5 presents the forecasts of the conditional dependence parameters. It is worth noting that there is an increase in the fitted correlation coefficients among oil, gold, and equity from October 2008, especially for oil and equity returns (see Exhibit 1). In addition, the dependence coefficients seem to evolve more similarly in the latter part of the sample. The degrees-of-freedom forecasts decrease after August 2007, indicating rising tail dependence since then (see Exhibit 2). In addition, the asymmetry parameter of oil ranges between -0.6 and -0.2, which implies that extreme dependence seems to be stronger during large depreciations of oil, compared with large drops in gold or equity, whose asymmetry parameters range between -0.2 and +0.2 (see the forecast of the asymmetry parameter vector in Exhibit 3).

In general, during our reestimation of the copula models, we found no evidence to contradict skewed and fat-tailed marginal distributions and asymmetric and extreme conditional dependence, but strong evidence indicated that Gaussian distribution and elliptical dependence are not the best-fitting models. These results over the allocation period are consistent with the sample statistics we reported previously.

In summary, the skewed t copula provides a more informative measure of the dependence between commodities and equity-index returns, even taking into account that part of the tail behavior is captured by the skewed fat-tailed marginal distribution models. Therefore, possibly time-varying tail thickness and asymmetry are key factors not taken into account in a standard elliptical, (*à la Markowitz*), approach. The extent to which these factors have a significant impact on the portfolio choice decision is addressed in the next section.

C. Optimal portfolio results

In this section, we compute optimal portfolio decisions driven by the copula models we presented in the previous sections. We also analyze the portfolio performance of these strategies over the whole out-of-sample period. We mainly focus on six model-driven portfolio strategies, which can be analyzed from the perspective of copula models and therefore estimated using the multistage procedure from Section III.C. We also include the equally weighted portfolio (with and without rebalancing), which is a common benchmark in prior literature.

First, we consider the unconditional multivariate Gaussian model (pure Markowitz), a constant Gaussian copula with unconditional Gaussian marginal distributions. Second, we generalize this case by considering two conditional multivariate Gaussian distributions: the constant conditional correlation (CCC) and the dynamic conditional correlation (DCC). Both

CCC and DCC specifications are formed by conditional Gaussian marginal distributions with the conditional means and variances defined in Equations (6) and (8). Third, we compute portfolio strategies using the conditional copula models of Section III. Thus, we consider the generalized Student's t distribution for the marginal models (Equations 6-9) and three types of conditional dependence functions: the Gaussian, t, and skewed t copulas (Equations 13-14). With this set of models, we can compare the gains of including more flexible distributions as a means to compute portfolio decisions.

For each copula model, we obtain the optimal portfolio weights and thus the optimal portfolio return series, maximizing the investor's three-moment utility function from Equation 3. One of the advantages of using these parametric models is that the optimization problem can be solved numerically using Monte Carlo simulations. Thus, for each period of the allocation sample, we employ the parameter forecasts to generate 10,000 independent paths, according to Equation 12. Then, we compute the moments of the portfolio and maximize the three-moment utility subject to the non linear budget constraint.[12]

We solve the optimization problem for different values of the parameters φ_{V} and φ_{S}, which define the investor's three-moment utility (see Equation 3). Following Harvey et al. (2010), using the different parameterizations of utility, we can account for different impacts of the portfolio variance and skewness on the investor's preferences. Some specifications of the investor's three-moment utility can be interpreted as the expected value of the third-order Taylor series approximation of the power utility function; in that case, the value of parameters φ_{V} and φ_{S} depend on the coefficient of relative risk aversion $\mathcal{A}$.[13]

[12]We conduct the maximization using a sequential quadratic programming method for nonlinearly constrained optimizations (Nocedal and Wright (2006)).

[13]As we mentioned in Section II, under the third-order Taylor approximation, the parameters φ_{V} and φ_{S} are given by the second and third partial derivatives of the power utility function (Guidolin and Timmermann (2008); Jondeau and Rockinger (2012)).

C.1. Portfolio weights

We first compute and analyze the portfolio strategies over the out-of-sample period driven by the different copula models. To consider the impact of risk and loss aversions, the optimal portfolio weights are obtained for different specifications (φ_{V} and φ_{S}) of the three-moment preferences. Table 1 reports the mean, standard deviation, and percentiles of the optimal weights for oil, gold, and equity, obtained using our most general model, the conditional skewed t copula. In the online Appendix F (Figure F.6), we plot the time series corresponding to these optimal weights.

<Insert Table 1 about here>

Regarding these results, we observe that under the skewed t copula model, holdings of oil and gold are positive and their sum is generally greater than 1, leveraged by short positions in equity. Over the out-of-sample period, mean weights in oil range from 0.41, for the preferences specification with parameters $\varphi_{\mathrm{V}} = 1$ and $\varphi_{\mathrm{S}} = 1/2$, to 1.06, when $\varphi_{\mathrm{V}} = 1/4$ and $\varphi_{\mathrm{S}} = 1/2$; the average weights of gold range from 0.82 (for $\varphi_{\mathrm{V}} = 5/2$ and $\varphi_{\mathrm{S}} = 5$) to 3.87 (for $\varphi_{\mathrm{V}} = 1/4$ and $\varphi_{\mathrm{S}} = 1/2$); and the mean of equity holdings is between -4.03 (for $\varphi_{\mathrm{V}} = 1/4$ and $\varphi_{\mathrm{S}} = 1/2$) and -1.16 (for $\varphi_{\mathrm{V}} = 5/2$ and $\varphi_{\mathrm{S}} = 5$).

Thus we find that on average, when the impact of skewness on investor's preferences (φ_{S}) increases (larger loss aversion), the position in equity diminishes, increasing the holdings of commodities in the portfolio. For example, when the preference parameter φ_{S} doubles its value from 1/2 to 1, leaving φ_{V} equal to 1, the mean weights of oil and gold exhibit a relative increase of 23% and 13%, whereas the short position in equity increases 27%. In addition, when the impact of portfolio variance on preferences (φ_{V}) increases (larger variance aversion), the short positions in equity decline on average, and consequently, exposure to commodities decreases. When the preference parameter φ_{V} increases in value from 1/4 to 1, the remaining φ_{S} equal 1/2, the long positions in oil and gold decrease 61% and 59% on average, and the short position in equity decreases 73%. The intuition is that both commodities are likely to experience extreme moves (low degrees of freedom in the marginal distribution) and thus may be perceived as riskier when variance aversion increases, whereas they become safer investments, especially gold, in terms of skewness (positive jumps) as loss aversion rises.

From the point of view of a power utility investor, we observe that when the coefficient of relative risk aversion $\mathcal{A}$ increases, the leverage of the portfolio declines, at the expense of diminishing long positions in gold. The distribution of weights is altered by the combined effect of larger variance (φ_{V}) and loss (φ_{S}) aversions: The median of equity weights rises substantially from -1.29 for $\mathcal{A}$=1 (φ_{V}=1/2 and φ_{S}=1/3) to 0.21 for $\mathcal{A}$=5 (φ_{V}=5/2 and φ_{S}=5), while that of gold weights diminishes from 1.65 to 0.43.

Next we compare the optimal portfolio weights obtained using the different multivariate copula models. Table 2 reports the summary statistics of the optimal portfolio weights over the allocation period for the six model-driven portfolio strategies. In this case, the comparison refers to a fixed level of variance and loss aversion, the preference parameters are set to φ_{V}=1 and φ_{S}=1 ($\mathcal{A}$ = 2). To address the effects of skewness preferences on the asset allocation with commodity futures (Panel A), we also report the optimal portfolio weights for mean-variance preferences (Panel B), that is, when φ_{S} is equal to 0. In addition, in Appendix Figure F.7, we plot the time-series of the allocation differences between the skewed t copula model and other multivariate specifications.

<Insert Table 2 about here>

The bulk of the difference between portfolios strategies depends on the use of different marginal distribution models. Significant discrepancies arise when using time-varying Gaussian marginal distributions (CCC and DCC models) instead of unconditional Gaussian ones (Uncond. Gaussian). There are also relevant differences between using Gaussian (CCC and DCC) and generalized Student's t distributions (conditional copulas) for modeling the conditional univariate distributions. The quantiles presented in Table 2 and the plots in Figure F.6 show that the weights of conditional copula models with skewed and fat-tailed marginal distributions are more extreme than those with conditional Gaussian univariate distributions.

A second source of allocation differences is driven by the various specifications of the conditional dependence. These discrepancies in optimal weights arise, first, from introducing a dynamics equation in the dependence structure (e.g., DCC vs. CCC); second, from considering tail dependence (e.g., t copula vs. Gaussian copula); and, third, from capturing asymmetric dependence (e.g., skewed t vs. t copula). The statistics in Table 2 show that the portfolio decisions obtained from the skewed t copula are more aggressive than those from the Gaussian and t copulas.

For an investor with mean-variance preferences (Panel B of Table 2), there are fewer allocation differences related to the type of conditional dependence; portfolio weights of conditional copulas are more similar. In addition, optimal weights from t and skewed t copula models are less extreme than matched weights under the three-moment preferences. Comparing the portfolio weights of the skewed t copula under both preference specifications, we find that long positions in oil and gold decrease 57% and 25% on average when the investor shows no loss aversion; the short position in equity decreases 60%. That is, there is less leverage in equity under the mean-variance preferences.

Finally, in Figure 2, we compare the optimal portfolio weights for different copula models and for different preference parameterizations. This comparison refers to the first period of the out-of-sample window; to simplify, we just consider three-moment preference specifications corresponding to the third-order Taylor expansion of the power utility function with coefficient $\mathcal{A}$. The main differences between the skewed t copula and other multivariate models emerge at low levels of $\mathcal{A}$ (low levels of variance and loss aversion). For Gaussian models, such as Uncond. Gaussian and DCC, these differences are more pervasive along the $\mathcal{A}$ line. The impact of tail and asymmetric dependence is less significant as long as $\mathcal{A}$ is increasing. Thus, variance aversion seems to dominate loss aversion when $\mathcal{A}$ increases, consistent with the Taylor expansion interpretation, such that the impact of the second-order factor is larger than that of the third-order one.

<Insert Figure 2 about here>

C.2. Investment ratios and portfolio performance

We turn to analyze the moments, investment ratios, and performance of the portfolio decisions. Table 3 contains the summary statistics and investment ratios of the realized optimal portfolios for each allocation strategy and for different specifications of a three-moment investor's preferences. We consider three investment ratios: Sharpe, Sortino, and Omega, given respectively by $(\mu_P - r_f)/\sigma_P$, $(\mu_P - r_f)/\sqrt{q_2^l(r_f)}$, and $q_1^u(r_f)/q_1^l(r_f)$, where μ_P is the average portfolio return, σ_P is the portfolio volatility, r_f is the risk-free rate, and $q_m^l(r_f)$ and $q_m^u(r_f)$ are the lower and upper partial moments of order m with a given target value equal to the risk-free rate.[14] The Sortino ratio modifies the Sharpe ratio by dividing the excess return of the portfolio by the downside standard deviation or square root of semi-variance. The Omega ratio can be interpreted as the probability weighted ratio of gains to losses, relative to the risk-free rate. The higher the values of these ratios, the better portfolio performance.

The Sharpe, Sortino, and Omega ratios of the equally weighted portfolios are equal approximately to 0.05, 0.07 and 1.14 (see Table 3). The poor performance of equity markets and the boom of gold and oil during the 2006-2010 period reveals that these portfolios, with constant holdings in oil and gold futures, perform remarkably well compared other strategies based on distribution models. In particular, for almost all three-moment preferences considered, the mean, skewness, and investment ratios of the unconditional Gaussian model (Markowitz model) are substantially smaller than those corresponding to the equally weighted portfolios.

<Insert Table 3 about here>

Yet one of the three conditional copula models with generalized Student's t marginal distributions has generally the largest investment ratios, performing better than the multivariate conditional Gaussian models. In addition, the conditional t and skewed t copulas

[14]The lower and upper partial moments of order m for a given target θ are defined as $q_m^l(\theta) = \int_{-\infty}^{\theta}(\theta - r)^m f_p(x)\mathrm{d}x$ and $q_m^u(\theta) = \int_{\theta}^{\infty}(r-\theta)^m f_p(x)\mathrm{d}x$, where $f_p(x)$ is the probability density function of the portfolio returns.

have greater Sortino and Omega ratios than do the Gaussian copulas for all the preferences considered. When we take into account downside risk and the ratio of gains to losses, conditional copulas that capture tail dependence obtain higher investment measures across our allocation sample. For example, the Sortino and Omega ratios of the skewed t copula range from 0.10 and 1.20 for the preference parameters $\varphi_v = 5/2$ and $\varphi_v = 5$ (more risk averse) to 0.18 and 1.38 for the mean-skewness preferences with parameters $\varphi_v = 0$ and $\varphi_v = 1/4$ (only loss aversion). For these preferences, the Sortino and Omega ratios of the DCC model range from 0.05 and 1.09 to 0.16 and 1.31.

To compare each model-driven portfolio strategy with a benchmark allocation rule related to the equally-weighted portfolio, we consider three measures of relative performance: the performance fee (*Fee*), the Graham-Harvey (GH) measure, and the certainty equivalent (CEQ). The performance fee or opportunity cost is the amount that must be added to the return of the benchmark strategy, such that it leaves the investor indifferent to both strategies. The Graham and Harvey (1997) measure is the difference between the alternative portfolio returns on the volatility-matched benchmark portfolio. Thus, we lever up/down the benchmark to match the alternative portfolio's volatility over the evaluation period to make both portfolios comparable. Finally, the CEQ denotes the additional wealth, per dollar of investment, needed to raise the utility of an investor using the equally weighted portfolio such that it reaches the level of an investor who employs an alternative strategy.

Table 4 reports the relative performance measures (in basis points per week) of the realized portfolio returns over the allocation period (2006-2010) for different specifications of the investor's three-moment preferences. For all the preference parameterizations, the three relative performance measures indicate the same best performing strategy (see values in bold in Table 4). We can obtain substantial gains using the portfolio rules based on conditional copulas with generalized t marginal distributions. In particular, the skewed t copula outperforms the Gaussian copula in five of the six preference specifications we report. The opportunity costs of an investor holding the equally weighted portfolio instead of the portfolio based on the skewed t copula are between 3 and 68 basis points per year. The opportunity costs with respect to the Gaussian copula strategy range from -1.5 to 33 basis points per year.

<Insert Table 4 about here>

The GH measure, which helps us compare alternative strategies with different levels of risk with respect to the benchmark, shows that conditional copulas that take into account tail dependence outperform Gaussian dependence in five of the six preferences considered. Regarding the CEQ, we find similar conclusions about the out-performance of the alternative portfolios compared with that obtained using the performance fee.

In general, for the skewed t copula model, relative performance measures increase as variance and loss aversion decrease. As variance and loss aversion increase, skewed t copula strategies are less likely to produce large differences in terms of relative performance with respect to the benchmark. These results are consistent with findings previously reported in portfolio choice literature (e.g., Das and Uppal (2004) for the case of systemic risk and international portfolio choice).

According to the investment ratios and performance measures, we can conclude that the univariate higher moments seem to be a key feature in terms of better investment ratios and out-of-sample performance of the optimal portfolios. The dynamics in dependence among marginal functions also have an important role in the allocation decision process. Finally, the asymmetric and extreme dependence modeled in conditional t copulas make a difference for certain types of risk and loss aversion.

C.3. Robustness checks

In this section, we evaluate the robustness of our results. To test for the significance of the differences between portfolio strategies in terms of their economic performance, we adopt the out-of-sample test developed by Hansen (2005) and check jointly the superior performance ability of the different portfolio strategies during the allocation period. Using the realized utility as a performance metric or loss function, we can evaluate the various portfolio decisions with respect to a chosen benchmark over the entire out-of-sample period (Patton (2004)). Under the null hypothesis, the benchmark is as good as any alternative model in terms of performance. Following Hansen (2005), we employ the stationary bootstrap of Politis and Romano (1994) to derive the distribution of the test statistics. Thus we can obtain a consistent estimate of the p-value, as well as upper and lower bounds for that value.

Appendix Table F.7 reports these p-values for various benchmark models. When the equally weighted portfolio and unconditional Gaussian model serve as benchmarks, the null hypothesis is rejected. We therefore conclude that these portfolio strategies do not perform as well as the best competing alternative strategy.

In a second analysis, we solve the portfolio decision problem again when a risk-free asset is part of the investment opportunities. In Table F.8 of the Appendix, we report the investment ratios of the new portfolio allocations. In this case, we also find that conditional copulas with tail dependence perform better than traditional Gaussian strategies, when the investor's preferences indicate aversion to variance and negative skewness. These differences in performance are even larger for the Sortino and Omega ratios.

V. Conclusion

This article investigates the portfolio selection problem of an investor with time-varying three-moment preferences when commodity futures are part of the investment opportunity set. In our specification, the portfolio returns' skewness provides a measure of the investor's loss aversion. We model the joint distribution of asset returns using a flexible multivariate copula setting that can disentangle the specific properties of each asset process from its dependence structure. The more general model we posit consists of a conditional skewed t copula with generalized Student's t marginal distributions and time-varying moments. Thus we can capture the specific distributional characteristics of commodity-futures returns and focus on their implications for the portfolio selection problem.

The empirical application employs weekly data for oil and gold futures and for the S&P

500 equity index, from June 1990 to September 2010. Our preliminary statistics and in-sample estimation suggest the presence of individual skewness in the univariate processes, as well as evidence of both tail and asymmetric dependence among equity, oil, and gold. When computing the optimal portfolio weights, we find substantial discrepancies between the holdings obtained from our conditional copula models and those from more traditional Gaussian models. These discrepancies translate into economically significant differences in terms of better investment ratios and relative performance measures.

In order of relevance, the key factors underlying these differences are the proper specification of the time-varying univariate processes, followed by the conditional dynamics of the dependence among marginal distributions, and finally the extreme and asymmetric dependence modeling.

References

Abramowitz, M. and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (Dover Publications, New York, 1965).

Ang, A. and J. Chen, "Asymmetric correlations of equity returns." *Journal of Financial Economics* 63 (2002), 443–494.

Barberis, N. C. and M. Huang, "Stocks as lotteries: The implications of probability weighting for security prices." *The American Economic Review* 98 (2008), 2066–2100.

Büyüksahin, B. and M. Robe, "Speculators, commodities, and cross-market linkages." *Working paper, Commodity Futures Trading Commission* (2010).

Campbell, J. and L. Hentschel, "No news is good news:: An asymmetric model of changing volatility in stock returns." *Journal of Financial Economics* 31 (1992), 281–318.

Casassus, J. and P. Collin-Dufresne, "Stochastic convenience yield implied from commodity futures and interest rates." *Journal of Finance* 60 (2005), 2283–2331.

Das, S. R. and R. Uppal, "Systemic risk and international portfolio choice." *Journal of Finance* 59 (2004), 2809–2834.

Demarta, S. and A. J. McNeil, "The t copula and related copulas." *International Statistical Review* 73 (2005), 111–130.

Diebold, F. X., T. A. Gunther and A. S. Tay, “Evaluating density forecasts with applications to financial risk management.” *International Economic Review* 39 (1998), 863–883.

Dusak, K., “Futures trading and investor returns: An investigation of commodity market risk premiums.” *Journal of Political Economy* (1973), 1387–1406.

Embrechts, P., F. Lindskog and A. McNeil, “Modelling dependence with copulas and applications to risk management.” in S. T. Rachev, ed., *Handbook of heavy tailed distributions in finance* (Elsevier/North-Holland, Amsterdam, 2003), 329–384.

Engle, R. F., “Dynamic conditional correlation: a simple class of multivariate generalized autoregressive conditional heteroskedasticity models.” *Journal of Business and Economic Statistics* 20 (2002), 339–350.

Engle, R. F. and K. Sheppard, “Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH.” *NBER Working Paper* (2001).

Erb, C. B. and C. R. Harvey, “The strategic and tactical value of commodity futures.” *Financial Analysts Journal* 62 (2006), 69–97.

Etula, E., “Broker-dealer risk appetite and commodity returns.” *Federal Reserve Bank of New York Staff Reports* 406 (2010).

Goffe, W., G. Ferrier and J. Rogers, “Global optimization of statistical functions with simulated annealing.” *Journal of Econometrics* 60 (1994), 65–99.

Gorton, G. and K. G. Rouwenhorst, “Facts and fantasies about commodity futures.” *Financial Analysts Journal* 62 (2006), 47–68.

Graham, J. R. and C. R. Harvey, “Grading the performance of market-timing newsletters.” *Financial Analysts Journal* 53 (1997), 54–66.

Guidolin, M. and A. Timmermann, “International asset allocation under regime switching, skew, and kurtosis preferences.” *Review of Financial Studies* 21 (2008), 889–935.

Hansen, B. E., “Autoregressive conditional density estimation.” *International Economic Review* 35 (1994), 705–730.

Hansen, P. R., “A test for superior predictive ability.” *Journal of Business and Economic Statistics* 23 (2005), 365–380.

Harvey, C. R., J. C. Liechty, M. W. Liechty and P. Müller, “Portfolio selection with higher moments.” *Quantitative Finance* 10 (2010), 469–485.

Harvey, C. R. and A. Siddique, "Autoregressive conditional skewness." *Journal of Financial and Quantitative Analysis* 34 (1999), 465–487.

———, "Conditional skewness in asset pricing tests." *Journal of Finance* 55 (2000), 1263–1295.

Hilliard, J. E. and J. Reis, "Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions in the spot." *Journal of Financial and Quantitative Analysis* 33 (1998), 61–86.

Hong, H. and M. Yogo, "What does futures market interest tell us about the macroeconomy and asset prices?." *Journal of Financial Economics, forthcoming* (2012).

Irwin, S. and D. Sanders, "Index funds, financialization, and commodity futures markets." *Applied Economic Perspectives and Policy* 33 (2011), 1–31.

Jondeau, E. and M. Rockinger, "Conditional volatility, skewness, and kurtosis: existence, persistence, and comovements." *Journal of Economic Dynamics and Control* 27 (2003), 1699–1737.

———, "The copula-garch model of conditional dependencies: an international stock market application." *Journal of International Money and Finance* 25 (2006), 827–853.

———, "On the importance of time variability in higher moments for asset allocation." *Journal of Financial Econometrics* 10 (2012), 84–123.

Kat, H. M. and R. C. A. Oomen, "What every investor should know about commodities. Part II: multivariate return analysis." *Journal of Investment Management* 5 (2007), 16–40.

Kraus, A. and R. H. Litzenberger, "Skewness preference and the valuation of risk assets." *Journal of Finance* 31 (1976), 1085–1100.

Longin, F. and B. Solnik, "Extreme correlation of international equity markets." *Journal of Finance* 56 (2001), 649–676.

Mardia, K., "Measures of multivariate skewness and kurtosis with applications." *Biometrika* 57 (1970), 519–530.

McNeil, A. J., R. Frey and P. Embrechts, *Quantitative risk management: concepts, techniques, and tools* (Princeton University Press, New Jersey, 2005).

Mitton, T. and K. Vorkink, "Equilibrium underdiversification and the preference for skewness." *Review of Financial Studies* 20 (2007), 1255–1288.

Nelsen, R. B., *An introduction to copulas* (Springer, New York, 2006).

Nocedal, J. and S. J. Wright, *Numerical Optimization* (Springer, New York, 2006).

Patton, A. J., "On the out-of-sample importance of skewness and asymmetric dependence for asset allocation." *Journal of Financial Econometrics* 2 (2004), 130–168.

———, "Estimation of multivariate models for time series of possibly different lengths." *Journal of Applied Econometrics* 21 (2006a), 147–173.

———, "Modelling asymmetric exchange rate dependence." *International Economic Review* 47 (2006b), 527–556.

Politis, D. and J. Romano, "The stationary bootstrap." *Journal of the American Statistical Association* (1994), 1303–1313.

Routledge, B. R., D. J. Seppi and C. S. Spatt, "Equilibrium forward curves for commodities." *Journal of Finance* 55 (2000), 1297–1338.

Samuelson, P., "The fundamental approximation theorem of portfolio analysis in terms of means, variances and higher moments." *Review of Economic Studies* 37 (1970), 537–542.

Shefrin, H., *A behavioral approach to asset pricing* (Academic Press, San Diego, 2008).

Singleton, K. J., "Investor flows and the 2008 boom/bust in oil prices." *Working paper, Stanford University* (2011).

Sklar, A., "Fonctions de répartition à n dimensions et leurs marges." *Publications de l'Institut Statistique de l'Université de Paris* 8 (1959), 229–231.

Tang, K. and W. Xiong, "Index investment and financialization of commodities." *NBER Working Paper* (2010).

Table 1. Summary statistics of the optimal portfolio weights for the skewed t copula

This table reports the statistics for the allocation period related to the optimal portfolio weights for the conditional skewed t copula model. We consider different investors' preferences. $\mathcal{A}$ is the coefficient of relative risk aversion of power utility functions.

	$\varphi_{\mathrm{V}} = 1/2$, $\varphi_{\mathrm{S}} = 1/3$ ($\mathcal{A}$=1)			$\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A}$=2)			$\varphi_{\mathrm{V}} = 5/2$, $\varphi_{\mathrm{S}} = 5$ ($\mathcal{A}$=5)		
	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}
Mean	0.694	2.588	-2.382	0.506	1.775	-1.381	0.440	0.821	-1.161
Std. Dev.	1.676	3.169	3.359	1.318	2.607	2.896	2.700	2.509	3.432
pct. 5%	0.000	0.000	-9.454	0.000	0.000	-9.680	0.000	0.000	-10.00
pct. 25%	0.000	0.000	-4.886	0.000	0.000	-2.235	0.000	0.000	-0.681
pct. 50%	0.001	1.654	-1.289	0.017	0.921	-0.421	0.071	0.426	0.206
pct. 75%	0.846	4.647	0.515	0.565	2.994	0.579	0.348	1.121	0.643
pct. 95%	3.031	9.919	1.000	1.483	9.158	1.000	7.048	2.831	0.997
	$\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/4$			$\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/2$			$\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1/2$		
	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}
Mean	0.895	3.862	-3.758	1.058	3.868	-4.025	0.412	1.572	-1.084
Std. Dev.	1.859	4.084	4.302	2.077	4.234	4.265	1.295	2.201	2.555
pct. 5%	0.000	0.000	-10.00	0.000	0.000	-10.00	0.000	0.000	-9.000
pct. 25%	0.000	0.000	-9.000	0.000	0.000	-9.000	0.000	0.000	-2.115
pct. 50%	0.000	2.052	-2.576	0.004	2.327	-3.176	0.012	0.966	-0.322
pct. 75%	1.000	8.460	0.999	1.006	8.286	0.906	0.417	2.643	0.807
pct. 95%	4.796	10.00	1.000	5.698	10.00	1.000	1.444	5.445	1.000

Table 2. Summary statistics of the optimal portfolio weights for different strategies

This table reports the statistics for the allocation period related to the optimal portfolio weights for different multivariate models and two preference specifications. In Panel A, $\varphi_{\mathrm{V}} = 1$ and $\varphi_{\mathrm{S}} = 1$ (i.e., third-order Taylor's approximation for power utility function with $\mathcal{A} = 2$); in Panel B, $\varphi_{\mathrm{V}} = 1$ and $\varphi_{\mathrm{S}} = 0$ (i.e., second-order Taylor's approximation for power utility function with $\mathcal{A} = 2$).

Panel A: Mean-variance-skewness preferences.

	Uncond. Gaussian			CCC			DCC			Gaussian Copula			t copula			skewed t copula		
	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}
Mean	0.19	0.30	0.52	0.47	0.63	-0.10	0.53	0.65	-0.18	0.51	1.37	-0.97	0.56	1.54	-1.19	0.51	1.78	-1.38
Std. Dev.	0.14	0.29	0.30	0.56	0.84	0.93	0.64	0.89	1.01	1.28	1.96	2.30	1.62	2.29	2.93	1.32	2.61	2.90
pct. 5%	0.00	0.00	0.08	0.00	0.00	-1.86	0.00	0.00	-2.13	0.00	0.00	-4.58	0.00	0.00	-10.00	0.00	0.00	-9.68
pct. 25%	0.07	0.00	0.26	0.00	0.00	-0.70	0.00	0.00	-0.60	0.00	0.00	-1.69	0.00	0.00	-2.00	0.00	0.00	-2.24
pct. 50%	0.18	0.23	0.49	0.29	0.23	0.09	0.18	0.15	-0.19	0.11	0.80	-0.34	0.00	0.77	-0.22	0.02	0.92	-0.42
pct. 75%	0.27	0.54	0.80	0.77	1.07	0.60	1.17	1.13	0.66	0.69	2.32	0.37	0.46	2.41	0.96	0.57	2.99	0.58
pct. 95%	0.45	0.77	1.00	1.49	2.40	1.00	1.63	2.56	1.00	1.64	4.42	1.00	2.62	6.07	1.00	1.48	9.16	1.00

Panel B: Mean-variance preferences.

	Uncond. Gaussian			CCC			DCC			Gaussian Copula			t copula			skewed t copula		
	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}	w_{oil}	w_{gold}	w_{equity}
Mean	0.18	0.30	0.52	0.51	0.63	-0.13	0.51	0.68	-0.18	0.23	1.29	-0.52	0.26	1.28	-0.54	0.22	1.33	-0.55
Std. Dev.	0.14	0.29	0.31	0.56	0.83	0.89	0.60	0.90	0.98	0.40	1.44	1.47	0.39	1.47	1.47	0.38	1.50	1.53
pct. 5%	0.00	0.00	0.06	0.00	0.00	-1.83	0.00	0.00	-2.15	0.00	0.00	-3.41	0.00	0.00	-3.48	0.00	0.00	-3.69
pct. 25%	0.06	0.00	0.26	0.00	0.00	-0.66	0.00	0.00	-0.56	0.00	0.00	-1.51	0.00	0.00	-1.42	0.00	0.00	-1.40
pct. 50%	0.17	0.23	0.49	0.35	0.23	-0.02	0.21	0.26	-0.20	0.01	0.79	-0.21	0.01	0.81	-0.12	0.00	0.91	-0.28
pct. 75%	0.27	0.54	0.81	0.86	1.04	0.49	1.05	1.20	0.60	0.33	2.31	0.81	0.54	2.23	0.56	0.34	2.30	0.80
pct. 95%	0.46	0.79	1.00	1.44	2.41	1.00	1.48	2.57	1.00	1.03	3.90	1.00	1.10	3.99	1.00	1.03	4.08	1.00

Table 3. Investment ratios of the realized portfolio returns.

This table reports the investment ratios of the realized portfolio returns over the allocation (out-of-sample) period for different specifications of the investor's preferences (φ_{V} and φ_{S}) and for different strategies. We present the Sharpe, Sortino, and Omega ratios, given respectively as $(\mu - r_f)/\sigma$, $(\mu - r_f)/\sqrt{q_2^l(r_f)}$, and $q_1^u(r_f)/q_1^l(r_f)$, where $q_m^u(\theta)$ and $q_m^l(\theta)$ are the upper and lower partial moments of order m for a given target θ. $\mathcal{A}$ is the coefficient of relative risk aversion for power utility functions.

	Sharpe	*Sortino*	*Omega*	*Sharpe*	*Sortino*	*Omega*
Equally Weighted	0.050	0.071	1.142	0.050	0.071	1.142
E.W. Buy & hold	0.048	0.067	1.133	0.048	0.067	1.133
	pref.: $\varphi_{\text{V}} = 1/2$, $\varphi_{\text{S}} = 1/3$ ($\mathcal{A} = 1$)			pref.: $\varphi_{\text{V}} = 1/4$, $\varphi_{\text{S}} = 1/4$		
Uncond. Gaussian	0.022	0.029	1.062	-0.011	-0.014	0.967
CCC	0.049	0.083	1.160	0.030	0.048	1.099
DCC	0.061	0.099	1.203	0.063	0.102	1.209
Gaussian copula	**0.070**	**0.115**	1.241	0.047	0.072	1.166
t Copula	0.067	0.109	**1.246**	0.063	0.099	1.227
skewed t copula	0.063	0.104	1.216	**0.090**	**0.147**	**1.338**
	pref.: $\varphi_{\text{V}} = 1$, $\varphi_{\text{S}} = 1$ ($\mathcal{A} = 2$)			pref.: $\varphi_{\text{V}} = 1/4$, $\varphi_{\text{S}} = 1/2$		
Uncond. Gaussian	0.035	0.047	1.100	-0.022	-0.029	0.935
CCC	0.047	0.077	1.144	0.042	0.066	1.137
DCC	0.032	0.051	1.094	0.052	0.085	1.177
Gaussian copula	0.023	0.036	1.070	0.049	0.075	1.172
t Copula	**0.080**	**0.130**	**1.268**	0.042	0.065	1.141
skewed t copula	0.078	0.121	1.253	**0.071**	**0.111**	**1.259**
	pref.: $\varphi_{\text{V}} = 5/2$, $\varphi_{\text{S}} = 5$ ($\mathcal{A} = 5$)			pref.: $\varphi_{\text{V}} = 1$, $\varphi_{\text{S}} = 1/2$		
Uncond. Gaussian	0.047	0.065	1.135	0.031	0.041	1.087
CCC	0.033	0.047	1.091	0.041	0.066	1.124
DCC	0.034	0.049	1.093	0.032	0.049	1.092
Gaussian copula	0.060	0.088	1.173	0.062	0.100	1.201
t Copula	0.045	0.066	1.129	**0.089**	**0.149**	**1.312**
skewed t copula	**0.067**	**0.100**	**1.198**	0.072	0.118	1.235

Table 4. Performance measures of the realized portfolio returns

This table reports performance measures (in basis points per week) of the realized portfolio returns over the allocation (out-of-sample) period for different specifications of the investor's preferences and for six strategies based on different multivariate conditional and unconditional models. We present the management fee (*Fee*), the Graham-Harvey measure (*GH*), the utility, and the certainty equivalent (*CEQ*). All relative performance measures are computed with respect to the equally weighted strategy. $\mathcal{A}$ is the coefficient of relative risk aversion for power utility functions.

	Fee	*GH*	*Utility*	*CEQ*	*Fee*	*GH*	*Utility*	*CEQ*
	$\varphi_{\mathrm{V}} = 1/2$, $\varphi_{\mathrm{S}} = 1/3$ ($\mathcal{A}$=1)				$\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/4$			
Uncond. Gaussian	-0,076	-0,094	0,055	-0,088	-0,193	-0,260	-0,053	-0,218
CCC	0,302	-0,012	0,103	-0,040	0,372	-0,339	-0,075	-0,240
DCC	0,442	0,104	0,195	0,052	0,985	0,230	0,482	0,317
Gaussian copula	**0,506**	**0,183**	**0,291**	**0,148**	0,608	-0,060	0,190	0,025
t Copula	0,458	0,151	0,265	0,122	0,906	0,218	0,455	0,290
skewed t copula	0,436	0,118	0,229	0,086	**1,304**	**0,638**	**0,917**	**0,752**
	$\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A}$=2)				$\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/2$			
Uncond. Gaussian	-0,053	-0,041	0,060	-0,040	-0,241	-0,306	-0,102	-0,267
CCC	0,088	-0,015	0,041	-0,059	0,568	-0,143	0,209	0,044
DCC	0,023	-0,097	-0,059	-0,159	0,759	0,027	0,415	0,250
Gaussian copula	-0,030	-0,137	-0,086	-0,186	0,636	-0,017	0,269	0,104
t Copula	**0,242**	**0,145**	**0,202**	**0,102**	0,534	-0,137	0,149	-0,016
skewed t copula	0,118	0,010	0,060	-0,041	**0,932**	**0,310**	**0,616**	**0,451**
	$\varphi_{\mathrm{V}} = 5/2$, $\varphi_{\mathrm{S}} = 5$ ($\mathcal{A}$=5)				$\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1/2$			
Uncond. Gaussian	-0,032	-0,007	0,003	0,033	-0,064	-0,054	0,048	-0,053
CCC	-0,048	-0,054	-0,093	-0,064	0,060	-0,046	-0,002	-0,102
DCC	-0,047	-0,050	-0,082	-0,052	0,014	-0,097	-0,057	-0,157
Gaussian copula	0,024	0,026	0,005	0,035	0,175	0,061	0,098	-0,002
t Copula	-0,014	-0,014	-0,041	-0,011	**0,308**	**0,197**	**0,237**	**0,137**
skewed t copula	**0,049**	**0,050**	**0,024**	**0,054**	0,214	0,109	0,155	0,055

Exhibit 1: Contour plots

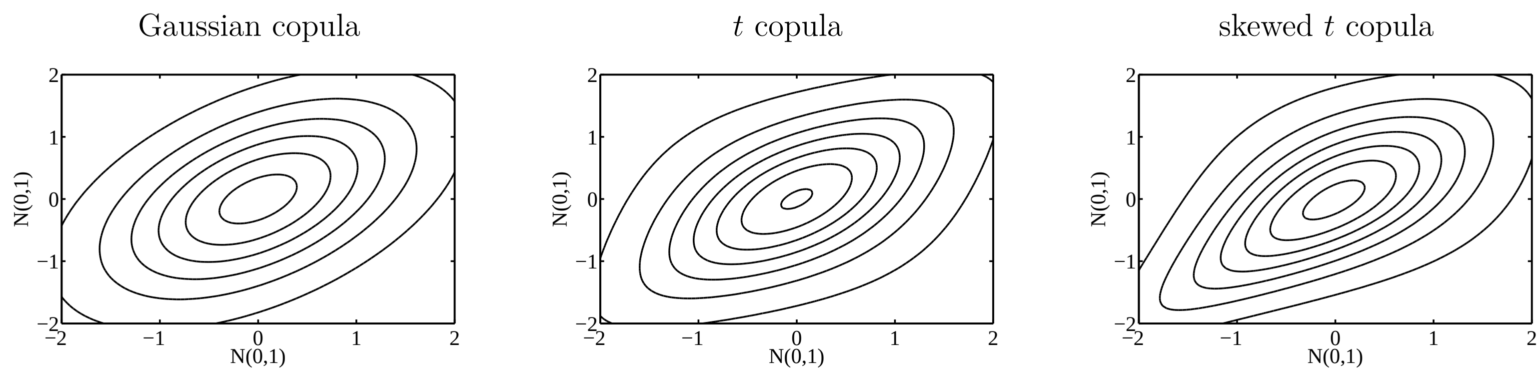

Exhibit 2: Probability density function

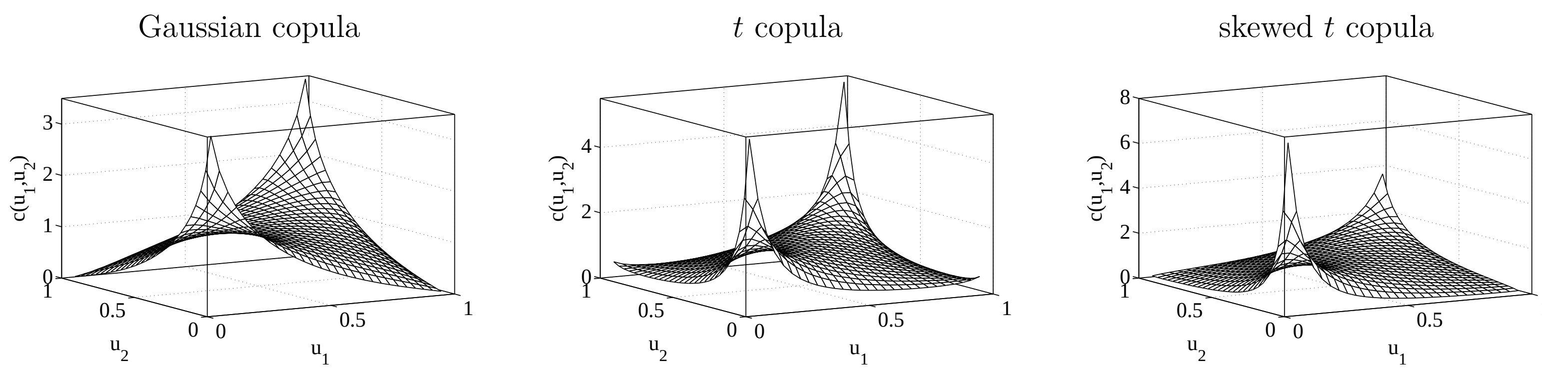

Figure 1. Copula functions.

In Exhibit 1, we show the contour plots of the distribution for three different copulas: Gaussian, t, and skewed t. To compare just the copula function, all of them are evaluated using standard normal marginal distributions (N(0,1) in xy-axis). In Exhibit 2, we show a bivariate representation of the probability density function $c(u_1, u_2)$ for the three copulas. In both exhibits we have employed the next set of parameters: $\rho = 0.5$ for the Gaussian copula, $\nu = 5$ and $\rho = 0.5$ for the t copula, and $\nu = 5$, $\boldsymbol{\gamma} = (-0.5, -0.5)'$, and $\rho = 0.5$ for the skewed t copula.

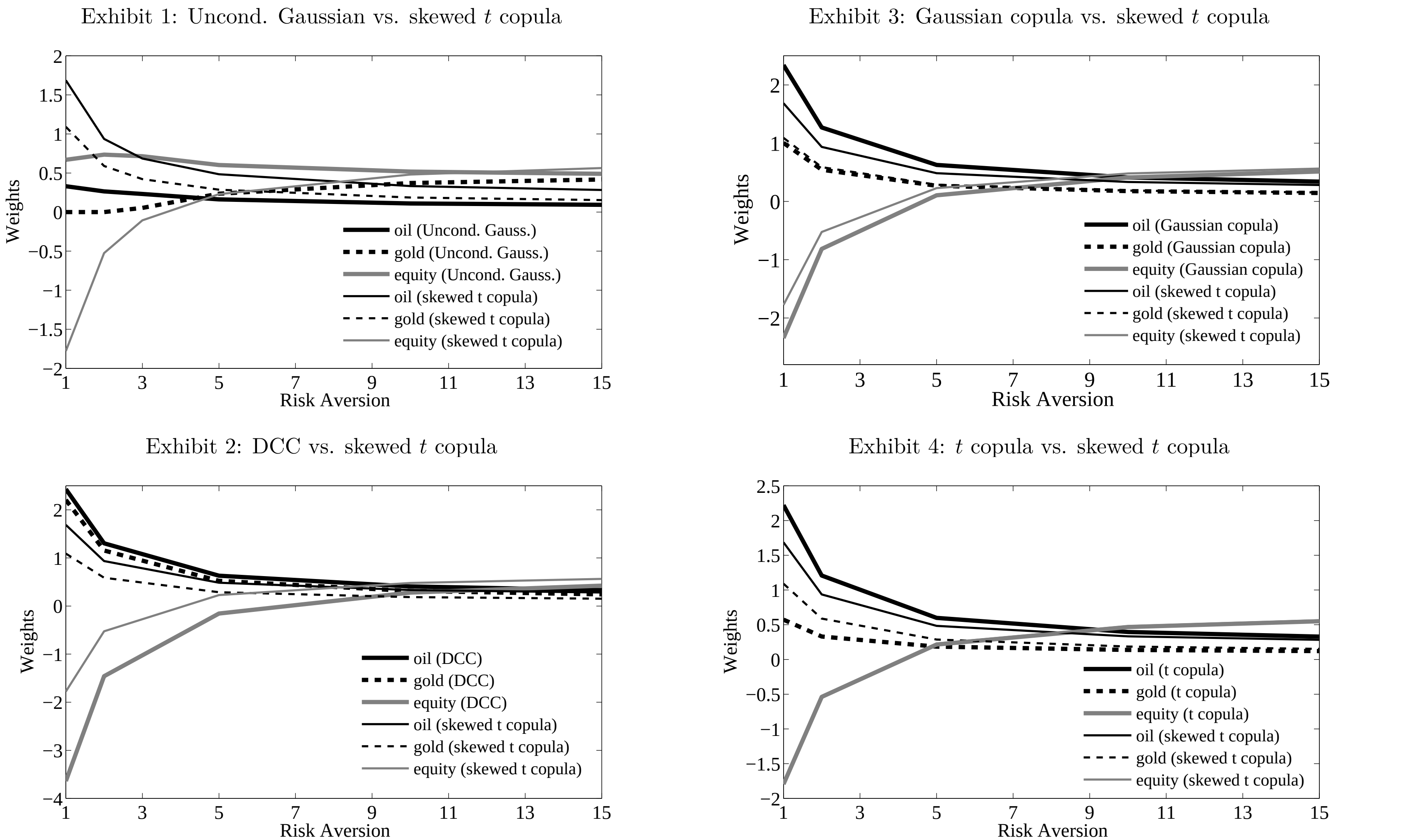

Figure 2. Optimal weights with respect to risk aversion.

In Exhibits 1 to 4, we compare, for some levels of risk aversion, the optimal portfolio weights of different models with those obtained with the conditional skewed t copula model. This comparison is realized at time $t = 1$ of the allocation period.

Online Appendix

A. The generalized t univariate distribution

In this appendix, we summarize some useful results related to the generalized t distribution introduced by Hansen (1994) and posteriorly analyzed by Jondeau and Rockinger (2003), among others. The following presentation is based on these works. Consider a random variable z that follows a generalized t distribution. Its probability density function $g(z;\, \nu, \lambda)$ is defined as

$$g(z;\, \nu, \lambda) = \begin{cases} b\,c\left[1 + \dfrac{1}{\nu-2}\left(\dfrac{b\,z+a}{1-\lambda}\right)^2\right]^{-(\nu+1)/2} & z < -a/b \\ b\,c\left[1 + \dfrac{1}{\nu-2}\left(\dfrac{b\,z+a}{1+\lambda}\right)^2\right]^{-(\nu+1)/2} & z \geqslant -a/b \end{cases}, \tag{A.1}$$

where $2 < \nu < +\infty$ and $-1 < \lambda < 1$, and the constants a, b and, c are given by

$$a = 4\,\lambda\,c\,(\nu-2)/(\nu-1), \quad b^2 = 1 + 3\lambda^2 - a^2, \quad \text{and} \quad c = \frac{\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi(\nu-2)}\,\Gamma\left(\nu/2\right)}. \tag{A.2}$$

Furthermore, $\Gamma(y)$ denotes the Gamma function, defined as $\Gamma(y) = \int_0^\infty t^{y-1}\mathrm{e}^{-t}\mathrm{d}t$ for $\Re(y) > 0$ (see (Abramowitz and Stegun, 1965)).

According to Jondeau and Rockinger (2003, Proposition 1), we can express the the generalized t cumulative distribution function, $G(p\,;\nu, \lambda)$, as a function of the standard Student's t distribution with ν degrees of freedom, $T(p;\, \nu)$, as follows

$$G(p;\, \nu, \lambda) = \begin{cases} (1-\lambda)\,T\left(\sqrt{\frac{\nu}{\nu-2}}\,\frac{bp+a}{1-\lambda}\,;\, \nu\right) & p < -a/b \\ (1+\lambda)\,T\left(\sqrt{\frac{\nu}{\nu-2}}\,\frac{bp+a}{1+\lambda}\,;\, \nu\right) & p \geqslant -a/b \end{cases}, \tag{A.3}$$

where the standard Student's t distribution is defined as

$$T(p;\, \nu) = \int_{-\infty}^{p} \frac{\Gamma\left((\nu+1)/2\right)}{\sqrt{\pi\nu}\,\Gamma\left(\nu/2\right)}\left(1 + x^2/\nu\right)^{-(\nu+1)/2}\,\mathrm{d}x. \tag{A.4}$$

Finally, the inverse function of the generalized t distribution can be obtained by inverting the standard Student's t distribution, such that

$$G^{-1}(u;\,\nu,\lambda) = \begin{cases} \frac{1}{b}\left[(1-\lambda)\sqrt{\frac{\nu-2}{\nu}}T^{-1}\left(\frac{u}{1-\lambda};\,\nu\right) - a\right] & u < \frac{1-\lambda}{2} \\ \frac{1}{b}\left[(1+\lambda)\sqrt{\frac{\nu-2}{\nu}}T^{-1}\left(\frac{u+\lambda}{1+\lambda};\,\nu\right) - a\right] & u \geq \frac{1-\lambda}{2} \end{cases}, \tag{A.5}$$

where $u \in [0, 1]$.

B. Copula functions

This section describes the three implicit copulas we propose as dependence functions of our multivariate model: the Gaussian, t, and skewed t copulas (see Section III). These three copulas correspond to the dependence functions contained in three multivariate normal mixture distributions (see McNeil et al. (2005)). This class of distributions adopts the following representation:

$$\boldsymbol{X} = \boldsymbol{\mu} + W\boldsymbol{\gamma} + \sqrt{W}\boldsymbol{Z} \tag{B.1}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are parameter vectors in $\mathbb{R}^d$, $\boldsymbol{Z} \sim N(\boldsymbol{0}, \boldsymbol{\Sigma})$, and W is a random variable independent of $\boldsymbol{Z}$. When the mixing random variable W satisfies $\nu/W \sim \chi^2(\nu)$, then the resulting mixture distribution of the random vector $\boldsymbol{X}$ is denoted as the asymmetric or skewed t distribution, $H(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \nu, \boldsymbol{\gamma})$, which belongs to the the wider family of multivariate generalized hyperbolic distributions.

Applying Sklar's theorem in Equation (5), we can obtain the skewed t copula function from the generalized hyperbolic skewed t distribution, $H(\boldsymbol{0}, \boldsymbol{P}, \nu, \boldsymbol{\gamma})$, defined by $\boldsymbol{\mu} = \boldsymbol{0}$, and the correlation matrix $\boldsymbol{P}$ implied by the dispersion matrix $\boldsymbol{\Sigma}$.[15]Then, the skewed t copula is defined as

$$C^{\mathrm{SK}}(\boldsymbol{u}; \boldsymbol{P}, \nu, \boldsymbol{\gamma}\,) = H\Big(\, H_1^{-1}(u_1\,; \nu, \gamma_1),\, \ldots,\, H_d^{-1}(u_d\,; \nu, \gamma_d)\,; \boldsymbol{P}, \nu, \boldsymbol{\gamma}\,\Big)\,, \tag{B.2}$$

where the $H_i(\cdot\,;\,\nu, \gamma_i)$ are the d univariate skewed t distribution functions, the H_i^{-1} are the corresponding quantile functions, and $\boldsymbol{u} = (u_1, \ldots, u_d)'$ is the probability transformed vector.

[15]The copula function is invariant under any strictly increasing transformation of the marginal distributions, including the standardization of the components of the random vector $\boldsymbol{X}$.

Special cases can be obtained from the normal mixture representation in Equation (B.1). When $\boldsymbol{\gamma} = \mathbf{0}$, we have the multivariate Student's t distribution; obviously, when W is constant, we obtain the multivariate Gaussian distribution. Thus, the unique t copula of a d-variate Student's t distribution can be expressed as

$$C^{\mathrm{T}}(\boldsymbol{u}; \boldsymbol{P}, \nu) = T(\, T^{-1}(u_1;\, \nu),\, \dots,\, T^{-1}(u_d; \nu)\,;\, \boldsymbol{P}, \nu)\,, \tag{B.3}$$

where $T(\cdot\,;\, \boldsymbol{P}, \nu)$ is the joint distribution function of a d-variate Student's t distribution with ν degrees of freedom and correlation matrix $\boldsymbol{P}$, and $T^{-1}(u_i; \nu)$ is the inverse function of the univariate Student's t distribution with ν degrees of freedom. In the same way, we can define the d-variate Gaussian copula as

$$C^{\mathrm{G}}(\boldsymbol{u};\, \boldsymbol{P}) = \Phi(\Phi^{-1}(u_1),\, \dots,\, \Phi^{-1}(u_d)\,;\, \boldsymbol{P})\,, \tag{B.4}$$

where $\Phi(\cdot\,;\, \boldsymbol{P})$ denotes the joint distribution function of the d-variate standard normal distribution with correlation matrix $\boldsymbol{P}$, and Φ^{-1} denotes the inverse of the univariate standard normal distribution.

We proceed to compute the density functions of the three copulas. The density function of a parametric copula that is absolutely continuous is given by

$$c(\boldsymbol{u}) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \cdots \partial u_d}. \tag{B.5}$$

For the case of the three implicit copulas we consider here, the density functions can be obtained from differentiating Equations (B.2), (B.3), and (B.4). Thus, the density function of the d-variate skewed t copula can be expressed as

$$c^{\mathrm{SK}}(\boldsymbol{u};\, \boldsymbol{P}, \nu, \boldsymbol{\gamma}) = \frac{h(\, H_1^{-1}(u_1;\, \nu, \gamma_1),\, \dots,\, H_d^{-1}(u_d;\, \nu, \gamma_d)\,;\, \boldsymbol{P}, \nu, \boldsymbol{\gamma}\,)}{h_1(H_1^{-1}(u_1;\, \nu, \gamma_1);\, \nu, \gamma_1) \cdots h_d(H_d^{-1}(u_d;\, \nu, \gamma_d);\, \nu, \gamma_d)}. \tag{B.6}$$

where $h(\cdot;\, \boldsymbol{P}, \nu, \boldsymbol{\gamma})$ is the joint density of the multivariate skewed t distribution H, and the $h_i(\cdot;\, \nu, \gamma_i)$ are its corresponding marginal density functions. Using the results from McNeil et al. (2005, Section 3.2.3) for the density functions of generalized hyperbolic distributions, and some algebra, we explicitly obtain the density function of the d-variate skewed t copula, given by

(B.7)

$$c^{\mathrm{SK}}(\boldsymbol{u};\, \boldsymbol{P}, \nu, \boldsymbol{\gamma}) = \frac{K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu + \boldsymbol{x}'\boldsymbol{P}^{-1}\boldsymbol{x})\boldsymbol{\gamma}'\boldsymbol{P}^{-1}\boldsymbol{\gamma}}\right)\left(\sqrt{(\nu + \boldsymbol{x}'\boldsymbol{P}^{-1}\boldsymbol{x})\boldsymbol{\gamma}'\boldsymbol{P}^{-1}\boldsymbol{\gamma}}\right)^{\frac{\nu+d}{2}} \mathrm{e}^{\boldsymbol{x}'\boldsymbol{P}^{-1}\boldsymbol{\gamma}}}{\prod_{i=1}^{d} K_{\frac{\nu+1}{2}}\left(\sqrt{(\nu + x_i^2)\gamma_i^2}\right)\left(\sqrt{(\nu + x_i^2)\gamma_i^2}\right)^{\frac{\nu+1}{2}} \mathrm{e}^{x_i\gamma_i}}$$
$$\times |\boldsymbol{P}|^{-1/2}\left(\frac{\Gamma(\frac{\nu}{2})}{2^{1-\nu/2}}\right)^{d-1} \frac{\prod_{i=1}^{d}\left(1 + \frac{x_i^2}{\nu}\right)^{(\nu+1)/2}}{\left(1 + \frac{\boldsymbol{x}'\boldsymbol{P}^{-1}\boldsymbol{x}}{\nu}\right)^{(\nu+d)/2}},$$

where $\boldsymbol{x} = (x_1, \ldots, x_d)'$ and $x_i = H_i^{-1}(u_i\,;\, \nu, \gamma_i)$.[16] In addition, K_η is the modified Bessel function of the second kind with order η, which can be implemented numerically (see Abramowitz and Stegun (1965) for more details about the modified Bessel functions of second kind and their properties). A similar expression to that in Equation (B.5) can be derived for the t and Gaussian copulas from their respective joint and marginal density functions. Thus the density function of the d-variate t copula is given by

(B.8)
$$c^{\mathrm{T}}(\boldsymbol{u}; \boldsymbol{P}, \nu) = |\boldsymbol{P}|^{-1/2} \frac{\Gamma\left(\frac{\nu+d}{2}\right)\Gamma\left(\frac{\nu}{2}\right)^{d-1}}{\Gamma\left(\frac{\nu+1}{2}\right)^{d}} \frac{\prod_{i=1}^{d}\left(1 + \frac{x_i^2}{\nu}\right)^{(\nu+1)/2}}{\left(1 + \frac{\boldsymbol{x}'\boldsymbol{P}^{-1}\boldsymbol{x}}{\nu}\right)^{(\nu+d)/2}},$$

where $x_i = T^{-1}(u_i; \nu)$. Finally, the density function of the d-variate Gaussian copula is expressed as

(B.9)
$$c^{\mathrm{G}}(\boldsymbol{u}; \boldsymbol{P}) = |\boldsymbol{P}|^{-1/2} \exp\left(-\frac{1}{2}\,\boldsymbol{x}'\left(\boldsymbol{P}^{-1} - \mathbb{I}_d\right)\boldsymbol{x}\right),$$

where $x_i = \Phi^{-1}(u_i)$ and $\mathbb{I}_d$ denotes the unit matrix of size d.

C. The two-stage log-likelihood function

Following Nelsen (2006, Theorem 2.10.9) and Patton (2006b, Theorem 1), Equation (5) presents a multivariate and conditional extension to Sklar's theorem. Then, according to

[16] In the empirical part of the article, to improve the feasibility of the computations related to the cumulative density and quantile functions, H_i and H_i^{-1}, respectively, we approximate the univariate skewed t density function h_i using cubic splines.

Equation (5), the conditional density function of the joint distribution $F_t(r_{1,t+1}, \ldots, r_{d,t+1};\, \boldsymbol{\theta})$ is given by

$$
\text{(C.1)} \qquad f_t(\boldsymbol{r}_{t+1}; \boldsymbol{\theta}) = \frac{\partial^d F_t(\boldsymbol{r}_{t+1}; \boldsymbol{\theta})}{\partial r_{1,t+1} \cdots \partial r_{d,t+1}} = \prod_{i=1}^{d} f_{i,t}(r_{i,t+1}\, ; \boldsymbol{\theta}_{i,M}) \cdot c_t(u_{i,t+1}, \ldots, u_{i,t+1}\, ; \boldsymbol{\theta}_C),
$$

where the $u_{i,t+1} = F_{i,t}(r_{i,t+1}\, ; \boldsymbol{\theta}_{i,M})$ are the marginal conditional distributions; $f_{i,t}(r_{i,t+1}\, ; \boldsymbol{\theta}_{i,M})$ are the marginal conditional density functions; and $c_t(u_{1,t+1}, \ldots, u^d_{d,t+1}\, ; \boldsymbol{\theta}_C)$ is the conditional copula density function (defined in Equation (B.5)).

Taking logarithms in Equation (C.1) and summing for all observations in the sample, $\bar{\boldsymbol{r}}_T = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_T\}$, we determine that the log-likelihood function of the joint model $\mathcal{L}(\boldsymbol{\theta}\, ; \bar{\boldsymbol{r}}_T)$ in Equation (15) can be divided in two terms, as follows:

$$
\begin{aligned}
\text{(C.2)} \qquad \mathcal{L}(\boldsymbol{\theta}\, ; \bar{\boldsymbol{r}}_T) &= \sum_{i=1}^{d} \sum_{t=1}^{T} \log f_{i,t}(r_{i,t+1}\, ; \boldsymbol{\theta}_{i,M}) + \sum_{t=1}^{T} \log c_t(u_{1,t+1}, \ldots, u_{d,t+1}\, ; \boldsymbol{\theta}_C) \\
&= \sum_{i=1}^{d} \mathcal{L}_i(\boldsymbol{\theta}_{i,M}; \bar{\boldsymbol{r}}_T) + \mathcal{L}_C(\boldsymbol{\theta}_C; \boldsymbol{\theta}_M, \bar{\boldsymbol{r}}_T)\,,
\end{aligned}
$$

where $\mathcal{L}_i$ and $\mathcal{L}_C$ are the log-likelihood functions for the i-th marginal model and for the copula function, respectively. Moreover, $\boldsymbol{\theta}_M = (\boldsymbol{\theta}_{1,M} \ldots \boldsymbol{\theta}_{d,M})'$ is the parameter set of the d marginal conditional distributions, and $\boldsymbol{\theta}_C$ is the parameter set of the conditional copula. Therefore, using Equation (C.2), we can separate the maximum likelihood estimation of the joint model parameters $\boldsymbol{\theta}$ into two stages: one for the d marginal conditional distributions, and the other for the conditional copula. Appendix B provides the explicit expressions of the copula densities, which we need for the maximum likelihood estimation.

D. Explanatory variables for the mean equation

We include the following explanatory factors in the conditional mean equation of our marginal distribution model: short rate, default spread, momentum, basis, and growth in open interest.

- The short rate is a proxy for the expected shocks in the economy. We use the weekly average yield of the three-month T-bills.

- The default spread is the difference between Moody's Baa and Aaa corporate bond yields and should capture the variation in the risk premium. We also employ the weekly average of this variable.

- Momentum is computed as the moving average of previous returns, and it is employed to measure the market sentiment. In our empirical application, we compute it using the weekly returns of previous eight weeks.

- The basis is the relative difference between the six-month-maturity future prices and the month-ahead futures prices; the basis measures the futures curve slope with respect to a six-month delivery horizon. For the empirical application, we employ the weekly average of this measure.

- The dollar open interest is the month-ahead futures price multiplied by the number of contracts outstanding; thus, growth in open interest measures the capital flow into commodity markets. Again, to reduce the noise in the measure, we use the weekly average of this variable.

Although all these variables are considered for the mean equation, we ultimately just select the exogenous regressors that are statistically significant in our time-series analysis.

E. Multivariate tests

Engle and Sheppard (2001) test for constant correlation Panel B of Table F.2 presents the results of Engle and Sheppard (2001) test for constant correlation for 5, 10, and 20 lags. This test requires a consistent estimate of the constant conditional correlation and a vector autoregression. We use the standardized residuals of GARCH(1,1) processes to estimate the correlation matrix, $\boldsymbol{P}$, and the diagonal matrix of standard deviations, $\boldsymbol{D}_{t+1}$. Then, under the null hypothesis of constant correlation, all the coefficients in

$$\text{vech}^{\text{u}}(\boldsymbol{Y}_{t+1}) = \alpha + \beta_1 \text{vech}^{\text{u}}(\boldsymbol{Y}_{t+1-1}) + \ldots + \beta_s \text{vech}^{\text{u}}(\boldsymbol{Y}_{t+1-s}) + \eta_t \tag{E.1}$$

should be 0; vech^{u} is an operator that selects the upper off-diagonal elements, and $\boldsymbol{Y}_{t+1}$ is a symmetric matrix defined by

$$(\boldsymbol{P}^{-1/2}\boldsymbol{D}_{t+1}^{-1}\boldsymbol{r}_{t+1})(\boldsymbol{P}^{-1/2}\boldsymbol{D}_{t+1}^{-1}\boldsymbol{r}_{t+1})' - \mathbb{I}.$$

Under the null hypothesis, the statistic $\frac{\hat{\beta}\boldsymbol{X}'\boldsymbol{X}\hat{\beta}'}{\hat{\sigma}_\eta^2}$ is asymptotically distributed as χ^2_{s+1}, where $\boldsymbol{X}$ is the vector of regressors, $\boldsymbol{\beta}$ is the vector of coefficients in the latter regression Equation (E.1), and $\hat{\sigma}_\eta^2$ is the unbiased sample variance of the estimated residuals $\hat{\eta}_t$. All test p-values, which represent the probability of constant correlation, are all less than 0.05; we reject the hypothesis of constant correlation for all lags considered and for all sample periods.

Mardia (1970) test of multivariate normality We implement the Mardia (1970) test of multivariate normality, based on d-variate measures for the skewness and kurtosis of the vector of returns. These measures are computed using the so-called *Mahalanobis angle*, defined as

$$D_{tt'} = (\boldsymbol{r}_t - \bar{\boldsymbol{r}})'\bar{\boldsymbol{S}}^{-1}(\boldsymbol{r}_{t'} - \bar{\boldsymbol{r}}),$$

where $\bar{\boldsymbol{r}}$ and $\bar{\boldsymbol{S}}$ are the sample mean and covariance estimators. Under this framework, d-variate skewness and kurtosis are computed as

$$s_d = 1/T^2 \sum_{t=1}^{T}\sum_{t'=1}^{T} D_{tt'}^3 \quad \text{and} \quad k_d = 1/T\sum_{t=1}^{T}(D_{tt}^{1/2})^4\,.$$

Under the null hypothesis of multivariate normality, $1/6Ts_d$ and k_d are asymptotically distributed as a $\chi^2(d(d+1)(d+2)/6)$ and a $N(d(d+2), 8d(d+2)/T)$, respectively. Panel C of Table F.2 reports these multivariate measures, s_d and k_d, for our three-dimensional vector of returns and their corresponding statistics, rejecting the null hypothesis of multivariate normality for the three sample periods considered.

Test of ellipticity of the vector of returns Following McNeil et al. (2005), we test for the ellipticity of the vector of returns. This test considers if standardized returns are consistent with a spherical distribution. Standardized returns $\boldsymbol{z}_t$ are defined by the sample mean and covariance as follows

$$\boldsymbol{z}_t = \bar{\boldsymbol{S}}^{-1/2}(\boldsymbol{r}_t - \bar{\boldsymbol{r}})$$

If $\boldsymbol{z}_t$ is consistent with the spherical assumption, then the statistic T_{ellip} will be distributed according to a Beta distribution; that is,

$$T_{ellip} = \sum_{i=1}^{k} z_i^2 / \sum_{i=1}^{d} z_i^2 \sim \text{Beta}(k/2, (d-k)/2)\,,$$

where d is the dimension of the returns vector (i.e. $d = 3$), and k is chosen to roughly equal $d - k$ (see McNeil et al. (2005)). We analyze the results of this test graphically, through the qq-plot in Exhibit 2 of Figure F.1, and numerically, implementing a Kolmogorov-Smirnov (KS) test with the data, whose results are reported in Panel C of Table F.2. The curvature in the qq-plot suggests that the vector of returns is not elliptically distributed for any of the sample periods considered (we just report the plot for the full-sample period). For the full-sample period, the KS test statistic equals 0.174, above the critical value (0.048). Therefore, we reject the elliptical hypothesis. The same conclusion is inferred for the other subsamples.

Exceedance correlation The exceedance correlation is defined as the correlation between the returns above or below a given quantile. Following Longin and Solnik (2001), Ang and Chen (2002), and Patton (2004), we use exceedance correlation to investigate the dependence structure among commodities and equity returns, checking for the presence of possibly asymmetric interactions. The exceedance correlation at a threshold level q is given by

$$\varrho(q) = \begin{cases} \text{Corr}\Big[r_i, r_j \,\Big|\, r_i \leqslant Q_i(q) \cap r_j \leqslant Q_j(q)\Big] & \text{if } q \leqslant 0.5 \\ \text{Corr}\Big[r_i, r_j \,\Big|\, r_i > Q_i(q) \cap r_j > Q_j(q)\Big] & \text{if } q > 0.5 \end{cases}\,,$$

where $Q_i(q)$ and $Q_j(q)$ are the q-th quantiles of returns r_i and r_j. Figure F.2 plots exceedence correlation as a function of returns quantiles. The shape of the exceedance correlation function depends on the bivariate distribution between each pair of returns; it provides a means to measure the degree of asymmetry in the joint distribution of these returns. The exceedance correlation for the extreme returns is 0 for a bivariate normal distribution.

Patton (2006b)'s Symmetrized Joe-Clayton copula To attend to the tail dependence of the returns vector, we fit the symmetrized Joe-Clayton (SJC) copula proposed by Patton (2006b) to our unfiltered sample of returns. The SJC copula is given by:

$$C^{\mathrm{SJC}}(u_1, u_2; \tau^U, \tau^L) = 1/2 \left(C^{\mathrm{JC}}(u_1, u_2; \tau^U, \tau^L) + C^{\mathrm{JC}}(1 - u_1, 1 - u_2; \tau^L, \tau^U) + u_1 + u_2 - 1 \right).$$

This copula is a modification of the Joe-Clayton copula C^{JC}:

$$C^{\mathrm{JC}}(u_1, u_2; \tau^U, \tau^L) = 1 - \left(1 - \left(\frac{1}{[1 - (1 - u_1)^\kappa]^\xi} + \frac{1}{[1 - (1 - u_2)^\kappa]^\xi} - 1 \right)^{-1/\xi} \right)^{1/\kappa}$$

where $\kappa = 1/\log_2(2 - \tau^U)$, $\xi = -1/\log_2(\tau^L)$, and $\tau^U, \tau^L \in (0, 1)$. The parameters τ^U and τ^D are measure of dependence in the extremes, that is,

$$\lim_{\epsilon \to 0} \mathbb{P}[U_1 \leq \epsilon; U_2 \leq \epsilon] = \tau^L \quad \text{and} \quad \lim_{\epsilon \to 1} \mathbb{P}[U_1 > \epsilon; U_2 > \epsilon] = \tau^U$$

By construction, the SJC copula is symmetric when $\tau^U = \tau^L$ and exhibits no upper tail dependence if $\tau^U = 0$, or no lower tail dependence if $\tau^L = 0$. In Panel D of Table F.2, we report the estimates of the upper and lower tail dependence parameters, τ^U and τ^D, corresponding to the SJC copula.

F. Additional Tables and Figures

Table F.1. Descriptive statistics for oil, gold, and equity weekly returns

This table reports sample statistics of the weekly returns for the crude oil futures (NYMEX), gold futures (COMEX), and equity index (SP500). The full sample period ranges from June 1990 to September 2010, and includes 1056 observations. The in-sample period runs from June 1990 to June 2006 (836 observations) and the out-of-sample period from June 2006 to September 2010 (220 observations). *Mean, Std. Dev., Min., Max., and VaR 5%* are expressed in weekly percentages. *Sharpe* is the ratio of mean returns over the standard deviation. *JB* and *KS* refer to the Jarque-Bera and Kolmogorov-Smirnov normality test statistics, respectively. *LB(10)* and *LM(10)* are the Ljung-Box and the Lagrange-Multiplier test statistics, both conducted using 10 lags, to test for the presence of autocorrelation in returns and squared returns, respectively. The *p-values* are reported in parentheses.

	Full sample period			In-sample period			Out-of-sample period		
Interval	20-Jun-1990 / 08-Sep-2010			20-Jun-1990 / 21-Jun-2006			21-Jun-2006 / 08-Sep-2010		
Assets	oil	gold	equity	oil	gold	equity	oil	gold	equity
Mean	0.136	0.120	0.104	0.163	0.061	0.147	0.033	0.343	-0.059
	(0.341)	(0.082)	(0.145)	(0.285)	(0.353)	(0.045)	(0.929)	(0.114)	(0.766)
Std. Dev.	4.634	2.236	2.326	4.401	1.896	2.129	5.441	3.210	2.958
Min.	-36.53	-13.21	-16.45	-36.53	-11.04	-9.04	-16.63	-13.21	-16.45
Max.	23.98	12.88	10.18	14.55	12.88	10.18	23.98	10.92	9.639
Sharpe	0.029	0.054	0.045	0.037	0.032	0.069	0.006	0.107	-0.020
VaR 5%	6.892	3.294	3.744	6.494	2.737	3.483	7.717	4.753	4.931
Skewness	-0.598	0.007	-0.552	-0.929	0.102	-0.134	0.114	-0.205	-1.024
	(0.000)	(0.927)	(0.000)	(0.000)	(0.229)	(0.115)	(0.489)	(0.215)	(0.000)
Kurtosis	8.259	7.343	7.238	9.860	8.258	5.000	4.914	4.536	8.012
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
JB	1280	829.8	843.9	1759	964.3	141.9	34.06	23.18	268.7
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.002)	(0.000)
KS	0.446	0.467	0.469	0.450	0.473	0.470	0.442	0.459	0.467
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
LB(10)	27.93	25.97	29.98	14.27	29.12	24.05	35.99	17.08	22.55
	(0.002)	(0.004)	(0.001)	(0.161)	(0.001)	(0.007)	(0.000)	(0.073)	(0.013)
LM(10)	90.38	189.6	138.7	54.76	72.11	84.29	39.40	64.06	31.52
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)

Table F.2. Descriptive multivariate statistics for oil, gold, and equity weekly returns

This table reports the descriptive multivariate statistics of crude oil, gold, and equity index weekly returns. The full sample period ranges from June 1990 to September 2010 and includes 1056 observations. The in-sample period runs from June 1990 to June 2006, and the out-of-sample period from June 2006 to September 2010. Panel A shows the sample correlation for each period. In Panel B, we present the results of the Engle and Sheppard (2001) test for constant correlation. The *p-values* reported in parentheses indicate the probability of constant correlation. In Panel C, we report the results of the Mardia (1970) test of joint normality, which is based on the multivariate measures of skewness and kurtosis, denoted by s_3 and k_3. In Panel C, we also present the Kolmogorov-Smirnov (KS) statitics associated with the test of ellipticity (see McNeil et al. (2005)). In Panel D, we present the estimates of the upper and lower tail dependence parameters, τ^U and τ^D, of the symmetrized Joe-Clayton (SJC) copula (Patton (2006b)) for each pair of asset returns: oil-gold (o-g), oil-equity (o-e), and gold-equity (g-e).

	Full sample period			In-sample period			Out-of-sample period		
	20-Jun-1990 / 08-Sep-2010			20-Jun-1990 / 21-Jun-2006			21-Jun-2006 / 08-Sep-2010		
Panel A: Unconditional correlation									
i	*oil*	*gold*	*equity*	*oil*	*gold*	*equity*	*oil*	*gold*	*equity*
$\rho_{i,\text{oil}}$	1.000	0.268	0.080	1.000	0.205	-0.061	1.000	0.394	0.393
		(0.000)	(0.009)		(0.000)	(0.079)		(0.000)	(0.000)
$\rho_{i,\text{gold}}$		1.000	0.012		1.000	-0.079		1.000	0.165
			(0.698)			(0.022)			(0.014)
Panel B: Test of dynamic correlation									
s lags	5	10	20	5	10	20	5	10	20
stat.	28.64	41.79	45.19	15.12	28.22	32.85	16.09	24.84	42.49
(*p-val.*)	(0.000)	(0.000)	(0.002)	(0.019)	(0.003)	(0.048)	(0.013)	(0.001)	(0.004)
Panel C: Mardia's test and test of ellipticity									
	s_3	k_3	KS	s_3	k_3	KS	s_3	k_3	KS
coeff.	1.318	33.73		1.436	31.62		2.166	26.63	
stat.	232.0	55.57	0.174	199.7	43.86	0.168	79.41	15.75	0.216
(*p-val.*)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Panel D: Tail dependence estimates of SJC copula for each pair of returns									
	o-g	o-e	g-e	o-g	o-e	g-e	o-g	o-e	g-e
τ^U	0.042	0.000	0.000	0.030	0.000	0.000	0.044	0.073	0.041
(*p-val.*)	(0.157)	(0.282)	(1.000)	(0.267)	(0.808)	(0.704)	(0.773)	(0.368)	(0.684)
τ^L	0.167	0.061	0.001	0.080	0.000	0.000	0.380	0.321	0.102
(*p-val.*)	(0.000)	(0.031)	(0.093)	(0.031)	(0.902)	(0.767)	(0.000)	(0.000)	(0.214)

Table F.3. Results for the marginal distribution models

This table reports the maximum likelihood parameter estimates of the marginal distribution model for oil, gold, and equity-index returns with generalized Student's t distribution and time-varying moments. Parameters of the mean, variance, degrees of freedom, and asymmetry are defined in Equations (6), (8), (10), and (11), respectively. The results corresponds to the estimation period from June 1990 to June 2006 (836 observations). The *p-values* of the estimates appear in parentheses and are computed using the robust standard errors. The *logL* is the sample log-likelihood of the marginal distribution model. η_c and λ_c are the degrees of freedom and asymmetry parameters of the constant version of the model.

	oil		gold		equity	
	coeff.	*(p-val.)*	*coeff.*	*(p-val.)*	*coeff.*	*(p-val.)*
mean equation						
μ (/100)	0.140	(0.316)	0.474	(0.037)	0.130	(0.037)
$basis_{t-1}$	-0.031	(0.216)				
$momentum_{t-1}$			-0.175	(0.153)		
r^f_{t-1}			-0.111	(0.013)		
r_{t-1}					-0.108	(0.002)
r_{t-2}			-0.041	(0.213)		
r_{t-3}	0.050	(0.110)				
variance equation						
α_0 (/1000)	0.025	(0.175)	0.013	(0.012)	0.016	(0.022)
α_1^+	0.085	(0.001)	0.183	(0.001)	0.000	(0.956)
α_1^-	0.060	(0.030)	0.029	(0.189)	0.145	(0.000)
α_2	0.920	(0.000)	0.870	(0.000)	0.890	(0.000)
degrees-of-freedom equation						
δ_0	0.100	(0.000)	0.025	(0.042)	-0.200	(0.054)
δ_1^+	-0.732	(0.576)	-3.633	(0.000)	21.138	(0.049)
δ_1^-	5.514	(0.007)	-3.257	(0.002)	-2.489	(0.383)
δ_2	0.998	(0.000)	1.009	(0.000)	0.966	(0.000)
η_c	11.24	(0.002)	4.792	(0.000)	12.25	(0.009)
asymmetry parameter equation						
ζ_0 (/10)	0.085	(0.177)	0.088	(0.111)	-0.250	(0.099)
ζ_1^+	-0.357	(0.313)	-0.903	(0.041)	-0.374	(0.798)
ζ_1^-	0.218	(0.702)	0.592	(0.222)	-2.553	(0.034)
ζ_2	0.998	(0.000)	1.001	(0.000)	0.981	(0.000)
λ_c	-0.093	(0.070)	0.018	(0.577)	-0.230	(0.000)
logL	1,515.4		2,262.7		2,149.6	

Table F.4. DGT and LR tests of the marginal distribution models

In Panel A, we report the statistics and p-values of the Diebold et al. (1998) (DGT) test for the marginal distribution model for oil, gold, and equity-index returns with generalized Student's t distribution and time-varying moments (estimates are in Table F.3). The DGT test consists of two stages: (1) Lagrange multiplier statistics (LM) over 20 lags for the first four moments of the residuals to test for serial correlation, and (2) a goodness-of-fit test for the adequacy of the distribution model (see also Figure F.3). In Panel B, we provide the likelihood ratio test statistics (LR) with respect to different restrictive versions of the marginal distribution model. All the restrictive models have the same conditional mean and variance dynamics, described in Equations (6) and (8), but we consider four conditional distributions: (i) the generalized Student's t distribution with constant degree-of-freedom and asymmetry parameters, the Student's t distribution with (ii) time-varying and (iii) constant degrees of freedom, and (iv) the univariate Gaussian distribution. The *dof* are the number of constraints under the null condition. All the tests results corresponds to the in-sample period from June 1990 to June 2006 (836 observations).

Panel A: DGT test for the generalized Student's t

		oil		gold		index	
		stat.	(*p-val.*)	*stat.*	(*p-val.*)	*stat.*	(*p-val.*)
(1)	1st moment LM(20)	25.39	(0.187)	21.02	(0.396)	21.57	(0.364)
	2nd moment LM(20)	23.96	(0.244)	21.49	(0.369)	18.72	(0.54)
	3rd moment LM(20)	18.45	(0.558)	21.74	(0.355)	12.97	(0.879)
	4th moment LM(20)	24.45	(0.223)	17.66	(0.610)	17.00	(0.653)
(2)	Goodness-of-fit test	7.66	(0.990)	11.30	(0.913)	7.80	(0.989)

Panel B: Loglikelihood ratio tests (LR)

		oil		gold		index	
	dof	*LR stat.*	(*p-val.*)	*LR stat.*	(*p-val.*)	*LR stat.*	(*p-val.*)
(i) generalized t const. par.	6	24.50	(0.000)	19.48	(0.003)	18.16	(0.006)
(ii) t time-varying df.	4	9.794	(0.044)	12.13	(0.016)	31.23	(0.000)
(iii) t constant df.	7	27.71	(0.000)	19.64	(0.006)	38.57	(0.000)
(iv) Gaussian	8	47.00	(0.000)	92.89	(0.000)	52.38	(0.000)

Table F.5. Results and LR tests for the copula models

This table presents the maximum likelihood parameter estimates of the copula models under different assumptions of the conditional joint dependence. The results corresponds to the period from June 1990 to June 2006 (836 observations). In each case, the copula is defined by the next set of parameters: the correlation matrix ($\{\rho_{i,j}\}$), the degrees of freedom (ν), the asymmetry vector ($\boldsymbol{\gamma}$), and the parameters of the dynamics (ω_0, ω_1, and ω_2) (see Equation (14)). For each parameter estimate, we report in parentheses the p-values computed from the asymptotic covariance matrix. The conditional copula likelihood at the optimum is denoted by CL, whereas CL_{uncond} reports the likelihood at the optimum of the corresponding unconditional version of the copula. We also report the likelihood ratio test statistics (LR) for different restrictive specifications of the copula models.The LR(vs. uncond.) corresponds to the LR test with respect to the corresponding unconditional version of the copula model. In LR(vs. symmetric), we test with respect to the conditional t copula. In LR(vs. gaussian) the restrictive model is the conditional Gaussian copula. With LR(vs. uncond. symmetric) and LR(vs. uncond. gaussian), we test the conditional copulas with respect to the unconditional t and Gaussian copulas. Finally, with LR(uncond. vs. symmetric) and LR(uncond. vs. gaussian), we test the unconditional versions of the copulas with respect to the unconditional t and Gaussian copulas.

	Conditional copulas					
	Gaussian		t copula		Skewed t	
	coeff.	*(p-val.)*	*coeff.*	*(p-val.)*	*coeff.*	*(p-val.)*
$\rho_{\text{oil,gold}}$	0.159	(0.000)	0.157	(0.000)	0.161	(0.000)
$\rho_{\text{oil,equity}}$	-0.020	(0.556)	-0.016	(0.653)	-0.013	(0.653)
$\rho_{\text{gold,equity}}$	-0.064	(0.064)	-0.058	(0.108)	-0.057	(0.108)
ν			18.998	(0.025)	19.050	(0.031)
γ_{oil}					-0.268	(0.027)
γ_{gold}					-0.018	(0.934)
γ_{equity}					-0.141	(0.139)
ω_0	0.136	(0.223)	0.128	(0.192)	0.127	(0.265)
ω_1	0.079	(0.055)	0.069	(0.054)	0.060	(0.082)
ω_2	1.647	(0.000)	1.666	(0.000)	1.676	(0.000)
CL	18.680		20.828		23.136	
CL_{uncond}	12.453		15.222		18.134	
LR(vs. uncond.)	12.454	(0.006)	11.212	(0.011)	10.004	(0.019)
LR(vs. gaussian)			4.296	(0.038)	8.913	(0.063)
LR(vs. symmetric)					4.617	(0.202)
LR(vs. uncond. gaussian)			16.750	(0.002)	21.367	(0.003)
LR(vs. uncond. symmetric)					15.829	(0.015)
LR(uncond. vs. gaussian)			5.538	(0.019)	11.363	(0.023)
LR(uncond. vs. symmetric)					5.826	(0.120)

Table F.6. Moments and investment ratios of the realized portfolio returns.

This table reports the summary statistics of the realized portfolio returns over the allocation (out-of-sample) period for different specifications of the investor's preferences and for different strategies. We present the weekly mean (in %), standard deviation (in %), skewness, and two risk measures: 1% Value at Risk and 1% expected shortfall. We also present the Sharpe, Sortino, and Omega ratios, given respectively as $(\mu - r_f)/\sigma$, $(\mu - r_f)/\sqrt{q_2^l(r_f)}$, and $q_1^u(r_f)/q_1^l(r_f)$, where $q_m^u(\theta)$ and $q_m^l(\theta)$ are the upper and lower partial moments of order m for a given target θ. $\mathcal{A}$ is the coefficient of relative risk aversion for power utility functions.

	Mean	*Std.Dev.*	*Skew.*	*VaR 1%*	*ES 1%*	*Sharpe*	*Sortino*	*Omega*
Equally Weighted	0.187	2.937	-0.240	8.31	9.66	0.050	0.071	1.142
E.W. Buy & hold	0.176	2.832	-0.432	8.63	9.72	0.048	0.067	1.133
	preferences: $\varphi_{\mathrm{V}} = 1/2$, $\varphi_{\mathrm{S}} = 1/3$ ($\mathcal{A} = 1$)							
Uncond. Gaussian	0.110	3.294	-0.730	11.10	12.19	0.022	0.029	1.062
CCC	0.489	9.203	1.276	19.68	25.16	0.049	0.083	1.160
DCC	0.628	9.662	0.924	21.97	27.50	0.061	0.099	1.203
Gaussian copula	0.692	9.360	1.150	23.00	30.05	**0.070**	**0.115**	1.241
t Copula	0.645	9.045	0.991	22.04	31.80	0.067	0.109	**1.246**
skewed t copula	0.622	9.268	1.160	19.88	29.61	0.063	0.104	1.216
	preferences: $\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A} = 2$)							
Uncond. Gaussian	0.134	2.702	-0.690	9.18	9.68	0.035	0.047	1.100
CCC	0.274	4.974	0.966	10.68	14.31	0.047	0.077	1.144
DCC	0.209	5.311	0.776	12.90	14.37	0.032	0.051	1.094
Gaussian copula	0.156	5.069	0.975	11.31	15.07	0.023	0.036	1.070
t Copula	0.429	4.880	0.760	10.88	15.96	**0.080**	**0.130**	**1.268**
skewed t copula	0.434	5.079	0.823	10.53	14.86	0.078	0.121	1.253
	preferences: $\varphi_{\mathrm{V}} = 5/2$, $\varphi_{\mathrm{S}} = 5$ ($\mathcal{A} = 5$)							
Uncond. Gaussian	0.154	2.440	-0.500	8.00	8.47	0.047	0.065	1.135
CCC	0.139	3.058	-0.037	9.07	9.70	0.033	0.047	1.091
DCC	0.140	3.001	0.100	8.07	8.31	0.034	0.049	1.093
Gaussian copula	0.211	2.887	0.073	8.20	8.59	0.060	0.088	1.173
t Copula	0.172	2.940	0.024	8.42	8.64	0.045	0.066	1.129
skewed t copula	0.236	2.930	0.014	8.45	9.32	**0.067**	**0.100**	**1.198**

Table F.6 (cont.). Moments and investment ratios of the realized portfolio returns.

	Mean	*Std.Dev.*	*Skew.*	*VaR 1%*	*ES 1%*	*Sharpe*	*Sortino*	*Omega*
	preferences: $\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/4$							
Uncond. Gaussian	-0.007	4.266	-0.697	16.41	18.31	-0.011	-0.014	0.967
CCC	0.559	17.099	0.662	41.10	51.83	0.030	0.048	1.099
DCC	1.172	17.972	0.620	38.28	50.26	0.063	0.102	1.209
Gaussian copula	0.794	16.233	0.357	44.34	58.25	0.047	0.072	1.166
t Copula	1.092	16.629	0.274	42.33	59.91	0.063	0.099	1.227
skewed t copula	1.491	16.204	0.529	38.73	56.58	**0.090**	**0.147**	**1.338**
	preferences: $\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/2$							
Uncond. Gaussian	-0.055	4.232	-0.751	16.46	17.91	-0.022	-0.029	0.935
CCC	0.754	17.089	0.604	41.88	53.23	0.042	0.066	1.137
DCC	0.945	17.497	0.715	37.84	50.23	0.052	0.085	1.177
Gaussian copula	0.822	15.921	0.240	42.75	58.37	0.049	0.075	1.172
t Copula	0.721	16.297	0.288	42.83	59.07	0.042	0.065	1.141
skewed t copula	1.119	15.317	0.253	37.08	55.23	**0.071**	**0.111**	**1.259**
	preferences: $\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1/2$							
Uncond. Gaussian	0.123	2.729	-0.686	9.20	9.69	0.031	0.041	1.087
CCC	0.247	5.054	0.847	12.21	14.26	0.041	0.066	1.124
DCC	0.201	5.146	0.785	11.42	14.23	0.032	0.049	1.092
Gaussian copula	0.362	5.206	0.833	12.56	15.81	0.062	0.100	1.201
t Copula	0.495	5.149	0.992	11.18	16.08	**0.089**	**0.149**	**1.312**
skewed t copula	0.401	5.025	0.898	10.02	14.99	0.072	0.118	1.235
	preferences: $\varphi_{\mathrm{V}} = 0$, $\varphi_{\mathrm{S}} = 1/4$							
Uncond. Gaussian	1.406	40.655	0.125	94.80	107.78	0.034	0.051	1.090
CCC	2.905	40.145	0.736	79.55	97.66	0.072	0.120	1.223
DCC	3.642	41.020	1.481	71.29	75.36	0.088	0.165	1.306
Gaussian copula	3.458	36.578	0.867	76.98	93.38	0.094	0.159	1.317
t Copula	2.782	36.909	0.962	74.73	77.01	0.074	0.126	1.242
skewed t copula	3.816	36.227	0.914	76.98	93.38	**0.104**	**0.181**	**1.379**

Table F.7. Test for superior portfolio performance using a stationary bootstrap

This table reports the reality check p-values of the Hansen (2005) test for superior portfolio performance for different benchmark models. We employ the management fee as the performance function in this case. For small p-values, we reject the hypothesis that the benchmark model performs as well as the best competing alternative model. The implementation is based on the stationary bootstrap of Politis and Romano (1994).

	$\varphi_{\mathrm{V}} = 1/2$, $\varphi_{\mathrm{S}} = 1/3$ ($\mathcal{A}$=1)			$\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A}$=2)			$\varphi_{\mathrm{V}} = 5/2$, $\varphi_{\mathrm{S}} = 5$ ($\mathcal{A}$=5)		
Benchmark	lower	consist.	upper	lower	consist.	upper	lower	consist.	upper
Equally Weighted	**0.039**	**0.039**	**0.039**	**0.052**	**0.057**	**0.057**	**0.057**	**0.063**	**0.063**
Uncond. Gauss.	**0.039**	**0.039**	**0.039**	**0,050**	**0,052**	**0,052**	**0.062**	**0.062**	**0.062**
CCC	0.139	0.189	0.189	**0.090**	**0,091**	**0,091**	**0.076**	**0.093**	**0.093**
DCC	0.105	0.129	0.129	**0.071**	**0,071**	**0,071**	**0.042**	**0.076**	**0.076**
Gaussian copula	0.538	0.726	0.726	0.485	0.679	0.679	1.000	1.000	1.000
t copula	0.294	0.442	0.442	0.288	0.438	0.438	0.146	0.206	0.206
skewed t copula	1.000	1.000	1.000	1.000	1.000	1.000	0.345	0.480	0.580
	$\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/4$			$\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/2$			$\varphi_{\mathrm{V}} = 0$, $\varphi_{\mathrm{S}} = 1/2$		
Benchmark	lower	consist.	upper	lower	consist.	upper	lower	consist.	upper
Equally Weighted	**0.011**	**0.011**	**0.011**	**0.040**	**0.040**	**0.040**	**0.084**	**0.085**	**0.085**
Uncond. Gauss.	**0.041**	**0.043**	**0.043**	**0.067**	**0.070**	**0.070**	0.106	0.106	0.106
CCC	**0.095**	0.121	0.121	0.105	0.144	0.144	0.209	0.329	0.329
DCC	**0.094**	0.125	0.125	0.123	0.164	0.164	0.495	0.658	0.658
Gaussian copula	1.000	1.000	1.000	1.000	1.000	1.000	0.520	0.683	0.683
t copula	**0.060**	0.143	0.143	0.479	0.722	0.798	0.462	0.606	0.606
skewed t copula	**0.076**	0.138	0.138	0.393	0.568	0.605	1.000	1.000	1.000

Table F.8. Investment ratios of the realized portfolio returns with risk-free asset.

This table reports the investment ratios of the realized portfolio returns over the out-of-sample period when a risk-free asset is part of the set of investment opportunities. We consider different specifications of the investor's preferences and seven strategies. We present the Sharpe, Sortino, and Omega ratios, given respectively as $(\mu - r_f)/\sigma$, $(\mu - r_f)/\sqrt{q_2^l(r_f)}$, and $q_1^u(r_f)/q_1^l(r_f)$, where $q_m^u(\theta)$ and $q_m^l(\theta)$ are the upper and lower partial moments of order m for a given target θ. $\mathcal{A}$ is the coefficient of relative risk aversion for power utility functions.

	Sharpe	*Sortino*	*Omega*	*Sharpe*	*Sortino*	*Omega*
Equally Weighted	0.050	0.071	1.142	0.050	0.071	1.142
	pref.: $\varphi_{\mathrm{V}} = 1/2$, $\varphi_{\mathrm{S}} = 1/3$ ($\mathcal{A}$=1)			pref.: $\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/4$		
Uncond. Gaussian	0.007	0.009	1.022	0.011	0.014	1.031
CCC	0.077	0.114	1.240	0.057	0.081	1.168
DCC	0.068	0.102	1.204	0.067	0.096	1.204
Gaussian copula	0.103	0.165	1.362	0.118	0.192	1.437
t Copula	**0.170**	**0.300**	**1.688**	0.101	0.165	1.351
skewed t copula	0.122	0.237	1.549	**0.199**	**0.429**	**1.883**
	pref.: $\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A}$=2)			pref.: $\varphi_{\mathrm{V}} = 1/4$, $\varphi_{\mathrm{S}} = 1/2$		
Uncond. Gaussian	0.005	0.006	1.014	0.012	0.016	1.036
CCC	0.070	0.104	1.219	0.058	0.084	1.171
DCC	0.046	0.068	1.137	0.049	0.070	1.145
Gaussian copula	0.121	0.201	1.519	0.167	0.277	1.682
t Copula	**0.151**	**0.318**	**1.778**	0.145	0.260	1.563
skewed t copula	0.123	0.260	1.689	**0.168**	**0.327**	**1.732**
	pref.: $\varphi_{\mathrm{V}} = 5/2$, $\varphi_{\mathrm{S}} = 5$ ($\mathcal{A}$=5)			pref.: $\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1/2$		
Uncond. Gaussian	0.007	0.010	1.022	0.010	0.013	1.031
CCC	0.066	0.097	1.211	0.066	0.098	1.198
DCC	0.090	0.134	1.285	0.064	0.094	1.201
Gaussian copula	0.124	0.241	1.840	0.101	0.178	1.432
t Copula	0.158	0.311	2.362	0.091	0.167	1.418
skewed t copula	**0.166**	**0.483**	**2.750**	**0.109**	**0.228**	**1.584**

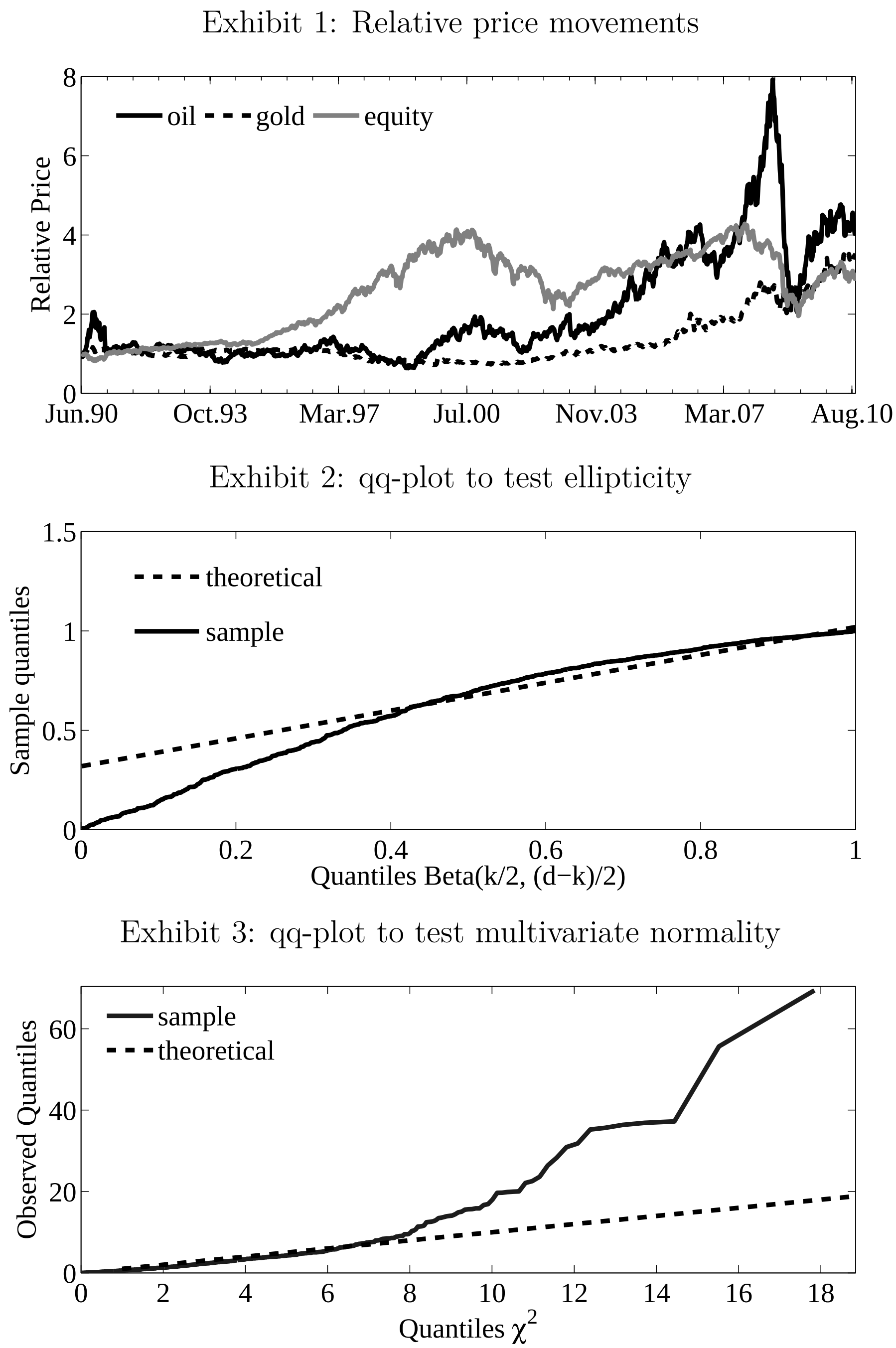

Figure F.1. Descriptive statistics: relative price performace and qq-plots.
Exhibit 1 plots the relative price movement of each asset to compare their performance over the full-sample period. Exhibit 2 compares the qq-plot of sample against the beta distribution, where $d = 3$ is the dimension of the vector of returns, and k is chosen to be roughly equal to $d - k$ (see McNeil et al. (2005)). The empirical observations are denoted by the solid line; the theoretical quantiles are represented by the dashed line. Exhibit 3 shows the qq-plot associated with Mardia's (1970) test of multivariate normality.

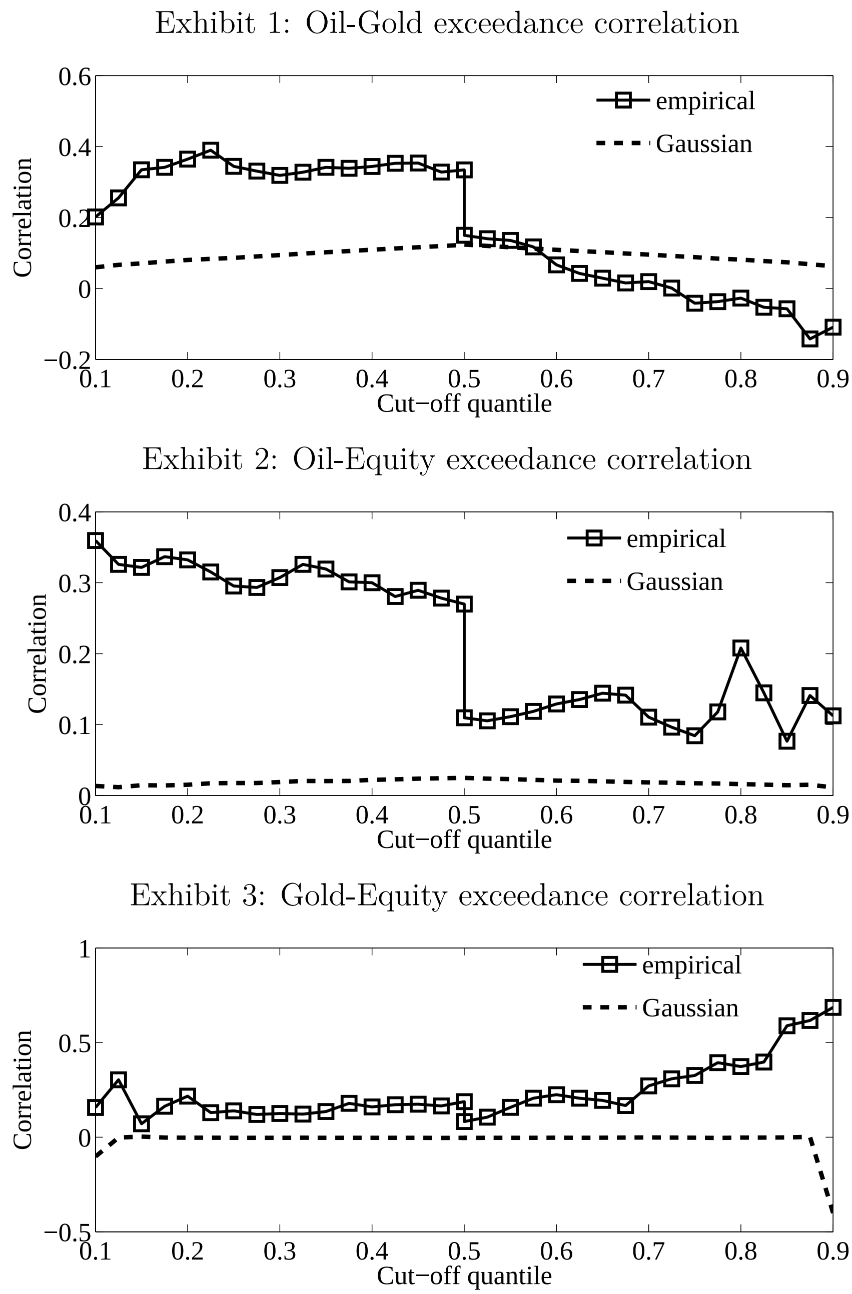

Figure F.2. Descriptive statistics: exceedance correlations.

Exhibits 1, 2, and 3 present the exceedance correlations for each pair of returns (see Longin and Solnik (2001) and Ang and Chen (2002)). The line with squares represent the actual exceedence correlation, whereas the dotted line represents the theoretical correlation between simulated normal return exceedances, assuming a Gaussian return distribution with parameters equal to the sample means and covariance matrix of the weekly returns (see Table F.1). The x-axis shows the cutoff quantile, and the y-axis presents the correlation between the two returns, given that both exceed that particular quantile.

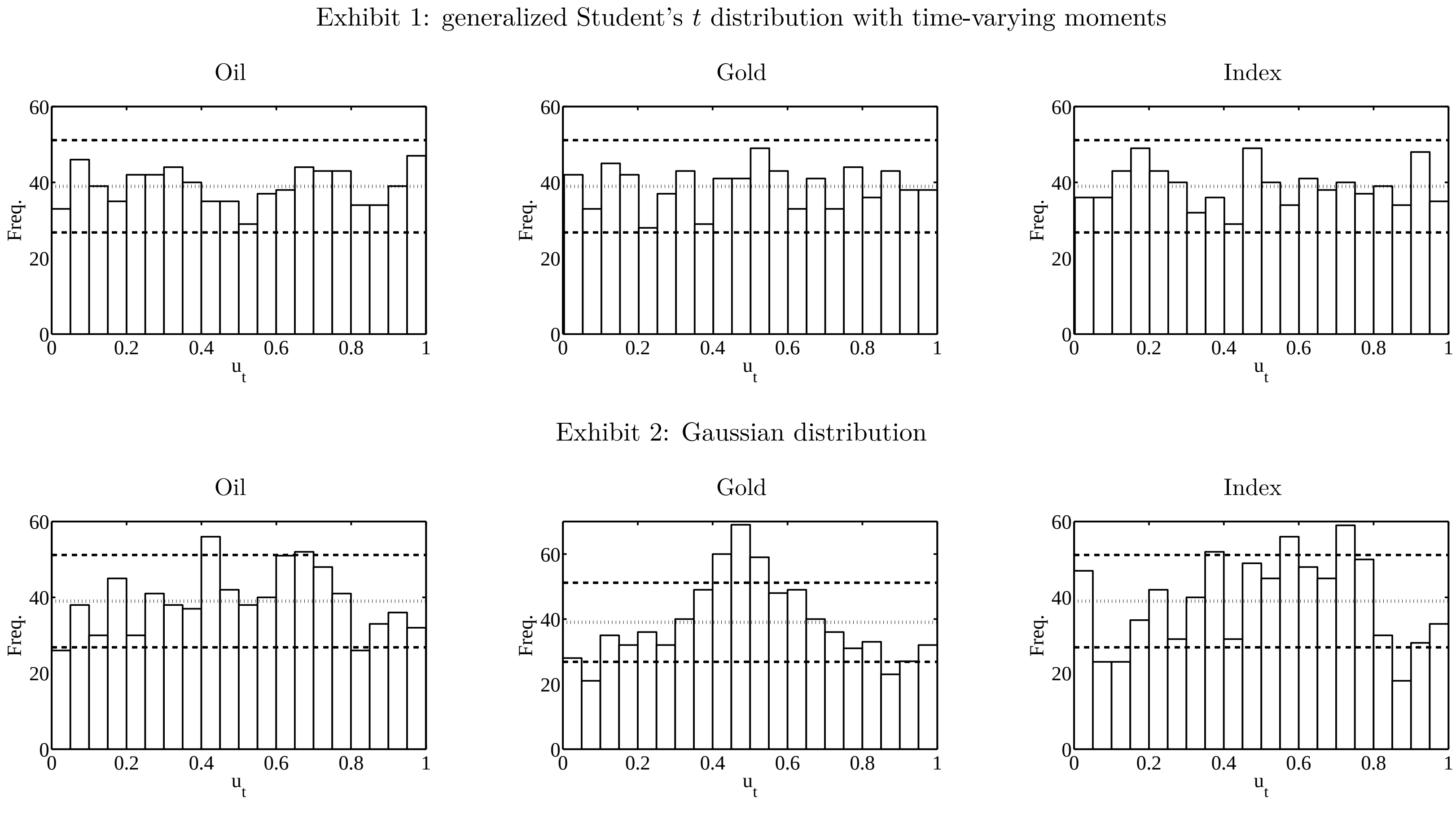

Figure F.3. Goodness-of-fit test

In this figure, we plot the goodness-of-fit test for the estimates of the density of $u_{i,t}$, $i = 1, 2, 3$ and $t = 1, ..., T$ (Diebold et al. (1998)) for two marginal distribution models for oil, gold, and equity-index weekly returns: one with a generalized Student's t distribution and time-varying moments (Exhibit 1), and the other with a Gaussian distribution (Exhibit 2). Parameter estimates of the generalized Student's t distribution are shown in Table F.3. Dotted line and horizontal dashed lines represent the mean and 95% confidence interval, respectively.

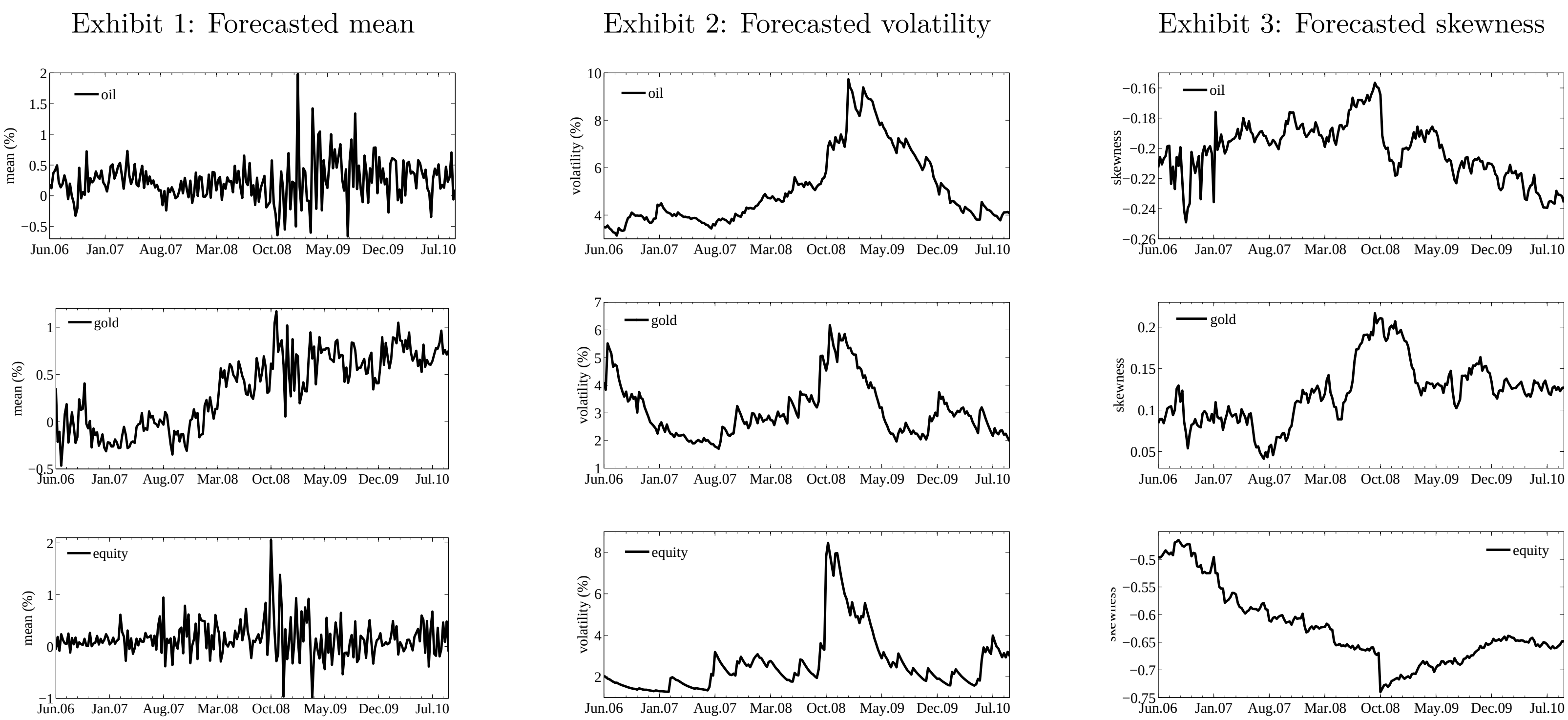

Figure F.4. Conditional parameters of the marginal distribution model.

This figure shows the one-step ahead forecasts over the out-of-sample period for the conditional mean, volatility, and skewness of the marginal distribution model with generalized Student's t distribution and time-varying moments.

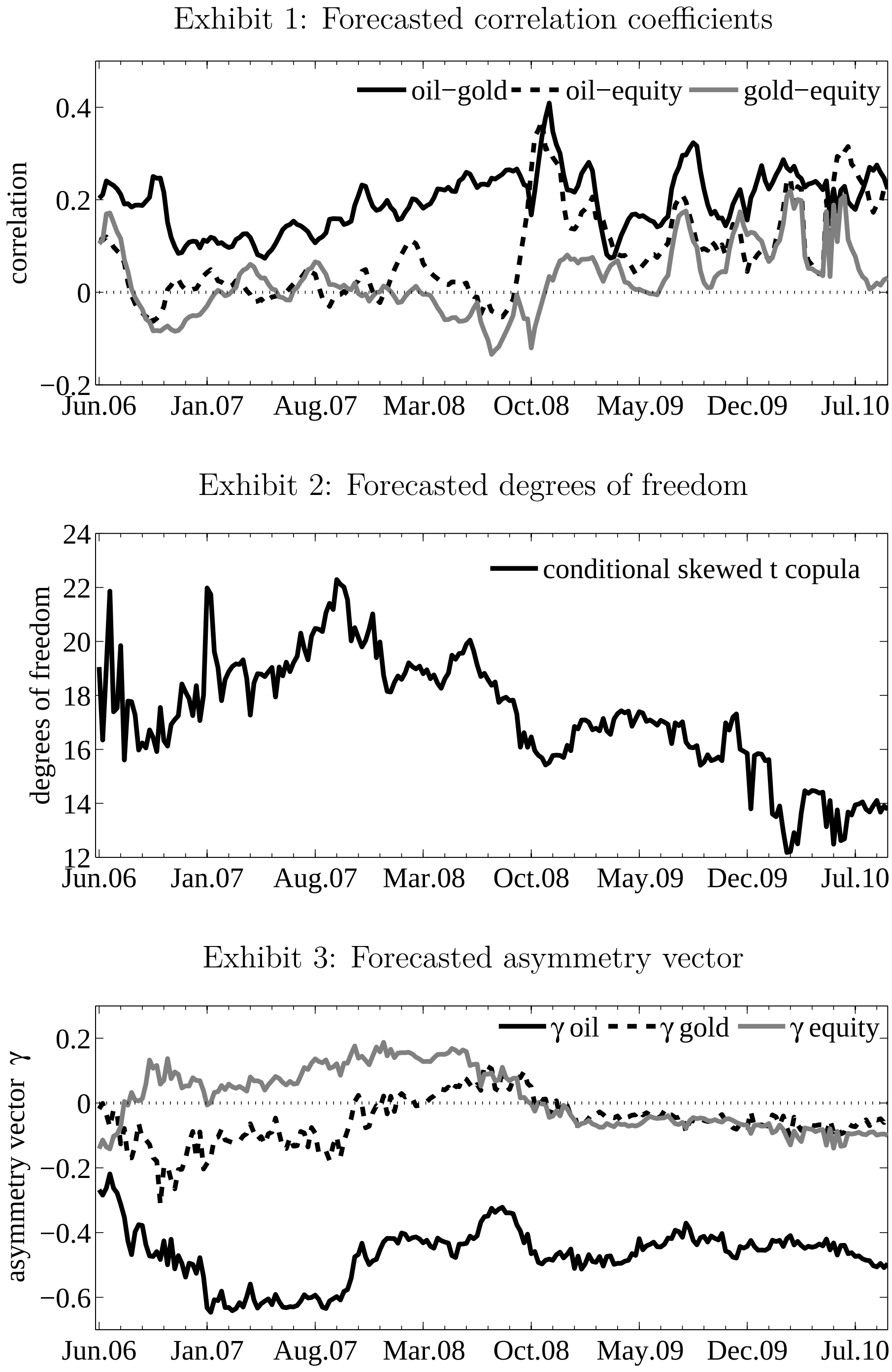

Figure F.5. Conditional parameters of the conditional skewed t copula.
This figure shows the one-step ahead forecasts over the out-of-sample period for the correlation coefficients, degrees of freedom, and asymmetry vector components of the conditional skewed t copula model.

Exhibit A.1: preferences: $\varphi_{\mathrm{V}} = 1/2$, $\varphi_{\mathrm{S}} = 1/3$ ($\mathcal{A}$=1)

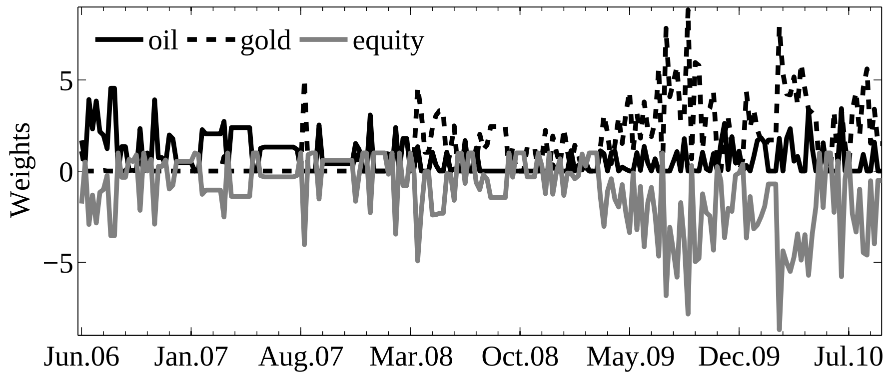

Exhibit A.2: preferences: $\varphi_{\mathrm{V}} = 1$, $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A}$=2)

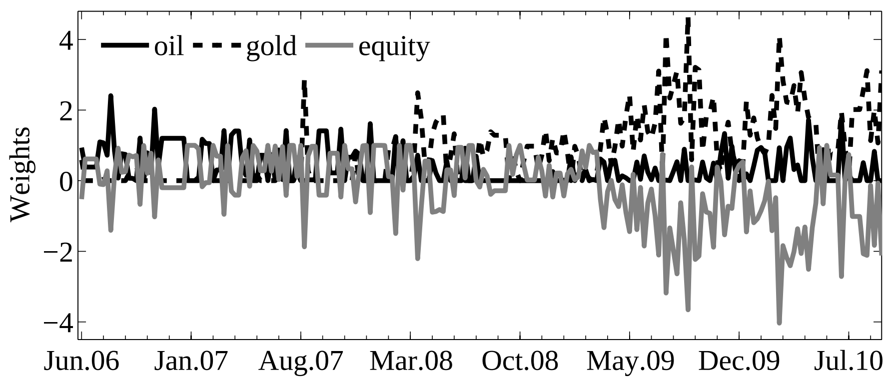

Exhibit A.3: preferences: $\varphi_{\mathrm{V}} = 5/2$, $\varphi_{\mathrm{S}} = 5$ ($\mathcal{A}$=5)

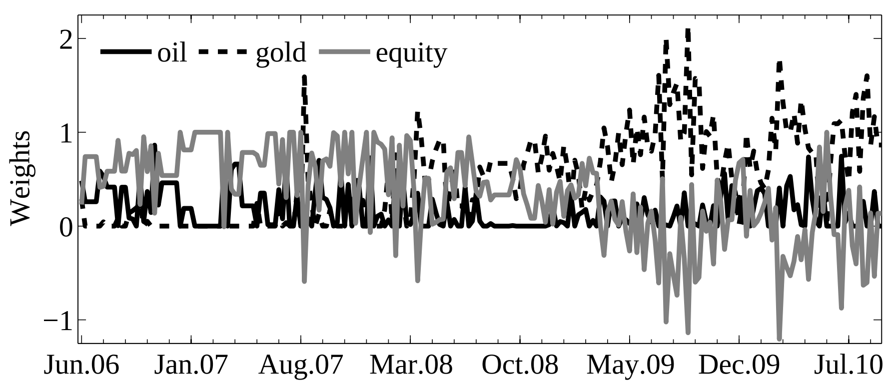

Figure F.6. Optimal weights for the skewed t copula.

We show the optimal weights over the out-of-sample period for the conditional skewed t copula model.

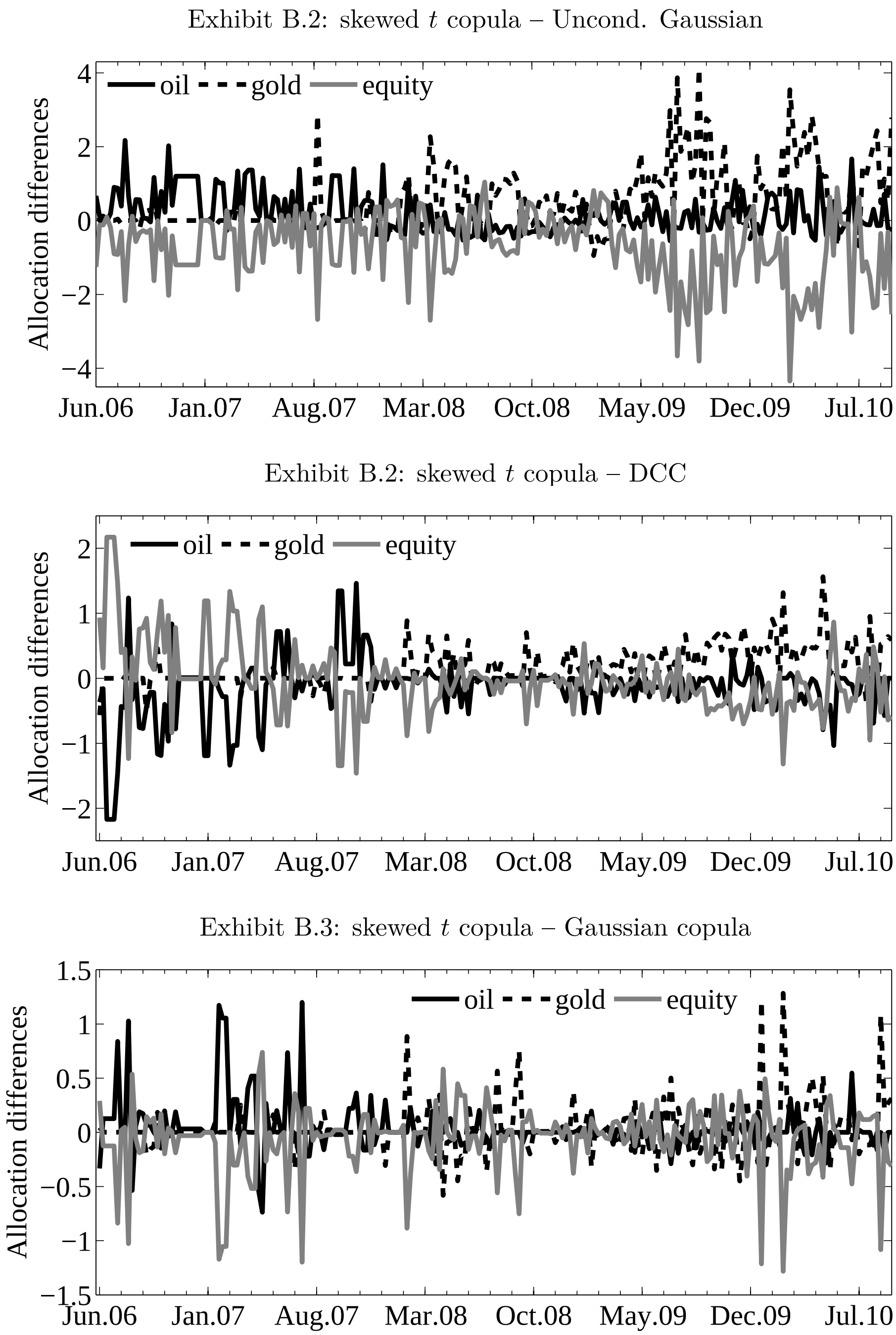

Figure F.7. Allocation differences with respect to the skewed t copula.
We present the allocation differences between various strategies and the conditional skewed t copula model, when the investor's preferences are given by the parameters $\varphi_{\mathrm{V}} = 1$ and $\varphi_{\mathrm{S}} = 1$ ($\mathcal{A}$=2).