Capital Market Equilibrium with Competition among Institutional Investors

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Despite a substantial evidence of competition among institutional investors, little is known about how competition affects stock market return and volatility. We aim to fill this gap and investigate this question within a dynamic general equilibrium model. We consider an economy in which fund managers strategically interact with each other, as each manager tries to increase her perfomance relative to the other managers. We fully characterize an equilibrium in this econony, and find that a more intense competition is associated with a higher level of the market, lower expected market return, while market volatility is not affected by competition. Empirical evidence supports our key predictions.

JEL: G12, G29

Keywords: Competition; Money Managers; Asset Pricing; Optimal Portfolios

This paper studies asset pricing implications of competition among institutional investors (mutual funds, hedge fund,s etc), arising as each investor tries to perform well relative to the other managers. Not so long ago, the macro-finance literature did not consider relative wealth concerns to have notable effects on financial markets, and so theoretical works routinely assumed that investors care only about their own wealth when choosing an investment strategy. More recently, there is a

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growing understanding that relative concerns do actually play an important role, and one of the key reasons that led to this view is the extensive evidence showing that instutational investors—major players in modern markets—are strongly affected by relative performance concerns.

The literature on delegated asset management identifies two main types of relative performance concerns affecting fund managers' behavior. First, a manager cares about relative performance with respect to performance of her peers. The reason is that by outperforming the peers the manager increases money flows to her fund (Chevalier and Ellison (1997), Sirri and Tufano (1998)), giving rise to a competition (or tournament) among managers (Brown, Harlow, and Starks (1996)). Second, relative concerns may play a role in that a manager's performance is benchmarked to some index. As for portfolio choice implications, each of these two features, competition and benchmarking, are analyzed quite extensively. [1] As for general equilibrium, however, while there is a big literature studying asset pricing implications of benchmarking, the implications of competition remain virtually unexplored.[2] What makes this even more surprising is the observation that competition plays a bigger role in real markets than benchmarking. Indeed, every manager has incentives to outperform the peers so as to increase money inflows, whereas benchmarking considerations do not affect a sizeable fraction of managers—as stated in Cuoco and Kaniel (2011, p. 265), only "9% of all U.S. mutual funds used [as of 2004] performance-based fees."

In this paper, we aim to start filling this gap in the literature by addressing a simple, yet fundamental question: How competition among fund managers affects expected market return and market volatility. We consider a dynamic general

[1] A complete list of related papers is very long, and so to save space we mention only several models that are cast, like ours, in continuous time, leaving aside many important static models. Examples of dynamic works studying portfolio choice implications of competition include Browne (2000) and Basak and Makarov (forthcoming); the effects of benchmarking are studied in Carpenter (2000), van Binsbergen, Brandt, and Koijen (2007), and Basak, Pavlova, and Shapiro (2007).

[2] Examples of works studying asset prices under benchmarking include (but not restricted to) Gomez and Zapatero (2003), Petajisto (2009), Cuoco and Kaniel (2011), Basak and Pavlova (2011), Brennan, Cheng, and Li (2012), Leippold and Rohner (forthcoming), . A notable exception is Kapur and Timmerman (2005) who study the competition among managers, though in a two-period setting with mean-variance investors.

equilibrium economy populated by multiple risk-averse fund managers. While in many respects our setting is standard, the key innovation is that fund managers seek to maximize their performance relative to the other managers. Because each manager wants to outperform the peers, the interaction between managers is of a strategic nature, in that every manager recognizes that her competitors do not follow some pre-determined investment rules, but rather respond strategically to what others are doing. This economic mechanism is not present in models where relative performance is with respect to an exogenous benchmark.

We solve analytically for the equilibrium level of the stock market, expected market return and market volatility, and study how these quantities depend on strength of relative performance concerns, which determines how intense the competition is. Our main result is that higher competition intensity is associated with a higher level of the market and lower expected market return, while market volatility is unaffected. Several empirical studies, looking at the data for different time periods, find that the sensitivity of money flows to relative performance is higher at a later than that at a prior period. Interpreting this sensitivity as a proxy for competition intensity, our model predicts that the market premium is expected to decline over time, while market volatility should not exhibit a notable trend. Empirical evidence supports this prediction.

I. Economic Setting

To better clarify which features of our model are standard and which are novel, we lay out our economic setting in two parts. Subsection I.A presents the assumptions that are standard in that they are commonly used in related dynamic models (Cuoco and Kaniel (2011), Basak and Pavlova (2011)), and subsection I.B focuses on the novel features whose effects on market return and volatility have not yet been studied.

A. Basic Set-up

We consider a standard continuous-time economy with a finite horizon $[0, T]$. The uncertainty is driven by a Brownian motion ω. The investment opportunities are given by a risk free bond and a risky stock representing the stock market. The bond return is normalized to zero without loss of generality. The stock market represents a claim to the terminal dividend D_T to be received at time T. We assume that D_T is determined as time-T value of a dynamic dividend process D_t, where D_t follows a geometric Brownian motion

$$dD_t = D_t \mu_D dt + D_t \sigma_D d\omega_t, \tag{1}$$

where the dividend mean growth rate, $\mu_D > 0$, and volatility, $\sigma_D > 0$, are constant.

The stock market level, S, follows the process

$$dS_t = S_t \mu_{S,t} dt + S_t \sigma_{S,t} d\omega_t, \tag{2}$$

where the expected market return $\mu_{S,t}$ and volatility $\sigma_{S,t}$ are endogenous processes to be determined in equilibrium. Because the riskless return is normalized to zero, the terms "market return" and "market premium" denote the same thing, and so in what follows we use them interchangeably.

There are M fund managers in the economy. Each manager i is endowed with e_i units of the stock. We assume that the stock is in unit supply, and so $\sum_{i=1}^{M} e_i = 1$. Manager i chooses a dynamic investment strategy $\theta_{i,t}$, the fraction of wealth invested in the stock at time t. Manager i's wealth at time t, $W_{i,t}$, follows the process

$$dW_{i,t} = \theta_{i,t} W_{i,t} \mu_{S,t} dt + \theta_{i,t} W_{i,t} \sigma_{S,t} d\omega_t. \tag{3}$$

B. Modeling Competition among Managers

When modelling fund managers' objective function, we take into account two considerations. First, a manager has incentives to maximize the absolute return on their investment because this increases her assets under management, and hence her compensation. Second, it is also rational for a manager to care about her return relative to the peers, because the higher the relative return is, the more money the manager is likely to receive from retail investors who largely use relative performance when choosing fund, as documented empirically (Chevalier and Ellison (1997), Sirri and Tufano (1998)). To capture these two features, we postulate that each manager i's utility function u_i is

$$u_i = \frac{1}{1-\gamma}\left(W_{i,T}^{1-\alpha}\left(\frac{W_{i,T}}{\overline{W}_{-i,T}}\right)^{\alpha}\right)^{1-\gamma}, \tag{4}$$

where $\overline{W}_{-i,T}$ is the geometric average of wealth of all investors excluding i:

$$\overline{W}_{-i,T} = \left(\prod_{j\neq i} W_{j,T}\right)^{\frac{1}{M-1}} \tag{5}$$

In specification (4), $\alpha \in [0,1]$ measures *competition intensity*, the strength with which managers care about outperforming the peers. In line with the discussion above, our leading interpretation for relative performance concerns is fully rational—increasing relative return leads to higher money inflows. Consequently, it is natural to think that the value of α is determined by the sensitivity of the flow-performance relation. The more sensitive money flows are to performance, the higher is α. Parameter $\gamma > 0$ reflects relative risk aversion.[3] As documented by empirical works and as often assumed in theoretical works, in what follows we assume $\gamma > 1$.

[3] In this paper, we want to isolate the effects of competition on economic variables of interest, and for this reason we do not introduce confounding features such as preference heterogeneity., and for this reason we consider homogeneous preferences. Moreover, if we allowed parameters α and γ to differ across managers, the model would no longer be analytically tractable.

The equilibrium in this economy is straightforwardly defined as follows. Taking the stock price characteristics S_t,$\mu_{S,t}$ and $\sigma_{S,t}$ as given, we compute managers' Nash equilibrium strategies: a collection of M trading strategies $(\theta_{1,t}, \ldots, \theta_{M,t})$ such that $\theta_{i,t}$ is manager i's best response to the other managers' strategies, i.e., $\theta_{i,t}$ yields the maximum of the expected utility (4) subject to the budget constraint (3). The equilibrium S_t, $\mu_{S,t}$ and $\sigma_{S,t}$ are such that markets clear after managers play this Nash game.

II. Equilibrium

The main focus of this paper is to examine how competition among fund managers affects the stock market expected return and volatility. However, to better understand the economic mechanisms behind these general equilibrium results, it is helpful to start with a partial equilibrium question: taking as given constant parameters μ_S and σ_S of the stock price dynamics (2), we determine what investors' optimal portfolios are. We focus on the case of constant μ_S and σ_S because this is what happens in equilibrium (as established in Proposition 2). Proposition 1 reports managers' optimal portfolios. Here and throughout the paper, a variable with a hat $\hat{}$ denotes an equilibrium quantity in the economy with competition, $\alpha > 0$, while a variable with a superscript ^{B} ("Benchmark") – an equilibrium quantity in the benchmark case of no competition, $\alpha = 0$.

PROPOSITION 1: *When expected return and volatility of the stock market, μ_S and σ_S, are constant, the optimal portfolios of fund managers are also constant and given by*

$$\hat{\theta}_i = \frac{1}{\gamma - \alpha(\gamma - 1)} \frac{\mu_S}{\sigma_S^2}, \quad i = 1, \ldots, M. \tag{6}$$

In the benchmark economy with no competition, $\alpha = 0$, the optimal portfolios are

$$\theta_i^B = \frac{1}{\gamma} \frac{\mu_S}{\sigma_S^2},\ i = 1, \ldots, M. \tag{7}$$

Consequently, competition causes managers to increase the riskiness of their portfolios, $\hat{\theta}_i > \theta_i^B$.

The main result of Proposition 1 is that the presence of competition leads to a higher risk taking in equilibrium. To understand why, note from utility specification 4 that each manager cares about the composite of own wealth and relative wealth, and so, being risk averse, she seeks to minimize the variance of both these components. Importantly, manager i choose the benchmark portfolios θ_i^B in (7) not only when she has no relative concerns ($\alpha = 0$), but also when she cares about relative wealth but the other managers invest fully in the bond. Indeed, in this case the average wealth $\overline{W}_{-i,T}$ is constant, and so can be dropped from manager i's utility without affecting her behavior, leading her to choose the benchmark portfolio θ_i^B. If, however, the other managers invest a positive amount in the stock, manager i has incentives to increase her stock investment over the benchmark level so as to hedge against the increased volatility of the term $\overline{W}_{-i,T}$ in her utility. As a result, in the presence of competition all managers increase their stock investments relative to the no competition case.

We now turn to the general equilibrium implications of managers' competition.

PROPOSITION 2: *The equilibrium stock market level $\hat{S}_t$, expected market return $\hat{\mu}_S$, and market volatility $\hat{\sigma}_S$ are*

$$\hat{S}_t = D_t e^{\left(\mu_D + (\alpha(\gamma-1)-\gamma)\sigma_D^2\right)(T-t)},\ \hat{\mu}_S = (\gamma - \alpha(\gamma-1))\sigma_D^2,\ \hat{\sigma}_S = \sigma_D. \tag{8}$$

The corresponding benchmark values with no competition, $\alpha = 0$, are:

$$S_t^B = D_t e^{\left(\mu_D - \gamma\sigma_D^2\right)(T-t)},\ \mu_S^B = \gamma\sigma_D^2,\ \sigma_S^B = \sigma_D. \tag{9}$$

Consequently, a higher competition intensity is associated with a higher stock market level, a lower expected market return, and constant market volatility.

Proposition 2 reveals that the stock market level S_t increases with the competition intensity α. The reason is that a higher α increases the demand for the stock market, as discussed above, and so the market level increases. More interesting economic mechanisms are at play behind are the two other results of Proposition 2 concerning the market premium $\hat{\mu}_S$ and volatility $\hat{\sigma}_S$. As is well known, the market premium reflects the compensation for risk associated with holding the market. However, as discussed after Proposition 1, investing in the market allows managers to control the volatility of their relative wealth component of utility function, and in this respect the market plays an important role from the viewpoint of risk averse managers who value the ability to minimize the volatility of this component. The more managers care about relative wealth, the more valuable this ability is. Hence, a higher α is associated with a lower compensation for holding the market, implying a lower $\hat{\mu}_S$. That α has no effect on volatility $\hat{\sigma}_S$ follows the result that the market level S_t is proportional to the contemporaneous dividend D_t, where the coefficient of proportionality is deterministic, as seen from the the first equation in (8). Intuitively, the market is a claim to the future dividend payment D_T, and so the market level S_t is given by (appropriately discounted) time-t expectation of D_T, and so is proportional to D_t because the dividends follow a geometric Brownian motion. Hence, the market volatility equals the dividend volatility, regardless of how intense the competition is.

Another way to look at the results of Propositions 1 and 2 is as follows. From equation (6) for the stock weight, observe that a higher α increases the first fraction on the right-hand side of (6). Given that in equilibrium the stock weight has to, by market clearing, remain the same, the second fraction μ_S/σ_S^2 must decrease. Obviously, there are many ways in general to alter μ_S and σ_S so that to decrease μ_S/σ_S^2, and what Proposition 2 finds is that the actual way this happens in equilibriu is rather special—it is only the expected market return μ_S that changes in response to a change in competition intensity, while the market volatility σ_S does not change.

A. Empirical Evidence

As established above, the effect of a higher competition intensity operates through lowering the expected market return, while the market volatility is unaffected. A natural empirical implication is as follows. Consider a financial market at two time periods, let's call them the *past* and the *present*, and suppose that the competition at the present is more intense relative to the past. Then, our model predicts that the market premium in the present should be lower than in the past, while there should be no notable difference in market volatilities.[4]

This prediction is borne out in the data. First, to see that the premise of the above argument—that the competition is becoming more intense with time—is realistic, recall our earlier discussion in subsection I.B that competition intensity α reflects the sensitivity of the flow-performance relation. Fant and O'Neal (2000) and Huang, Wei, and Yan (2007) document that the sensitivity of this relation has increased over time, and so it is indeed reasonable to assume that α has also increased. Consistent with our model, empirical evidence shows that the market premium has been decreasing over time (Blanchard, Shiller and Siegel (1993), Fama and French (2002), Jagannathan, McGrattan and Scherbina (2000)), while market volatility does not seem to have a significant trend (Campbell, Lettau, Malkiel, and Xu (2001)).

III. Conclusion

We develop a dynamic general equilibrium model to study how competition among fund managers affects the expected market return, market volatility, and market level. We find that a more intense competition is associated with a higher level of the market, lower expected market return, and constant market volatility. Empirical evidence supports our key predictions.

[4]When talking about how the two variables are expected to change, we, of course, talk about secular trends, and not short-term fluctuations.

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Mathematical Appendix

PROOF OF PROPOSITION 1:

Given that markets are complete, there exists a state price density process ξ_t given by

$$d\xi_t = -\xi_t \kappa d\omega_t, \tag{A1}$$

where

$$\kappa \equiv \mu_S/\sigma_S \tag{A2}$$

is the market price. As is well-known (see, e.g., Karatzas and Shreve (1998)), the dynamic budget constraint (3) is equivalent to

$$E_t[\xi_T W_{i,T}] = \xi_t W_{i,t}. \tag{A3}$$

The first-order condition for maximizing the expected utility function (4) subject to (A3)is

$$\begin{aligned} 0 &= \hat{W}_{i,T}^{-\gamma}\overline{W}_{-i,T}^{\alpha(\gamma-1)} - y_i\xi_T, \\ \hat{W}_{i,T} &= (y_i\xi_T)^{-1/\gamma}\overline{W}_{-i,T}^{\alpha(\gamma-1)/\gamma}, \end{aligned} \tag{A4}$$

where y_i is the Lagrange multiplier attached to the budget constraint (A3). Considering M equations (A4) for each investor $i = 1, \dots, M$, we obtain a system of M equations with M unknowns defining Nash equilibrium wealth profiles ($\hat{W}_{1,T}$,..., $\hat{W}_{M,T}$). To solve it, let us consider the first two equations of this system, and substitute (5) in them. This gives

$$\hat{W}_{1,T} = (y_1\xi_T)^{-1/\gamma}(\hat{W}_{2,T} * \ldots * \hat{W}_{M,T})^{\frac{\alpha(\gamma-1)}{\gamma(M-1)}}, \tag{A5}$$

$$\hat{W}_{2,T} = (y_2\xi_T)^{-1/\gamma}(\hat{W}_{1,T} * \hat{W}_{3,T} * \ldots * \hat{W}_{M,T})^{\frac{\alpha(\gamma-1)}{\gamma(M-1)}}. \tag{A6}$$

Dividing (A5) by (A6), we get

$$\frac{\hat{W}_{1,T}}{\hat{W}_{2,T}} = \left(\frac{y_1}{y_2}\right)^{-\frac{1}{\gamma}} \left(\frac{\hat{W}_{2,T}}{\hat{W}_{1,T}}\right)^{\frac{\alpha(\gamma-1)}{\gamma(M-1)}} \Rightarrow \hat{W}_{2,T} = \left(\frac{y_1}{y_2}\right)^{\frac{1}{\gamma\left(\frac{\alpha(\gamma-1)}{\gamma(M-1)}+1\right)}} \hat{W}_{1,T}. \tag{A7}$$

Replacing subscript 2 in (A7) by $j = 3, \ldots, M$, we obtain the relations between Nash equilibrium wealth of manager j, $\hat{W}_{j,T}$, and manager 1, $\hat{W}_{1,T}$, and substituting all of them into (A5) yields

$$\begin{aligned}\hat{W}_{1,T} =&(y_1\xi_T)^{-\frac{1}{\gamma}}\left(\frac{y_1}{y_2}\right)^{\frac{1}{\gamma\left(1+\frac{\gamma(M-1)}{\alpha(\gamma-1)}\right)}}\hat{W}_{1,T}^{\frac{\alpha(\gamma-1)}{\gamma(M-1)}} * \ldots * \left(\frac{y_1}{y_M}\right)^{\frac{1}{\gamma\left(1+\frac{\gamma(M-1)}{\alpha(\gamma-1)}\right)}}\hat{W}_{1,T}^{\frac{\alpha(\gamma-1)}{\gamma(M-1)}} \\ =&y_1^{-\frac{1}{\gamma}+\frac{M-1}{\gamma\left(1+\frac{\gamma(M-1)}{\alpha(\gamma-1)}\right)}}(y_2 * \ldots * y_M)^{-\frac{M-1}{\gamma\left(1+\frac{\gamma(M-1)}{\alpha(\gamma-1)}\right)}}\hat{W}_{1,T}^{\frac{\alpha(\gamma-1)}{\gamma}}\xi_T^{-\frac{1}{\gamma}}, \\ \hat{W}_{1,T} =& K_1\xi_T^{-\frac{1}{\gamma-\alpha(\gamma-1)}},\end{aligned}$$

from which we obtain

$$\hat{W}_{1,T} = K_1\xi_T^{-\frac{1}{\gamma-\alpha(\gamma-1)}}, \tag{A8}$$

where

$$K_1 = \left(y_1^{-\frac{1}{\gamma}+\frac{M-1}{\gamma\left(1+\frac{\gamma(M-1)}{\alpha(\gamma-1)}\right)}}(y_2 * \ldots * y_M)^{-\frac{M-1}{\gamma\left(1+\frac{\gamma(M-1)}{\alpha(\gamma-1)}\right)}}\right)^{\frac{\gamma}{\gamma-\alpha(\gamma-1)}}. \tag{A9}$$

Analogously to (A8), we can obtain Nash equilibrium wealth of manager i, $i = 1, \ldots, M$:

$$\hat{W}_{i,T} = K_i\xi_T^{-\frac{1}{\gamma-\alpha(\gamma-1)}}, \tag{A10}$$

where K_i is obtained from K_1 in A9 by switching subscripts 1 and i. To derive manager 1's equilibrium portfolio, we substitute (A8) into a no-arbitrage condition $\xi_t\hat{W}_{1,t} = E_t[\xi_T\hat{W}_{1,T}]$:

$$\xi_t\hat{W}_{1,t} = K_1E_t[\xi_T^{1-\frac{1}{\gamma-\alpha(\gamma-1)}}] = C_t\xi_t^{1-\frac{1}{\gamma-\alpha(\gamma-1)}},$$

$$\hat{W}_{1,t} = C_t\xi_t^{-\frac{1}{\gamma-\alpha(\gamma-1)}}. \tag{A11}$$

In (A11), for brevity we use C_t to denote a certain deterministic function of time which, as will be seen momentarily, does not affect managers' Nash equilibrium investment strategies. Applying Ito's Lemma to (A11) and using (A1), we get that the diffusion term of $d\hat{W}_{1,t}$ is equal to $\frac{\kappa}{\gamma-\alpha(\gamma-1)}\hat{W}_{1,t}$. Equating this term to the diffusion term $\hat{\theta}_{1,t}\hat{\sigma}_S\hat{W}_{1,t}$ in (3), and using (A2), we get

$$\hat{\theta}_{1,t} = \frac{1}{\gamma-\alpha(\gamma-1)}\frac{\mu_S}{\sigma_S^2} \tag{A12}$$

For other managers, the derivations are analogous, and so (6) obtains. Plugging $\alpha = 0$ in (6) yields (7).

PROOF OF PROPOSITION 2:

In the above proof of Proposition 1, we relied on μ_S and σ_S being constants only when deriving manager 1's investment strategy (A12), and all the analysis before equally holds when these parameters are stochastic. The analysis below does not rely on managers' investment policies, and so in what follows we do *not* assume μ_S and σ_S are constant, and hence we do not assume that the market price of risk κ in (A1) is constant. Rather, we establish that *in equilibrium* these parameters are constant. Substituting (A10) in time-T budget constraint yields

$$D_T = \sum_{i=1}^{M}\hat{W}_{i,T} = \left(\sum_{i=1}^{M}K_i\right)\xi_T{}^{-\frac{1}{\gamma-\alpha(\gamma-1)}}, \tag{A13}$$

and so time-T value of the state price density is

$$\xi_T = \left(\sum_{i=1}^{M}K_i\right)^{\gamma-\alpha(\gamma-1)} D_T^{\alpha(\gamma-1)-\gamma}. \tag{A14}$$

From (A1), ξ_t is a martingale, and so using (A14), we get

$$\begin{aligned}\xi_t = E_t[\xi_T] &= \left(\sum_{i=1}^{M}K_i\right)^{\gamma-\alpha(\gamma-1)} E_t\left[D_T^{\alpha(\gamma-1)-\gamma}\right]\\ &= \left(\sum_{i=1}^{M}K_i\right)^{\gamma-\alpha(\gamma-1)} E_t\left[D_T^{\alpha(\gamma-1)-\gamma}\right].\end{aligned} \tag{A15}$$

Applying Ito's lemma to $D_t^{\alpha(\gamma-1)-\gamma}$ and using (1), it is easy to get that $D_t^{\alpha(\gamma-1)-\gamma}$ follows a geometric Brownian motion with drift $(\alpha(\gamma-1)-\gamma)\mu_D + \frac{1}{2}(\alpha(\gamma-1)-\gamma)(\alpha(\gamma-1)-\gamma-1)\sigma_D^2$, substituting which into (A15) yields

(A16)

$$\xi_t = \left(\frac{\sum_{i=1}^M K_i}{D_t}\right)^{\gamma-\alpha(\gamma-1)} e^{\left((\alpha(\gamma-1)-\gamma)\mu_D + \frac{1}{2}(\alpha(\gamma-1)-\gamma)(\alpha(\gamma-1)-\gamma-1)\sigma_D^2\right)(T-t)}.$$

The equilibrium time-t stock price $\hat{S}_t$ is given by a no-artbitrage condition

$$\hat{S}_t = E_t[\xi_T D_T]/\xi_t,$$

and plugging in it (A14) and (A16) and cancelling $\left(\sum_{i=1}^M K_i\right)^{\gamma-\alpha(\gamma-1)}$ in the numerator and denominator, we get

$$\begin{aligned} \hat{S}_t &= \frac{E_t[D_T^{\alpha(\gamma-1)-\gamma+1}]}{D_t^{\alpha(\gamma-1)-\gamma} e^{\left((\alpha(\gamma-1)-\gamma)\mu_D + \frac{1}{2}(\alpha(\gamma-1)-\gamma)(\alpha(\gamma-1)-\gamma-1)\sigma_D^2\right)(T-t)}} \\ &= D_t e^{\left(\mu_D + (\alpha(\gamma-1)-\gamma)\sigma_D^2\right)(T-t)}. \end{aligned} \tag{A17}$$

Applying Ito's lemma to (A17), we get that the stock price dynamics in equilibrium is

$$d\hat{S}_t = (\gamma - \alpha(\gamma-1))\sigma_D^2 \hat{S}_t dt + \sigma_D \hat{S}_t d\omega_t. \tag{A18}$$

Equilibrium characterization (8) follows from (A17) and (A18). Substituting $\alpha = 0$ into (8) yields (9).