A MEASURE OF LIQUIDITY RISK IN A SOVEREIGN DEBT MARKET

Emma Berenguer(*)
Universidad Pablo de Olavide

Ricardo Gimeno
Banco de España

Juan M. Nave
Universidad Castilla La Mancha

Abstract

Since the seminal paper of Vasicek and Fong (1982) term structure models are estimated assuming that yields are cross-sectionally homokedastic. In this paper, we show that this hypothesis does not hold even for bonds from the same issuer, when there are differences in their level of liquidity. Those bonds with a lower daily turnover would experiment a higher volatility around the expected yield determined by the term structure. The existence of a minimum tick size on the bond price negotiation would also produce a higher volatility for those bonds approaching their expiration term. In order to show these effects, we use data from Spanish sovereign bonds from 1988 to 2010, covering more than 700 bonds and 5000 days. With these data we have estimated the out-of-sample error for each bond and day. The variance of these errors is negatively correlated with the turnover of each bond and its duration, while the mean of the error is directly correlated with the estimated variance. Taking into account these features we propose, for fitting the term structure, a modified Svensson (1994) yield curve model where an additional liquidity term is added and parameters are estimated by weighted least squared errors to take into account the liquidity-induced heterokedasticity.

Keywords: liquidity risk; liquidity premium; yield curve; Spanish Sovereign Bonds

JEL classification: G12; C58; E43.

EFM classification codes: 340; 550.

(*)Corresponding Author: ebercar@upo.es

The authors acknowledge financial support from the Ministry of Science and Innovation (grant ECO2009-13616/ECON)

1. INTRODUCTION

Since the paper of Vasicek and Fong (1982), the term structure of interest rates is estimated assuming that yields are homokedastic, regardless of the model considered (i.e. Nelson and Siegel, 1987; Svensson, 1994; Bliss, 1997; Jordan and Mansi, 2003). In fact, a vast majority of Central Banks use this error correction in their model estimations (Bank of International Settlements, 2005). Nevertheless, the evidence presented in this paper show that this Vasicek and Fong (1982) assumption of homokedastic yield errors no longer holds when the bonds considered have different levels of liquidity.

Differences in the level of liquidity would cause wider movements for less liquid bonds, both in the upside and in the downside. Therefore, liquidity considerations should imply differences in the variance of their yields, even for those bonds from the same issuer. The main direct implication of this heterokedasticity is that parameter estimation of the yield curve would be inefficient.

In the financial literature, there is little disagreement that liquidity is the second most important factor after credit risk that affects the yields of bonds. However, it is one of the least understood areas of finance. Since the pioneer work of Fisher (1959) pointed out that it was liquidity the variable responsible for the existing differentials of profitability between the titles of private equity and the titles of Government Securities (GS), many authors have studied the liquidity factors on the debt markets.

Amihud and Mendelson (1986) stated that the first consequence of the liquidity factor is the major or minor exigency of a return by investors. These differences are known as liquidity premiums and are deviations from the yields of the different assets to compensate differences in liquidity. This liquidity premium has been frequently treated in the literature (i.e. Amihud and Mendelson, 1986; Elton and Green, 1998; Alonso et al., 2004; Diaz et al., 2006).

In the recent financial crisis, where spreads between Sovereign bonds issued by different countries has been used as a measure of credit risk differences, preferences for more liquid bonds[1] may have been distorting the supposed meaning of these spreads, imposing an upward bias. Liquidity considerations may have also implied an undesired role in the interpretation of the differences between nominal bonds and inflation-linked bonds as inflation compensation, since liquidity differences in favour of nominal bonds produce a downward bias in the supposed market inflation expectations. Lately, the proposal for Eurobonds[2] has been the object of intense political debate due to the threat of default of Greek sovereign bonds. The argument in favour of these assets goes in the line of the liquidity improvement over individual sovereign bonds, which could reduce the cost of issuance for all GS (Delpla and Weizsäcker, 2010). Therefore, although liquidity spreads have received increasing attention in the literature, the effect of liquidity factors on yield variance has been sidelined.

Alonso et al. (2004) define liquidity as the ease of its conversion into money whereas Díaz et al. (2006) defines liquidity as a feature of financial assets related to the ease with which a security can be traded within a short period of time period of time without causing significant impact on prices. The main consequence of the lack of liquidity is that, in the case of a trader willing to either buy or sell a given asset, the direction of the trade will have a sizeable effect on the price, been this movement upward in the case of a buy order or downward for a sell order. Therefore, price changes will be higher for illiquid assets, which imply differences in the associated variance of the bonds due to their different degree of liquidity.

An additional issue raised in the literature, is the way liquidity is measured. Some studies make inferences about bond liquidity or about the valuation implications of liquidity differences using such proxies for liquidity as securities age (Sarig and Warga, 1989), security type (Amihud and Mendelson, 1991; Kamara, 1994), ontherun/offtherun status (Warga, 1992), trading volume (Elton and Green, 1998) and term to maturity (Shen and Starr, 1998).

For GS, Alonso et al. (2004) stated that liquidity should be closely linked to the market-makers' inventory risk and order-processing costs which ultimately depend on the level

[1] Bond spreads increase during recessions due to a phenomenon called flight-to-liquidity and flight-to-quality (Goyenko *et al.*, 2011).
[2] Defined as "pooled" sovereign debt instruments of the member States of the euro area.

of risk of the asset (duration) and the frequency with which a transaction will be executed (turnover). The *on-the-run* issues are those more recently auctioned and tend to be more liquid than previous issued bonds (*off-the-run*) maturing on similar dates (Pasquariello and Vega, 2009). Nevertheless, even among these bonds there might be differences on liquidity, and therefore, heteroskedasticity among the yields.

By contrast, Amihud and Mendelson (1991) proposed a model for short term interest rates where yields had a variance that was conditional on the time to maturity. Although this approach deal with the duration component of liquidity does nothing with the trading volume factor. In this sense, Elton and Green (1998) proposed a model for the term structure estimated minimizing the mean root squared error, but the errors are then used in a liquidity model where the parameters are estimated using a heteroskedasticity and autocorrelation consistent estimate of the variance (HAC estimator). Díaz et al. (2011) considered the heteroskedasticity of the interest rates but used a time series approach rather than a cross-sectional liquidity related approach.

Elton and Green (1998) propose the trading volume as the main variable producing liquidity differences among bonds with the same issuers. However, as this variable is not always available, some authors use proxies of this variable. The most common one is the classification of the bonds in on-the-run (the most recently auctioned issue), off-the-run (next to the most recently auctioned issue) and off-off-the-run (older issued bonds). That is the case for Alonso et al. (2004) and Diaz et al. (2006). Both Alonso et al. (2004) and Diaz et al. (2006) also include the pre-benchmark category that includes the first days of a new issue, where there is not enough trading volume on that issue yet.

Alonso et al. (2004) verify the existence of liquidity premiums in the prices of titles negotiated in the Spanish GS market. The methodology used is that of Elton and Green (1998), based on the estimation of the Term Structure of Interest Rates. These authors incorporate the effect of the liquidity in the estimation introducing dummy variables for the different categories (*on-the-run/off-the-run)* of the bonds. The instantaneous forward remains defined according to the method of Svensson (1994).

Díaz *et al*. (2006) analyse the liquidity structure of the Spanish Treasury bond market using trading volume market share and "auction status" as proxies for liquidity to determine if the entry of Spain in the European Monetary Union (EMU) has had some repercussion in liquidity. They also analyse the impacts of EMU on volatility of yields in the Spanish treasury market[3], finding a dramatically decline after the market began pricing EMU.

The main goal of this paper is to present a model able to explaining the role played by liquidity considerations in the departure of sovereign bond yields from a theoretical liquidity-free term structure of interest rates. In order to do so, we propose a heterokedastic model for the yields, where the variance equation is function of the trading volume and the duration, allowing for consistent estimators of the yield curve, in the sense of White (1980).

Following both Elton and Green (1998) and Alonso et al. (2004), we add a term of liquidity on the estimation of the Svensson (1994) term structure model. But we depart from both papers in the sense that we use the heteroskedasticity variable estimated for the variance equation for the term structure, instead of the trading volume like Elton and Green (1998) or the *on-the-run/off-the-run* quality of the bonds like Alonso et al. 2004). In this way, we use a model similar to a garch-in-mean model.

The rest of the paper is organized as follows. In the next section we introduce the problem of efficiency in a term structure estimation. Section 3 describes the data. On Section 4 we show that liquidity factors produce heteroskedasticity in the GS yields. On Section 5 we modified Svensson (1994) model in order to incorporate liquidity constrains when estimating the term structure, both in the mean and the variance equation, and estimate it for Spanish GS. On section 6 we sum up the main conclusions of the paper.

[3] Díaz *et al*. (2006) use Nelson and Siegel (1987) exponential model to fit the daily term structures. They do not incorporate any specific liquidity effects.

2. The estimation of the Term Structure

The price ($\hat{P}$) of a coupon-bearing bond, as it is typical of long term GS, is equal to,

$$P = \sum_{j=1}^{n} C_j \cdot e^{-s\left(t_j|\varphi\right) t_j} + N \cdot e^{-s\left(m|\varphi\right) \cdot m} \tag{1}$$

where C_j are the coupons paid in t_j (j=1,…,n), and N is the nominal paid at the final term (m). Each payment is discounted using an interest rate (s) that is a function of time (t) and a set of parameters (φ). This function $s\left(t|\varphi\right)$ is also known as the term structure. Nevertheless, the term structure is not observable. Given a security tradable in a secondary market, we could see the price paid for the security (P), or equivalently, the yield (y), defined as the constant interest rate that solve equation 1,

$$P = \sum_{j=1}^{n} C_j \cdot e^{-y \cdot t_j} + N \cdot e^{-y \cdot m} \tag{2}$$

For the estimation of the parameter set φ in the function $s\left(m|\varphi\right)$, we have two options,

1. Compare the observed Price (P) with the one derived of $s\left(m|\varphi\right)$ ($\hat{P}(\varphi)$),

$$P = \hat{P}(\varphi) + \varepsilon_P \tag{3}$$

2. Compare yields, the ones obtained in equation 2 using the observed price (P) and the ones computed from the term structure ($\hat{P}(\varphi)$),

$$y = \hat{y}(\varphi) + \varepsilon_y \tag{4}$$

Since the paper of Vasicek and Fong (1982), the error term in equation 4 is assumed to be homokedastic ($E\left[(y_i - \hat{y}_i(\varphi))^2\right] = \sigma^2$). They also showed that under that assumption, the variance of price errors would be given by:

$$E\left[\left(P-\hat{P}\right)^2\right]=\sigma^2\cdot\left(\frac{dP_i}{dy}\right)^2 \tag{5}$$

where the derivative of the bond price with respect to yield is equal to the bond duration, when price is normalized to 1. Therefore, the term structure can be estimated by minimizing, either the sum squared errors in yields,

$$\phi=\min_{\varphi}\sum_{i=1}^{k}\left(y_i-\hat{y}_i(\varphi)\right)^2 \tag{6}$$

Or the sum of squared weighted (by the inverse of durations) errors in prices,

$$\phi=\min_{\varphi}\sum_{i=1}^{k}\left(P_i-\hat{P}_i(\varphi)\right)^2\cdot\frac{1}{D_i} \tag{7}$$

Both solutions are considered equivalent. For instance, Gurkaynak et al. (2007) uses the in inverse of the duration for weighting the errors in prices for their estimation of the US Treasury Yield curve. Also, the Bank of International Settlements (2005) reports that, regardless of the type of term structure estimated, 5 out of 11 Central Banks estimate the term structure minimizing the error in yields (Germany, Norway, Sweden, Switzerland and UK), whereas the other 6 Central Banks use the weighted errors in prices to estimate the term structure (Belgium, Canada, Finland, France, Italy, Spain and US). Nevertheless, if the Vasicek and Fong (1982) assumption were rejected, and the errors in yield were heteroskedastic, this would imply that al those estimation methods would be inefficient.

3. Data

In order to check the validity of the homokedasticity of the errors in yields, we are going to estimate the term structure of interest rates for a long set of data. In order to avoid distortions of credit risk, we need to use bonds from the same issuer (with a similar level of credit risk), and GS are ideal for this purpose, since the number of securities traded at the same time is higher enough to estimate the parameters in the term structure model.

Therefore, we will use Spanish GS. This securities are classified, depending on their maturities in *Letras del Tesoro* (equivalent to US Treasury Bills, short-term zero-coupon bonds, with maturities up to eighteen months), *Bonos del Estado* (coupon-bearing bonds equivalents to US Treasury notes with maturities ranging from 3 to 5 years that earn a fixed rate of interest every year until maturity) and *Obligaciones del Estado* (similar to *Bonos* but with larger maturities of 10, 15 and 30 years). In all cases, their nominal value is of one thousand Euros.

Spanish GS are generally sold via auctions, where some dealers (market makers) have some privileges and usually buy a big share of the securities sold in the auction that later sell in the secondary market. Since January 1987, the Bank of Spain uses a unique pricing mechanism (called Spanish auction) that is a hybrid between the uniform and discriminatory auctions (see Abbink *et al.*, 2006). The Spanish Treasury performs several auctions every month. Long term securities (*Bonos* and *Obligaciones*) are auctioned by tranche, whereby the issue of a determined security is kept open over several consecutive auctions (three at least), in order to improve their liquidity. The securities allocated at such auctions are fungible, because they share the same nominal coupon, the same interest payment and redemption dates. When the total nominal amount issued reaches enough outstanding volume, the corresponding security issuance is closed and a new one opened. In the case of *Letras*, liquidity is enhanced by a mechanism that made that, once the 12-month bill is close to the 6 month term, it is auctioned again as a 6-month bill, and again when it is closed to the 3-month term to redemption. Despite their different denominations, hereafter we will refer to all *Letras, Bonos* and *Obligaciones* as Bonds.

The database is provided by Banco de España public webpage and contains all transactions in the secondary market called Spanish Public Debt Market[4]. As explained by Diaz et al. (2006), this database reports the number of transactions and both the nominal and effective volumes for each issue, as well as the maximum, minimum and the average price on a daily base. The database provides data for a period that goes from 1988 until 2010, and supposes a total of almost 5.000 trading days, and 700 issues and a total of 121.758 observations. On a separate database from the same web page we gather information for each issue (coupons, date of first issuance and redemption) that we need for the pricing function (equation 1). There are other databases like Reuters or Bloomberg, were some dealers report their bid and ask offers for each bond. Nevertheless, they have no information on actual transactions (they have to be reported to the Spanish Public Debt Market, and gathered by Banco de España's database), and offers reported are not binding, so information is of a lesser quality.

From this data, similarly to Gurkaynak et al. (2007), we have produced daily estimations of the yield curve using Svensson (1994) model implementing the genetic algorithm (GA) proposed by Gimeno and Nave (2009), to ensure the stability of the nonlinear optimization. Svensson (1994) model modifies the original work of Nelson and Siegel (1987). To do so, it uses a second term (the one that Nelson and Siegel (1987) had abandoned in their work) and added two additional parameters (φ_5 and φ_6). The equation for the instantaneous *forward* rate is:

$$f(m) = \varphi_1 + \varphi_2 \cdot e^{-\frac{m}{\varphi_5}} + \varphi_3 \cdot \frac{m}{\varphi_5} e^{-\frac{m}{\varphi_5}} + \varphi_4 \cdot \frac{m}{\varphi_6} e^{-\frac{m}{\varphi_6}} \tag{8}$$

Equation 8 generates a complete family of *forward* curves that reflects a great variety of term structure shapes (Gurkaynak et al., 2007; Gimeno and Nave, 2009). Integrating equation 8 between $[0,m]$ and dividing into m, results a equation that relates spot interest rate to time to maturity:

[4] Since 1988 the Bank of Spain has been building a database from closing prices. http://www.bde.es/webbde/es/secciones/informes/banota/series.html

$$s(m)=\varphi_1+\varphi_2\frac{1-e^{-m/\varphi_5}}{\left(m/\varphi_5\right)}+\varphi_3\left[\frac{1-e^{-m/\varphi_5}}{\left(m/\varphi_5\right)}-e^{-m/\varphi_5}\right]+\varphi_4\left[\frac{1-e^{-m/\varphi_6}}{\left(m/\varphi_6\right)}-e^{-m/\varphi_6}\right] \tag{9}$$

And replacing the discount function, the pricing equation of a zero-coupon bond would be:

$$P_m=e^{\left[-\varphi_1 m-(\varphi_2+\varphi_3)\varphi_5\left(1-\exp\left(-\frac{m}{\varphi_5}\right)\right)+\varphi_3 m\exp\left(-\frac{m}{\varphi_5}\right)-m\varphi_4\left(\frac{\varphi_6}{m}\left(1-\exp\left(-m/\varphi_6\right)\right)-\exp\left(\frac{-m}{\varphi_6}\right)\right)\right]} \tag{10}$$

where m is the term and φ_1, φ_2, φ_3, φ_4,, φ_5 and φ_6 the parameters to be estimated. In the case of coupon-bearing bonds, the pricing equation becomes more complex since multiple payments have to be discounted from different dates.

We have chosen this model based on its presence in a considerable number of studies, the great number of Central Banks that use it[5] (including the Bank of Spain); and the best performance the model has shown in the Spanish Government Debt Market, compared to other parametric models[6].

Following Bliss (1996), for each of the 4996 days in the sample, we have computed the *out-of-the-sample* errors in yields of each bond traded. Since it is an out-of-sample error (each estimated yield is obtained from a term structure recovered from the rest of bonds traded on this day), this implies that we have to estimate 121.758 term structures (one for each bond and day). Term Structure estimations using coupon-bearing bonds are extremely non-linear (see Gimeno and Nave, 2009), so for each of the term structures we run 30 GAs to ensure that the estimated parameters do not correspond to a local minimum, raising the total number of term structure estimations above three million and a half of GAs. For these estimations, we used, as a target function, the min squared weighted errors in prices of equation 7, equivalent to min squared errors in yields but faster to compute.

---

[5] See Bank of International Settlements (2005).
[6] See Núñez (1995) and Berenguer (2009).

4. Analysis of the liquidity factor

Once we have estimated the deviations (ε_{it}) of observed yields from the ones implied by the estimated term structure of interest rates: the difference between the quoted yield of a bond (i) and its yield implied by the *out-of-sample* Svensson term structure model for a given day (t). If these deviations were affected by liquidity considerations, they should be strongly influenced by two factors, one related with the easiness of closing trades, and the other with the market microstructure.

The first factor is the turnover or trading volume (T_{it}). If a bond is rarely traded, opposite offers would be difficult to match, and the willing seller (buyer) would have to accept a lower (higher) price in order to fulfill the operation. Warga (1992) and Alonso et al. (2004), among others, classify securities as *on-the-run/off-the-run* as a proxy for trading volume, when the turnover data is unavailable. Nevertheless, Elton and Green (1998) signaled that trading volume was a more robust measure of asset liquidity than these other proxies.

Another alternative that is sometime used in the literature is the bid/ask spread. However, there are two main drawbacks for this measure. First of all, there is not a single bid/ask spread for each trading day, since this is changing along the day. Although we could compromise for the spread at closing time, the second and more relevant issue is the quality on bid/ask data. As we mentioned in previous section, providers of information on bid/ask spreads include non-binding quotes, that may distort the real bid/ask spread. Sometimes, it might be possible to discern binding from non-binding quotes but given the magnitude of our study make unfeasible to distinguish them for each bond and day.

In our sample, as can be seen in Figure 1, error dispersion is higher for those days when trading volume of the bond is lower. The figure shows the typical funnel shape that is a characteristic signal of heteroskedasticity.

The second factor related to liquidity consideration is in the market microstructure of the bond: the tick size. Although all bond prices had the same tick size, changes in prices have completely different effects in yield terms. To illustrate this, let's suppose we have a zero-coupon bond where prices changes at a minimum Δ,

$$P = e^{-r \cdot m}$$

$$P + \Delta = e^{-r_\Delta \cdot m}$$

A change in prices, will suppose a change in yields equal to

$$r_\Delta - r = \frac{1}{m} \cdot \ln \frac{P+\Delta}{P} \quad (11)$$

Although the logarithm of the price change is the same for all bonds, the change in yields is inversely related with the term to maturity of the bond (*m*). In the case of coupon-bearing bonds, equation 11 is much more complex, but the effect can be proxied by the time to maturity (d_{it}), so those close to maturity will experience, for the same price change, higher return swings than the rest. In this sense, Amihud and Mendelson (1991) found evidence that there was a liquidity premia that was decreasing and convex function of the time to maturity.

Even in the case that prices were allowed to change in a continuum rather than at discrete values, the effect will still be present. Dealers only will trade on a bond if the expected profit from the trade compensates them from the trading costs. This profit will be related with the difference between the prices of buying and selling. So, when a bond is close to maturity, dealers will accept higher deviations from their bond valuation before it compensate them from the cost of taking a position on that bond.

As can be seen in Figure 2, we also observe the funnel pattern when comparing the error in yields with the duration of the bond. Although, this is a further signal of the heteroskedasticity in yields, this outcome might not be a consequence of a second factor, but caused by the differences in turnover previously mentioned: bonds close to maturity tend to be also the lest traded ones. We need a multivariate model to consider both factors jointly.

As stated above, liquidity constrains would produce wider movements for less liquid bonds, both in the upside and in the downside. Therefore, liquidity considerations should imply differences in the variance of ε_{it} (heteroskedasticity). In this sense, Amihud and Mendelson (1991) proposed a model where yields had a variance that was conditional on the time to maturity. The unobserved variance variable (h_{it}) would depend, both on turnover (there would be a negative relationship between turnover and variance) and duration (there would be a positive relationship between the inverse of the duration and the variance). Thus, a heteroskedastic model for the yield errors (l_{it}) would be equal to the one in equation 12.

$$\varepsilon_{it} = \beta_0 + \sqrt{h_{it}} \cdot u_{it} \qquad u_{it} \sim N[0,1]$$

$$\log h_{it} = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \tag{12}$$

In the variance equation, we would expect that $\gamma_1 < 0$ and $\gamma_2 > 0$, if both turnover and the tick size explain the variance of ε_{it}. In Table 1, model 1 represents the estimations of parameters of equation (12). As can be seen, both turnover and duration coefficients have the expected sign (negative for the turnover and positive for the duration). This result confirms that yields are not homokedastic, a feature that we will use in the models of next section.

Although these variables affect primarily the variance of ε_{it} (h_{it}), this is far to be the only expected effect. A difference in the variance of the yields of different bonds implies a different level of risk for an investor. Therefore, we would expect that investors would ask for higher return in the case of bonds that are susceptible of higher liquidity volatilities. Thus, the level of the variance (h_{it}) would also affect the level of the yield (the level equation).

$$\varepsilon_{it} = \beta_0 + \beta_1 \cdot \log h_{it} + \sqrt{h_{it}} \cdot u_{it} \qquad u_{it} \sim N[0,1]$$

$$\log h_{it} = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \tag{13}$$

The variance equation remains invariant with respect specification of equation (12), where we expected that $\gamma_1 < 0$ and $\gamma_2 > 0$, if both turnover and the tick size explain the range of movements in ε_{it}. Nevertheless, in the level equation, the heterokedastic behavior of ε_{it} would be derived from the first term ($\sqrt{h_{it}} \cdot u_{it}$), while these differences in volatility would be compensated by a higher liquidity premium ($\beta_1 \cdot \log h_{it}$). Therefore, we can call the parameter β_1 as the price of liquidity risk. We would expect that $\beta_1 > 0$, implying that investors demand a premia for the risk they are assuming[7].

In the second model of table 1, we present the parameters estimations of model (13). As can be seen, parameters estimated in the variance equation are similar to the ones we obtained in model 1, and both turnover and duration coefficients have the expected sign (negative for the turnover and positive for the duration). The main difference is in the case of the level equation, where we also find the expected positive (and significant) price of risk.

[7] Although it is reasonable to suppose that the price of risk change in time (increasing with the crisis and decreasing in normal times) for simplicity we have suppose in this section that the price of risk is constant. In next section, we present a model where the price of risk is estimated in a daily base.

5. A proposal for a modified Svensson model

Liquidity weighting

In previous section, we have shown that Vasicek and Fong (1982) assumption of homokedastic yield errors no longer holds when the bonds considered have different levels of liquidity. Therefore, the usual optimization functions (equations 6 and 7) aggravate the observed heterokedasticity instead of correcting it. In fact, the variance for the price error would be better described by equation 14, where the constant yield variance (σ^2) has been replaced by the heteroskedastic version proposed in equations 12 and 13 (h_i),

$$E\left[\left(P-\hat{P}\right)^2\right]=h_i\cdot\left(\frac{dP}{ds}\right)^2 \quad (14)$$

$$\log h_{it}=\gamma_0+\gamma_1\cdot\log T_{it}+\gamma_2\cdot\frac{1}{D_{it}} \quad (15)$$

Therefore, to estimate the term structure we will require to jointly estimate the parameters of both the term structure (equation 9) and the variance equation (equation 15), using, as the function we want to minimize, the squared errors in prices weighted by the standard deviation of yields times the bond duration.

$$\left(\hat{\varphi},\hat{\gamma}\right)=\min_{\varphi,\gamma}\cdot\sum_{i=1}^{k}\left(P_i-\hat{P}_i(\varphi)\right)^2\cdot\frac{1}{D_i\sqrt{h_i(\gamma)}} \quad (16)$$

Although we apply the optimization criteria of equation 16 to the estimation of a Svensson model, this modification can be used also for any other term structure definition you consider, parametric (e.g. Nelson and Siegel (1987) model) or non-parametric (e.g. Vasicek and Fong (1982) splines).

Liquidity premium

Previous model specification does not take into account the presence of an eventual liquidity premium. It just takes into account the variability in prices derived from these liquidity considerations. In order to include this premium in the model, we should vary the pricing equation (equation 1) to add an additional term:

$$\hat{P} = \sum_{j=1}^{n} C_j \cdot e^{-s(t_j,\varphi)\cdot t_j - \alpha\cdot h} + N \cdot e^{-s(m,\varphi)\cdot m - \alpha\cdot h} \tag{17}$$

This modification is equivalent to multiply the estimated price by $e^{-\alpha\cdot h}$ or to add a premium ($\alpha\, h/t$) to the bond yield. The variance equation (equation 15) would remain valid, and the function we want to optimize now will be:

$$(\varphi,\gamma) = \min_{\varphi,\gamma,\alpha} \sum_{i=1}^{k} \left(P_i - \hat{P}_i(\varphi,\gamma,\alpha)\right)^2 \cdot \frac{1}{D_i \sqrt{h_i(\gamma)}} \tag{18}$$

The introduction of the liquidity model proposed in equation 17 is similar to the one in Elton and Green (1998), where the log of the trading volume was added for the pricing equation, although they did not take into account the liquidity-induce heteroskedasticity to modify their weighting.

Term Structure estimation

In tables 3a-3c we present term structure estimations for three different days: April 20th, 2010 (in the way to the first Euro Area Sovereign Debt Crisis), May 11th, 2010 (Just in the middle of the first Greek sovereign debt crisis), and July 7th, 2010 (in the middle of the market easing after the publishing of the first European banks' stress tests). For each day, we compute four different estimations:

1. A traditional Svensson model (Equation 9), minimizing equation 7, that is, the traditional weighting error prices by the modified bond duration.

2. A Svensson model (Equation 9), but using as minimizing equation16. In order to get the estimated variances ($\hat{h}_{it}$) we need to compute the weights. We approach this in a two steps process. Firstly we compute the squared differences between observed and estimated yields (y) from previous model, and once we have those errors, we estimate a regression similar to the one we would use in a White heterokedasticity test (equation 19). Finally we use this modeled variance to estimate the term structure minimizing equation 18.

$$\log(y_{it} - \hat{y}_{it})^2 = \gamma_0 + \gamma_1 \cdot \log T_{it} + \gamma_2 \cdot \frac{1}{D_{it}} \tag{19}$$

3. Instead of two steps, we proceed to do a joint estimation of the mean (Svensson equation 9) and variance (equation 15) estimation (minimizing equation 18). In this case, we do not need to rely on the traditional duration-weighted estimators to obtain in the first step the variance equation.

4. A model with a liquidity premium (equation 17), where we include a compensation for liquidity risk (α) in the price equation as well as a variance equation (equation 15), that we jointly estimate using optimization program 18.

As can be seen in the attached tables, yield curves can vary a lot depending on the specification used. Furthermore, the liquidity premia can change between days: itwas higher in the middle of the Greek debt crisis (May 11th, 2010) than before, or after those days.

6. Conclusions

In this paper we have tried to explain the role played by liquidity on the deviations of sovereign bonds yields from a theoretical liquidity-free term structure of interest rates. In this process, we observed that differences in the level of liquidity of the bonds would cause wider movements of prices for less liquid bonds. Therefore, liquidity considerations should imply differences in their associated variance.

In this sense, we propose a heterokedastic model for the yields of the bonds, where the variance equation is function of the trading volume and the duration. After estimation of this model we obtained the expected sign of the coefficients (negative for the turnover and positive for the duration). This seems to confirm that liquidity differences among bonds from the same issuer can produce heterokedasticity.

Main direct implication of this heterokedasticity appears for the estimation of the yield curve. Vasicek and Fong (1982) estimated the term structure of interest rates assuming that bonds returns were homokedastic, and they proposed an error correction that consisted in weighting the price squared errors by the inverse of the duration. In this paper we have showed that this hypothesis does not hold, even for bonds from the same issuer, when there are differences in their level of liquidity. Therefore, cross-sectional models for the term structure should be corrected for liquidity differences.

Finally, to take into account the presence of a liquidity premia, we propose a Svensson model modified by adding a liquidity risk premium.

References

Abbink, K., Brandts, J. and Pezanis-Christou,P., 2006."Auctions for governments securities: A laboratory comparison of uniform, discriminatory and Spanish designs". *Journal of Economic Behavior and Organization,* vol.61 pp. 284-303.

Alonso, F., Blanco, R., Río, A.D. and Sanchís, A., 2004. "Estimating Liquidity Premia in the Spanish Government Securities Market". *The European Journal of Finance*, vol. 10, no. 6, pp. 453-474.

Amihud, Y. and Mendelson, H., 1986. "Asset Pricing and the Bid-Ask Spread". *Journal of Financial Economics*, vol. 17, no. 2, pp. 223-249.

Amihud, Y. and Mendelson, H., 1991. "Liquidity, Maturity, and the Yields on US Treasury Securities". *The Journal of Finance*, vol. 46, no. 4, pp. 1411-1425.

Bank for International Settlements, 2005. *Zero Coupon Yield Curves: Technical Documentation*. BIS Papers n.25.

Bliss, R.R., 1997. "Testing Term Structure Estimation Methods". *Advances in Futures and Options Research*, vol. 9, pp. 197-232.

Delpla J. and Weizsäcker, J. von, 2010, "The blue bond proposal", Bruegel Policy Brief n. 2010/03.

Díaz, A, Merrick, J.J and Navarro, E., 2006. "Spanish Treasury bond market liquidity and volatility pre- and post-European Monetary Union". *Journal of Banking y Finance.* Vol.30. 1309-1332

Díaz, A, Jareño, F and Navarro, E., 2011. "Term structure of volatilities and yield curve estimation methodology". *Quantitative Finance,* 11: 4, 573-586.

Elton, E.J. and Green, T.C., 1998. "Tax and Liquidity Effects in Pricing Government Bonds". *The Journal of Finance*, vol. 53, no. 5, pp. 1533-1562.

Fisher, L., 1959. "Determinants of Risk Premiums on Corporate Bonds". *The Journal of Political Economy*, vol. 67, no. 3, pp. 217-237.

Gimeno, R. and Nave, J.M., 2009. "A Genetic Algorithm Estimation of the Term Structure of Interest Rates". *Computational Statistics & Data Analysis*, vol. 53, no. 6, pp. 2236-2250.

Goyenko, R., Subrahmanyam, A. and Ukhov, A., 2011. "The Term Structure of Bond Market Liquidity and Its Implications for Expected Bond Returns". *Journal of Financial and Quantitative Analysis*, vol. 46, pp. 111-139.

Gurkaynak, R.S., Sack, B. and Wright, J.H., 2007, "The US Treasury Yield Curve: 1961 to the Present", *Journal of Monetary Economics*, vol. 54 (8), pp. 2291-2309.

Jordan, J. and S. Mansi, 2003. "Term structure estimation from on-the-run Treasuries,". *Journal of Banking and Finance*, vol. 27, pp. 1487-1509.

Kamara, A., 1994. "Liquidity, Taxes, and Short-Term Treasury Yields". *Journal of Financial and Quantitative Analysis*, vol. 29, no. 03, pp. 403-417.

Nelson, C.R. and Siegel, A.F., 1987. "Parsimonious Modeling of Yield Curves". *The Journal of Business*, vol. 60, no. 4, pp. 473-489.

Pasquariello, P. and Vega, C., 2009. "The on-the-run liquidity phenomenon," *Journal of Financial Economics*, vol. 92(1), pp. 1-24.

Sarig, O. and A. Warga, 1989. "Bond Price Data and Bond Market Liquidity", *Journal of Financial and Quantitative Analysis*, vol. 24, pp. 367-378.

Shen, P. and Starr, R.M., 1998. "Liquidity of the Treasury Bill Market and the Term Structure of Interest Rates". *Journal of Economics and Business*, vol. 50, no. 5, pp. 401-417.

Svensson, L.E.O., 1994. "Estimating and Interpreting Forward Interest Rates: Sweden 1992-1994". IMF Working Paper No. WP/94/114.

Vasicek, O.A. and Fong, H.G., 1982. Term Structure Modeling using Exponential Splines. *The Journal of Finance*, vol. 37, no. 2, pp. 339-348.

Warga, A., 1992. "Bond Returns, Liquidity, and Missing Data". *Journal of Financial and Quantitative Analysis*, vol. 27, no. 04, pp. 605-617.

White, H., 1980. "A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity", *Econometrica*, vol. 48, pp. 817-838.

Fig 1. Errors in yields of Spanish GS vs. their turnover
Out-of-sample errors have been estimated using Svensson (1994) model from 1988 to 2010.

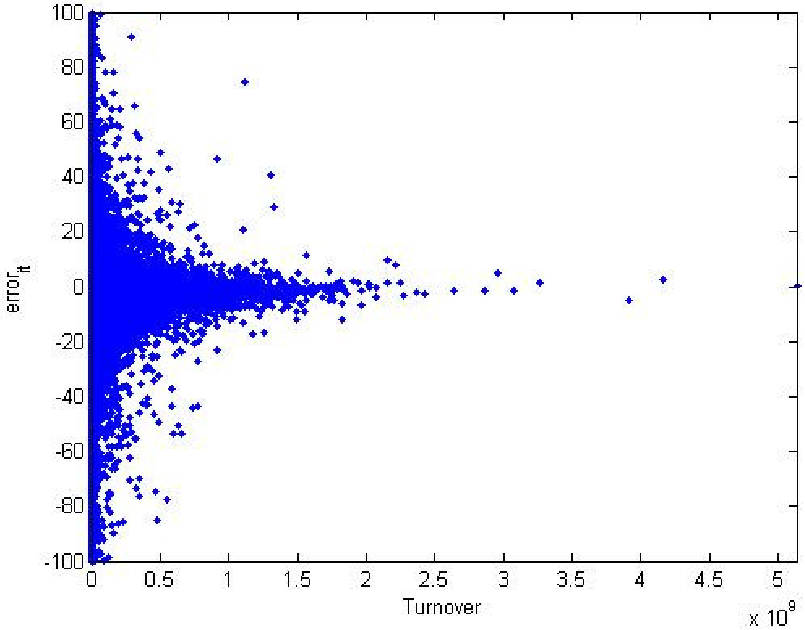

Figure 2. Errors in yields of Spanish GS vs. duration
Out-of-sample errors have been estimated using Svensson (1994) model from 1988 to 2010.

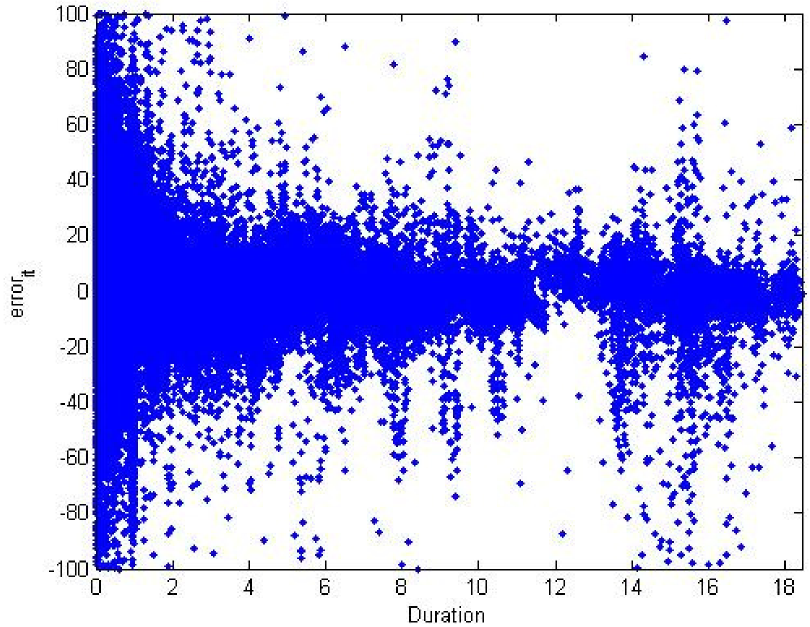

TABLE 1: Heteroskedastic models for the out of sample yield errors.

	Model 1		Model 2	
Level Equation				
Intercept	-0,989		-0.240	
$\log(h_{it})$			0.184	***
Variance Equation				
Intercept	3,373		3.290	
$\log(Turnover_{it})$	-0,133	***	-0.129	***
$1/Duration_{it}$	0,139	***	0.142	***
# of observations	121758		121758	
# of days	4996		4996	
# of bonds	662		662	

Individual LR tests have been computed for each parameter (outside the intercepts) under the null hypothesis of non significant variable. *** denotes rejection of the null at a 1% level, ** at 5% and * at 10%.

TABLE 2a: Term Structure of Spanish Government Bonds for April 20th, 2010.

	Duration-Weighted	Liquidity-Weighted	Mean-Variance Joint Estimation	Liquidity Premium Model
Svensson model				
φ_1	0.022	0.022	0.024	0.023
φ_2	-0.019	0.000	0.000	0.000
φ_3	0.041	0.039	0.028	0.031
φ_4	0.085	0.086	0.087	0.092
φ_5	5.575	5.238	4.059	4.142
φ_6	20.945	17.900	17.801	19.303
Variance equation				
γ_0	-	-12.412	9.344	10.869
γ_1	-	-0.237	-0.248	-0.313
γ_2	-	0.572	1.177	1.179
Price of Risk				
α	-	-	-	0.000121

Figure 3. Term Structure of Spanish Government Bonds for April 20th, 2010
The curves represented are zero-coupon and the dots are yields vs. time to maturity.

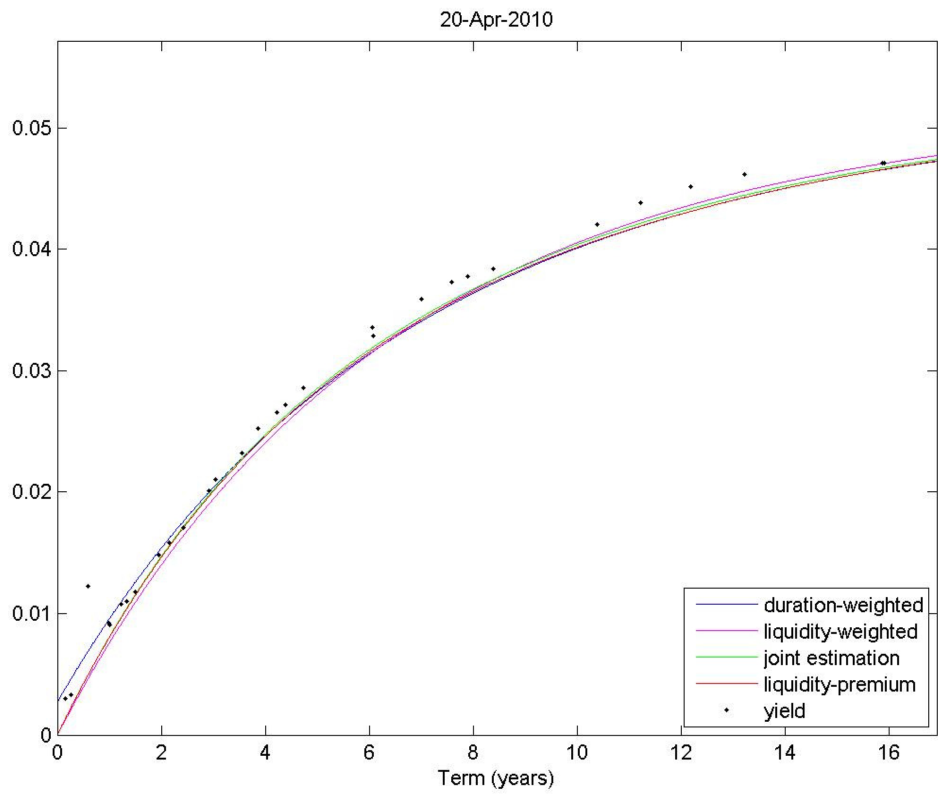

TABLE 2b: Term Structure of Spanish Government Bonds for May 11th, 2010.

	Duration-Weighted	Liquidity-Weighted	Mean-Variance Joint Estimation	Liquidity Premium Model
Svensson model				
φ_1	0.046	0.031	0.026	0.024
φ_2	-0.041	0.000	0.000	0.000
φ_3	-0.002	0.000	0.000	0.000
φ_4	0.067	0.084	0.105	0.110
φ_5	2.337	1.754	0.967	0.888
φ_6	51.435	16.800	22.228	21.716
Variance equation				
γ_0	-	-15.206	-0.012	0.000
γ_1	-	0.013	0.000	-0.043
γ_2	-	0.939	1.906	2.034
Price of Risk				
α	-	-	-	0.000599

Figure 4. Term Structure of Spanish Government Bonds for May 11th, 2010
The curves represented are zero-coupon and the dots are yields vs. time to maturity.

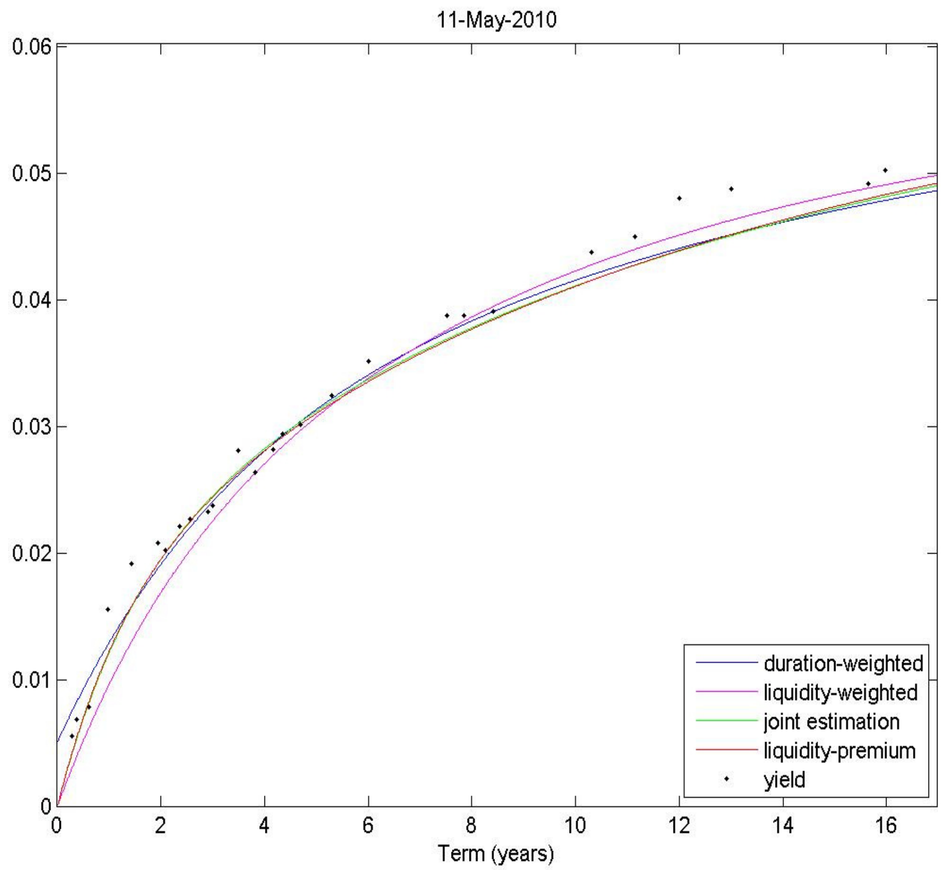

TABLE 2c: Term Structure of Spanish Government Bonds for July 7th, 2010.

	Duration-Weighted	Liquidity-Weighted	Mean-Variance Joint Estimation	Liquidity Premium Model
Svensson model				
φ_1	0.036	0.037	0.031	0.029
φ_2	-0.032	0.000	0.000	0.000
φ_3	0.002	0.014	0.001	0.003
φ_4	0.086	0.075	0.094	0.101
φ_5	0.729	1.198	0.416	0.374
φ_6	23.446	17.148	17.243	16.115
Variance equation				
γ_0	-	-10.860	-0.812	0.000
γ_1	-	-0.191	-0.082	-0.122
γ_2	-	-0.200	0.810	0.803
Price of Risk				
α	-	-	-	0.000000

Figure 5. Term Structure of Spanish Government Bonds for July 7th, 2010
The curves represented are zero-coupon and the dots are yields vs. time to maturity.