Markov-switching range-based volatility model and its application in volatility adjusted VaR estimation

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Abstract

We propose a more flexible range-based volatility model which can capture volatility process batter than conventional GARCH approach. Considering the regime switching process is appropriate for dealing the structure change embedded in the time series data. Range-based volatility CARR model with Markov-switching structure can assist us to describe the effect for exogenous shock to market data. After the data fitting and VaR estimation, we conclude that the range-based volatility method is better than the return-based GARCH model in volatility fitting. In particular, incorporating the possibility of regime switching into volatility process can boost the efficiency for VaR estimation. We also present an empirical application for demonstrating our model could characterize the unexpected switching of volatility process. Furthermore, comparing with non-regime switching volatility model, our model our model outperforms other alternatives on the estimation of volatility adjusted historical VaR.

JEL: C22; C53; G12

Keywords: CARR model, Markov-switching, range, exogenous shock, Volatility adjusted historical VaR estimation.

1. Introduction

Since the concerned financial market variables suffered dramatic shocks at times in relative steady state, the corresponding traditional linear fitting model should face the poor performance. It is natural to develop more realistic time series model to capture these phenomena. There are a number of nonlinear adjustment approaches to model financial time series with unexpected dramatic changes. Examples include threshold autoregressive, smooth transition and Markov-switching model. The advantage of using Markov switching approach is that there is no feedback from the observed information to the switching process. That is to say the switching process is dominated by the Markov chain. In this article we concern about the Markov-switching method proposed by Hamilton (1989, 1990), since this method treats the regime shift as an exogenous variable. The Markov-switching model has been applied from mean to variance equation. The combination of Markov-switching and volatility model can provide much more flexible estimation. The advantage is to allow the estimated coefficients change in different states. In financial econometrics, one of the classical volatility models is the ARCH/GARCH family pioneered by Engle (1982) and Bollerslev (1986). The extensions of regime switching approach for ARCH/GARCH model have been widely studied in financial literature.

Lamoureux and Lastrapes (1990) propose using Markov-switching approach to depict the shift of unconditional variance. Based on this suggestion, Cai (1994) and Hamilton and Susmel (1994) develop the Markov-switching ARCH model. Cai (1994) and Hamilton and Susmel (1994) employ the ARCH model since their model is restricted to the path dependence. In fact, the problem of path dependence may cause unattainable estimation. As a remedy, Gray (1996) integrates out the path dependence by conditional distribution of error term. Gray's notion is to use the GARCH term to skillfully evade the regime path. This method is undoubtedly workable for estimation but is harmful to multi-period-ahead volatility forecasting. Klaassen (2002), therefore, modifies Gray's model and provides a solution to this problem. Although Gray (1996) and Klaassen (2002) have overcome the problem of path dependent, their model has the problem of intractable likelihood function. Alexander (2008), Hamilton (2008) and Lange and Rahbek (2008) show that the Markov-switching GARCH model provided by Haas, Mittnik and Paolella (2004) could circumvent this problem. In addition to analytical tractability, Haas, Mittnik and Paolella's (2004) model has other advantages, such as the existence of stationarity conditions, and the possibility of allowing dynamic properties of process.

The foregoing are the return-based volatility models with Markov-switching. In this paper we replace the Markov-switching volatility model with the range-based ingredient, since they both belong to Multiplicative Error Model (MEM) of Engle (2002) and their estimated approach is of great similarity. The benefit of using range data is that they consider two elements of information: high and low asset prices; on the contrary, the traditional approach of using return data can only reflect the close price. As more information is contained in the range data, the estimation based on them is expected to give more efficient result. This conjecture has been supported by Parkinson (1980), Rogers and Satchell (1991), Gallant, Hsu, and Tauchen (1999), Yang and Zhang (2000), Alizadeh, Brandt, and Diebold (2002), Brandt and Jones (2006), Chou (2005, 2006), Chou and Cai (2009) and Chou, Wu, and Liu (2009). According to these studies, it is believed that the range-based volatility model could serve as an effectual alternative to the return-based case in depicting the dynamic volatility process. Therefore, this study is based on the range-based volatility model.

In the empirical result, it is shown that the regime switching phenomenon could be clearly depicted if it is really in presence. In addition, through our model specification, we see the volatility processes have contrastingly different dynamics under different regimes that incorporate the smooth and volatile pattern. We follow the suggestion of Hamilton (2008) and

Hamilton and Susmel (1994) to illustrate the Markov-switching range-based volatility model is superior to single regime volatility models. That is to compare models' forecasting abilities. Consequently, we apply the estimated volatility in measurement of historical VaR estimates. As a result of estimating the regime switching volatility model needs a longer sample period and that might cause estimation bias for unadjusted historical VaR estimates, especially for returns volatility containing regime switching, this study chooses the volatility adjusted historical simulation to measure the VaR of indices returns. Our empirical results show that using Markov-switching range-based volatility model for adjusted historical VaR estimation outperforms using other alternatives including CARR, GARCH and Markov-switching GARCH model.

The remainder of this paper is arranged as follows. In section 2 we introduce the Markov-switching CARR model (MS-CARR), the volatility adjusted historical VaR estimation and basic conservatism, accuracy and efficiency evaluation for VaR estimates. The Monte Carlo simulation with MS-CARR model is reported in section 3. Section 4 presents the main empirical results. Finally section 5 provides our concluding remarks.

2. Regime switching range-based volatility model and volatility adjusted VaR estimation

This section introduces the proposed model and discusses its own estimating procedure. In addition, we also report the volatility application for volatility adjusted VaR estimation and its assessment criteria.

2.1 The Markov-switching CARR model

The range-based volatility model (CARR) proposed by Chou (2005) can be express as,

$$\begin{aligned} &R_t = \lambda_t \varepsilon_t, \qquad \varepsilon_t | I_{t-1} \sim f(1;\cdot) \\ &\lambda_t = \omega + \alpha R_{t-1} + \beta \lambda_{t-1}, \end{aligned} \tag{1}$$

where R_t denotes the observed high low range in logarithm type during the time interval t, λ_t is the conditional mean of range, and ε_t is assumed to be distributed with a density function $f(.)$ with a unit mean.

To allow regime switching in dynamical range-based volatility process, we construct the MS-CARR model with general k-regime as below,

$$R_t = \lambda_{s_t,t}\varepsilon_t, \tag{2}$$

$$p_{ij} = \Pr(s_t = j | s_{t-1} = i) \quad i \quad j=1 \quad \dots \quad k, \tag{3}$$

$$\lambda_{s_t,t} = \omega_{s_t} + \alpha_{s_t} R_{t-1} + \beta_{s_t}\lambda_{s_t,t-1} \tag{4}$$

where R_t is the observed high low range in logarithm type during the time interval t, ε_t is assumed to be distributed with a density function $f(.)$ with a unit mean, and s_t follows a Markov chain with finite state space **S**$=\{1 \quad 2 \quad \dots \quad k\}$. The transition probability is illustrated in eq. (3). By probability axiom, the sum for probabilities have to satisfy $\sum_{j=1}^{k} p_{ij} = 1$ for i = 1, 2,…, k, and $0 \le p_{ij} \le 1$ for $i \quad j$ = 1, 2,…, k. According to the time varying transition probabilities, it is easy to infer the stationary distribution of the Markov chain as $\pi_{\infty}^{s_t}$. The regime variance, $\lambda_{s_t,t}$, follows the CARR(1,1) framework shown in eq. (4). The coefficients ($\omega_{s_t}, \alpha_{s_t}, \beta_{s_t}$) in the conditional range equation are all positive to ensure non-negative constraint for $\lambda_{s_t,t}$. Furthermore, eq. (4) can be rewritten as $\lambda_{s_t,t} = \omega_{s_t}(1-\beta_{s_t})^{-1} + \alpha_{s_t}\sum_{i=1}^{\infty}\beta_{s_t}^{i-1}R_{t-i}$, for s_t=1,…,k. By this specification, the conditional range with regime change, $\lambda_{s_t,t}$, depends only on the within-regime CARR parameters. In addition, the parameters $\omega_{s_t}(1-\beta_{s_t})^{-1}$, α_{s_t}, and β_{s_t} can be regarded as the total impact effect, the short-run impact effect, and the long-run effect of shocks to regime s_t conditional range respectively.

We derive the log likelihood function of this estimator as below. According to Hamilton (2008), we can make an inference of s_t through the observed high low range R_t. By the case of two probabilities, we can infer that,

$$\xi_{jt} = \Pr(s_t = j | I_t; \boldsymbol{\theta}), \ j=1,2, \tag{5}$$

where the information set is $I_t = \{R_t, R_{t-1}, ..., R_1, R_0\}$, the unknown vector parameters are $\boldsymbol{\theta} = (\lambda_{s_t}, \omega_{s_t}, \alpha_{s_t}, \beta_{s_t}, p_{11}, p_{22})'$, and the sum for two state probabilities is unity. We can obtain the state probabilities by iteration method. The Eq. (6) can be viewed as the input for generating Eq. (5).

$$\xi_{i,t-1} = \Pr(s_{t-1} = i | I_{t-1}; \boldsymbol{\theta}), \ i=1,2, \tag{6}$$

The distribution for R_t is assumed to follow the exponential distribution with unit mean, since the range data is positively valued. The density function for two regimes is,

$$\eta_{jt} = f(R_t | s_t = j, I_{t-1}; \boldsymbol{\theta}) = \lambda_{s_t} \exp[-\lambda_{s_t} R_t], \ j=1,2, \tag{7}$$

By the input Eq. (6), we can estimate the conditional density as,

$$f(R_t | I_{t-1}; \boldsymbol{\theta}) = \sum_{i=1}^{2} \sum_{j=1}^{2} p_{ij} \xi_{i,t-1} \eta_{jt} \tag{8}$$

The renewed joint probabilities can be expressed as,

$$\xi_{jt} = \frac{\sum_{i=1}^{2} p_{ij} \xi_{i,t-1} \eta_{jt}}{f(R_t | I_{t-1}; \boldsymbol{\theta})} . \tag{9}$$

After we execute the iteration, the log likelihood function can be written as,

$$\log f(R_1, R_2, ..., R_T | R_0; \boldsymbol{\theta}) = \sum_{t=1}^{T} \log f(R_t | I_{t-1}; \boldsymbol{\theta}) . \tag{10}$$

The unknown vector parameters ($\boldsymbol{\theta}$) can be obtained by maximizing the Eq. (10).

2.2 The estimation of the volatility adjusted historical VaR

It is hard to directly judge the performance of Markov-switching volatility model, and therefore we apply the estimated volatility to evaluate the VaR estimates. We consider that the well performance of VaR estimates expresses the better estimation of volatility model. It is essential to examine the practical performance for MS-CARR model. We employ the historical VaR volatility adjustment approach proposed by Duffie and Pan (1997) and Hull and White (1998) for the time being. It is reasonable to define the volatility adjusted returns series as,

$$\tilde{r}_{t,T} = \frac{\hat{\sigma}_T}{\hat{\sigma}_t} r_t, \quad t=1,\ldots,T, \tag{11}$$

where r_t and $\hat{\sigma}_t$ are the unadjusted returns and the estimated standard deviation series, respectively. The time index T is fixed, and $\hat{\sigma}_T$ is the constant estimated standard deviation at time T. We could calculate the historical VaR estimates by this adjusted returns series. Such on adjustment method could eliminate estimate bias from a longer historical sample period.

2.3 Assessment Criteria for the VaR estimation

A better VaR estimation could provide both risk control and profit. The conservatism and accuracy is to measure the risk controlling ability. The efficiency is to create the profit after bearing the risk. The assessment criteria for conservatism, accuracy and efficiency are measuring the performance of competing VaR estimations.

2.3.1 Conservatism

We employ the mean relative bias (MRB) proposed by Hendricks (1996) to compare the effect of conservatism for these competing VaR estimations. The MRB can judge the relative size of various VaR estimates. The larger value of MRB shows more conservatism for the corresponding VaR estimates. Given N VaR estimations, and T time periods, the mean relative bias can be defined as,

$$MRB_i = \frac{1}{T}\sum_{t=1}^{T}\frac{VaR_{i,t} - \overline{VaR_t}}{\overline{VaR_t}} \quad \text{where} \quad \overline{VaR_t} = \frac{1}{N}\sum_{i=1}^{N} VaR_{i,t} \ . \qquad (12)$$

2.3.2 Accuracy

As to the measurement for accuracy, one can adopt the binary loss function (BLF) developed by Lopez (1999), and the LR test for unconditional coverage (LR_{uc}) proposed by Kupiec (1995). [2] The accuracy criterion can explain whether the estimated VaR is able to cover the realized daily loss. The BLF takes account of whether the given days loss is greater or smaller than the estimated VaR. The BLF contains two possible daily realizations. When the estimated VaR fails to cover the realized loss, the $BLF_i = 1$. Otherwise, the $BLF_i = 0$ represents the estimated VaR is able to cover the profits and losses. The BLF for a method *i* is,

$$BLF_{i,t+1} = \begin{cases} 1, & if \quad r_{i,t+1} < VaR_{i,t} \\ 0, & if \quad r_{i,t+1} \geq VaR_{i,t} \end{cases}, \qquad (13)$$

where $r_{i,t+1}$ is the realized return in time *t+1*. It is intuitive to calculate the average binary loss function (ABLF) by collecting the BLF for every period. The value of the ABLF provides an indication of accuracy of VaR estimates, since that is the actual failure rate.

As to the LR_{uc} test for testing the VaR estimates' accuracy, the null hypothesis of LR_{uc} test is assuming that the probability of failure for each trial ($\hat{\alpha}_c$) is amount to the desired significance level (α_c). A rejection of this hypothesis indicates the inappropriateness of a VaR estimation under consideration. The LR statistics for unconditional coverage can be shown as,

$$LR_{uc} = -2\ln[\frac{\alpha_c^{n_1}(1-\alpha_c)^{n_0}}{\hat{\alpha}_c^{n_1}(1-\hat{\alpha}_c)^{n_0}}] \sim \chi^2_{1,\alpha_c} \ , \qquad (14)$$

[2] Although Christofferson (1998) develop a more complete LR test incorporating unconditional coverage, independence and conditional coverage test; our empirical results show the failure process contains no Markov process. Consequently, we apply the LR test of unconditional coverage to evaluate the accuracy of VaR estimation.

where α_c denotes the desired significance level, n_0 is the observed number of non-exceptions, n_1 denotes the observed number of exceptions, and the rate of failures can be calculated as $\hat{\alpha}_c = n_1/(n_0 + n_1)$.

2.3.3 Efficiency

The efficiency criterion is essential in that it provides an objective judgment on the performance of VaR estimations, since it can point out whether the accurate model is too conservative to abandon profits for investors. We use the mean relative scaled bias (MRSB) proposed by Hendricks (1996) as the efficiency assessment statistic. The MRSB is aimed at determining which VaR estimation providing adequate risk coverage contains smallest average risk measure. The computation of this statistic consists of the following two steps. The first step is to calculate the scaling factor (X_i) for each model *i* so that,

$$F_i = T\alpha_c, \qquad F_i = \sum_{t=1}^{T} \begin{cases} 1, & if \quad r_{i,t+1} < X_i VaR_{i,t} \\ 0, & if \quad r_{i,t+1} \geq X_i VaR_{i,t} \end{cases}, \tag{15}$$

where F_i denotes the total number of exceptions for model *i*, $r_{i,t+1}$ is the realized return in time *t+1*, *T* is the sample size, and α_c is the desired significance level. The second step is to compare the scaled VaR figures with their relative average sizes by using the mean relative bias calculation,

$$MRSB_i = \frac{1}{T}\sum_{t=1}^{T}\frac{X_{i,t} - \overline{X}_t}{\overline{X}_t}, \quad \overline{X}_t = \frac{1}{N}\sum_{i=1}^{N} X_{i,t} \tag{16}$$

where *N* is the number of competing VaR estimations. The VaR estimation with smallest MRSB is superior to others in that it is the most efficient one.

3. Monte Carlo simulation with MS-CARR model

In this section we use the Monte Carlo simulation to illustrate the property of MS-CARR model. According to the previous literatures of Markov switching volatility model, the experimental design of Monte Carlo simulation needs to discuss the simulated performances under the different degrees of Markov effect.[3] We could directly comprehend the advantage of considering Markov switching approach in CARR model by this experimental design. Our simulation experiment considering two scenarios follows a two-state MS-CARR(1,1) model by eq. (2)-(4). The given parameters are shown as follows,

$$(\omega_1,\omega_2)=(0.05,0.25)\,, \tag{17a}$$

$$(\alpha_1,\alpha_2)=(0.05,0.05)\,, \tag{17b}$$

$$(\beta_1,\beta_2)=(0.7,0.7)\,, \tag{17c}$$

$$(p_{11},p_{22})\in\{(0.5,0.5),(0.999,0.999)\}\,. \tag{17d}$$

The scenario 1 of $(p_{11},p_{22})=(0.5,0.5)$ denotes the MS-CARR model with the weakest Markov effect, but the opposite scenario 2 of $(p_{11},p_{22})=(0.999,0.999)$ shows that with the strongest Markov effect. Figure 1 plots the simulation results including the simulated range, volatility, and state for these two different scenarios.

[Figure 1]

There are some main findings in Figure 1. Firstly, the scenario 1 of weakest Markov effect shows that it is difficult to recognize the regular pattern between volatilities and its regimes. However, this regular pattern is obvious in the scenario 2. This indicates the degree of Markov effect plays an important role for Markov-based model as the Markov effect is strong. Secondly, the different range levels display various volatility regimes in scenario 2. Finally, by the bottom panel of Figure 1, the MS-CARR model with strongest Markov effect could

[3] Also see Chen and Tsay (2011), Alexander and Lazar (2006), Hass et al. (2004) and Carvalho and Lopes (2007).

produce volatility clustering that perfectly reflects the behavior of financial data.

4. Empirical analysis

In this section we use empirical data to examine our proposed MS-CARR model and a few related models on their performance in VaR estimation, and give comparison according the assessment criteria set out in the preceding section.

4.1. Data description

The data we use in our analysis are the indices of three most representative markets in the USA and UK, including the Nasdaq index, the S&P 500 stock index and the FTSE 100 index. These data are collected from Yahoo Finance (http://finance.yahoo.com/) for the period starting from February 3, 2003 to January 29, 2010. Over this period of time, the close price of each market trading day is collected for the return based models (GARCH, MS-GARCH), whereas the daily high and low prices are collected for the range based models (CARR, MS-CARR).

Descriptive statistics for daily returns and ranges data are reported in Table 1. The high kurtosis for all market returns and ranges exhibit fat-tailed shapes, and the range distribution is even more fat-tailed. In addition, the skewness for all market returns is negative skewness. This indicates that the left tail is considerably extreme for all market returns. Figure 2 further shows how the daily indices, ranges, and returns evolve through the period under study. At first glance, the GARCH family model seems appropriate to fit these trading data, since the pattern of daily returns for all markets represents volatility clustering. Furthermore, it can be obviously seen that for all the three markets, the indices, returns, and ranges have undergone a dramatic (structural) change since 2008 when the financial crisis broke out.

[Table 1]

[Figure 2]

4.2. Empirical results

According to the signal of ARCH effect in Figure 2, it seems that the concept for conditional heteroskedasticity volatility might be proper to be incorporated. Table 2 states the results of data fitting for the simple CARR(1,1) and GARCH(1,1) model. This study adopts the suggestion of Engle (2001) to use the quasi-maximum likelihood estimation proposed by Bollerslev and Wooldridge (1992) to estimate these models.[4] The estimation results show that the coefficient estimation for α in CARR model are 0.074 for the Nasdaq index, 0.086 for the S&P 500 index and 0.133 for the FTSE 100 index, but those of GARCH model are 0.058 for the Nasdaq, 0.067 for the S&P 500 and 0.088 for the FTSE 100. This indicates the CARR model could depict more sensitivity of volatility transitory shocks. In addition, the estimated results conform to the stationarity and non-negative conditions of CARR and GARCH volatility models.

[Table 2]

It is critical to detect whether there is a structural change in the volatility process. If the structure change of volatility does not actually exist, fitting to a volatility model with regime switching nature could be meaningless. Consequently, it is natural to adopt two simple tests to check the existence of structure change in the static volatility processes. We use the financial credit crisis as the tentative break point in the time horizon, and separate the entire

[4] Engle (2001) interprets using the estimation of robust standard errors can reduce the trouble of over heteroskedasticity.

sample into two periods for the ease of comparison.[5] Table 3 reports the static volatility equality test for these two sub-periods. We test the variance and mean range for examining whether the different sub-periods static volatilities have a significant change. It is evident to realize that the variances for two sub-samples are totally different by Panel A of Table 3, and further the crisis event really changes the variance by the result of F-test. According to Panel B of Table 3, the mean range has two fold increases after the episode.

[Table 3]

Although the above testing evidence the crisis event could change the variance and mean range, we further anticipate the outbreak of the crisis could also have caused a structural change in the dynamic volatility process. Hence, we estimate the CARR and GARCH model with a time break dummy variable. The results of model estimation are given in Table 4, where we see all the estimated coefficients are significant. Based on these results of Table 3 and Table 4, we could infer that this crisis not only could change the static volatility but dynamic volatility. In other words, the necessity of considering the regime switching nature in the volatility model is perfectly justified.

[Table 4]

Billio and Caporin (2005) state an essential inference that in large systems the Markov switching dynamic conditional correlation model may have some convergence problems if a high number of parameters are involved. Naturally, the MS-CARR model may have the same problem, too. To avoid over parameterization, it is appropriate to estimate the MS-CARR

[5] Preston (2009) has clear discussion on the date of financial credit crisis.

structure with two different regimes. The estimated results for two-state MS-CARR and MS-GARCH model are shown in Table 5. By the specification of two regime states, we could calculate two different volatility regime processes, the smooth and volatile volatility regime. If the estimated parameters for both regime processes follow the stationarity condition of $\alpha_1 + \beta_1 \leq 1$ and $\alpha_2 + \beta_2 \leq 1$, we define the volatility process containing relative smaller coefficient of α_i as the smooth volatility regime, since the α_i is the estimated parameter of short-term effect. Nevertheless, as one of the estimated parameters of volatility regime shows $\alpha + \beta > 1$, we define that as the volatile volatility regime. By the MS-CARR model, the persistent rate ($\hat{\alpha} + \hat{\beta}$) of smooth volatility regime is 0.978 for the Nasdaq, 0.975 for the S&P 500 and 0.951 for the FTSE 100, but that of volatile volatility regime is 0.930 for the Nasdaq, 0.951 for the S&P 500 and 0.946 for the FTSE 100. This indicates that the smooth volatility regime has higher volatility persistent rate for all market indices than volatile volatility case. In other words, the smooth volatility regime shows the more persistent sensitivity of the volatility expectation to market shocks. The components of persistent rates of different regimes are contrasting different for all market volatilities. The estimated transition probability from smooth volatility to volatile volatility regime ($1 - \hat{p}_{11}$) is 0.001 for the US market and 0.002 for the UK market; nevertheless, the estimated transition probability from volatile volatility situation to smooth volatility regime ($1 - \hat{p}_{22}$) is 0.005 for both markets. This indicates that the transition probability of smooth volatility regime is lower than that of volatile state. We could infer the expected transition period through the estimated transition probability. The expected transition period from smooth volatility to volatile volatility regime ($1/(1 - \hat{p}_{11})$) is approaching 3.97 year for both Nasdaq and S&P 500, and 1.98 year for the FTSE 100; nevertheless, the expected transition period from volatile volatility to smooth volatility regime ($1/(1 - \hat{p}_{22})$) is about 0.79 year for all market indices.

Based on the previous statistics, we could conclude that the volatility for all market indices contains the regime switching behavior and the expected transition period from volatile volatility to smooth volatility regime is shorter than that from smooth volatility to volatile volatility regime. It means that the volatility is relative stable in the long term although big shocks may cause it violent oscillation and change its regime from smooth to volatile. We also calculate the stationary regime probabilities. According to the estimated result of MS-CARR model, it shows that the probability for volatility of Nasdaq contributing to smooth (volatile) regime in next time is 0.796 (0.204), and that of S&P 500 contributing to smooth (volatile) regime in next time is 0.784 (0.216). As to the volatility of FTSE 100, the probability for that contributing to smooth (volatile) regime in next time is 0.772 (0.228). In brief, the probability of expected volatility for all market indices contributing to smooth regime is over 75%.

[Table 5]

[Figure 3]

In Figure 3 we plot the estimated range-based volatility, the estimated Markov-switching range-based volatility and the smoothed probability for all market indices. The estimated Markov-switching range-based volatility characterizes more sharply increasing pattern than that of range-based volatility from 2008 to 2009. We could attribut this phenomenon to the flexible specification of regime switching process. The volatility pattern of smoothed probability starts from smooth regime and then switches to volatile regime but finally it back to smooth regime. Although the pattern of smoothed probability for FTSE 100 has a little more fluctuation than Nasdaq and S&P 500 during 2008 to 2009, it is obvious that the smoothed probabilities for all market indices show a rather similar pattern. These results illustrate that the phenomenon of regime switching volatility is indeed existent in all market indices, and the MS-CARR model could depict this pattern reasonably better. In addition, we also find that the smooth (volatile) volatility regime could match the relative stable (dramatic changing) volatility for all market indices.

We use the estimated volatility fitting from CARR, GARCH, MS-GARCH and MS-CARR model to calculate the 1-day historical VaR estimates at 1% significant level for a position on all market indices for 300 trading days. Table 6 reports the forecasting performance summary for different VaR estimations. Firstly, from the conservatism measure, MRB, we see that the GARCH based VaR estimation is the most conservative for all the three market indices. The least conservative model varies: it is MS-CARR for Nasdaq and S&P 500, while it is MS-GARCH for FTSE 100. Secondly, from the accuracy criterion ABLF and LR_{uc} test results, we see all the four models performs reasonably well in that the null hypothesis of appropriate VaR estimation is not rejected for all market indices. In brief, all the four volatility models for VaR estimation are appropriateness through the conservatism and accuracy criteria. However, a conservative and accurate but inefficient VaR estimation would tend to overestimate VaR during low risk period. Therefore, the efficiency aimed at deciding which VaR estimation provides better assets allocation could provide further judgment criterion based on conservatism and accuracy. Thirdly and more importantly, according to the efficiency assessment measure, the MS-CARR based VaR estimation gives the smallest MRSB value of -0.089 for Nasdaq, -0.101 for S&P 500, and -0.063 for FTSE 100. This indicates the MS-CARR based VaR estimation could provide more precise resource allocation than other competing models. In addition, the MRSB value for the models with regime switching nature is smaller than that for the models with single regime, implying that the volatility model considering the regime switching approach could nicely depict the volatility dynamic process. As a summary, from these assessment criteria we see that the MS-CARR based VaR estimation is the most efficient method than those based on CARR, GARCH and MS-GARCH. In addition, we also see the dynamic volatility model incorporating Markov-switching process could capture volatility process more accurately than the single regime volatility model. In short, the Markov-switching range-based volatility model outperforms than other competing volatility models.

[Table 6]

[Figure 4]

We compute not only the daily historical VaR estimates at 1% significant level for 300 trading days on all market indices but their returns, and then graph them in Figure 4 for checking the exceptions points easily. It is more obvious to find out that the pattern of all the four VaR estimates responds to changes in the underlying returns very quickly. Besides, the results of accuracy analysis for all VaR estimations are almost equivalent by Figure 4. This result is in line with the outcome of accuracy criterion on Table 6.

5. Conclusion

In this paper we present a new approach to modeling the range-based volatility by allowing for Markov chain transition in the parameters of CARR model. Introducing the nonlinear method into the CARR model could enhance the flexibility in estimating the dynamic volatility process. The property of Markov chain is to capture the endogenous changes in the data generating process. Hence, we insert this method to explain the dynamic volatility process may not always follow just one estimated dynamic process in the specific period. In the empirical part of this study we collect the Nasdaq, the S&P 500 and the FTSE 100 to estimate the MS-CARR model. Though the estimated transition probability, we could conclude that the volatility regime switching process can be accurately captured and illustrated by our model specification for all market indices and the switching pattern is from smooth to volatile and then back to smooth volatility regime. Overall, the MS-CARR model could precisely depict the different dynamic volatility process across regimes when the endogenous shocks really make the break of single regime volatility process. We also calculate the expected transition period between different regimes, and evidence that the volatility staying smooth regime is longer than volatile case during our research period. It shows that the volatility is relative stable even though the uncertain shocks may change its regime to the volatile state. Considering the regime switching process into volatility model for measuring the volatility adjusted historical VaR outperforms these single regime volatility models, and furthermore we demonstrate that the Markov-switching range-based volatility model for VaR estimate provides the highest efficiency among all the models under study. This result illustrates that MS-CARR model is superior not only to single regime CARR and GARCH model but to MS-GARCH model.

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Table 1. Descriptive statistics for the returns and ranges of daily S&P 500 and FTSE 100 index (2003.2.3-2010.1.29).

	Nasdaq		S&P 500		FTSE 100	
	Return	Range	Return	Range	Return	Range
Mean	0.028	1.567	0.013	1.425	0.021	1.477
Median	0.088	1.301	0.085	1.085	0.061	1.132
Maximum	11.159	11.129	10.957	10.904	9.384	10.753
Minimum	-9.588	0.274	-9.470	0.239	-9.265	0.232
Std. Dev.	1.487	1.079	1.365	1.200	1.296	1.135
Skewness	-0.130	3.092	-0.256	3.481	-0.100	2.941
Kurtosis	9.851	17.858	14.503	20.433	12.142	16.248
observation	1760	1761	1767	1768	1767	1768

Table 2. CARR and GARCH model fitting for daily Nasdaq, S&P 500 and FTSE 100 index (2003.2.3-2010.1.29).

CARR model: $\lambda_t = \omega + \alpha R_{t-1} + \beta \lambda_{t-1} \qquad R_t | I_{t-1} \sim \exp(1,\cdot)$

GARCH model: $h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1} \qquad \varepsilon_t | I_{t-1} \sim N(0, h_t)$

	Nasdaq		S&P 500		FTSE 100	
	CARR	GARCH	CARR	GARCH	CARR	GARCH
$\hat{\omega}$	0.001 (<0.001)	0.014 (0.007)	0.002 (0.001)	0.010 (0.006)	0.002 (0.001)	0.009 (0.003)
$\hat{\alpha}$	0.074 (0.012)	0.058 (0.014)	0.086 (0.019)	0.067 (0.015)	0.133 (0.024)	0.088 (0.017)
$\hat{\beta}$	0.909 (0.014)	0.934 (0.013)	0.886 (0.028)	0.924 (0.013)	0.837 (0.031)	0.906 (0.015)
$Q^2(10)$	13.300 [0.207]	14.629 [0.102]	10.505 [0.397]	9.617 [0.382]	12.555 [0.250]	10.076 [0.434]
LLF	-148.720	-2828.457	-225.137	-2452.944	-189.584	-2445.295

Notes: h_t and λ_t is the return- and range-based conditional volatility, respectively. R_t and ε_t is the range and residual, respectively. *LLF* is the log likelihood function, p-values are in brackets and the numbers in parentheses are robust standard errors proposed by Bollerslev and Wooldridge (1992). $Q^2(10)$ is the statistics for serial correlation up to 10^{th} order in the squared standardized residuals.

Table 3. Static volatility equality test.

	Nasdaq	S&P 500	FTSE 100
Panel A: $H_0: \sigma_1^2 = \sigma_2^2$			
σ_1^2	1.241	0.801	0.965
σ_2^2	6.049	6.058	4.487
$H_1: \sigma_1^2 \neq \sigma_2^2$	[<0.001]	[<0.001]	[<0.001]
$H_1: \sigma_1^2 < \sigma_2^2$	[<0.001]	[<0.001]	[<0.001]
$H_1: \sigma_1^2 > \sigma_2^2$	[1.000]	[1.000]	[1.000]
Panel B: $H_0: R_1 = R_2$			
R_1	1.330	1.122	1.207
R_2	2.502	2.625	2.537
$H_1: R_1 \neq R_2$	[<0.001]	[<0.001]	[<0.001]
$H_1: R_1 < R_2$	[<0.001]	[<0.001]	[<0.001]
$H_1: R_1 > R_2$	[1.000]	[1.000]	[1.000]

Notes: The numbers in brackets are p-values. σ_1^2 and R_1 are the unconditional variance and mean range for February 2003 to August 2008, respectively. σ_2^2 and R_2 are the unconditional variance and mean range for September 2008 to January 2010, respectively.

Table 4. CARR/GARCH model with dummy variable for daily Nasdaq, S&P 500 and FTSE 100 index (2003.2.3-2010.1.29).

CARR-model with dummy:

$$R_t = \lambda_t \varepsilon_t$$
$$\lambda_t = \omega_1 + \alpha_1 R_{t-1} + \beta_1 \lambda_{t-1} + \omega_2 D_1 + \alpha_2 D_1 R_{t-1} + \beta_2 D_1 \lambda_{t-1}$$

GARCH-model with dummy:

$$r_t = \sqrt{h_t}\varepsilon_t$$
$$h_t = \omega_1 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1} + \omega_2 D_1 + \alpha_2 D_1 \varepsilon_{t-1}^2 + \beta_2 D_1 h_{t-1}$$

	Nasdaq		S&P 500		FTSE 100	
	CARR	GARCH	CARR	GARCH	CARR	GARCH
$\hat{\omega}_1$	0.002	0.015	0.003	0.011	0.005	0.009
	(<0.001)	(0.001)	(0.001)	(<0.001)	(<0.001)	(<0.001)
$\hat{\alpha}_1$	0.083	0.055	0.083	0.061	0.149	0.098
	(0.014)	(0.006)	(0.010)	(0.006)	(0.001)	(0.011)
$\hat{\beta}_1$	0.894	0.935	0.876	0.925	0.780	0.895
	(0.013)	(0.005)	(0.014)	(0.005)	(0.001)	(0.010)
$\hat{\omega}_2$	0.053	0.193	0.045	0.243	0.037	0.562
	(0.002)	(0.083)	(0.002)	(0.018)	(0.004)	(0.029)
$\hat{\alpha}_2$	- 0.266	- 0.091	- 0.222	- 0.108	- 0.403	- 0.264
	(0.031)	(0.007)	(0.090)	(0.007)	(0.004)	(0.021)
$\hat{\beta}_2$	0.159	0.110	0.086	0.154	0.353	0.269
	(0.008)	(0.031)	(0.015)	(0.003)	(0.115)	(0.017)
LLF	- 2346.861	- 4193.564	- 586.749	- 3764.242	- 4259.131	- 6958.667

Notes: λ_t is the range-based conditional volatility and R_t is the range. D_1 is the time break dummy variable for the financial credit crisis. Before 31, Aug., 2008, D_1 is 0. After 1, Sep., 2008, D_1 is 1. *LLF* is the log likelihood function, and the number in parentheses are robust standard error proposed by Bollerslev and Wooldridge (1992).

Table 5. Markov-switching CARR/GARCH model for daily Nasdaq, S&P 500 and FTSE 100 index (2003.2.3-2010.1.29).

MS-CARR model:

$$R_t = \lambda_{s_t,t}\varepsilon_t$$
$$\lambda_{s_t,t} = \omega_{s_t} + \alpha_{s_t} R_{t-1} + \beta_{s_t}\lambda_{s_t,t-1}$$
$$p_{ij} = \Pr(s_t = j \mid s_{t-1} = i) \quad i, j = 1,2.$$

MS-GARCH model:

$$r_t = \sqrt{h_{s_t,t}}\,\varepsilon_t$$
$$h_{s_t,t} = \omega_{s_t} + \alpha_{s_t}\varepsilon_{t-1}^2 + \beta_{s_t} h_{s_t,t-1}$$
$$p_{ij} = \Pr(s_t = j \mid s_{t-1} = i) \quad i, j = 1,2.$$

	Nasdaq		S&P 500		FTSE 100	
	MS-CARR	MS-GARCH	MS-CARR	MS-GARCH	MS-CARR	MS-GARCH
LLF	- 150.022	- 2810.276	- 223.706	- 2427.914	- 192.699	- 2425.839
Smooth Regime						
$\hat{\omega}_1$	0.001	0.008	0.001	0.005	0.003	0.005
	(0.001)	(0.004)	(0.001)	(0.002)	(0.001)	(0.001)
$\hat{\alpha}_1$	0.028	0.037	0.024	0.046	0.091	0.022
	(0.016)	(0.007)	(0.016)	(0.008)	(0.027)	(0.006)
$\hat{\beta}_1$	0.950	0.955	0.951	0.946	0.860	0.965
	(0.025)	(0.008)	(0.029)	(0.008)	(0.042)	(0.007)
$\hat{p}_{11}$	0.999	0.999	0.999	0.999	0.998	0.993
	(0.247)	(0.127)	(0.211)	(0.125)	(0.272)	(0.084)
$\hat{\pi}_\infty^1$	0.796	0.943	0.784	0.969	0.772	0.880
Volatile Regime						
$\hat{\omega}_2$	0.013	0.203	0.008	0.139	0.011	0.571
	(0.008)	(0.054)	(0.006)	(0.083)	(0.010)	(0.125)
$\hat{\alpha}_2$	0.249	- 0.028	0.217	0.033	0.252	0.172
	(0.132)	(0.009)	(0.108)	(0.033)	(0.139)	(0.082)
$\hat{\beta}_2$	0.681	0.996	0.734	0.982	0.694	0.759
	(0.149)	(0.002)	(0.110)	(0.017)	(0.145)	(0.072)
$\hat{p}_{22}$	0.995	0.978	0.995	0.958	0.995	0.948
	(0.260)	(0.220)	(0.331)	(0.233)	(0.275)	(0.144)
$\hat{\pi}_\infty^2$	0.204	0.057	0.216	0.031	0.228	0.120

Notes: *LLF* is the log likelihood function, and the numbers in parentheses are robust standard errors proposed by Bollerslev and Wooldridge (1992). The stationary regime probabilities, π_∞^1 and π_∞^2, are computed by the expression: $\pi_\infty^1 = (1 - p_{22})/(2 - p_{11} - p_{22})$ and $\pi_\infty^2 = (1 - p_{11})/(2 - p_{11} - p_{22})$, respectively.

Table 6. Forecasting performance comparison for different VaR estimates at 99% confidence level.

	Descriptive Statistics		Conservatism	Accuracy		Efficiency
	Mean VaR	S.D. VaR	MRB	ABLF	LR_{uc}	MRSB
Nasdaq						
MS-CARR	- 3.971	1.637	- 0.127	0.017	1.957	- 0.089
CARR	- 4.120	1.753	- 0.100	0.020	1.923	- 0.029
MS-GARCH	- 4.718	2.122	0.022	0.013	1.986	0.017
GARCH	- 5.159	1.704	0.205	0.000	0.000	0.101
S&P 500						
MS-CARR	- 3.934	1.517	- 0.128	0.013	1.986	- 0.101
CARR	- 5.389	1.709	- 0.105	0.010	2.000	- 0.084
MS-GARCH	- 5.354	2.537	- 0.032	0.010	2.000	- 0.038
GARCH	- 4.615	1.659	0.265	0.003	1.761	0.223
FTSE 100						
MS-CARR	- 3.881	1.417	- 0.048	0.010	2.000	- 0.063
CARR	- 3.905	1.407	- 0.041	0.010	2.000	0.013
MS-GARCH	- 3.784	1.489	- 0.078	0.007	1.969	- 0.021
GARCH	- 4.559	1.385	0.167	0.007	1.969	0.071

Notes: The conservatism criterion of MRB is the mean relative bias, and the larger MRB statistic shows that the corresponding model is more conservatism. The accuracy criteria of ABLF and LR_{uc} denote the average binary loss function and the LR test of unconditional coverage. The efficiency criterion of MRSB denotes the mean relative scaled bias, and the smaller MRSB statistic means that the corresponding model is more efficiency. The critical value of LR_{uc} statistic at 1% significance level is 6.63.

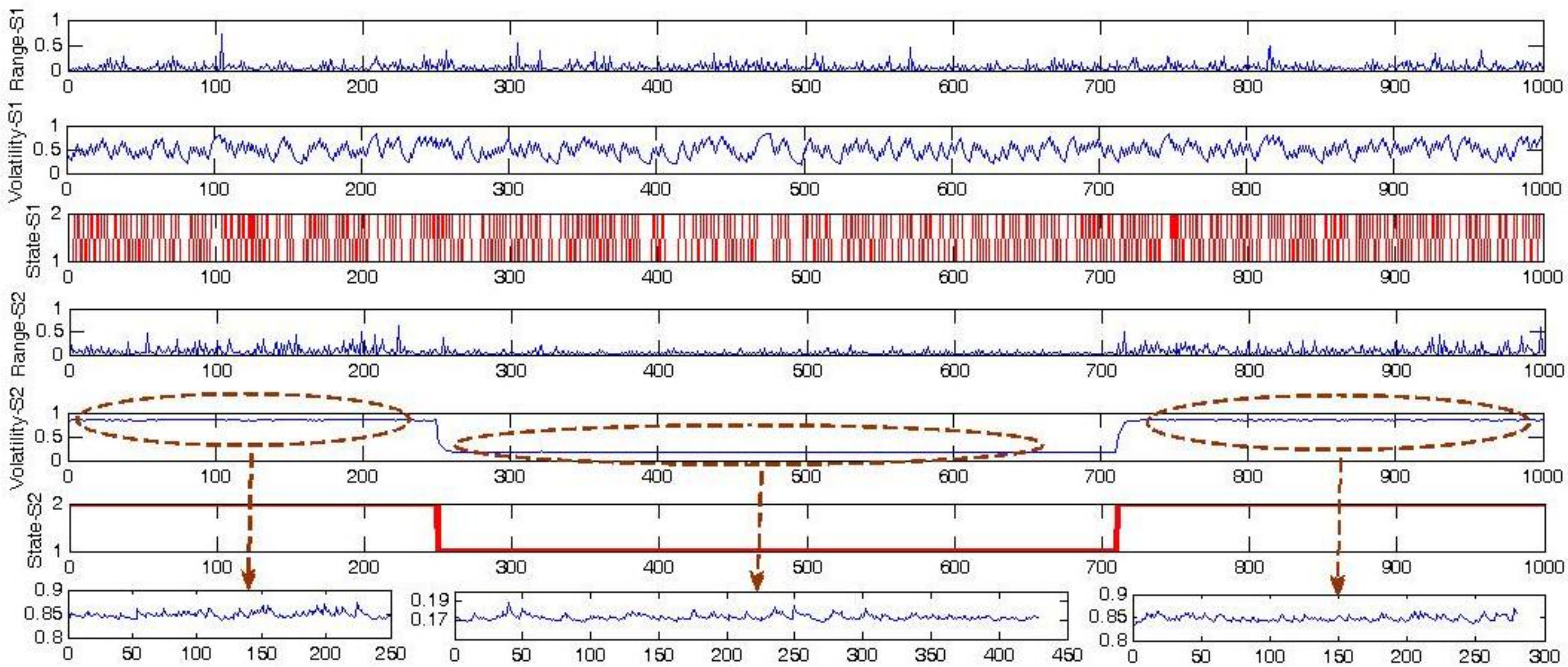

Figure 1: Markov-switching CARR simulation for different degrees of Markov effect.
There are two simulated scenarios including scenario 1 (S1) and scenario 2 (S2). S1 and S2 denote the lower- and higher-degree of Markov effect, respectively. Top panel is the simulated range series for S1. Second panel is the simulated volatility for S1. Third panel is the simulated state process for S1. Fourth panel is the simulated range series for S2. Fifth panel is the simulated volatility for S2. Sixth panel is the simulated state process for S2. Bottom panel is the amplification of simulated volatility process for S2 in different states.

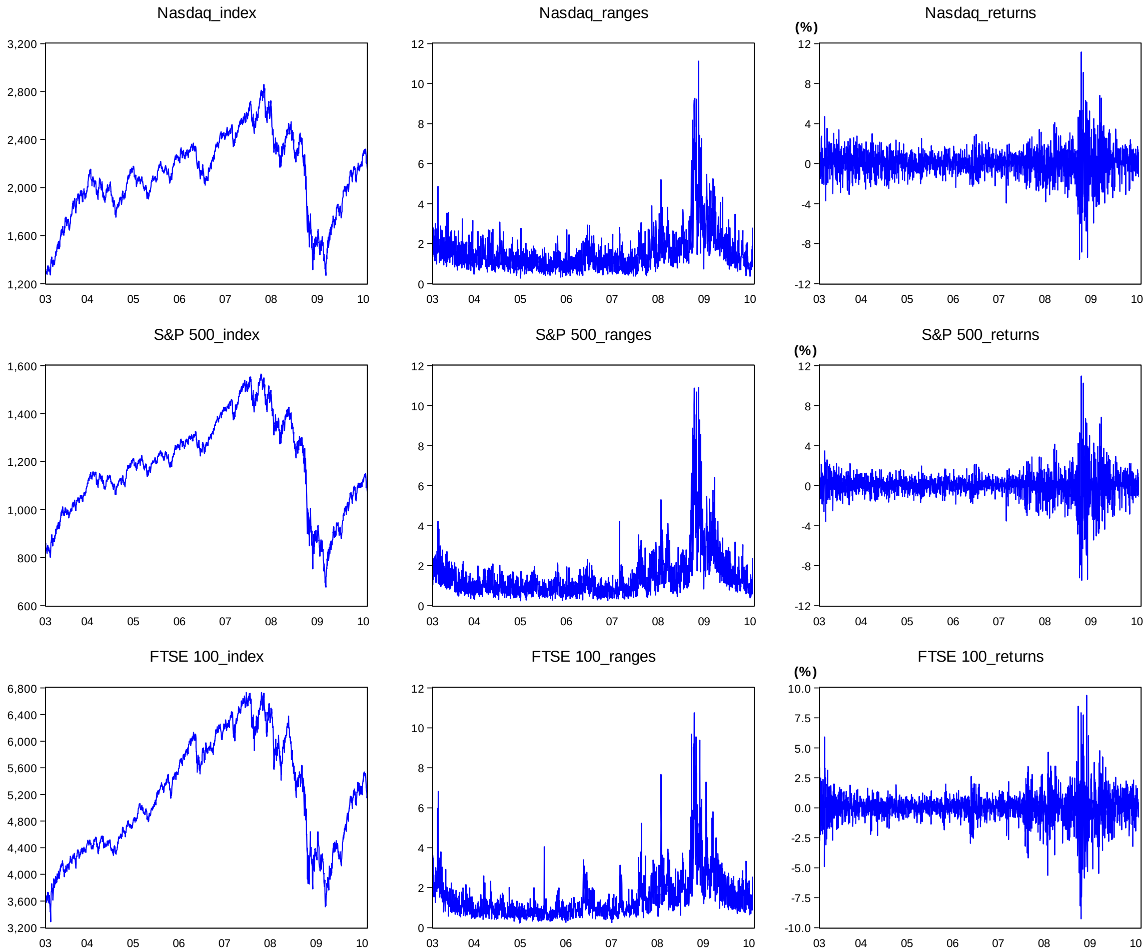

Figure 2: The daily index, ranges, and returns for Nasdaq, S&P 500 and FTSE 100 index (2003.2.3-2010.1.27)

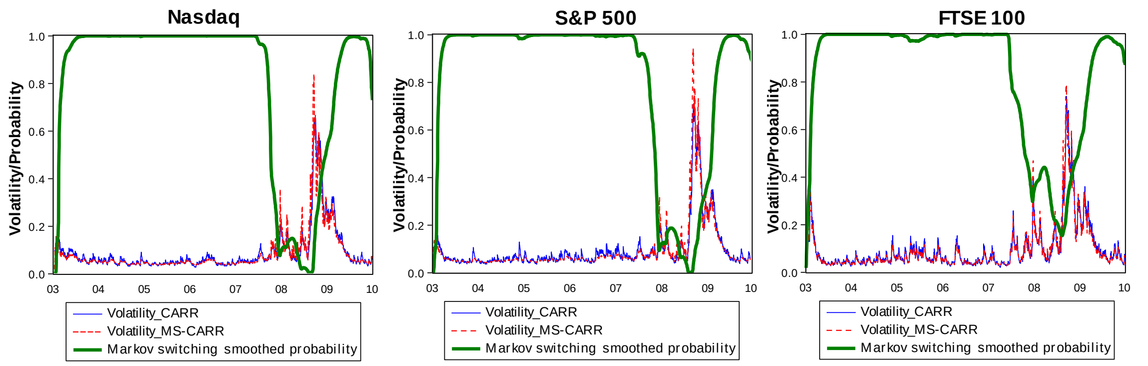

Figure 3: The range-based (CARR) volatility pattern, Markov switching range-based volatility pattern and smoothed probability for smooth volatility regime for Nasdaq, S&P 500 and FTSE 100 index (2003.2.3-2010.1.29).

This figure plots the estimated range-based volatility (thin solid line) pattern, the estimated Markov switching range-based volatility (dashed line) pattern and smoothed probability (thick solid line) at a daily frequency. The estimated range-based volatility is modeled by the CARR model. The estimated Markov switching range-based volatility and the smoothed probability are modeled by the MS-CARR model.

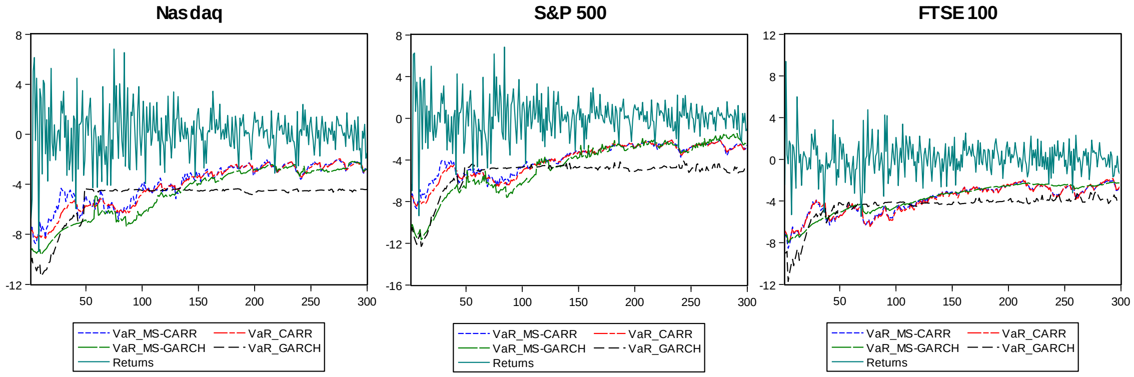

Figure 4: 1% daily VaR estimates and daily returns for Nasdaq, S&P 500 and FTSE 100 index (2009.1.30-2010.1.29).

This figure plots the different daily VaR estimates and daily returns at 1% significant level. VaR_MS-CARR and VaR_MS-GARCH are the VaR estimates calculated by the MS-CARR and MS-GARCH model, respectively. VaR_CARR and VaR_GARCH are the VaR estimates calculated by the CARR and GARCH model, respectively.