

Commodity derivatives pricing with an endogenous convenience yield market price of risk*

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ABSTRACT

We develop a partial equilibrium model of the term structure of storable commodity futures and options on futures, where the stochastic movements of the convenience yield are acknowledged. Moreover, as specified in recent papers about commodity derivatives pricing, interest rates and risk premia of primitives assets are assumed to evolve randomly over time. However, contrary to the existing literature, the risk premium of the convenience yield is derived endogenously. This framework makes it possible to analyze agent preference structure and investment horizon impact on this premium. Finally, closed form solutions for the prices of futures and options on futures obtain, making our model useful for practical commodity risk management.

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A growing literature is devoted to the pricing and hedging of commodity assets. We focus in this paper on storable commodities. Storable commodities can be held for a consumption purpose and this option generates a convenience yield¹. This stylized fact must be taken into account for modeling. Existing models can be classified into two categories, Mellios (2007). In the first category, models rely on an equilibrium approach to derive this convenience yield endogenously. The second category, termed reduced-form models, takes the convenience yield as given, along with other factors such as interest rates.

In the former category, models, such as Brennan (1958), Deaton and Laroque (1992), Litzenberger and Rabinowitz (1995), and Routledge, Seppi, and Spatt (2000), put light on the relations between supply, demand and storage to justify the specific features of storable commodity assets, spot and derivatives, for instance the endogenous convenience yield, its mean reversion movement and backwardation. However, these models suffer from 1) a lack of tractability necessary to efficiently price and hedge commodity derivatives and 2) the assumption that interest rates are nil or at least not stochastic, and 3) a restrictive analysis of risk premia. In particular, these models are silent about the convenience risk premium. Since the convenience yield is stochastic (see Fama and French (1988)), a risk premium must be introduced and as this convenience yield is imperfectly correlated with the spot price of the commodity, the former bears a specific risk premium.

The second category of models aims at pricing commodity derivatives and therefore fully describes risk premia. However, since there is no primitive assets perfectly correlated with the convenience yield, the risk premium of the latter is exogenously specified. So far as commodity derivatives pricing, *stricto sensu* is involved, specifying the risk premium is not compulsory since pricing is achieved under the risk neutral probability measure, nevertheless, if one resorts to empirical analysis or for hedging purposes, the specification of the model's risk premia is necessary.

¹See, Kaldor (1939), Working (1948, 1949), Telser (1958), and Brennan (1958) for a thorough explanation of the convenience yield and the underlying theory of storage.

The first model that falls in this category and where the stochastic movements of the convenience yield are acknowledged is that of Gibson and Schwartz (1990). Improvements of the Gibson and Schwartz model have been achieved in several directions. Schwartz (1997) used stochastic interest rates, Yann (2002) introduced jumps and stochastic volatility, and Richter and Sorensen (2007) took into account of the seasonality. However, contrary to the convenience yield, all these extensions are not typical to commodity markets in that they also characterize other financial markets. In the Schwartz model, the risk premium of the convenience yield is assumed constant. This assumption is relaxed in the recent work of Casassus and Collin-Dufresne (2005) and Richter and Sorensen (2007) who acknowledge that the convenience yield varies over time. However, the authors rely on an ad-hoc exogenous functional form that embeds agent preferences to ensure tractability.

The present paper belongs to the second class of models where the stochastic movements of the underlying spot commodity, the convenience yield and the interest rate are exogenously specified. However, contrary to the existing literature, our model derives endogenously the risk premium of the convenience yield through equilibrium on the spot and contingent claim commodity markets. Since we eventually focus on the pricing of commodity derivatives, we postulate that our investor is a price taker on the interest rate market since he (she) cannot, through holding a single commodity, affect prices in the interest rate market.

First, this paper assesses the impact of the stochastic spot price of the underlying commodity as well as the stochastic interest rate on the risk premium of the convenience yield. Second, the impact of investors' preference and investment horizon on the risk premium of the convenience yield, is explicitly assessed. Third, By deriving the functional form of the risk premium of the convenience yield endogenously, this article tells under which specifications of investors' preferences and that of the underlying state variables, namely the commodity spot price, the convenience yield and the interest rate, the exogenous functional form of this risk premium assumed in Casassus and Collin-Dufresne (2005) is valid.

Indeed, when investors' preferences are assumed to exhibit constant relative risk aversion, while the risk premia of the spot commodity and the interest rate are affine in the state variables, the risk premium of the convenience yield is also an affine function of these three state variables. Fourth, we derive closed form formula for commodity futures and commodity futures options, a feature that is of a very practical interest for commodity risk management. In particular, this paper is the first² to examine investors' risk preferences with respect to the pricing of commodity contingent claims.

The remainder of the paper is organized as follow. In Section I, the general underlying economic framework is described. Section II derives and analyzes the convenience yield risk premium as well as the partial differential equation of a commodity contingent claim under general market and preference structure. Section III examines the convenience yield market price of risk under constant relative risk aversion and affine state variables framework. Section IV is devoted to the pricing of futures and options on futures under this specialized framework. Section V offers some conclusion and possible extensions.

1 The economy

Consider a continuous-time economy in which the underlying uncertainty is generated by a complete filtered probability space $(\Omega, \mathbf{F}, \mathbf{P})$. We also consider a three dimensional imperfectly correlated Wiener process $Z(t) = (Z_X(t), Z_r(t), Z_\delta(t))$. In addition, we represent the augmented filtration of the paths of $Z(t)$ by $\mathbf{F} = \{\mathcal{F}_t : t \in [0, T]\}$, with T designates the terminal date of the economy.

Let $S(t)$ the spot price of a storable commodity, $\delta(t)$ is the convenience yield linked to this commodity, and $r(t)$ designates the instantaneous interest rate. Furthermore, we define Let $X(t) \equiv \ln(S(t))$. General diffusion processes are assumed for the spot commodity, convenience yield and the interest rate. In addition, since our investors in commodity assets are supposed to be price takers on the interest rates market, the dynamics of the interest

²To the authors'knowledge.

rates is not affected by the the other state variables, namely, the (log) spot price of the commodity and the convenience yield :

$$dX(t) = \left(\mu_X(Y, t) - \delta(t) \right) dt + \sigma_X(Y, t) dZ_X(t) \quad (1)$$

$$dr(t) = \mu_r(r, t) dt + \sigma_r(r, t) dZ_r(t) \quad (2)$$

$$d\delta(t) = \mu_\delta(Y, t) dt + \sigma_\delta(Y, t) dZ_\delta(t) \quad (3)$$

with $Y'(t) = (X(t), r(t), \delta(t))$ the vector of the state variables and $'$ the transpose symbol. The dynamics of the state variables can be stated in matrix form as follows :

$$dY(t) = \mu_Y(Y, t) dt + \Sigma_Y'(Y, t) dZ(t) \quad (4)$$

where $\mu_Y(Y, t)$ is a 3×1 vector with $\mu_Y(Y, t)' = (\mu_X(Y, t) - \delta(t), \mu_r(r, t), \mu_\delta(Y, t))$, $\sigma_Y(Y, t)$ is 3×3 diagonal matrix with $\Sigma_Y(Y, t) = \begin{pmatrix} \sigma_X(Y, t) & 0 & 0 \\ 0 & \sigma_r(r, t) & 0 \\ 0 & 0 & \sigma_\delta(Y, t) \end{pmatrix}$.

$Z_X(t), Z_r(t)$ and $Z_\delta(t)$ are correlated Wiener processes with correlation matrix given by :

$$\rho(Y(t)) = \begin{pmatrix} 1 & \rho_{rX}(Y, t) & \rho_{\delta X}(Y, t) \\ \rho_{Xr}(Y, t) & 1 & \rho_{\delta r}(Y, t) \\ \rho_{X\delta}(Y, t) & \rho_{r\delta}(Y, t) & 1 \end{pmatrix} \quad (5)$$

We decorrelate the Wiener processes $Z(t) = (Z_X(t), Z_r(t), Z_\delta(t))'$ with a Cholesky transformation , to obtain : $z(t) = (z_X(t), z_u(t), z_v(t))'$ given by $z(t) = \varpi^{-1}(Y, t) Z(t)$ with

$$\varpi(t, Y) = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{Xr}(Y, t) & \rho_{ur}(Y, t) & 0 \\ \rho_{X\delta}(Y, t) & \rho_{u\delta}(Y, t) & \rho_{v\delta}(Y, t) \end{pmatrix} \quad (6)$$

such that $\rho(t, Y) = \varpi(t, Y)\varpi(t, Y)'$ with :

$$\begin{aligned}\rho_{ur}(Y, t) &= \sqrt{1 - \rho_{Xr}^2(Y, t)} \\ \rho_{u\delta}(Y, t) &= \frac{\rho_{r\delta}(Y, t) - \rho_{Xr}(Y, t)\rho_{X\delta}(Y, t)}{\rho_{ur}(Y, t)} \\ \rho_{v\delta}(Y, t) &= \frac{\sqrt{1 - \rho_{Xr}^2(Y, t) - \rho_{X\delta}^2(Y, t) - \rho_{r\delta}^2(Y, t) + 2\rho_{Xr}(Y, t)\rho_{X\delta}(Y, t)\rho_{r\delta}(Y, t)}}{\rho_{ur}(Y, t)}\end{aligned}$$

with $z_u(t)$ and $z_v(t)$ the idiosyncratic risk pertained to the interest rate and the convenience yield, respectively.

We define the risk premia associated with the state variables by $\Lambda(*, t) = (\lambda_X(Y, t), \lambda_r(r, t), \lambda_\delta(*, t))$.³

The orthogonal risk premium associated with $z(t)$ is defined as follows : $\lambda(t) = \varpi^{-1}(Y, t)\Lambda(t)$

$$\lambda_v(*, t) = \frac{\rho_{Sr}(Y, t)\rho_{u\delta}(Y, t) - \rho_{S\delta}(Y, t)\rho_{ur}(Y, t)}{\rho_{ur}(Y, t)\rho_{v\delta}(Y, t)}\lambda_X(Y, t) - \frac{\rho_{Sr}(Y, t)}{\rho_{ur}(Y, t)\rho_{v\delta}(Y, t)}\lambda_r(r, t) + \frac{1}{\rho_{v\delta}(Y, t)}\lambda_\delta(*, t) \quad (7)$$

Eq. (7) states that finding the market price of risk of the convenience yield is equivalent to finding the market price of risk of its idiosyncratic risk because the other parameters are exogenous variables.

We consider an investor that, in addition to the spot commodity market, has access to the interest rate market where she can invest the proportions $\pi_\beta(t)$ and $\pi_B(t)$ of the time- t wealth $W(t)$ in a money market account and a zero-coupon bond, respectively.⁴ Moreover, our agent can invest $\pi_G(t)$ in a contingent claim asset written on the spot commodity. Thus, we have $1 = \pi_S(t) + \pi_\beta(t) + \pi_B(t) + \pi_G(t)$. Let $\pi(t)$ represents the vector of proportions of the time- t wealth $W(t)$ invested in risky assets, i.e. $\pi(t) = (\pi_S(t), \pi_B(t), \pi_G(t))'$. Hence, the proportion invested in the risk-free asset is $1 - \pi(t)'1_3$.

The prices of the money market account $\beta(t)$ and the zero-coupon bond $B(t, T_B)$ are

³Because the risk premium of the convenience yield depend on investor aggregate wealth, which will be defined below, we let its functional form unspecified for now.

⁴For presentation clarity, we drop in the sequel the dependence on state variables and wealth.

governed by the following dynamics :

$$d\beta(t) = \beta(t)r(t)dt \quad (8)$$

$$dB(t, T_B) = B(t, T_B) \left[\mu_B(t, T_B)dt + \sigma_B(t, T_B)dZ_r(t) \right] \quad (9)$$

with $\beta(0) = 1$. T_B denotes the maturity of the zero-coupon bond such that $B(T_B, T_B) = 1$. Let $G(t, Y, W, T_G)$ denotes the time- t price of the contingent claim written on the spot commodity. We, further, assume that this contingent claim is not marked-to market. Its dynamics is as follows :

$$dG(t, Y, W, T_G) = G(t, Y, W, T_G) \left[\mu_G(t, Y, W, T_G)dt + \Sigma_G(t, Y, W, T_G)'dZ(t) \right] \quad (10)$$

where $\mu_G(t, Y, W, T_G)$ denotes the instantaneous anticipated change in the contingent claim and $\Sigma_G(t, Y, W, T_G)$ represents a 3×1 vector of volatilities with respect to the state variables. Specifically, $\Sigma_G(t, Y, W, T_G)' = (\sigma_{GX}(t, Y, W, T_G), \sigma_{Gr}(t, Y, W, T_G), \sigma_{G\delta}(t, Y, W, T_G))$. For later references, we summarize the dynamics of the three assets, the spot commodity, the zero-coupon bond and the contingent claim, as follows :

$$dA(t) = I_A(t) \left[\mu_A(t)dt + \Sigma_A'(t)dZ(t) \right] \quad (11)$$

where $I_A(t)$ is a 3×3 diagonal matrix with $I_A(t) = \begin{pmatrix} S(t) & 0 & 0 \\ 0 & B(t, T_B) & 0 \\ 0 & 0 & G(t, Y, W, T_G) \end{pmatrix}$,

$\mu_A(t)$ is a 3×1 vector with $\mu_A(t) = (\mu_X(t) + \frac{1}{2}\sigma_S^2(t) - \delta(t), \mu_B(t, T_B), \mu_G(t, Y, W, T_G)')$,

and $\Sigma_A(t)$ is a 3×3 matrix with $\Sigma_A(t) = \begin{pmatrix} \sigma_X(t) & 0 & \sigma_{GX}(t, Y, W, T_G) \\ 0 & \sigma_B(t, T_B) & \sigma_{Gr}(t, Y, W, T_G) \\ 0 & 0 & \sigma_{G\delta}(t, Y, W, T_G) \end{pmatrix}$. Since

our goal is to identify the components of the idiosyncratic part of the convenience yield risk premium, $\lambda_v(t)$, we apply the preceding orthogonal change of basis for the different

volatility matrices as follows :

$$\sigma_A(t) = \varpi'(t)\Sigma_A(t) \quad (12)$$

$$\sigma_Y(t) = \varpi'(t)\Sigma_Y(t) \quad (13)$$

Given the dynamics of G , X , B and β as well as the self-financing condition, the dynamics of the wealth constraint writes :

$$dW(t) = W(t) \left[\left(r(t) + \underline{\sigma}_W(t)' \lambda(t) \right) dt + \underline{\sigma}_W(t)' dz(t) \right] \quad (14)$$

where

$$\underline{\sigma}_W(t) = \sigma_A(t)\pi(t) \quad (15)$$

Our agent maximizes the expected utility of her wealth over an investment horizon T_I :

$$\begin{aligned} & \max_{\pi(t)} \mathbf{E}_t \left(U(W_{T_I}) \right) \\ \text{s.t.} \quad & dW(t) = W(t) \left[\left(r(t) + (\sigma_A(t)\pi(t))' \lambda(t) \right) dt + (\sigma_A(t)\pi(t))' dz(t) \right] \end{aligned} \quad (16)$$

2 Commodity contingent claim pricing

To solve the problem in Eq. (16), we use the dynamic programming approach where we define the indirect function $J^*(t, Y, W) \equiv \max_{\pi(t)} \mathbf{E}_t \left(U(W_{T_I}) \right)$ which solves the following Hamilton-Jacobi-Bellman equation :

$$\max_{\pi(t)} \mathcal{D}J(t, Y, W) = 0 \quad (17)$$

with \mathcal{D} denotes the Dynkin operator.

$$\begin{aligned} & J_t + W(t)J_W(t)r(t) + W(t)J_W(t)(\sigma_A(t)\pi(t))' \lambda(t) + \frac{1}{2}W^2(t)J_{WW}(t)(\sigma_A(t)\pi(t))' \sigma_A(t)\pi(t) \\ & + J_Y'(t)\mu_Y(t) + \frac{1}{2}Tr(J_{YY}(t)\sigma_Y'(t)\sigma_Y(t)) + W(t)(\sigma_A(t)\pi(t))' \sigma_Y(t)J_{WY}(t) = 0 \end{aligned} \quad (18)$$

Applying first order condition we obtain the optimal demand for risky assets :

$$\pi^*(t) = \frac{-J_W(t)}{W(t)J_{WW}(t)} \sigma_A^{-1}(t)\lambda(t) + \sigma_A^{-1}(t)\sigma_Y(t) \frac{-J_{WY}(t)}{W(t)J_{WW}(t)} \quad (19)$$

Inversing the preceding equation, we can deduce the orthogonal market prices of risk as a function of optimal risky proportions :

$$\lambda(t) = -\frac{W(t)J_{WW}(t)}{J_W(t)}\sigma_A(t)\pi^*(t) - \sigma_Y(t)\frac{J_{WY}(t)}{J_W(t)} \quad (20)$$

The equilibrium on the contingent claim market, the market clearing condition, states that $\pi_G^*(t) = 0$. Therefore, we obtain the following proposition :

Proposition 1 *The idiosyncratic part of the convenience yield's risk premium is given by :*

$$\lambda_v(t) = \sigma_{\delta v}(t)\frac{-J_{W\delta}(t)}{J_W(t)} \quad (21)$$

where $\sigma_{\delta v}(t) = \sigma_{\delta}(t)\rho_{v\delta}(t)$ denotes the idiosyncratic component of the convenience yield volatility.

We notice from Eq. (21) that the agent's preferences affect the idiosyncratic part of the risk premium of the convenience yield. In addition, Eq. (21) states that the idiosyncratic market price of risk of the convenience yield is, whatever is the preference of the representative agent, proportional to the idiosyncratic volatility of the convenience yield, $\sigma_{\delta v}(t)$. *Ceteris paribus*, the more the convenience yield is volatile, the more this market price of risk is high. In the same manner, everything else being equal, less correlated the convenience yield is with the primitive assets, the higher is $\rho_{v\delta}(t)$, and the higher is $\lambda_v(t)$. In the extreme case, when the convenience yield is perfectly correlated with the primitive assets or when the volatility of the convenience yield is nil, $\sigma_{\delta v}(t) = 0$, and the orthogonal market price of risk of the convenience yield is nil $\lambda_v(t) = 0$. Moreover, $\lambda_v(t) = 0$ is (negatively) proportional to the sensitivity of marginal utility of wealth to the movements of the convenience yield, $\frac{-J_{WY}(t)}{J_W(t)}$. Indeed, the value function, $J(t)$, stands for the total time t satisfaction of agents in this economy, so $J_W(t)$, the marginal utility of wealth, is the satisfaction gained by the last unity of wealth. Therefore, $\frac{-J_{WY}(t)}{J_W(t)}$ measures the opposite of the relative influence of the convenience yield on this satisfaction. For example, when $J_{W\delta}(t) > 0$ and since $\frac{J_{WY}(t)}{J_W(t)}$ is also positive because obviously $J_W(t)$ is always positive too, a rise (drop) in the

convenience yield increases (decreases) globally the representative agent's satisfaction and, therefore, the orthogonal market price of risk the convenience yield is negative (positive) due to the fact that $\sigma_{\delta v}(t)$ is always positive. Conversely when $J_{W\delta}(t) > 0$, the convenience yield and the agent's satisfaction move in opposite directions. Moreover, when the impact of the convenience yield on aggregate satisfaction is low (high) in absolute value, that is, when $\frac{-J_{WY}(t)}{J_W(t)}$ is low (high) in absolute value, the orthogonal market price of risk will also be low (high) in absolute value. Finally, we can conclude that the absolute of the convenience yield market price of risk is proportional to the volatility of the convenience yield, to how much the convenience yield is poorly correlated with primitive assets, and to its influence, in absolute value, on market aggregate satisfaction. However, its sign is opposite to that of the influence the convenience yield on market satisfaction.

The (vector) volatility of the agent (Eq. (15)) can be rewritten, using Eq. (19) in terms of the exogenous market prices of idiosyncratic risk :

$$\sigma_W(t) = \frac{-J_W(t)}{J_{WW}(t)}\lambda_e(t) + M\sigma_Y(t)\frac{-J_{WY}(t)}{J_{WW}(t)} \quad (22)$$

where $\lambda_e(t) \equiv (\lambda_X(t), \lambda_u(t), 0)'$ represents a 3×1 vector of prices of the orthogonal market prices of risks pertained to the primitive assets, and $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Note that, because

of the contingent claim market clearing condition, we check that volatility of optimal wealth is perfectly correlated with primitive assets. Moreover, substituting Eq. (22) into (14) gives the dynamics of the wealth where only primitive assets are used, i.e. innovations in wealth are governed by the following stochastic differential equation :

$$dW(t) = W(t) \left[\left(r(t) + \sigma'_W(t)\lambda(t) \right) dt + \sigma'_W(t)dz(t) \right] \quad (23)$$

Let \mathbf{P} the historical probability measure. \mathbf{Q} , the risk neutral probability associated with $\beta(t)$, is uniquely defined by the following Radon-Nikodym derivative :

$$\frac{d\mathbf{Q}}{d\mathbf{P}} \Big|_{\mathcal{F}_t} = \exp \left(-\frac{1}{2} \int_0^t \lambda'(s)\lambda(s)ds - \int_0^t \lambda'(s)dz(s) \right) \quad (24)$$

Since it depends on $\lambda_v(t)$, the risk neutral probability measure embeds investors' risk preferences and investment horizon. Using the Girsanov theorem we give the expressions of the drifts of the state variables and the aggregate wealth under the probability measure \mathbf{Q} :

$$\mu_Y^Q(t) = \mu_Y(t) - \sigma_Y'(t)\lambda(t) \quad (25)$$

$$\mu_W^Q(t) = r(t) \quad (26)$$

Now, we can give the fundamental equations of a contingent claim price that is not marked-to-market, $G(t, Y, W, T_G)$, and of a marked-to-market contingent claim, such as futures, price, $H(t, Y, W, T_H)$:

$$\begin{aligned} G_t(t) + G_Y'(t)\mu_Y^Q(t) + G_W(t)r(t)W(t) + \frac{1}{2}Tr[G_{YY}(t)\sigma_Y'(t)\sigma_Y(t)] \\ + \frac{1}{2}G_{WW}(t)\sigma_W'(t)\sigma_W(t) + G_{YW}(t)\sigma_Y'(t)\sigma_W(t) - G(t)r(t) = 0 \end{aligned} \quad (27)$$

with terminal condition $G(T_G, Y, W, T_G) = P_G(Y, W, T_G)$.

$$\begin{aligned} H_t(t) + H_Y'(t)\mu_Y^Q(t) + H_W(t)r(t)W(t) + \frac{1}{2}Tr[H_{YY}(t)\sigma_Y'(t)\sigma_Y(t)] \\ + \frac{1}{2}H_{WW}(t)\sigma_W'(t)\sigma_W(t) + H_{YW}(t)\sigma_Y'(t)\sigma_W(t) = 0 \end{aligned} \quad (28)$$

with terminal condition $H(T_H, Y, W, T_H) = P_H(Y, W, T_H)$.

So, in order to compute contingent claims prices, see Eqs. (27) and (28), we must compute the indirect utility function, which requires solving a four dimensional PDE, and solve for another four dimensional PDE to get the prices. We see below that by making assumptions on the preference structure of investors and on the underlying state variables dynamic, we can considerably improve the tractability of our model.

3 The case of CRRA preferences

We assume in this section that our investor exhibits constant relative risk aversion. Her utility function is then given by :

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma} \quad (29)$$

where γ defines her constant relative risk aversion. It is well-known that under this specification, the value function is then separable in aggregate wealth and state variables :

$$J(t, Y, W) = \frac{W^{1-\gamma}}{1-\gamma} \exp\left(\gamma h(\tau, Y)\right), \quad \tau = T_I - t \quad (30)$$

We rewrite Eq. (18) in terms of the exogenous parameters only, and use Eq. (30) as well as the results of Eq. (19). We obtain :

$$\begin{aligned} h_t(t) &= kr(t) + \frac{1}{2}g\lambda'_e(t)\lambda_e(t) + (k\lambda'_e(t)\sigma_Y(t) + \mu_Y(t))h_Y(t) \\ &+ \frac{1}{2}h'_Y(t)\sigma'_Y(t)M\sigma_Y(t)h_Y(t) + \frac{1}{2}Tr[h_{yy}(t)\sigma'_Y(t)\sigma_Y(t)] \end{aligned} \quad (31)$$

with the initial condition $h(0, Y) = 0$, and $k \equiv \frac{1-\gamma}{\gamma}$, $g \equiv \frac{1-\gamma}{\gamma^2}$.

Using Eq. (30) and Proposition (1), the expression of the idiosyncratic part of the convenience yield reduces to :

$$\lambda_v(t) = -\sigma_{\delta v}(t)\gamma h_{\delta}(t) \quad (32)$$

Therefore, the idiosyncratic risk premium of the convenience yield does not depend on aggregate wealth. Because of this feature, the contingent claim prices are solely functions of the state variables. Thus, Eqs. (27) and (28) become :

$$G_t(t) + G'_Y(t)\mu_Y^Q(t) + \frac{1}{2}Tr[G_{YY}(t)\sigma'_Y(t)\sigma_Y(t)] - G(t)r(t) = 0 \quad (33)$$

and,

$$H_t(t) + H'_Y(t)\mu_Y^Q(t) + \frac{1}{2}Tr[H_{YY}(t)\sigma'_Y(t)\sigma_Y(t)] = 0 \quad (34)$$

respectively.

We assume that the underlying dynamics of state variables are affine, see Casassus and Collin-Dufresne (2005). Specifically, we assume that this dynamics are Gaussian affine, but, this assumption is not binding and our findings also work with other affine frameworks. We consider the following dynamics :

$$dX(t) = \left(r(t) - \delta(t) + \sigma_X(\lambda_{X0} + \lambda_{XX}X(t) + \lambda_{Xr}r(t) + \lambda_{X\delta}\delta(t) - \frac{1}{2}\sigma_X^2) \right) dt + \sigma_X dz_X(t) \quad (35)$$

$$dr(t) = \alpha(\theta - r(t))dt + \sigma_r dz_r(t), \quad \lambda_r(t, r) = \lambda_{r0} + \lambda_{rr}r(t) \quad (36)$$

$$d\delta(t) = \kappa(\bar{\delta} - \delta(t))dt + \sigma_\delta dz_\delta(t) \quad (37)$$

In Eq. (35) the instantaneous volatility of the (log) spot commodity price is assumed to be constant whereas the market price of risk is an affine function of the state variables (see Casassus and Collin-Dufrene (2005)). Eq. (36) is the classical Vasicek (1977) dynamics of the short term interest rate. However, the interest rate risk premium is assumed to depend on the level of interest rates. Finally, Eq. (37) specifies the dynamics of the convenience yield which is assumed to follow a mean reverting process (see Fama and French (1988), Gibson and Schwartz (1990), Schwartz (1997) and Collin-Dufrene (2005)). Nevertheless, we haven't specified the convenience yield risk premium since it is endogenously obtained in our approach as it will be shown in the sequel.

We specify the exogenous market prices of risk as well as the drifts of the state variables in matrix using the affine structure :

$$\lambda_e(t) = \lambda_{e0} + \lambda_{eY}Y(t) \quad (38)$$

$$\text{with } \lambda_{e0} = \begin{pmatrix} \lambda_{X0} \\ -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{X0} + \frac{1}{\rho_{ur}}\lambda_{r0} \\ 0 \end{pmatrix} \text{ and } \lambda_{eY} = \begin{pmatrix} \lambda_{XX} & \lambda_{Xr} & \lambda_{X\delta} \\ -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{XX} & -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{Xr} + \frac{1}{\rho_{ur}}\lambda_{rr} & -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{X\delta} \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\mu_Y = \mu_{Y0} + \mu_{YY}Y(t) \quad (39)$$

$$\text{where } \mu_{Y0} = \begin{pmatrix} \sigma_X \lambda_{X0} - \frac{1}{2} \sigma_X^2 \\ \alpha \theta \\ \kappa \bar{\theta} \end{pmatrix} \text{ and } \mu_{YY} = \begin{pmatrix} \sigma_X \lambda_{XX} & \sigma_X \lambda_{Xr} + 1 & \sigma_X \lambda_{X\delta} - 1 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\kappa \end{pmatrix}.$$

Moreover, the (conditional) Gaussian structure requires that not only the instantaneous volatilities be constant, but also the instantaneous correlation between state variables. This implies that ϖ does not depend on the state variables and that the instantaneous covariances are constant. The instantaneous volatility matrix of state variables is then given in the orthogonal basis :

$$\sigma_Y = \begin{pmatrix} \sigma_X & \sigma_r \rho_{Xr} & \sigma_\delta \rho_{X\delta} \\ 0 & \sigma_r \rho_{ur} & \sigma_\delta \rho_{u\delta} \\ 0 & 0 & \sigma_\delta \rho_{v\delta} \end{pmatrix} \quad (40)$$

Under these assumptions, the function h , which determines the idiosyncratic risk of the convenience yield and therefore the risk neutral probability drifts, is a solution of the following partial differential equation :

$$\begin{aligned} h_t = & kr + \frac{1}{2}g \left(\lambda'_{e0} \lambda_{e0} + 2\lambda'_{e0} \lambda_{eY} Y + Y' \lambda'_{eY} \lambda_{eY} Y \right) + \frac{1}{2} h'_Y \sigma'_Y M \sigma_Y h_Y + \frac{1}{2} Tr(h_{YY} \sigma'_Y \sigma_Y) \\ & + \left(k \lambda'_{e0} \sigma_Y + k Y' \lambda'_{eY} \sigma_Y + (\mu_{Y0} + \mu_{YY} Y)' \right) h_Y \end{aligned} \quad (41)$$

We notice the quadratic structure of Eq. (41) implies that h is also quadratic in state variables :

$$h(\tau, Y) = A(\tau) + B'(\tau)Y + \frac{1}{2}Y' C(\tau)Y \quad (42)$$

where $A(\tau)$ is a function of time, and $B(\tau)$ and $C(\tau)$ are a vector and a symmetric matrix that depend solely on of time to horizon, respectively. Substituting (42) into (41) and since our goal is to compute the first derivative of h with respect to the state variables Y (see Eq. (32)) to determine the orthogonal component of convenience yield risk premium, we only give the system of ordinary equations followed by $C(\tau)$ and $B(\tau)$:

$$C_\tau = g \lambda'_{ey} \lambda_{ey} + (k \lambda'_{ey} \sigma_y + \mu'_{yy}) C + (k \lambda'_{ey} \sigma_y + \mu'_{yy})' C + C' \sigma'_y M \sigma_y C \quad (43)$$

$$B_\tau = \frac{1}{2}g\lambda'_{ey}\lambda_{e0} + C'(k\sigma'_y\lambda_{e0} + \mu_{y0}) + (C'\sigma'_yM\sigma_y + k\lambda'_{ey}\sigma_y + \mu'_{yy})B + kV \quad (44)$$

where $V = (0, 1, 0)'$. To compute the orthogonal market price of risk of the convenience yield, $\lambda_v(t) = -\sigma_{\delta v}\gamma h_\delta(t)$, we only need to compute the sensitivity of h regarding the movements of the convenience yield. This means that if we note the symmetric matrix $C(\tau)$, $C(\tau) \equiv C_{i,j}(\tau)$, $i, j \in \{X, r, \delta\}$, and the vector $B(\tau)$, $B(\tau) \equiv B_i(\tau)$, $i \in \{X, r, \delta\}$, then we only has to compute the scalar functions $B_\delta(\tau)$, $C_{X\delta}(\tau)$, $C_{r\delta}(\tau)$, $C_{\delta\delta}(\tau)$ so the orthogonal market price of risk of the convenience yield is given by :

$$\lambda_v(t) = -\sigma_{\delta v}\gamma \left[B_\delta(\tau) + C_{X\delta}(\tau)X(t) + C_{r\delta}(\tau)r(t) + C_{\delta\delta}(\tau)\delta(t) \right] \quad (45)$$

Nevertheless, the structure of the system, Eqs. (43) and (44), shows that these functions cannot be computed while the other being not. So, the calculation of the risk premium of the convenience yield requires the computation of nine scalar functions of time only. Moreover, a closer look at equations Eqs. (43) and (44) reveals that the functions $B_\delta(\tau)$, $C_{X\delta}(\tau)$, $C_{r\delta}(\tau)$, $C_{\delta\delta}(\tau)$ will be nil, and so will be the orthogonal market price of risk of the convenience yield in only two cases. The first one is tight to the description of the risk premium of primitive assets. Indeed, when $\lambda_{eY} = \mathbf{0}_{3 \times 3}$, we can get from Eq. (43) that $C(\tau) = \mathbf{0}_{3 \times 3}$, thus Eq. (44) reduces to $\partial_\tau B_r(\tau) = k$ and $B_X(\tau) = B_\delta(\tau) = 0$. This feature is due to the special form of the matrix μ_{YY} which indicates that neither the spot commodity, nor the convenience yield influences the interest rates. This assumption is at the heart of our partial equilibrium model. Indeed, at a macroeconomic level, it would be true that interest rates depend on aggregate prices of commodities. However, in our microeconomic framework, we are concerned with the pricing of a single commodity which in itself does not influence interest rates. The second case is tight to the preference structure of the agents : when the latter is the Bernoulli investor with relative risk aversion $\gamma = 1$, then all of these scalar functions are nil. In our model, the Bernoulli representative agent is also blind in the sense that it does not give any value to the convenience yield risk, and that, whatever the underlying economy description. In any other cases, the functions $B_\delta(\tau)$, $C_{X\delta}(\tau)$, $C_{r\delta}(\tau)$,

$C_{\delta\delta}(\tau)$ are in general not nil. Therefore, Eq. (45) reveals that the orthogonal market price of risk of the convenience yield is also an affine functions of the three state variables : the (log) spot commodity price, interest rates and the convenience yield. In particular, this equation gives a meaning to $C_{X\delta}(\tau)$, $C_{r\delta}(\tau)$, $C_{\delta\delta}(\tau)$. We indeed reverse Eq. (45) to find :

$$\frac{\lambda_v(t)}{\sigma_{\delta v}} = -\gamma \left[B_{\delta}(\tau) + C_{X\delta}(\tau)X(t) + C_{r\delta}(\tau)r(t) + C_{\delta\delta}(\tau)\delta(t) \right] \quad (46)$$

Furthermore, combining Eqs. (??) and (45) gives :

$$\frac{J_{W\delta}(t)}{J_W(t)} = B_{\delta}(\tau) + C_{X\delta}(\tau)X(t) + C_{r\delta}(\tau)r(t) + C_{\delta\delta}(\tau)\delta(t) \quad (47)$$

hence,

$$C_{i\delta}(\tau) = \partial_i \left(\frac{J_{W\delta}(t)}{J_W(t)} \right), \quad i \in \{X, r, \delta\} \quad (48)$$

$C_{i\delta}(\tau)$ is the sensitivity to state variable i of the influence of the convenience yield on aggregate satisfaction. While $B_{\delta}(\tau)$ defines a level of the influence of the convenience yield on aggregate satisfaction that is independent of state variables. Finally, we notice from Eqs. (43) and (44) that the functions C and B depend on all parameters of the model.

Now that the market price of risk of the convenience yield is precisely determined, we can rewrite the market price of risk in matrix form using our affine framework :

$$\lambda(t) = \lambda_0(t) + \lambda_Y(t)Y(t) \quad (49)$$

$$\text{with } \lambda_0(\tau) = \begin{pmatrix} \lambda_{X0} \\ -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{X0} + \frac{1}{\rho_{ur}}\lambda_{r0} \\ B_{\delta}(\tau) \end{pmatrix} \text{ and } \lambda_Y(\tau) = \begin{pmatrix} \lambda_{XX} & \lambda_{Xr} & \lambda_{X\delta} \\ -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{XX} & -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{Xr} + \frac{1}{\rho_{ur}}\lambda_{rr} & -\frac{\rho_{Xr}}{\rho_{ur}}\lambda_{X\delta} \\ C_{X\delta}(\tau) & C_{r\delta}(\tau) & C_{\delta\delta}(\tau) \end{pmatrix}$$

The state variables are then also (conditionally) Gaussian affine under the risk neutral probability, however, they depend on investors' preferences and horizon :

$$dY(t) = \left[\mu_{Y0}^Q(\tau) + \mu_{YY}^Q(\tau)Y \right] dt + \sigma'_Y dz^Q(t) \quad (50)$$

where $\mu_{Y0}^Q(\tau) = \mu_{Y0} - \sigma'_Y \lambda_0(\tau)$ and $\mu_{YY}^Q(\tau) = \mu_{YY} - \sigma'_Y \lambda_Y(\tau)$ through Girsanov theorem : $dz^Q(t) = dz(t) + [\lambda_0(\tau) + \lambda_Y(\tau)Y(t)]dt$. We now can rewrite the contingent claims pricing

equations (34) and (35) in light of our new framework :

$$G_t(t) + G'_Y(t)[\mu_{Y_0}^Q(\tau) + \mu_{YY}^Q(\tau)Y] \frac{1}{2} Tr[G_{YY}(t)\sigma'_Y(t)\sigma_Y(t)] - G(t)r(t) = 0 \quad (51)$$

and,

$$H_t(t) + H'_Y(t)[\mu_{Y_0}^Q(\tau) + \mu_{YY}^Q(\tau)Y] + \frac{1}{2} Tr[H_{YY}(t)\sigma'_Y(t)\sigma_Y(t)] = 0 \quad (52)$$

4 Closed form solution of commodity derivatives

Let $F(t, Y, T_F) \equiv F(t, T_F)$ the time- t price of a futures contract of maturity T_F written on the spot commodity. Since the futures contract is a marked-to-market asset, its price obeys an equation similar to Eq. (52), that is :

$$F_t(t) + F'_Y(t)[\mu_{Y_0}^Q(\tau) + \mu_{YY}^Q(\tau)Y] + \frac{1}{2} Tr[F_{YY}(t)\sigma'_Y(t)\sigma_Y(t)] = 0 \quad t \in [0, T_F] \quad (53)$$

with $F(T_F, T_F) = \exp(u'Y(T_F))$ and $u' = (1, 0, 0)$. Similarly, let $C(t, Y, T_C) \equiv C(t, T_C)$ the time- t price of a European call written on the futures contract with strike K and maturing at $T_C \leq T_F$. We assume that this option is not marked to market. It then obeys to an equation similar to (51) :

$$C_t(t) + C'_Y(t)[\mu_{Y_0}^Q(\tau) + \mu_{YY}^Q(\tau)Y] \frac{1}{2} Tr[C_{YY}(t)\sigma'_Y(t)\sigma_Y(t)] - C(t)r(t) = 0, \quad t \in [0, T_C] \quad (54)$$

and $C(T_C, T_C) = \left(F(T_C, T_F) - K \right)^+$. Solution of Eq. (53) has the following form :

$$F(t, T_F) = \exp(A_F(\tau_F) + B'_F(\tau_F)Y), \tau_F = T_F - t \quad (55)$$

where the scalar and vector functions $A_F(\tau_F)$ and $B_F(\tau_F)$ are, by identification, solutions to the following system of ordinary equations :

$$\partial_t B_F(t) = \mu_{YY}^Q(\tau_{IF} + t)' B_F(t), \quad B_F(0) = u \quad \tau_{IF} \equiv T_I - T_F \quad (56)$$

$$\partial_t A_F(t) = \mu_{0Y}^Q(\tau_{IF} + t)' B_F(t) + \frac{1}{2} B'_F(t)\sigma'_Y\sigma_Y B_F(t), \quad A_F(0) = 0 \quad (57)$$

As in many other reduced-form models, the future price is an exponential affine function of the three state variables. However, because it depends on the risk neutral probability \mathbf{Q} , through the vector and matrix functions, μ_{0Y}^Q and μ_{YY}^Q , it depends on aggregate preference in the market and on investors' horizon T_I .

The price of the call under the risk-neutral measure is given by :

$$\frac{C(t, T_C)}{\beta(t)} = \mathbb{E}_t^Q \left[\frac{C(T_C, T_C)}{\beta(T_C)} \right] \quad (58)$$

For later reference, we note D the event the call finishes in the money, that is, $D = \{\omega \in \Omega / [F(T_C, T_F) \geq K]\}$. Using the linearity of the conditional expectation, the price of the call then writes :

$$\frac{C(t, T_C)}{\beta(t)} = \mathbb{E}_t^Q \left[\frac{F(T_C, T_F) 1_D}{\beta(T_C)} \right] - K \mathbb{E}_t^Q \left[\frac{1_D}{\beta(T_C)} \right] \quad (59)$$

We then rely on two changes of numéraire that define two new probabilities. The first one associated with stochastic interest rates and a zero coupon maturing at T_C , the call's maturity, defines the forward neutral probability of maturity T_C :

$$\left. \frac{d\mathbf{Q}^C}{d\mathbf{Q}} \right|_{\mathbf{F}_t} = \frac{B(T_C, T_C)}{\beta(T_C)} \frac{\beta(t)}{B(t, T_C)} \quad (60)$$

The second change of numeraire uses the futures contract. However, because the contract is a marked-to-market asset, we need to multiply it by the riskless asset. Hence, we use $\beta(t)F(t, T_F)$ as a numeraire to define the probability \mathbf{Q}^F :

$$\left. \frac{d\mathbf{Q}^F}{d\mathbf{Q}} \right|_{\mathbf{F}_t} = \frac{\beta(T_C)F(T_C, T_F)}{\beta(T_C)} \frac{\beta(t)}{\beta(t)F(t, T_F)} = \frac{F(T_C, T_F)}{F(t, T_F)} \quad (61)$$

The option price is given in the following proposition :

Proposition 2 *The price of a European call option written on a marked-to-market commodity futures contract is given by :*

$$C(t, T_C) = \beta(t)F(t, Y, T_F)h(t, Y, T_C) - B(t, r, T_C)Kg(t, Y, T_C) \quad (62)$$

with

$$h(t, Y, T_C) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp(\alpha_{VR0}(t) + \alpha_{VR1}\nu^2 + \beta'_{VR}(t)Y) \sin(\nu(\alpha_{VI}(t) + \beta'_{VI}Y)) \frac{d\nu}{\nu} \quad (63)$$

$$g(t, Y, T_C) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp(\alpha_{CR}(t)\nu^2) \sin(\nu(\alpha_{CI}(t) + \beta'_{CI}Y)) \frac{d\nu}{\nu} \quad (64)$$

where $\alpha_{VR0}(t)$, $\alpha_{VR1}(t)$, $\beta_{VR}(t)$, $\alpha_{VI}(t)$, $\beta_{VI}(t)$, $\alpha_{CR}(t)$, $\alpha_{CI}(t)$, $\beta_{CI}(t)$ solve for :

$$\partial_t \beta_{VR}(t) = U - \mu_{YY}^Q(t)' \beta_{VR}(t), \quad \beta_{VR}(T_C) = 0_3, \quad U = (0, 1, 0)' \quad (65)$$

$$\partial_t \beta_{VI}(t) = -\mu_{YY}^Q(t)' \beta_{VI}(t), \quad \beta_{VI}(T_C) = B_F(T_F - T_C) \quad (66)$$

$$\partial_t \alpha_{VR0}(t) = -\left[\mu_{Y0}^{Qc}(t)' \beta_{VR}(t) + \frac{1}{2} \beta_{VR}(t)' \sigma_Y' \sigma_Y \beta_{VR}(t) \right], \quad \alpha_{VR0}(T_C) = 0 \quad (67)$$

$$\partial_t \alpha_{VR1}(t) = \frac{1}{2} \beta_{VI}(t)' \sigma_Y' \sigma_Y \beta_{VI}(t), \quad \alpha_{VR1}(T_C) = 0 \quad (68)$$

$$\partial_t \alpha_{VI}(t) = -\left[\mu_{Y0}^{Qc}(t)' \beta_{VI}(t) + \frac{1}{2} \beta_{VR}(t)' \sigma_Y' \sigma_Y \beta_{VI}(t) \right], \quad \alpha_{VI}(T_C) = A_F(T_F - T_C) - \log(K) \quad (69)$$

$$\partial_t \beta_{CI}(t) = -\mu_{YY}^Q(t)' \beta_{CI}(t), \quad \beta_{CI}(T_C) = B_F(T_F - T_C) \quad (70)$$

$$\partial_t \alpha_{CR}(t) = \frac{1}{2} \beta_{CI}(t)' \sigma_Y' \sigma_Y \beta_{CI}(t), \quad \alpha_{CR}(T_C) = 0 \quad (71)$$

$$\partial_t \alpha_{CI}(t) = -\mu_{Y0}^{Qc}(t)' \beta_{CI}(t), \quad \alpha_{CI}(T_C) = A_F(T_F - T_C) - \log(K) \quad (72)$$

The proposition shows that the option price is clearly dependent on the state variables, not only through the futures price, but also through the functions $h(t, Y, T_C)$ and $g(t, Y, T_C)$. This highlights possible non linearities in the state variables. Indeed, a closer look at $\beta_{VI}(t)$ shows that its components are not nil in general. This means that the function $h(t, Y, T_C)$ depends on interest rates and the spot commodity and the convenience yield. This result is in sharp contrast with the one given in Richter and Sorensen (2005) to price options on a futures contract.

5 Conclusion

In this paper, we construct a partial equilibrium to derive endogenously the market price of risk of the convenience yield. In particular, the quantity of interest in this market price of risk is the part of risk that is not correlated with primitive assets, that is, the idiosyncratic risk of the convenience yield. In addition, we have shown that this orthogonal risk is proportional to three quantities in the market. The first one is the volatility of the convenience yield : the more volatile is the convenience yield *ceteris paribus*, the higher is this market price of risk. The second one highlights the correlation between the convenience yield and the primitive assets : when the convenience yield is highly (poorly) correlated with primitive assets, then this orthogonal market price of risk is low (high). Moreover, the sign of this orthogonal market price of risk is given by the impact of the movements of the convenience yield on aggregate market satisfaction. When a rise (drop) in the convenience yield tends to increase (decrease) aggregate satisfaction, then this market price of risk is negative. Whereas when a rise (drop) in the convenience yield tends to decrease (increase) aggregate satisfaction then this market price of risk is positive. Moreover, by making a standard assumption on agents preference structure, (constant relative risk aversion), and on the underlying market (affine state variables and market price of risk), we obtain a closed form solution for this market price of risk. Therefore, we show that this market price of risk is also affine in all the three state variables. This closed form solution enables us to easily simulate this market price of risk. This affine expression of the market price of risk that embeds agents preference and investors horizon also let us derive in closed form solutions futures prices and options on futures prices. These derivatives' prices also depend on agents' preference and horizon. In particular, we find that, contrary to Richter and Sorensen (2005), the futures price is not a sufficient statistic to derive options prices : the latter depend on the futures price but also on a function of the state variables that could possibly be highly non-linear. Our framework can be extended to include stylized facts of commodities such as jumps or stochastic volatility. In particular, the latter would

be implemented easily because it is tractable under an affine framework. Moreover, we reduce our practical analysis to constant relative risk aversion preference in order to obtain analytical solutions. However, numerical methods could be used to solve partial differential equations in order to evaluate the impact of other preference structures on derivatives prices. Finally, due to its tractability, our framework could be extended and adapted to other markets where, in particular, a contingent claim market exists while the underlying asset bears a risk that it does not fully hedge. For example, we can think of markets covering inflation or exchange rates.

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