CONDITIONAL AFFINE TERM RISK OF JUMPS
(Q-Poisson single jump risk)

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Current problem in asset pricing - term risk (+ risk premium anomalies)

- Inter-temporal risk (term risk) premia generally heavily underestimated
- Lack of transparency in new financial instruments – lack of reliable fair value method or liquidity measurement
- Current model discredited – misapplication of models for tail risk and term structure (over-reliance on Gaussian-Wiener log normal)
- This is a generic problem

Why we believe in risk neutral (Q) jump risk (Poisson intensity model)

- Poison intensity model is key:

  Cumulative probability = $CDF_{Default} = e^{-\lambda T}$
- Empirical evidence confirms that term risk occurs as a market risk – Peso, the whole market moves, the autoregression (clustering in GARCH)
- Naturally we think of the Poisson distribution as discrete event arrivals – but from Physics we know that a dispersion (distribution) can exist

Limitations in existing theory to deliver term risk - examples

Stochastic volatility (Hull and White; Heston)

- Bivariate stochastic model over cross-sectional underlying (asset) and volatility ; non-conditional, ad hoc in volatility time dependence

Exponential Levy models (Cont and Tankov)

- Infinitely divisible Poisson intensities +stationarity, therefore no unique pricing martingale nor mechanism for volatility 'clustering'

Affine jump diffusion (Duffie Singleton)

- Reduced form modelling of rates difficult to conceptualise to structure; assumed discrete P probabilities of jumps not risk neutral

Econometric time series (ARCH/GARCH)

- Conditional time auto-correlations but in P not risk neutral; calibration easily smothered by non-stationarity

Limitations in existing theory to measure credit spread risk - examples

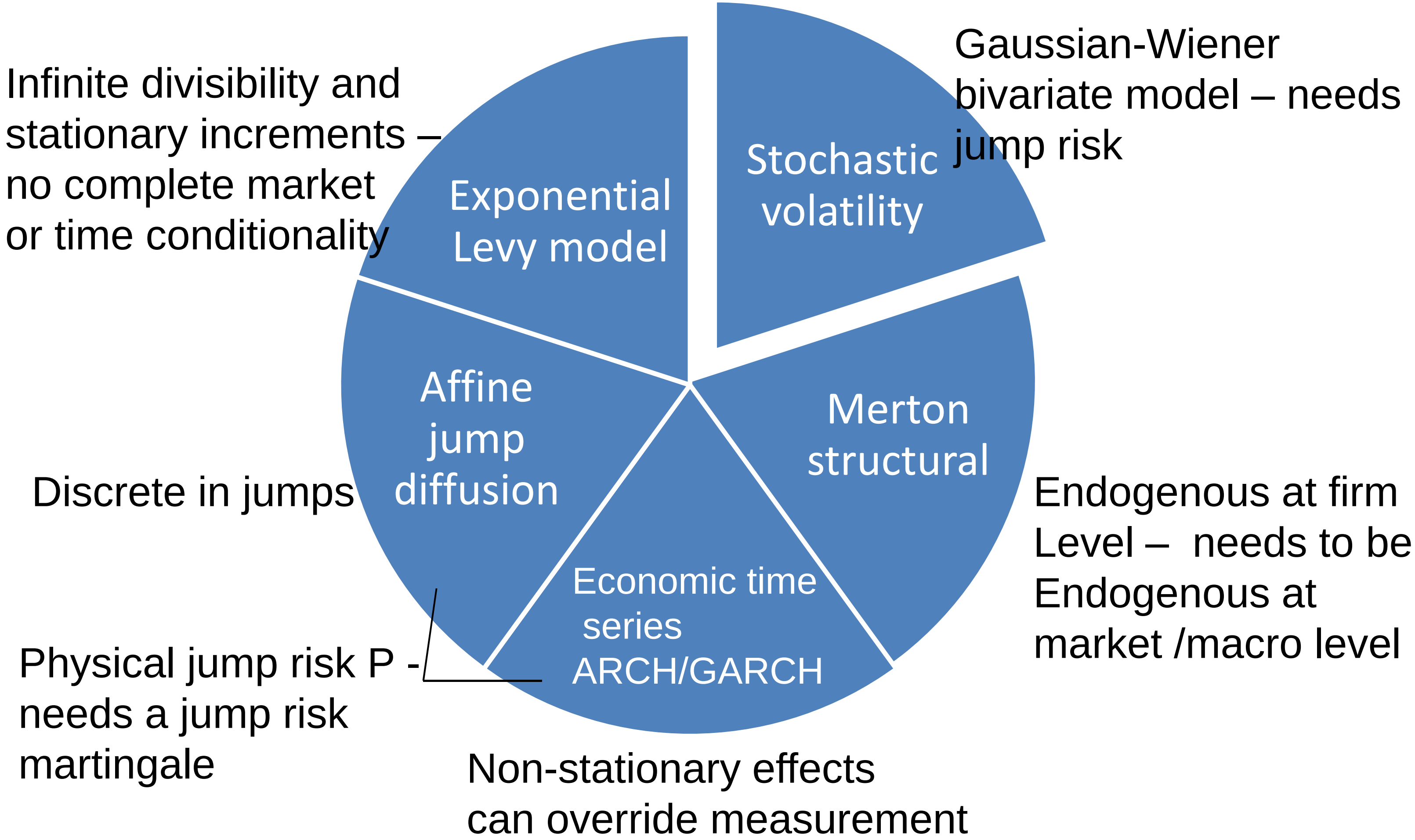
Merton (1974) structural model

- A conceptual model but focused exclusively on firm (entity) level variables; empirical results indicate missed conditionality exists besides capital structure; the risk premia given by Gaussian diffusion and P jump probabilities are much to small.

Exponential Levy models (Cont and Tankov 2004)

- A high degree of flexibility in parameterisation, analytically tractable, high fitting capabilities, lacking in structural interpretation or causal insights

Generic problems



Standard approach to solution is applied

Three steps:

- 1. Stochastic differential equation focused on a pricing kernel – but replace the cross-sectional log-normal kernel with a new pricing kernel for jumps

- 2. Find the P to Q transform using a Nikon Radodym dQ/dP - replace the Girsanov drift Gaussian drift change of measure with a new dQ/dP martingale transform for jumps

- 3. Then use a partial differential equation between the interest rate and the pricing kernel - replace the Feynman-Kac probabilistic representation by a simple first order equation between log rate and jump's pricing kernel

Conventional Stochastic Differential Equation (SDE)

consider a standard form of jump diffusion model. **Equation 1** below shows this in the mean-reversion jump-diffusion stochastic equation for interest rates with added jumps, as developed by Zhou (2001) for interest rates, with increments given by:

$$dr = \kappa(\theta - r)dt + \sigma dW + dZ(\lambda) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In practice, this SDE may be further elaborated with further stochastic features depending on application, e.g. with deterministic terms, on the grounds of improved fitting to observed market prices. Some of the basic features are: the mean reversion term for the central tendency of interest rates to a mean rate, θ, with rate of reversion κ. Randomness is modelled as a geometric Brownian diffusion dW (Wiener increment) with instantaneous volatility, σ. Jump risk in **equation 1** is would typically has two sources of variability: Poisson process increments, $dN(\lambda)$, with intensity λ to the Poisson process and stochastic size of jumps, J.

Solution for the single jump in P by Fourier Analysis

Substituting in the Fourier transform obtains the characteristic function $\phi(\omega)$:

$$\phi(\omega) = F_t[P(t)](\omega) = \frac{\sqrt{2}}{\pi}.\int_0^\infty e^{-i\omega t}.(-\lambda).\mathrm{e}^{-\lambda t}.\mathrm{dt} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(7)$$

With the integration range selected to give a starting time of $t = 0$, as required for a financial process

From **equation 7** we therefore obtain:

$$\phi''(\omega) = Im.\phi(\omega) = \frac{\sqrt{2}}{\lambda}.\frac{\omega/\lambda}{1 + (\omega/\lambda)^2}$$

The shapes of distributional components for the single jump can then be readily plotted from **equation 7**. These are shown in **figure 1a**.

Why a mathematical solution must exist

$$CDF(t) = e^{-\lambda t}.1_{t \geq 0}$$

$$\psi(\omega) = E^*\left[e^{-i\omega X(t)}\right] = \int_{-\infty}^{\infty} e^{-i\omega t}.\frac{d(CDF(t))}{dt}.dt$$

$$\psi(\omega) = E^*\left[e^{-i\omega X(t)}\right] = \int_{-\infty}^{\infty} e^{-i\omega t}.d(CDF(t))$$

- Single Poisson jump
- A solution in frequency = 1/time [but this is the differential of CDF is undefined
- The differential of CDF does exist by the Wiener-Khintchine theorem

P to Q Equivalent Martingale (single jump)

To find equivalent martingale expectation, $E^Q[1_A] = E^P\left[1_A.\frac{1}{\omega}\right]$, consider the transformation of the cumulative pricing distribution in P, i.e. $CDF^P(\omega) = E^P[1_A]$.

From **equation 8**, the cumulative P distribution $CDF^P(\omega)$ is given by:

$$CDF^P(\omega) = E^P[1_A] = \int_0^\infty \frac{\sqrt{2}}{\pi}.\frac{\phi''(\omega)}{\omega}.d\omega$$

$$= \int_0^\infty \frac{\sqrt{2}}{\pi}.\frac{\omega/\lambda}{1+(\omega/\lambda)^2}.\frac{d\omega}{\omega} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10)$$

Apply the proposed Radon Nikodym differential, $\frac{dQ}{dP} = \frac{1}{\omega} = t$ **equation 9** to **equation 10**, ~~to~~ obtains the cumulative pricing distribution in Q, $E^Q[1_A]$:

$$CDF^Q(\omega) = E^P\left[1_A\frac{1}{\omega}\right]$$

Gives:

$$= \int_0^\infty \frac{\sqrt{2}}{\pi}.\phi''(\omega).\frac{1}{\omega}.d\omega = \int_0^\infty \frac{\sqrt{2}}{\pi}.\frac{\omega/\lambda}{1+(\omega/\lambda)^2}.dln\omega \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (11)$$

Q Poisson pricing kernel
Q-PDF and -CDF for the single jump

The randomness in the BSM is a steady slowly varying risk due to noise by instantaneous volatility. The source randomness in the Q-jump on the other hand represents uncertainty due to a conditionality based over time, i.e. an anticipation of an uncertain-in-time rare event. The attraction of both models is their availability of analytical pricing kernels for the PDF and CDF, in the case of the BSM through *N(d)* the cumulative of the lognormal distribution, and for the Q-jump its log Cauchy PDF and CDF:

$$PDF^{Q}(jumps) = \frac{\sqrt{2}/\pi}{2 + (\ln(t) - \ln(\tau))^2}$$

$$CDF^{Q}(jumps) = 0.5 + (1/\pi).arctan((\ln(t/\tau))/\sqrt{2})$$

Comparison of P pricing kernel PDF/CDF heavy and nonlinear tails, with uniform martingale (Q) PDF/CDF

Figure 1a

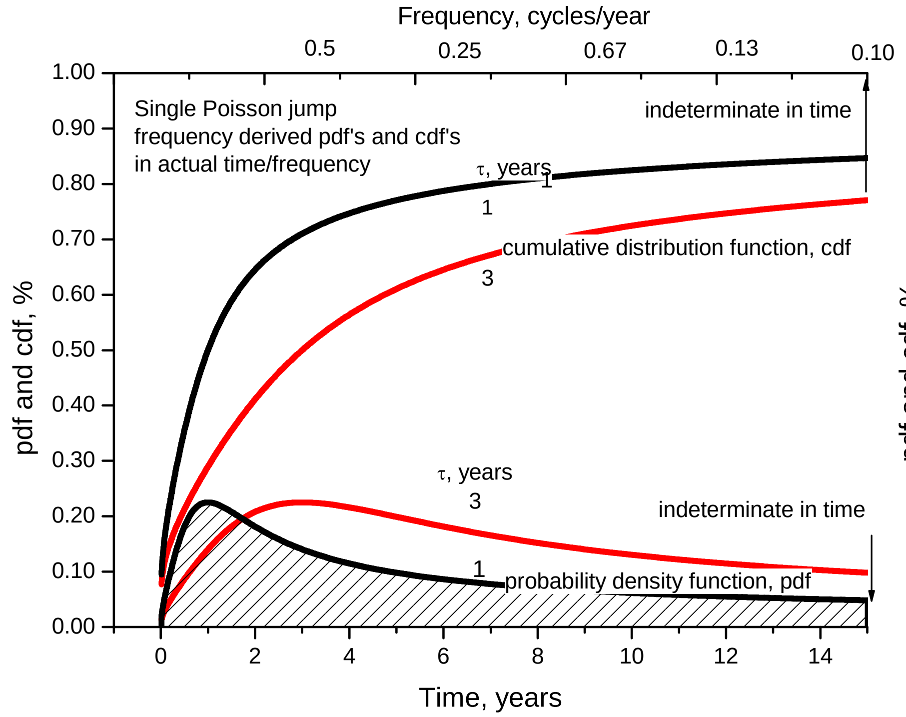
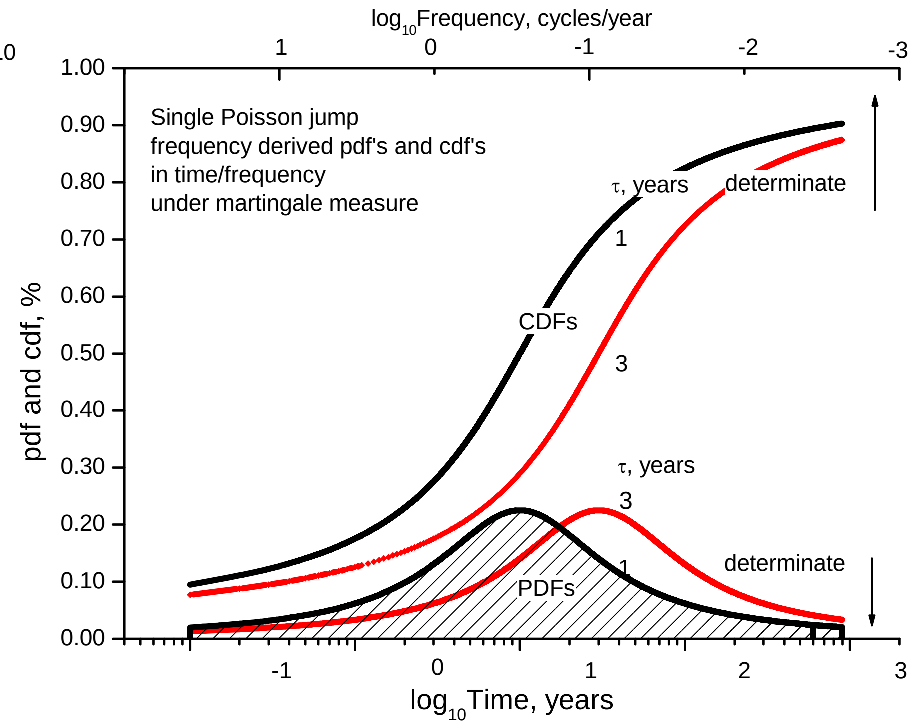


Figure 1b



Real probabilities P
→ Nonlinear tail risk problem

Risk neutral probabilities Q
→ Solves tail risk problem

Properties of the Q Poisson PDF/CDF

- Q Poisson PDF/CDF equations are analytical
  - if substituted in place of the log-normal N(.) of the geometric Brownian motion, the Q jump risk is given in place of the Weiner diffusion
- Q Poisson PDF/CDF are EMM in complete markets, therefore linear
  - the Q Poisson is a log Cauchy in time, orthogonal to the cross-sectional log normal of the Brownian diffusion
  - the linear superposition property is crucial when pricing spreads and term risk in portfolios
- Q jump risk time constants, τ's
  - realistic time constants are obtained, e.g. 6 months to 5 years, high risk premia e.g. Relevant to credit spreads and risk free term premium, are obtained

Testing methodology for Q Poisson jumps [yields and spreads]

- The Q Poisson theory derived above, now needs testing on term risk problems
- Ideal data for this are: yield curves
    - since yield curves give term structure over widest known time range; are reliable and very likely satisfy complete market conditions
- Credit spread curves
    - since these are very large markets but lacking in appropriate models for their risks. A new generic approach is sorely needed, which the Q Poisson might provide

First order partial differential equation for ln(interest rate) and ln(spread)

the **yield curve** follows an exponential affine equation of the type:

$$r = e^{-Z(\lambda,t)}$$

i.e. $$dlnr = dZ(\lambda, t)$$

To set the lower bound of very short time rates, assume the entire all the term risk is Q-jump risk driven. Then all but the lowest bound is determined by the model. Substituting for the Q-jump dynamic **equation 13**, obtains:

$$lnr = lnr_0 + \beta.\left[(0.5 + \frac{1}{\pi}.\operatorname{arctan}\left(\frac{\ln(t/\tau)}{\sqrt{2}}\right)\right] \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.\dots\dots\dots (14)$$

Keeping the initial rate lnr_0 separate as a given, there are two explanatory variables in **equation 3** for the behaviour of the yield curve.

To model the **credit spreads** compare the two sets of curve: risky and riskless, with the equation:

$$dln\lambda = \quad \beta_B\big(.5 + (1/\pi).arctan((\ln t - \ln\tau_B)/\sqrt{2}\,\big)$$
$$-\beta_G\big(.5 + \quad (1/\pi).arctan((\ln t - \ln\tau_G)/\sqrt{2}\,\big) \dots\dots\dots\dots\dots\dots\dots\dots\dots.\dots.(15)$$

Yield curve term structure parameterised first; Then credit spreads are calculated by comparing risky vs. riskless yield curves over full term

- Slide 17 shows how yield curve shape is obtained from the Q Poisson CDF
  - Mostly curves are upward sloping as a smooth sinusoidal S
- Slide 18 shows how when yields invert
  - They are modelled as both an up and a down Q Poisson S
- Credit spreads are calculated from the upward yields for the UK 2003
- Slides 19 and 20 show these using the Q Poisson kernel
  - and based on a first order partial diff. equation between log (interest rate) and the pricing kernel

Application of Q Poisson CDF as first order partial differential equation to log (interest rate) increments

Figure 2a

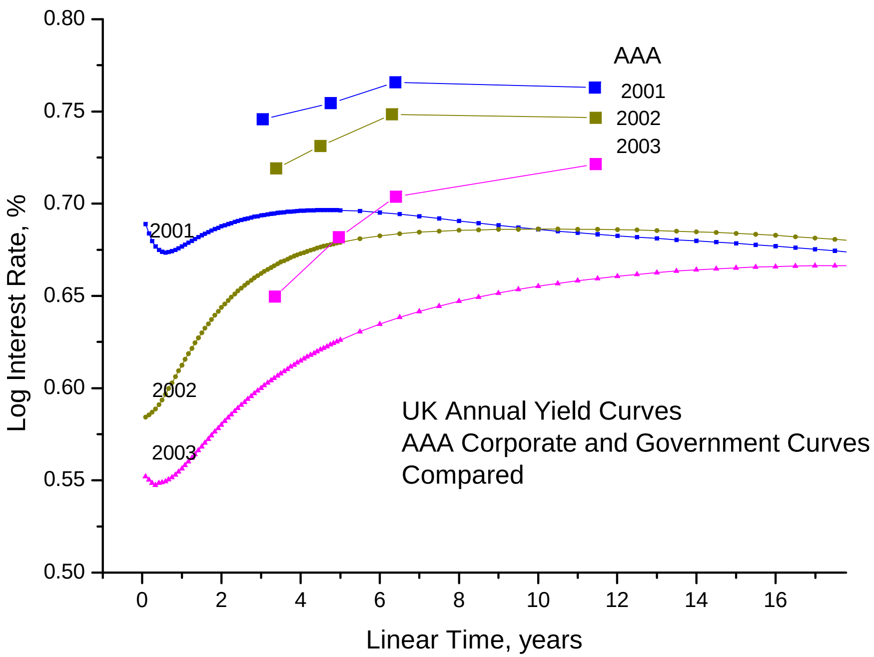
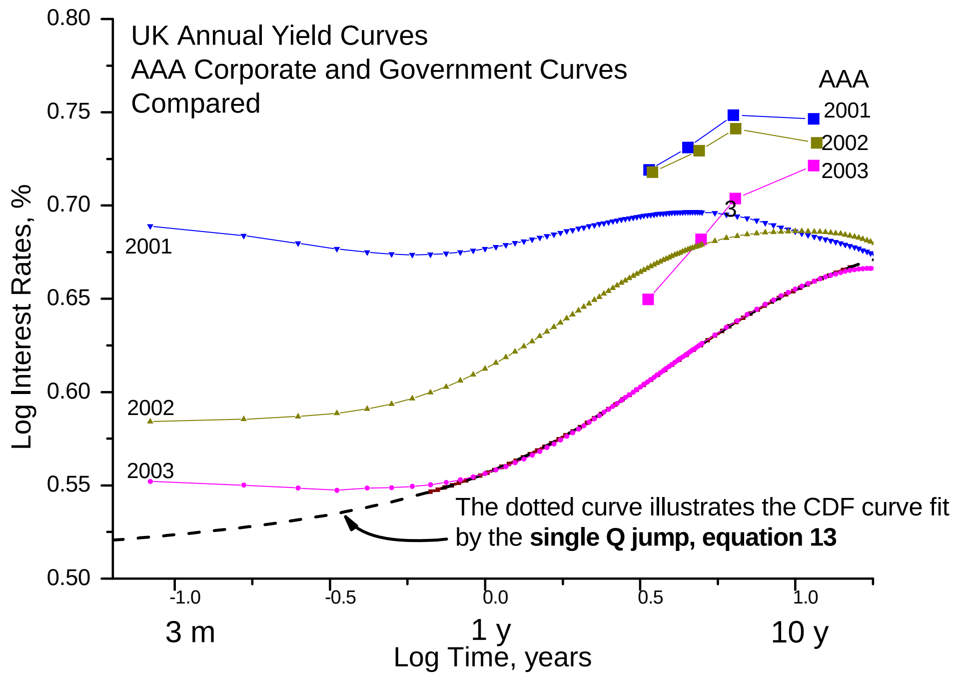


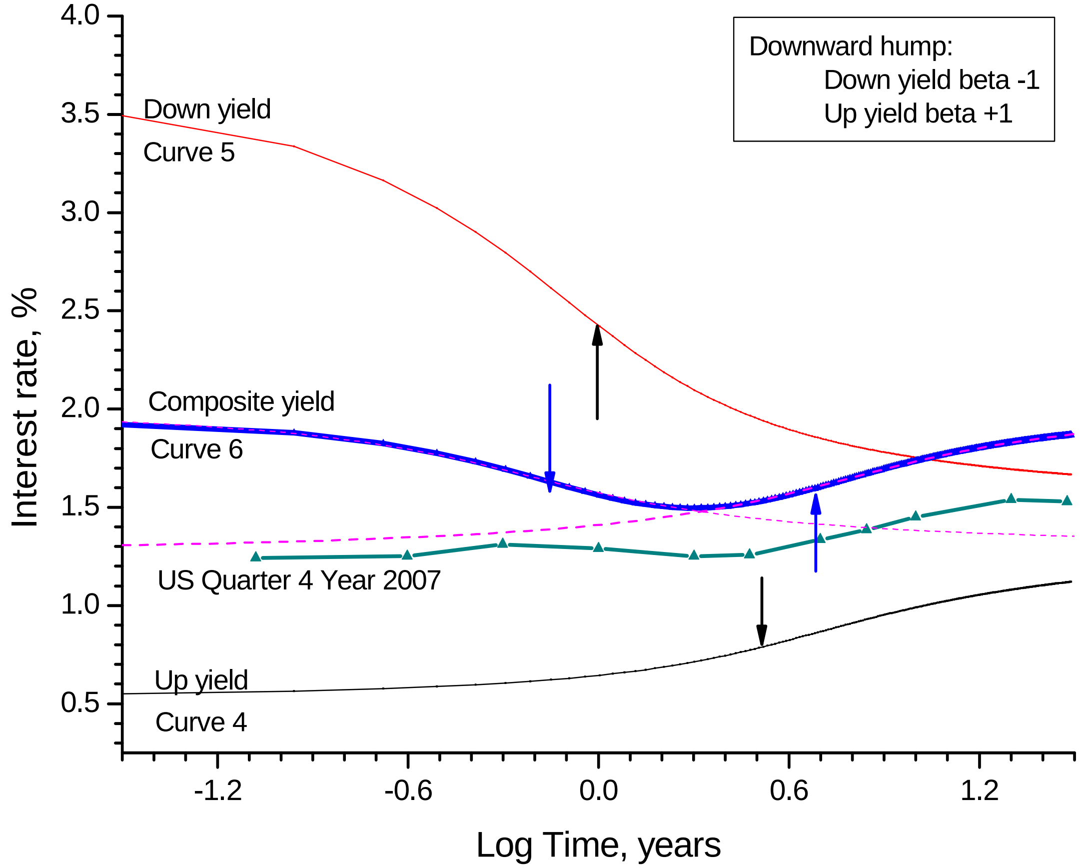
Figure 2b



Note how the risky yield curve is always a limited sample – some of its shape can be 'interpreted' from the more complete benchmark curve based on its fitted Q Poisson

RISKY AND 'RISKLESS' YIELD CURVES

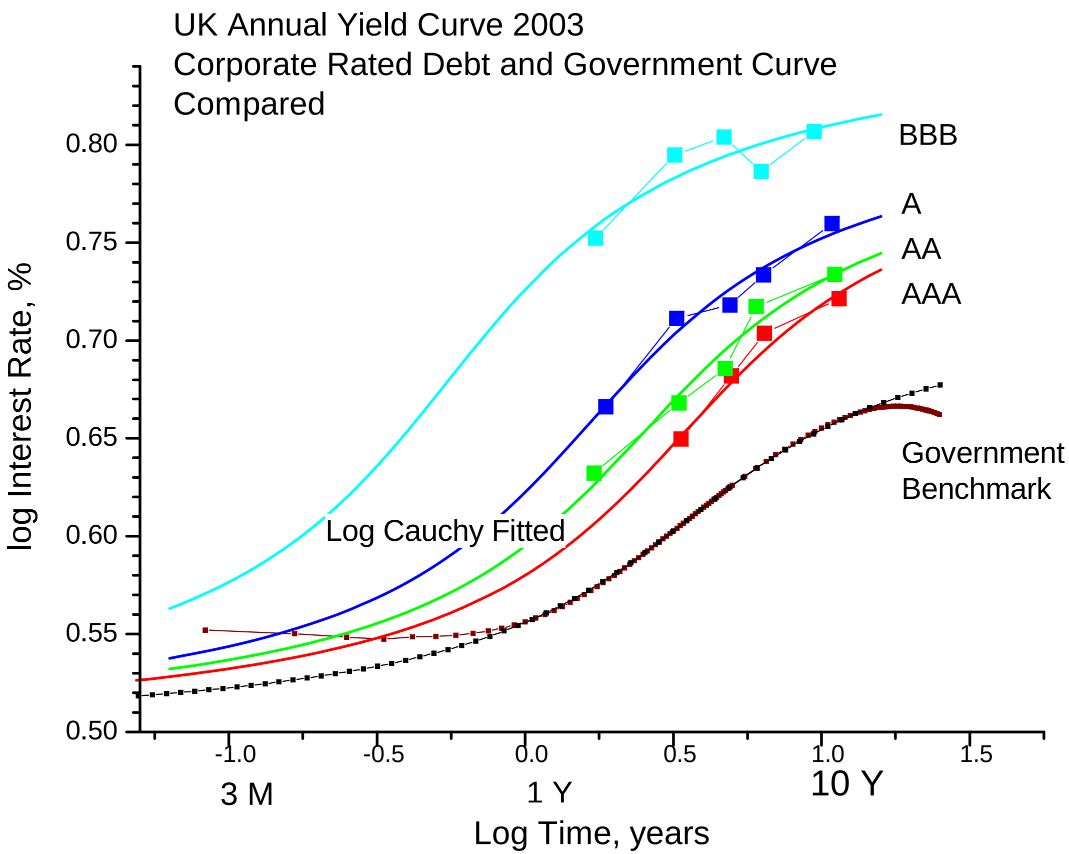
Parameterisation of yield curves shapes by 1 up and 1 down Q Poisson risk at different τ's



INVERTED YIELD CURVE FROM DOWN AND DOWN Q POISSON KERNEL

Parameterisation of credit spreads by Q Poisson jump

Figure 3



Characterisation of spreads by rating, UK 2003

Table 1 - UK 2003 annual average Q-jump spreads and calibration coefficients by rating, by Exponential Log Cauchy Model

Government bond or S&P Rating	Single Jump Intensity, β_G or β_B	Time constants τ_G, τ_B years, y months, m	Q-jump spread, bps
Government benchmark	0.52	3y 6m$\pm$0.3m	
AAA	0.66	2y 9m$\pm$3.2m	41.7$\pm$17.1
AA	0.68	2y 4m$\pm$2.3m	51.4$\pm$10.2
A	0.72	1y 8m$\pm$1.4m	86.6$\pm$7.2
BBB	0.81	4.8m$\pm$1.3m	190.2$\pm$29.3

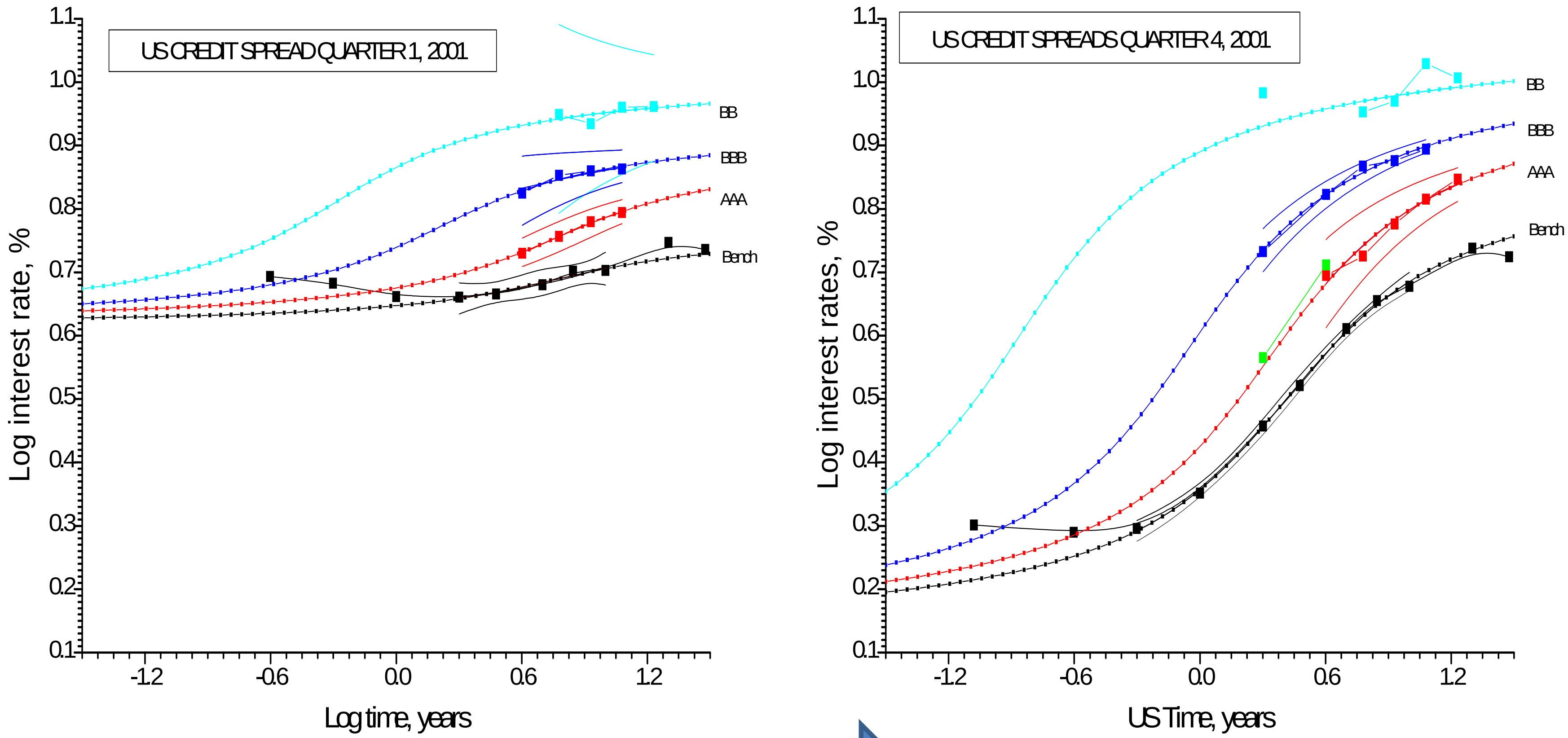
The lowest bound interest rate, r_0 = 3.1%
Errors to 95% confidence limits

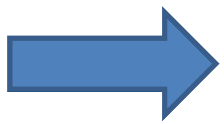
GOOD FITTING OF SPREADS IS OBTaINED ON PARAMETERS SHOWN

The Q Poisson is tested by interpreting the term risk effects revealed in kernel parameters over the economic cycle

- Credit spreads vary as the jump term risk varies over the cycle
  - Slide 22 shows how this suddenly changes in the 2001 market crash
- The term risk time constant, up/down beta of the Q Poisson kernel delineate change, e.g.
  - In Slide 24, tau shortens and beta steepens respectively going into a crash
- Slide 23 shows the lower bound interest rate (a given), and the Q Poisson spread
  - for the US rated corporate debt 2000 to 2009

US spreads as the dot.com market crashed into illiquidity





Parameterisation of credit spreads by Q Poisson jump over economic cycle

Figure 5a

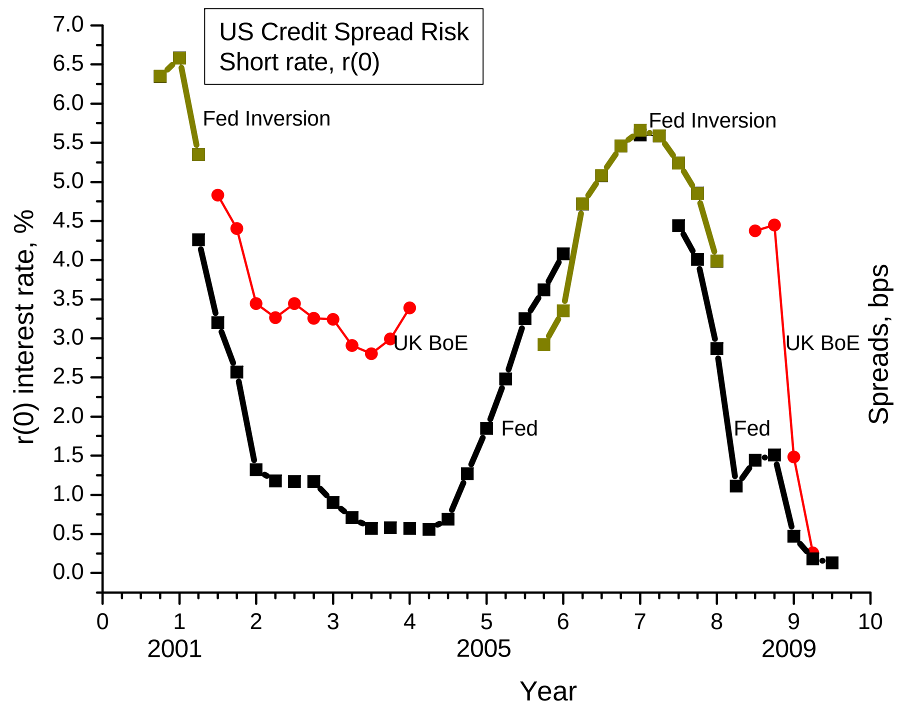
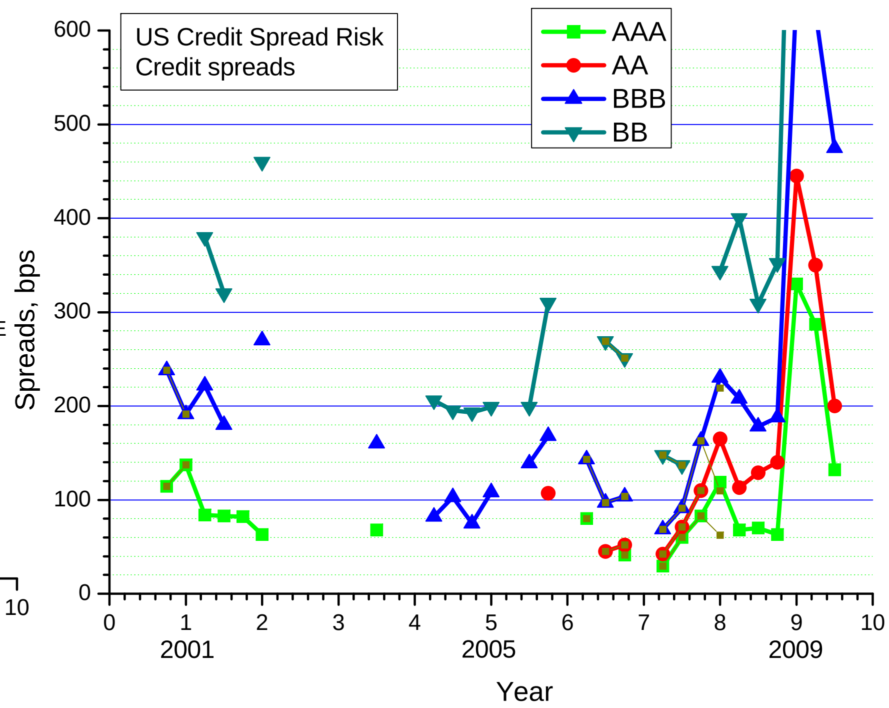


Figure 5d



SPREADS OVER THE CYCLE

Parsimonious parameterisation of credit spreads by Q jump variables

Figure 5b

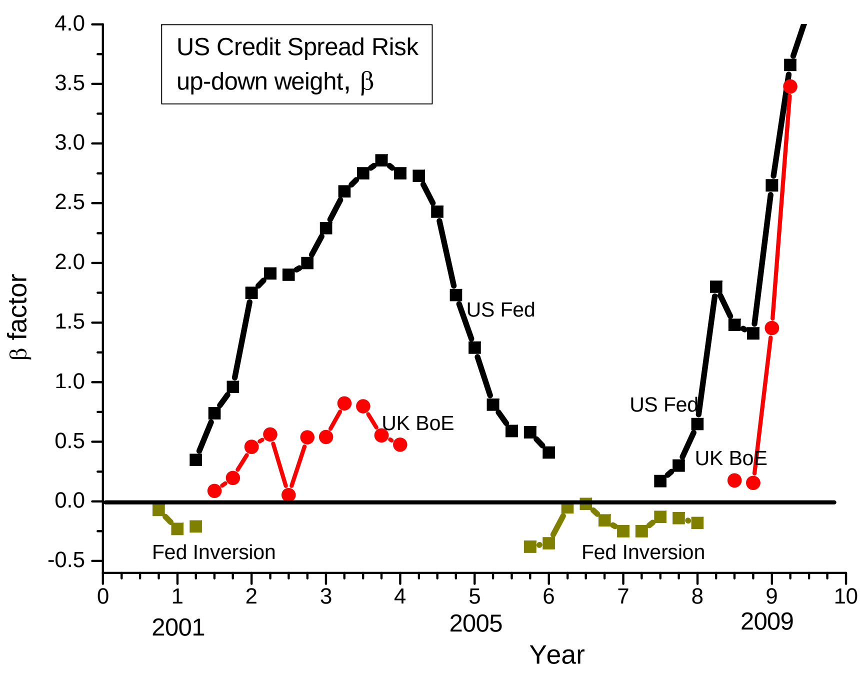
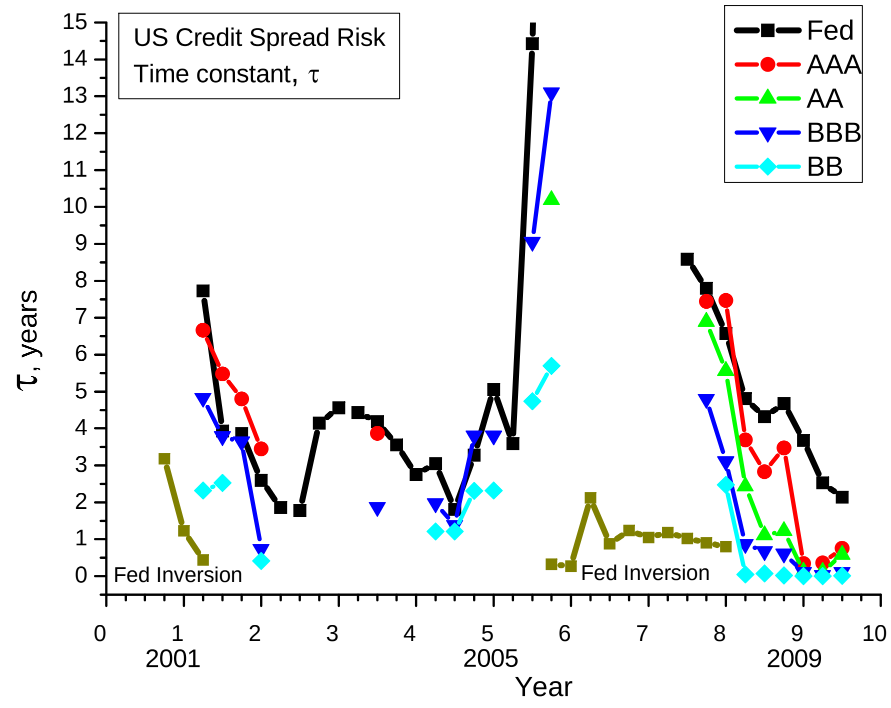


Figure 5c

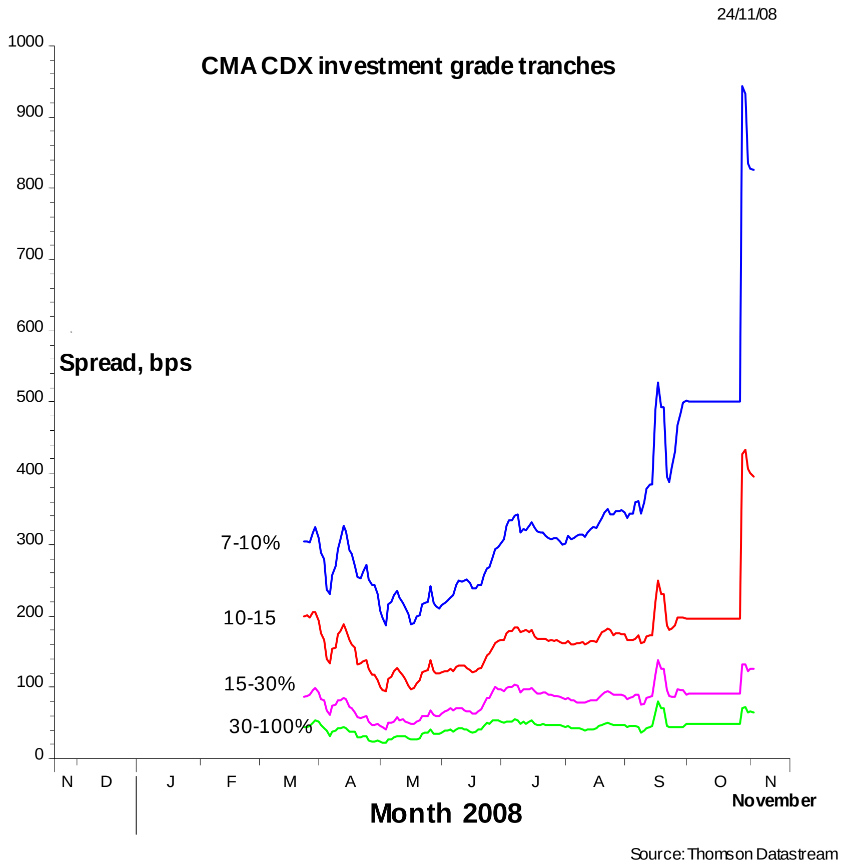


SPREADS OVER THE CYCLE

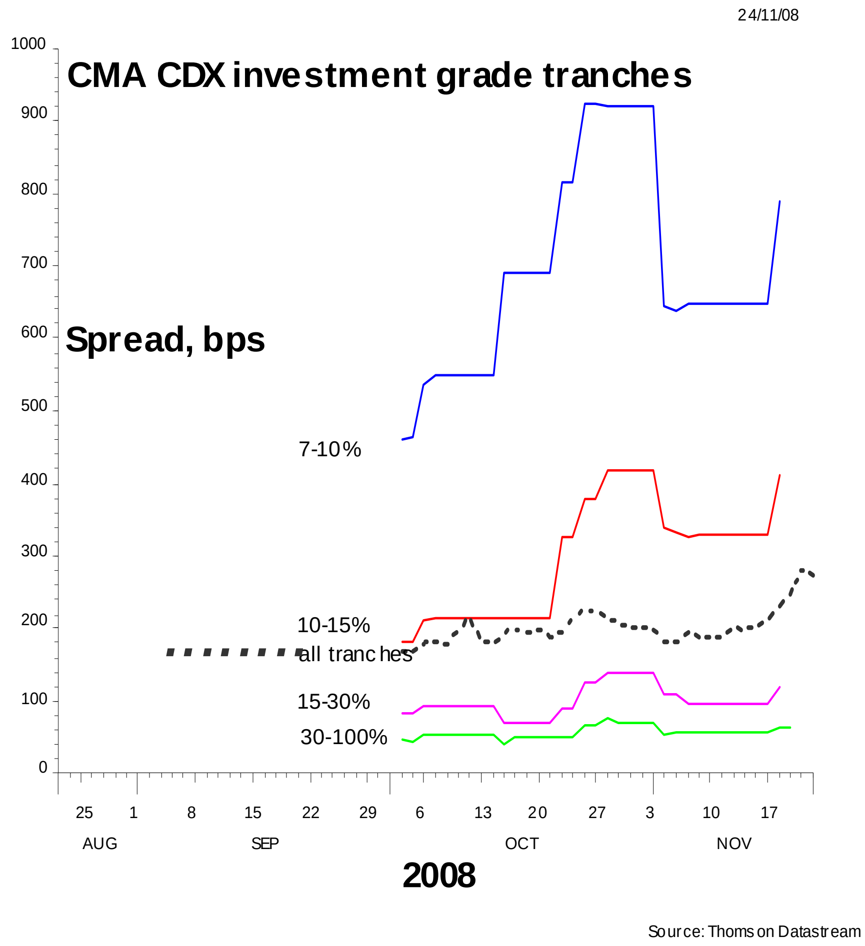
The Q Poisson jump theory adds transparency in tackling the 'new' risks (liquidity, term risk, credit spreads, credit portfolio and tail risk)

- Due to the term risk nature of these risks, the Q Poisson jump premia are very large therefore mark to market adjustments are very large
  - Slide 26 demonstrates these large movements
- The Q Poisson jump model provides an *a priori* method to estimate portfolio effects and study sovereign risks not accessible by equilibrium Gaussian risk models
  - Slide 27 indicates some of the work now in progress to evaluate these effects

DJ CDX Crash



24/11/08
CMA CDX investment grade tranches
Spread, bps
1000
900
800
700
600
500
400
300
200
100
0
7-10%
10-15
15-30%
30-100%
N D J F M A M J J A S O N
November
Month 2008
Source: Thomson Datastream



24/11/08
CMA CDX investment grade tranches
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1000
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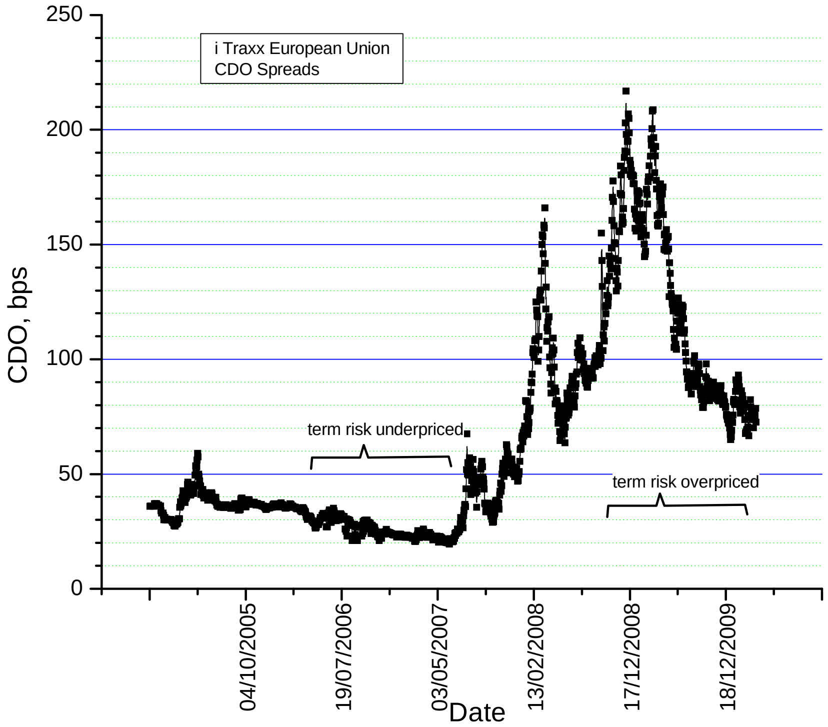
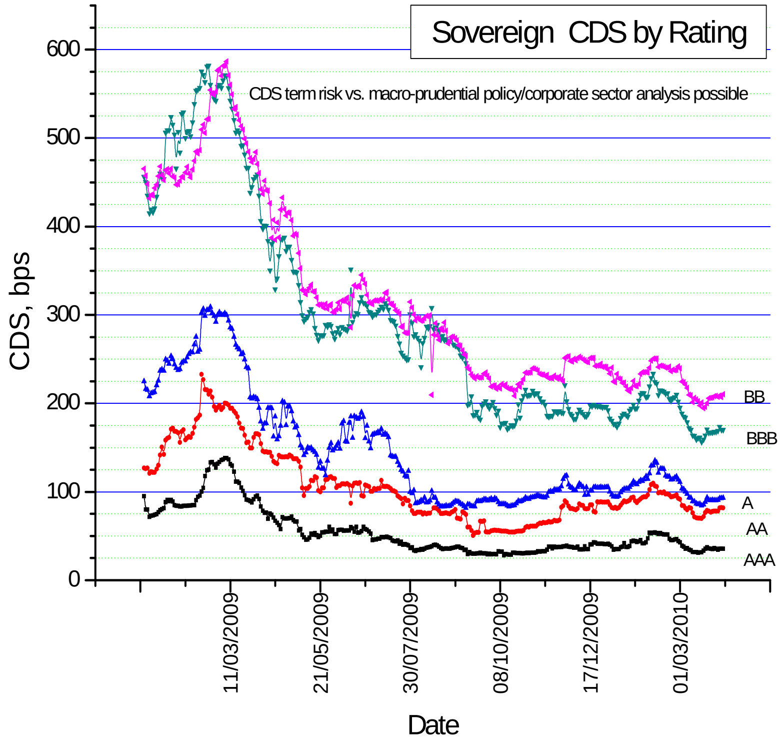
200
100
0
7-10%
10-15%
all tranches
15-30%
30-100%
25 1 8 15 22 29 6 13 20 27 3 10 17
AUG SEP OCT NOV
2008
Source: Thomson Datastream

Q Poisson applications to CDS and DCO structural market-pricing [work in process]

Current equilibrium models using cross-sectional correlations under and over price



Current equilibrium models fail to measure amount of term risk premium affecting spreads



TO CONCLUDE

The Q Poisson pricing kernel is given by:

$\phi''(\ln\omega)$ equates to a Cauchy distribution in log frequency (or log time, see below), for all values x to leading order:

$$PDF^Q(jumps) = \phi''(\ln(\omega)) = \frac{\sqrt{2}/\pi}{2 + (\ln(\omega) - \ln(\lambda))^2}$$
$$= \frac{\sqrt{2}/\pi}{2 + (\ln(t) - \ln(\tau))^2} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots..\dots\dots (12)$$

The PDF equation has an analytical cumulative Cauchy distribution of the form:

$$CDF^Q(jumps) = 0.5 + (1/\pi).arctan((\ln(\omega) - \ln(\lambda))/\sqrt{2})$$
$$= 0.5 + (1/\pi).arctan((\ln(t) - \ln(\tau))/\sqrt{2}) \dots\dots\dots\dots\dots\dots\dots\dots..\dots. (13)$$

TO CONCLUDE

Resolution risk management problems – practitioner issues

- The Q Poisson jump theory adds transparency in the ‘new’ risks (liquidity, term risk credit spread dynamics).
    - E.g. Large tail risk, portfolio correlation skew, intractable to Brownian Wiener diffusion
- Fair value of instruments affected by jump risk.
    - e.g. Genuine risk neutral model – not low premium physical jumps
- Allows time effects to be stochastically modelled.
    - E.g. Remove excess of mathematics in models, inappropriate assumptions, freedom to model from fundamentals (arbitrage free, parsimony in a priori assumption)

TO CONCLUDE

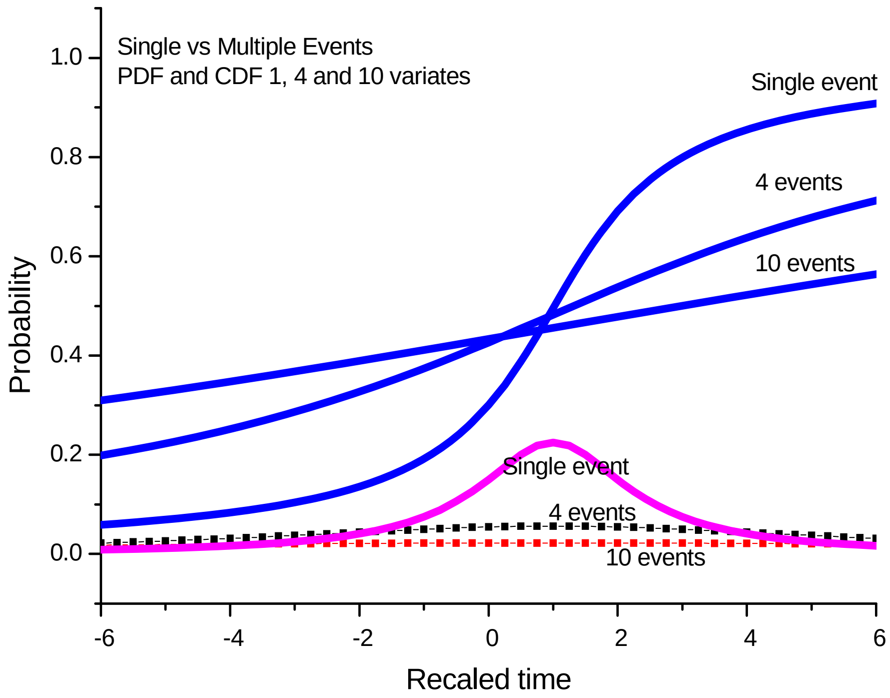
Resolution of conventional model problems – theory and practice

- The Q Poisson jump solves the gap between conventional Poisson models (reduced form) and the structural firm level (Merton) model.
    - E.g. Q Poisson structural in market not entity perspective
- Dynamics of the yield curve and the spreads curve from a priori perspective not reverse engineered from 'unpredictive' market data.
    - E.g. Focus on correct a priori assumption – Poisson intensity is a constant returns to scale, fractal theory
- Linear superposition due to Q Poisson martingale.
    - E.g. Term risk premium is high because the anticipated not historic risk is the relevant measure

TO CONCLUDE

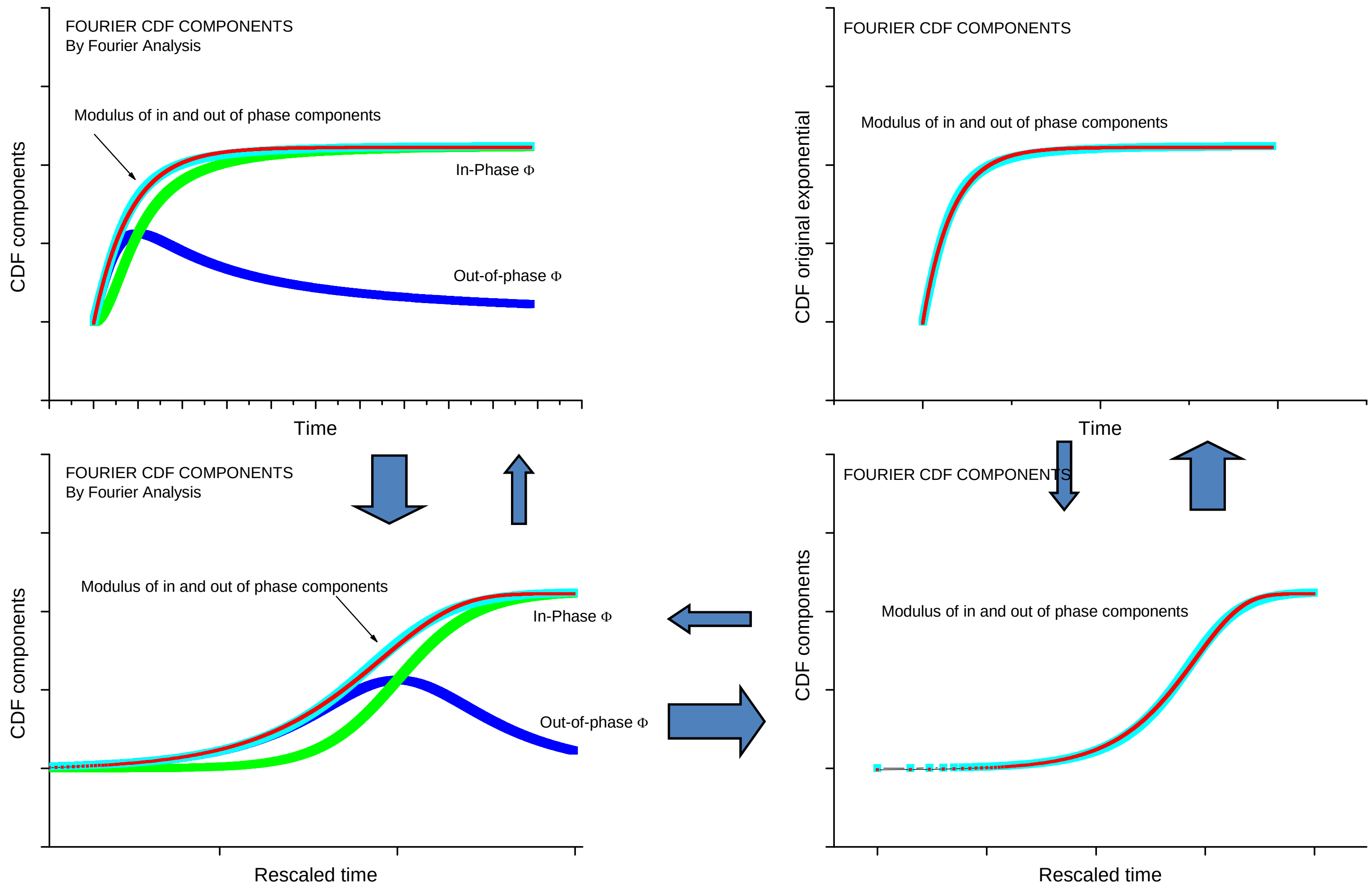
Appendix items

APPENDIX – Item 1
The Single Event Pathway Property for the Q Poisson



This demonstrates the single (rare) event property of the Q Poisson PDF and CDF. As Poisson events are added the combined log Cauchy distribution loses its skewness, becoming a normal distribution through the central limit theorem

APPENDIX – Item 2
Fourier Analysis of Jump (Poisson exponential) CDF



Demonstration of how the Fourier splits distributions harmonically into PDF and CDF (reversibility of transform assures linearity must apply)

Cox double stochastic process

In the Cox model it is assumed at the pathway level that counted jumps occur

$$p(N_t = n) = \frac{(\lambda t)^n . e^{-\lambda t}}{n!}$$

The financial modelling with Cox processes for example is founded on the principle that multiple jumps occur as iid processes with arrivals conditioned on information to the point in time (this is filtration of jumps conditional on information).
The sample paths follow the rule that sample paths are right continuous with left limits (cadlag). For any ω

the sample path is piecewise constant and increases by jumps of size 1.
The Cox process also allows for stochasticity in the jump size, given a jump arrival. An example of the Cox process with stochastic jumps sizes would be the Merton (1976) model, of the form:

$$p_t(x) = \sum_{n=1}^{\infty} \overbrace{f *^n (x)}^{Distribution\_over-jump\_size} \quad . \quad \underbrace{\frac{e^{-\lambda t}.(\lambda t)^n}{n!}}_{Distribution\_over\_multiple\_jumps} \qquad \forall x \neq 0$$

……… (3.5)

where p_t is jump size density, n is the count number of jumps, and $f(x)$

is the probability density at time t, conditional on the fact that the process has jumped at least once.
In the Merton (1976) model for discontinuous processes, the jumps size distribution is assumed to be lognormally distributed and the pathways are described by physical jump probabilities.