Tick Size Regulation, Intermarket Competition and Sub-Penny Trading

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Abstract

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1 Introduction

The minimum price change, or minimum tick size, for securities traded in financial markets is a timely issue in market design and has been at the center of the financial regulatory debate over the last decade. When reducing the minimum tick size, regulators have to ponder the trade-off between the benefits of the enhanced price competition for the provision of liquidity (and the resulting smaller spread), and the reduced incentive to submit limit orders caused by easier undercutting.

Nevertheless, over the last ten years stock exchanges around the World have persistently reduced the minimum price variations into small magnitudes (Table 1). The most recent change in the U.S. markets occurred between 2001 and 2004, when the minimum price change was gradually reduced to one penny. More precisely, today's quotations in National Market System (NMS) stocks must be priced in an increment of \$.01, unless the price of the quotation is less than \$1.00, in which case the minimum increment is \$.0001.

[**Insert Table 1 here**]

In this paper we model intermarket competition between two limit order books to investigate how the tick size should be regulated when trading platforms can compete with each other by reducing the tick size, and when some market participants can take advantage of internalization pools to undercut aggressively (by fractions of a penny) orders posted on top of visible limit order books.

As a vast body of empirical literature[1] has shown, when the minimum tick size is reduced, spread decreases, but depth at the top of the book deteriorates; for this reason to protect displayed limit orders from the practice of stepping ahead by trivial amounts, the Sub-Penny Rule (adopted Rule 612 under Regulation NMS) was introduced that prohibits market participants from displaying, ranking, or accepting quotations in NMS stocks that are priced at increments less than the minimum price variation.

During the last five years, however, two new elements - the development of dark markets and of fast trading facilities - have emerged that have deeply affected intermarket competition and have made Rule 612 ineffective to protect displayed limit orders. In particular, two features of the rule must be considered. First, the Security and Exchange Commission (SEC) Rule 612 prohibits market participant from quoting prices in sub-penny, but in the believe that sub-penny trading would not be as detrimental as sub-penny quoting, it expressly allows broker-dealers to "provide price improvement to a customer order that resulted in a sub-penny execution". Second, Rule 612 prohibition of sub-penny quoting does not apply to dark markets. This means that broker-dealers who can make use of dark pools to submit

[1] See Ahn et al. (1996 and 2007), Bacidore (1997), Bourghelle and Declerck (2004), Cai et al. (2008), Griffiths et al. (1998), Lau and McInish (1995), Porter and Weaver (1997) and Ronen and Weaver (2001).

their orders, have an advantage that permits them to jump the queue by a fraction of a penny and so preempt the National Best Bid Offer (NBBO); in addition, by drawing on fast algorithmic programmes to persistently replicate trading strategies, they can profit from the considerable amount of volume that they can create by buying price priority at negligible cost.[2]

The sub-penny practice should be more profitable for stocks that are either illiquid or low priced as broker-dealers can take advantage of gaining a wider relative spread when taking position at sub-penny on both sides of the market. This is confirmed by Delassus and Tyc (2010) who show that the practice of queue jumping is negatively related to the stock's price, resulting in 6% to 10% of total volume for NASDAQ stocks between $1 and $5.[3] Consistently, BATS (2009) shows that up to $20 the effective spread is held artificially wide and for lower priced stocks the use of dark markets is negatively related with stock prices.

It should be remarked, however, that the practice of queue jumping must not be confused with executions at the spread midpoint that take place in those dark pools designed to reduce the price impact of block trading (Buti, Rindi and Werner, 2010). Figure 1 shows in fact that over the last 10 years the proportion of queue jumping has dramatically increased compared to mid-quote trading.

[**Insert Figure 1 here**]

The negative consequence of sub-penny trading is that investors are discouraged from providing liquidity to the top of the book; hence, within the regulatory debate on the possible solutions to the sub-penny issue, the SEC has recently proposed the Trade-At rule[4] that would practically ban sub-penny trading by prohibiting "any trading center from executing a trade at the price of the NBBO unless the trading center was displaying that price at the time it received the incoming contra-side order." Quite the opposite is instead the proposed solution to level the playing field by reducing the minimum price increment of publicly displayed market centers to sub-pennies (BATS, 2009).

We build a model of intermarket competition to investigate the interaction of the minimum tick size rule with the decision of broker-dealers to sub-penny the liquidity provision on the top of the limit order book. Indeed the decision of a broker-dealer on whether to trade against a customer order (i.e. to internalize the order) by undercutting passive limit orders depends on the size of the minimum price variation. On the one hand the finer the price grid, the greater their profits from sub-penny trading as by offering a tiny price improvement they can seize considerable volume from the top of the public limit order book; on the other hand, supporters of tick size reduction argue that a smaller tick size can foster competition for the

[2] Jarnecic and Snape (2010) suggest that high frequency trading is negatively related to the tick size.

[3] Delassus and Tyc complete sample comprises 1800 NASDAQ stocks; their sample period includes 5 consecutive trading days.

[4] SEC concept release on Equity Market Structure No. 34-61358.

provision of liquidity and alleviate the negative effect of sub-penny trading. In addition one should also consider that a smaller tick size can negatively affect market depth. So the critical regulatory issue is to adjust the minimum tick size by taking into consideration the competitive environment that today's fragmented markets offer to financial operators.

Starting with a one-market model (Section 3 and 4), we show that the effect of a tick size reduction depends on both the liquidity of the stock and the asset value. For very illiquid stocks a reduction of the tick size discourages liquidity provision and worsens market quality, the opposite being true for liquid stocks, even though for very liquid stocks we find that inside depth decreases. These effects become stronger for lower priced securities.

We then extend the framework to a dual-market model (Section 5) and we find that, due to intermarket competition, liquidity concentrates on the small-tick limit order book. These results explain both the existing empirical evidence on the effects of a tick size reduction, and the current trend that sees exchanges competing to reduce the minimum tick size.

Finally, we investigate the sub-penny trading issue (Section 6) by embedding a group of broker-dealers who can choose between trading in a lit market or in an internalization pool where they can execute their customers' orders. Our model shows that for illiquid and low priced stocks the availability of internalization pools is detrimental as it worsens market quality; for liquid stocks it enhances price competition so that spread narrows but liquidity provision and trading volume substantially decline. These results allow us to discuss both the sub-penny rule and the recently proposed Trade-At Rule. We suggest that by setting the minimum price improvement, regulators should consider both the asset value and the bid-ask spread. This would reduce the incentive for broker-dealers to implement sub-penny trading in illiquid and low priced stocks and therefore prevent the effective bid-ask spread from being held unnaturally wide. Having adjusted the minimum tick size, regulators should consider a carefully designed Trade-At rule that banned sub-penny trading without unfairly impacting on dark venues (BATS, 2009) and commonly used hidden mid-point orders (Buti and Rindi, 2009). One possible waiver that could be introduced is to allow sub-penny trading and quoting to take place in dark markets at the prevailing inside spread midquote.

Our model is related to three strands of the existing theoretical literature (Section 2), respectively on intermarket competition[5], the optimal tick size[6] and broker-dealers' internalization of order flows (Battalio and Holden, 2001), and to the best of our knowledge it is the first one that allows researchers to investigate the minimum tick size rule within the context of intermarket competition. It also departs from existing theoretical works as it embeds sub-penny trading through an internalization pool.

[5] See, for example, Chowdhry and Nanda (1991) and Parlour and Seppi (2003).

[6] See Anshuman and Kalay (1998), Cordella and Foucault (1999), Foucault et al. (2005), Goettler et al. (2005), Kadan (2004) and Seppi (1997).

2 Literature Review

Financial literature extensively covers the relationship between the reduction of the tick size and market quality. Empirical findings from different markets concur that after reducing the tick size, spread and depth decline and that the spread is not equally affected across stocks.[7] These findings are also consistent with a recent pilot programme implemented by the major European platforms aimed at investigating the effect of a reduction of the tick size.[8] Theoretically Seppi (1997) points out that as the tick size decreases, cumulative depth is minimized. Goettler, Parlour and Rajan (2005) show that by reducing the tick size regulators do not achieve a Pareto improvement but rather an increase of total investors' surplus. Kadan (2004) demonstrates that the effects of a tick size reduction on dealers and investors' welfare depends on the number of dealers active in the market, being detrimental to dealers and beneficial to investors when the number of dealers is large.

As our paper also investigates how different markets compete for the provision of liquidity by reducing the tick size, it is also related to the literature on intermarket competition that documents an improvement in market efficiency brought about by competition.[9] Chowdhry and Nanda (1991) extend Kyle (1985) model to accommodate multi-market trading and show that markets with the lowest transaction costs attract liquidity. Closer to our framework Degryse et al. (2009) analyze the interaction between a dealer market and a crossing network and show that overall welfare is not necessarily enhanced by the introduction of a crossing network. Our framework substantially differs from this as we model competition between first two LOB and then a LOB and an Internalization Pool. Our model also departs from Buti, Rindi and Werner (2010) who model competition between a LOB and an Dark Pool, as it focuses on Internalization Pools.

Finally our model is related to the literature on broker-dealers internalization.[10] Battalio and Holden (2001) extends the Glosten and Milgrom (1985) protocol to consider the practice of payment for order flows and internalization. They show that brokers make profits by exploiting their direct relationships with customers. This is consistent with the related empirical works (Chung et al. (2004a and 2004b), Hansch et al. (1999), He et al. (2006), Hendershott and Jones (2005) and Porter and Weaver (1997)).

[7] See footnote 1.

[8] In December 2008, BATS Europe, in conjuction with Chi-X, Nasdaq OMX Europe and Turquoise, developed a proposal to standardize the tick size of the pan European trading platforms. Starting June 1, 2009, Chi-X, followed by Turquoise, BATS Europe and finally the LSE and Nasdaq OMX Europe reduced the tick size for a number of stocks. This pilot programme, aimed at studying the effect of a change in the tick size based on actual market data, showed that following the reduction of the tick size, effective spread, inside spread, inside depth and average trade size decreased (BATS, 2010).

[9] See, for example, Barclay et al. (2003), Bessembinder and Kaufman (1997), Fink et al. (2006) and Goldstein et al. (2008).

[10] Internalization is either the direction of order flows by a broker-dealer to an affiliated specialist, or the execution of order flows by that broker-dealer acting as a market maker.

3 Single Market Model

3.1 The Market

A market for a security is run over a trading day divided into T periods: $t = 1, ..., T$. At each period t a trader arrives and can submit orders of unitary size. Following Parlour (1998), traders are rational and have the following linear preferences:

$$U(C_1, C_2; \beta) = C_1 + \beta C_2$$

where C_1 is the cash inflow from selling or buying the security on day 1, while C_2 is the cash inflow from the asset payment on day 2 and is equal to $+v$ $(-v)$ in case of a buy (sell) order. Notice that traders are risk neutral and have a personal trade-off between consumption in the two days that is equal to β, a patience indicator drawn from the uniform distribution $U(\underline{\beta}, \overline{\beta})$ with $0 \leq \underline{\beta} < 1 < \overline{\beta}$. A patient trader has a β next to 1 while an eager one has extreme values of $\overline{\beta}$. This is a modelling device which captures the trade-off that traders' face between waiting costs and execution costs.

Upon arrival at the market in period t, the trader observes the state of the book that is characterized by the number of shares available at each level of the price grid. The latter assembles two prices on the ask (A_1, A_2) and two on the bid side of the market (B_1, B_2), symmetrically distributed around the asset value v. The difference between two adjacent prices, which we name τ, is equal to the minimum price increment and also corresponds to the minimum inside spread. Thus the available prices are equal to $A_2 = v + \frac{3\tau}{2}$, $A_1 = v + \frac{\tau}{2}$, $B_1 = v - \frac{\tau}{2}$ and $B_2 = v - \frac{3\tau}{2}$, and the state of the book is defined as $S_t = [Q_t^{A_2}, Q_t^{A_1}, Q_t^{B_1}, Q_t^{B_2}]$. As in Seppi (1997) and Parlour (1998), we assume that a trading crowd provides liquidity at the highest levels of the limit order book and prevents traders from bidding or asking prices that are too far away from the top of the book. Besides, traders are allowed to submit limit orders queuing in front of the trading crowd. In this parsimonious way, we can extend Parlour (1998) model to include two price levels where traders can submit orders, and, at the same time, keep the strategy space as small as possible. In addition we can investigate the effects of the tick size reduction on depth at different levels of the book.

Each trader can submit a unitary order that cannot be modified or cancelled thereafter; his strategy at time t is defined by H_t. The market permits two types of orders: limit orders (LO) represented by $+1$ and market orders (MO) represented by -1. Traders can hence submit limit orders to buy (sell) one share at different levels of the bid (ask) prices, or market orders which hit the bid (ask) prices and are executed immediately, or they can decide not to trade (no grade). More precisely, a trader's strategy space is $H = \{\pm 1^i, 0\}$, where $i = A_2, A_1, B_1$ and B_2. The change in the LOB induced by the trader's strategy H_t is indicated by h_t and defined as:

$$h_t = [h_t^{A_2}, h_t^{A_1}, h_t^{B_1}, h_t^{B_2}] = \begin{cases} [\pm1, 0, 0, 0] & \text{if } H_t = \pm1^{A_2} \\ [0, \pm1, 0, 0] & \text{if } H_t = \pm1^{A_1} \\ [0, 0, \pm1, 0] & \text{if } H_t = \pm1^{B_1} \quad \forall t = 1..T \\ [0, 0, 0, \pm1] & \text{if } H_t = \pm1^{B_2} \\ [0, 0, 0, 0] & \text{if } H_t = 0 \end{cases} \tag{1}$$

The state of the book is hence characterized by the following dynamics:

$$S_t = S_{t-1} + h_t, \, \forall t = 1..T \tag{2}$$

The expected state of the book at time t is given by:

$$E[S_{t|t-1}] = S_{t-1} + E[h_t], \, \forall t = 1..T \tag{3}$$

where $E[h_t^i] = \int\limits_{\beta \in \{\beta : H_t(\beta) = \pm 1^i\}} H_t(\beta) d\beta$ for $i = A_2, A_1, B_1, B_2$.

3.2 Order Submission Decision

To optimize his order submission strategy, a trader needs to choose an order type and a price. Hence traders have to maximize their utility, which in this risk neutral setting is equivalent to maximize their payoff, considering all the strategies available to them. Market orders guarantee immediate execution but higher price opportunity cost, while limit orders enable traders to get better prices at the cost of uncertain execution. Hence in this market traders face the standard trade-off between execution cost and price opportunity cost. The payoffs of the different strategies available to traders are listed in Table 2. Market equilibrium strategies are derived in the following Section.

[**Insert Table 2 here**]

In Table 2 we denote by A and B with no subscript the best available quotes, so that for example a market buy order executed at the best available price is indicated by -1^A. Notice that $p_t^*(A_k^{N_{-k}, N_k} | S_t)$ (or $p_t^*(B_k^{M_{-k}, M_k} | S_t)$) with $k = 1, 2$ is the equilibrium execution probability, conditional on the state of the limit order book, for a limit sell (or buy) order queuing at the N_k (M_k) position at the price level A_k (B_k), where $N_{-k} = \sum_{d<k} N_d$ ($M_{-k} = \sum_{d<k} M_d$) is the number of shares standing at lower price levels.[11] The execution probability depends both on the price level at which the order is posted and the depth available on the

[11]Notation with a star as superscript indicates for equilibrium values.

limit order book. An order posted at A_k and queueing at the $N_k - th$ position, is executed against the $(N_{-k} + N_k) - th$ market order only when $(N_{-k} + N_k - 1)$ market orders have already hit all the N_{-k} shares available at lower prices, and the $N_k - 1$ shares available at A_k with time priority. If $N_{-k} + N_k$ is larger than the number of remaining periods, additional limit orders at that price level will never be executed and $p_t(A_k^{N_{-k},N_k}|S_t) = 0$. Notice that the execution probability also depends on the state of the other side of the LOB: a deep LOB on the bid side increases the incentive for a seller to post limit orders as he knows that incoming buyers will be more inclined to post market orders (due to the long queue on the bid side). To facilitate the proof, when the best ask is A_k we indicate the execution probability of a limit order queuing at A_k by $p_t(A_k^{N_k}|S_t)$ instead of using $p_t(A_k^{0,N_k}|S_t)$ and the execution probability of the order standing fist by $p_t(A_k|S_t)$ instead of using $p_t(A_k^{0,1}|S_t)$.

3.3 Market Equilibrium

Traders first use information from the state of the limit order book to rationally compute different orders' execution probabilities, and then compare the expected payoffs from each order type to choose the optimal strategy consistent with their own β.[12] The model allows us to compute cutoff values for β from which we derive the probabilities of the equilibrium trading strategies. In this multi-period game, subgame perfect equilibrium is found by backward induction. At time T, the execution probability for limit orders is zero, hence traders will only submit market orders or decide not to trade. Hence, it can be easily shown that traders' equilibrium strategies are:

$$H_T^*(\beta, S_{T-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \frac{B}{v}) \\ 0 & \text{if } \beta \in [\frac{B}{v}, \frac{A}{v}) \\ -1^A & \text{if } \beta \in [\frac{A}{v}, \overline{\beta}] \end{cases}$$

where the best ask and bid prices are equal to $A = A_{1,2}$ $(B = B_{1,2})$ depending on the state of the LOB. By using these equilibrium strategies together with the distribution of β, we can calculate the equilibrium execution probabilities at the best quotes for limit orders submitted at $T - 1$:

$$p_{T-1}^*(A|S_{T-1}) = \int\limits_{\beta \in \{\beta : H_T^*|S_{T-1} = -1^A\}} 1 d\beta = \frac{\overline{\beta}v - A}{(\overline{\beta} - \underline{\beta})v} \tag{4}$$

$$p_{T-1}^*(B|S_{T-1}) = \int\limits_{\beta \in \{\beta : H_T^*|S_{T-1} = -1^B\}} 1 d\beta = \frac{B - \underline{\beta}v}{(\overline{\beta} - \underline{\beta})v} \tag{5}$$

[12] Notice that, differently from Parlour(1998), we do not assume that traders are ex ante buyers or sellers but rather endogenously solve for the trader's decision to buy or to sell the asset.

These execution probabilities are the dynamic link between period T and $T-1$. Notice that a trader arriving at $T-1$ can choose between a market and a limit order, and his choice is driven by his β value. The following Lemma holds:[13]

Lemma 1 *If at time $t \neq T$ at least one limit order strategy has positive execution probability, then there will always exist a β value for which a limit order is optimally selected by the incoming trader.*

As an example, if we solve the trader's maximization problem at $T-1$ after substituting the equilibrium execution probabilities at T given by (4) and (5) for the case $p^*_{T-1}(A_k^{N_{-k},N_k}|S_{T-1}) \neq 0$ and $p^*_{T-1}(B_k^{M_{-k},M_k}|S_{T-1}) \neq 0$, we obtain the following optimal strategies:

$$H^*_{T-1}(\beta, S_{T-2}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}) \\ +1^{A_k} & \text{if } \beta \in [\beta_{1,T-1}, \beta_{3,T-1}) \\ +1^{B_k} & \text{if } \beta \in [\beta_{3,T-1}, \beta_{5,T-1}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}, \overline{\beta}] \end{cases} \tag{6}$$

where $\beta_{1,T-1} = \frac{B}{v} - \frac{p^*_{T-1}(A_k|S_{T-1})}{1-p^*_{T-1}(A_k|S_{T-1})} \cdot \frac{A_k - B}{v}$, $\beta_{3,T-1} = \frac{p^*_{T-1}(A_k|S_{T-1}) + p^*_{T-1}(B_k|S_{T-1})B_k}{p^*_{T-1}(A_k|S_{T-1}) + p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{1}{v}$, and $\beta_{5,T-1} = \frac{A}{v} + \frac{p^*_{T-1}(B_k|S_{T-1})}{1-p^*_{T-1}(B_k|S_{T-1})} \cdot \frac{A-B_k}{v}$. Notice that $p^*_{T-1}(A_k^{N_{-k},N_k}|S_{T-1}) \neq 0$ only when $N_{-k} = 0$ and $N_k = 1$: a limit order posted at $T-1$ has a positive execution probability only if it undercuts all the orders resident on the LOB and gains execution priority, as only one trader can still arrive at the market at T. Moreover, the larger is the limit order execution probability, $p^*_{T-1}(A_k|S_{T-1})$, the smaller is the threshold between market sell orders and limit sell orders, β_1, and the more likely it is for traders to submit limit rather than market orders. More generally, if execution probabilities at time t are such that waiting costs are lower than execution costs, traders will submit limit orders. If instead execution probabilities are low, they will choose market orders. Notice also that the optimal price at which a trader will submit a limit order is the result of a trade-off between price risk and execution risk: a more competitive price implies a higher execution probability due to both the lower risk of being undercut by incoming traders and the greater attractiveness of the order for traders on the opposite side. However this is obtained at the cost of lower revenues once the order is executed. This trade-off crucially depends on the relative tick size $\frac{\tau}{v}$, as shown in the following Lemma:

Lemma 2 *At each time $t \neq T$ traders' aggressiveness in the provision of liquidity is positively related to the value of $\frac{\tau}{v}$.*

From the equilibrium strategies at $T-1$, we can derive the execution probabilities for limit orders submitted at $T-2$ and therefore we can compute the corresponding equilibrium

[13] All the proofs are presented in the Appendix.

strategies. We can then derive the execution probabilities of limit orders submitted in previous periods, compare the payoffs of different strategies, and finally compute equilibrium strategies back to period $t = 1$. Traders' equilibrium strategies shape the equilibrium books $S_{1:T}$. The equilibrium is defined as follows:

Definition 1 *Given an initial book S_0, a dynamic equilibrium is a set of order submission decisions $\{H_t^*\}$ and states of the limit order book $\{S_t\}$, for $t \in [1, T]$, such that at each period the trader maximizes his payoff $U(\cdot)$ (Table 2) according to his Bayesian belief over the execution probabilities $p^*(\cdot)$, i.e.*

$$
\begin{array}{ll}
\{H_t^* := \arg\max \ U(\cdot \mid S_{t-1}, p_{t-1}^*)\} & \forall t[1, T] \\
\{S_t := S_{t-1} + h_t^*\} & \forall t[1, T] \\
\text{where } h_t^* \text{ is defined by (1)} &
\end{array}
$$

4 Tick Size Reduction and Market Quality

To investigate the effects of a change in the tick size on traders' strategies and, as a result, on market quality, we start from the single market framework and then, in the next Section, we add intermarket competition by allowing agents to trade in a parallel LOB with a different tick size. This allows us to discuss in Section 6 how the tick size changes undertaken both in the US and in Europe during the last decade have interacted with the development of fast and dark trading facilities. We can finally compare the different recent proposals on equity market structure put forward both in US and in Europe by the SEC and the Committee of European Securities Regulators (CESR).

We start with the market characterized by a large tick size (LM) that we already presented in Section 3 and then we compare the resulting equilibrium trading strategies with those obtained when, all else equal, the tick size is reduced from τ to $\frac{1}{3}\tau$. Both price grids are shown in Table 3: on the LM market the price grid is still $P^{LM} = \{A_2, A_1, B_1, B_2\}$, while on the small tick market (SM) it has 5 levels on both the ask and the bid side, a_l and b_l, where $l = 1, .., 5$. Notice that for the SM the dynamics and expected state of the book are still characterized by equations (2) and (3) respectively, the main difference being that both S_t^{SM} and h_t^{SM} now consist of ten components instead of four. Notice also that the trader's strategy space is much richer thanks to the finer price grid, $H^{SM} = \{\pm 1^j, 0\}$ with $j = \{a_{1:5}, b_{1:5}\}$.

[**Insert Table 3 here**]

To compare the two markets, we build standard indicators of market quality using traders' equilibrium strategies. Depth is measured by the number of shares available on the LOB at different price levels. More precisely, for the LM we define average depth at price i as $DP_t^{i,LM} = E[Q_t^i]$, where $i = \{A_{1:2}, B_{1:2}\}$, and average depth at the best quotes, i.e. the inside depth, as $DPI_t^{LM} = E[Q_t^A + Q_t^B]$; total depth is measured by the sum of average depth at all price levels, $DPT_t^{LM} = \sum_i E[Q_t^i]$. Average spread is the expected difference between the best ask and bid prices, $SP_t^{LM} = E[A - B]$. Volume is measured by the number of orders executed, while liquidity provision is obtained by considering the number of limit orders submitted. As at each period only one trader arrives at the market who submits only one order of unitary size, expected volume, VL_t, and liquidity provision, LP_t, are computed as the probability that this trader will submit a market order or a limit order at all price levels:

$$VL_t^{LM} = E[\sum_i \int_{\beta \in \{\beta: H_t | S_{t-1} = -1^i\}} 1 d\beta]$$

$$LP_t^{LM} = E[\sum_i \int_{\beta \in \{\beta: H_t | S_{t-1} = +1^i\}} 1 d\beta]$$

Indicators of market quality for the SM are computed in a similar way, but using $j = \{a_{1:5}, b_{1:5}\}$. To illustrate the effects of a tick size reduction on different groups of stocks (liquid versus illiquid), we use the initial state of the book as a proxy for liquidity and consider there cases: an empty book for illiquid stocks, a book with either one or two units on the first (second) level of the LM (SM) price grid for liquid stocks. The following Proposition summarizes the effects of a tick size change on both traders' strategies and market quality:

Proposition 1 *When the tick size is reduced, changes induced on traders' order submission strategies and market quality depend on the initial state of the book.*

- *For liquid stocks*
  - *liquidity provision increases and spread and depth improve;*
  - *trading volume decreases;*
  - *for very liquid stocks the effects are the same except for inside depth that worsens.*
- *For illiquid stocks*
  - *the results are the opposite: liquidity provision decreases and spread, depth and inside depth deteriorate.*
- *All the above effects become stronger for low priced stocks.*

[**Insert Tables 4, 5 and 6 here**]

Tables 4 and 5 report results at $T-2$ for orders submission strategies and market quality for both the large and the small tick size regimes under two different opening states of the limit order book: with 1 share on $A_1(a_2)$ and $B_1(b_2)$ ($S_{T-3} = [01|10] = [00010|01000]$), and with 2 shares ($S_{T-3} = [02|20] = [00020|02000]$). By considering books that differ for market depth, we can offer insights on how the effects of a tick size change on market quality can be influenced by liquidity. Consider first the regime under which the book opens at $T-2$ with only 1 share on both $A_1(a_2)$ and $B_1(b_2)$ and notice that when the tick size is smaller (the price grid is finer), undercutting is cheaper and competition for the provision of liquidity becomes more intense so that in equilibrium traders switch from market to limit orders that they post to the new best price levels (a_1 and b_1). The result is that depth increases -both total and at the inside spread-, spread narrows and trading volume decreases (e.g. for $v = 1$, from .8994 to .7666). Notice that these effects become stronger as the stock price decreases: for example when v is equal to 1 following a reduction in the tick size, spread decreases by .02, whereas it only narrows by .0004 when v is equal to 50. Clearly, when the value of the tick size becomes relatively small compared to the stock price, the benefit of having a finer price grid decreases, and the probability that traders switch from market to limit orders posted at a_1 and b_1 becomes smaller.

Consider then the regime where the book opens full at $A_1(a_2)$ and $B_1(b_2)$ (Table 5) and notice that, compared to the previous case, the tick size reduction produces effects that are more intense and of the same direction except for depth at the inside spread, that decreases rather than increases. When at $T-2$ the book opens full at $A_1(a_2)$ and $B_1(b_2)$, in the large tick size protocol there is no room for limit orders and traders are forced either to use market orders or to refrain from trading; hence when the tick size is reduced, traders move even more aggressively than before to the top of the book (the probability of observing a limit order at $a_1(b_1)$ increases to .1167) and hence aggregate depth increases; however, depth at the top of the book decreases due to the fact that before the reduction of the tick size the liquidity pressure at the best bid and offer was very intense.

We can therefore conclude that the reduction of the tick size improves liquidity as it narrows the inside spread and increases total depth, but its effect on inside depth depends on the state of the book. If we believe that the regime with a deeper book is a good proxy for very liquid stocks, we can then suggest that for these stocks a tick size reduction can actually decrease depth at the inside spread. For illiquid stocks, however, that we proxy by the empty book, the effect of a tick size reduction is to worsen the inside spread as well. Indeed Table 6 shows that when the book opens empty, the inside spread widens and depth decreases. The reason being that traders do not have enough incentive to undercut aggressively by posting limit orders on the new top of the book: the higher execution probability they would obtain is in fact not large enough to compensate the lower execution price. Notice that, even in this case, when the stock price increases, the reduction of the tick size tends to produce effects that gradually drop off.

The results obtained so far are consistent with most of the existing empirical evidence on

tick size reduction[14] showing that when the tick size is reduced, inside spread decreases but depth not necessarily improves. The results are also consistent with Bourghelle and Declerck (2004) who investigate the effects of a reduction of the tick size and show that, as one moves to less liquid stocks, the percentage spread increases and the quoted depth decreases.
The results from Proposition 1 also show that the effect of the tick size reduction is stronger the lower priced securities are. Indeed, a relevant issue put forward by the most recent regulatory debate concerns the relation between tick size and stock price. It has been suggested that the tick size value established uniformly for all NMS stocks is not adequate for low priced stocks. More precisely, it is being observed that the current tick size is relatively too large for securities priced below $20 (BATS, 2009). Delassus and Tyc (2010) also suggest that for stocks between $1 and $5 the relatively high value of the minimum price change is playing a role in keeping the relative spread artificially wide. Our model captures this effect and explains how a wider tick size increases the bid-ask spread and, even more importantly, how this effect gets stronger as the price of the security decreases. Both Table 4 and Table 5 report equilibrium order submission probabilities and indicators of market quality for different stock prices and show that all the effects of a tick size reduction tend to lessen as the ratio between the tick size and the stock price decreases. In particular the positive effect that a reduction of the tick size produces on the inside spread steadily decreases with the increase in the stock price. Hence we can suggest that when the stock price is too small relative to the tick size, the inside spread is kept unnaturally wide so that, by curtailing the tick size, regulators can effectively reduce the spread. When instead the tick size is small relative to the stock price, a reduction of the tick size have marginal positive effects on market quality. This result bears important policy implications in that it suggests that what regulators should seek is not an unlimited reduction in the tick size, but rather an optimal ratio between the security price and the minimum price variation. We have shown that by increasing this ratio, the effect of a reduction of the tick size produces a smaller and smaller effect on the inside spread, and that for the most liquid stocks it can also reduce inside depth. We can then conjecture that the optimal tick to price ratio should be achieved by reducing the tick size down to the point where a further reduction does not produce any additional effect on the inside spread.

5 Dual-Market Model: Intermarket Competition

In this Section we extend the previous framework where we compare two markets with different tick size, by allowing these same markets to compete with each other: traders arriving at each period t can now choose not only their order type and aggressiveness, but

[14] See Ahn et al. (1996), Bacidore (1997), Harris(1994), and Porter and Weaver (1997).

also the trading venue where to submit their orders. Thus the strategy space expands into $H = \{H^{SM}, H^{LM}\}$, where $H^{LM} = \{\pm 1^i, 0\}$ with $i = \{A_{1:2}, B_{1:2}\}$, and $H^{SM} = \{\pm 1^j, 0\}$ with $j = \{a_{1:5}, b_{1:5}\}$. We also assume that if the two trading venues offer the same payoff, a trader will randomize his order submission and post his order to one of the two markets with equal probability. We indicate the number of shares available at each price level by $N_{1:2}$ ($n_{1:5}$) and $M_{1:2}$ ($m_{1:5}$) respectively for the ask and the bid side of the LM (SM). The dual-market model is solved by backward induction as the single market model, the following Lemma characterizes the equilibrium:

Lemma 3 *When the two markets open with the same depth at the common price levels, they have the same expected book dynamics at those same price levels.*

Intuitively, assume that one book is thinner than the other one at the common price levels. This implies that incoming traders have an incentive to submit limit orders at those price levels on the thinner book because of the higher execution probability. As a result, liquidity builds up in the thinner book up to the point in which limit order execution probabilities are equal in the two trading venues. So in equilibrium the two books should always have equal depth at the common price levels. Notice that, compared to the equilibrium strategies in the single market model, here traders need to consider the potential competition in liquidity provision coming from the other market. Hence they will take advantage of the finer price grid in the SM to undercut aggressively the quotes available on the LM, submitting orders at the price levels just below the common ones. For example a seller will submit orders on a_4, to avoid the competition from the trading crowd standing on the other market at A_2 or, if more impatient, on a_1 as no undercutting by incoming traders is possible at that price level. This aggressive liquidity supply is observed especially when competition is fierce, i.e. for liquid and/or low priced stocks. The following Proposition summarizes the main findings in the dual-market model.

Proposition 2 *When two trading venues with different tick size compete, no matter how liquid the markets and how priced the securities are, liquidity provision concentrates on the small tick size market as traders undercut orders standing on the large tick market.*

Competition for liquidity provision between markets with different tick size induces traders to submit limit orders exclusively to the small tick size market with the result that in the large tick market depth and inside spread deteriorate. Why do traders move to the new trading platform? Because when the tick size is smaller they have a better chance to fine tune the trade-off between execution risk and price risk, and more aggressive liquidity suppliers can submit their orders to the new inside quotes. Notice that, as in the single market case, this effect lessens significantly when the asset value increases from $v = 1$ to $v = 10$, and that it is stronger for more liquid stocks, i.e. when the book opens with 1 or more units posted at

A_1 (a_2) and B_1 (b_2).[15] The intuition here is the same as before: when the tick to price ratio decreases, any incentive from a tick size reduction becomes smaller and traders' advantage to undercut existing quotes decreases. However, by comparing the single market protocol with the dual market one, it can be noticed that market interaction fosters competition for the provision of liquidity so that the probability to observe orders posted to the top of the book is higher in the case with intermarket competition. And this effects -once again- becomes stronger for liquid securities. In fact, even if limit orders move to the SM, depth on the LM still attract liquidity demanders so that the execution probability of limit orders posted at the same price level on the SM is smaller than in the single market case. Clearly this is due to the assumption that traders randomize their market orders when expected profits from the two markets are the same, and it explains -for example- why in the dual market model traders post their orders to a_4 rather than a_5 as at a_5 the trading crowd is active on both markets an hence the execution probability of limit orders posted to this price level is halved compared to the single market case. Noticeably our three-period model does not capture the whole real dynamic interaction between the two markets, as presumably the LM would shut down as soon as the trading crowd were moved to the SM; however, our model markedly shows how the adjustment process starts and in which direction the equilibrium would converge had we assumed an "endogenous" rather than exogenous trading crowd.
These results are consistent with the empirical findings by Oppenheimer et al. (2003) who analyze the impact of US decimalization on the Canadian stocks and find that spread and inside depth decline by a greater amount in the US than in Canada. Similarly, Lin et al. (2009) look at the effect of US decimalization on stocks cross-listed on Euronext and NYSE and find that the NYSE proportion of trading of French firms declines markedly in the post-decimalization period. Our results are also useful to investigate the effects of competition between MTFs/ECNs and regulated markets. Consistently with Proposition 2, Bias et al. (2003) find that limit orders posted by traders to Island ECN undercut NASDAQ quotes. Similarly Hengelbrock and Theissen (2009) find that competition fostered by the entry of Turquoise led to a decrease in the inside spread.

6 Sub-Penny Trading and Internalization Pool

In this Section, we extend the dual-market model to investigate sub-penny trading. This practice is carried out by those broker-dealers who can access internalization pools (IP) to compete on price with the liquidity posted to the top of the book by limit order traders. IP are a special type of dark pools, that were initially designed to internalize order flows for cost-saving purposes (Degryse et al. 2009). Rosenblatt breakdown (Table 8) shows that

[15] Notice that having 1 or more shares on A_1 (a_2) and B_1 (b_2) is indifferent as, when traders can choose between two trading venues, depth at each price level is the result of the sum of the shares posted in the two markets - traders in fact can submit their market orders to both and obtain the same execution price.

volume made on IP has steadily increased over the last three years and that, for example, in August 2010 it attracted 8.55% of the consolidated US equity volume -the rest being drawn by public crossing networks, exchange and consortium based pools- scoring an increase of 25% over the previous year. One of the main features that characterizes IP is that they are controlled by broker-dealers and hence contain proprietary order flows. In practice broker dealers can use IP to internalize orders and execute them at sub-penny quotes. How is this possible? Rule 612 does not allow market participants to quote prices in sub-penny on lit markets, but it indeed allows broker-dealers to execute customers' orders at quotes that price improve even by only a fraction of the tick size (which in US markets is equal to 1 penny for stocks priced above $1). This is precisely what broker-dealers can achieve by posting sub-penny quotes in their internalization pools. Assume, for example, that the best bid and ask prices on a public limit order book are equal to $50.62 and $50.70 respectively; then a broker-dealer can post a limit order to sell at $50.6999 and a limit order to buy at $50.6201 so that when an investor sends a market buy order for -say- 500 shares he/she sells short $25349.95, and when another investor sends a market sell order for a further 500 shares, he can cover his short position by buying in front of the displayed bid for $25310.05. With this round trip transaction the broker-dealer captures the bid-ask spread and earns a profit of $39.90, whereas the two investors save $0.1 as their orders are executed at a $1/100 better than the prices quoted on the lit market. Who bears the cost of allowing dealers to use internalization pools to step in front of the NBBO? Liquidity suppliers who were offering displayed liquidity at the best bid offer suffer a reduction in the execution probability of their orders. This can lead to a reduced incentive to supply liquidity and in turn to a lower depth.

[**Insert Table 8 here**]

Given that the volume intermediated at sub-penny is steadily increasing (Figure 1), and that the estimated percentage of share volume in NMS Stocks intermediated by broker-dealers accounted up to 17.5% in September 2009 (SEC, 2010) regulators are worried about the ultimate effects of sub-penny trading on market quality. Does this practice foster competition for the provision of liquidity or it only allows highly sophisticated dealers to generate considerable returns from their activity? To answer this question we adapt our previous dual market model to embed sub-penny trading.

We assume that at each trading period one individual out of two groups of traders -rather than one- arrives at the market: with probability α the incoming trader is a broker-dealer and with the complementary probability he is a normal trader. While a normal trader can only post his orders to the public limit order book (PLB, same as previous LM) the incoming broker-dealer can use both the PLB and the internalization pool (IP) with a smaller tick size where he can undercut orders posted by other traders on the top of the PLB. So we assume as before that two markets compete with each other, one of which with a smaller tick size; however, differently from before, we assume that only a fraction α of the investors' population can trade on the small tick market. Furthermore, we assume that consistently with the very nature of real IP, the small tick market does not have -as before- a trading crowd that closes the book at A_2 and B_2. Finally, while only broker-dealers can post limit orders to the IP, all traders can take advantage of the liquidity offered by both trading platforms, which is consistent with the existence of a smart order routing technology[16] that allows all investors to simultaneously access multiple sources of liquidity (Butler, 2010). The degree of access to IP volume affects the visibility of this trading platform by normal traders. We will first assume perfect inference and then extend the model to include partial inference and Bayesian learning.[17] The effect of this assumption is that the information structure of the game can change: broker-dealers directly observe both markets as they are allowed to submit orders in both, while the other traders observe the public limit order book, but can only infer the state of the IP. The US market provides the National Best Bid Offer (NBBO) and therefore allows traders to search for the best execution on a consolidated limit order book. Accordingly we assume that traders can employ smart order routers to search the best quotes on the consolidated limit order book (PLB&IP): if the resulting inference is perfect, then investors' market orders will always get the best execution even though they do not necessarily observe the IP. If instead not all of them have access to sophisticated liquidity aggregators, then their overall inference will be based on their gradual learning process. We will start by assuming perfect inference and move later on to the gradual learning hypothesis. The following Proposition summarizes the results obtained under the assumption of perfect foresight.

Proposition 3 *When an Internalization Pool is added to a Public Limit Order Book that allows broker-dealers to sub-penny existing liquidity on the PLB, traders' order submission strategies and market quality change as follows.*

- *For illiquid stocks:*
  - *market quality deteriorates as liquidity provision decreases: both depth at the BBO and inside spread worsen;*
  - *these effects become stronger as the proportion of broker-dealers (α) increases;*
  - *for low priced stocks these effects are substantial, while they soften as the price of the stock increases.*
- *For liquid stocks, the effect of sub-penny trading is to foster price competition so that in the PLB spread and depth improve, liquidity provision worsens, and trading volume decline.*

[16]Examples are ITG Dark Aggregator and Smartrade Liquidity Aggregator.
[17]This version of the paper only includes results from the case with perfect inference.

The effect of sub-penny trading is detrimental to market quality for illiquid stocks, whereas it improves market quality for liquid ones; both effects are stronger for low priced stocks. Tables 9 and 10 provide the results for both $\alpha = 10\%$ and $\alpha = 50\%$.[18] Table 9 is focused on the case with an empty opening book at $T-2$, that we assume to proxy illiquid stocks, whereas Table 10 shows results for a book that opens with one share on $A_1(a_2)$ which should offer intuitions for more liquid stocks. Starting from Table 9, notice that comparing the two market protocols, without an internalization pool (PLB) and with IP (PLB&IP), both limit orders and depth decrease when an IP is added to the PLB, the reason being that traders perceive the potential competition of broker-dealers and react by either shifting to market orders or supplying less aggressive liquidity to the book. As a result, when this effect is substantial, i.e. for low priced stocks, the inside spread also worsens with the introduction of aggressive broker-dealers. Indeed, as the proportion of broker-dealers (α) increases, these effects become stronger: when the number of broker-dealers is low, there is not much competition among them and they tend to submit their orders to the PLB; when instead their number increases they tend to undercut at a_1 and as a result normal traders switch even more heavily to market orders or, when patient, become more conservative in the provision of liquidity. Notice that as before when the value of the stock is higher, traders are likely to use more market than limit orders as the price advantage offered by a limit order becomes too small to compensate for its higher execution risk. We can then conclude that for illiquid stocks sub-penny trading has a strong negative impact on market quality.
For more liquid stocks (Table 10) the main effect of sub-penny trading is to foster price competition: when the market opens with some depth on the PLB the effect on market quality is positive as both spread and depth improve. These improvements on the PLB are due to a reduction of the market orders submitted to this market venue: when the IP platform is introduced, traders allowed to trade on both markets submit limit orders to the IP at a_1 to undercut the existing depth at A_1, instead of posting market orders to the PLB at A_1. However, the switch of orders from PLB to IP implies that liquidity provision worsens and volume declines on the public venue, as documented respectively by the reduction in limit and market orders submissions. As before, all these effects get stronger when α increases, and tend to vanish when the value of the asset raises.
Which conclusions can we draw from these results? We show that sub-penny trading is detrimental for market quality when it takes place in illiquid stocks, especially low priced, whereas it benefits market quality for liquid ones. The existing preliminary empirical evidence (Delassus and Tyc, 2010) shows that as the stock value decreases and the relative spread increases, the percentage of sub-penny trading via internalization pool increases. This evidence combined with our theoretical results supports the SEC concern about sub-penny trading that was recently discussed in the April 2010 Concept Release on Equity Market Structure.

[18] Results for intermediate values of both α and v are available from the authors upon request.

[**Insert Tables 9 and 10 here**]

7 Policy Discussion and Conclusions

In light of the growing interest for tick size regulation and its effects in a global fragmented environment, this paper extends the existing literature on tick size and intermarket competition in a number of directions.

First of all, it discusses the effects of a reduction in the tick size within the context of a limit order book and shows that such effects depend on the liquidity of the stock and its underlying asset value. In this respect, we show that the market quality of illiquid stocks worsens with a reduction of the tick size, which on the other hand improves the liquidity of liquid stocks. We also show that the effect is relevant for low priced stocks, whereas it vanishes as the value of the security increases. These results, that are consistent with most of the existing empirical evidence, suggest that the objective of the tick size regulation should not be an indefinite reduction of the tick size, but rather the definition of a minimum price change that is consistent with the stock's main attributes and should be related to liquidity and asset value.

This paper also extends previous literature on tick size to include intermarket competition and shows that when two LOB compete on price by reducing their tick size, liquidity concentrates on the smaller-tick market, which is consistent with the prevailing tendency of exchanges to competitively reduce their minimum price change.

This extension constitutes the building block that allows us to investigate the issue of sub-penny trading, which is one of the main concerns expressed by the SEC in the April 2010 concept release on Equity Market Structure. Our model suggests that sub-penny trading undertaken by broker-dealers in their internalization pools can have dramatic effects on the quality of illiquid and low priced stocks, and as the existing very preliminary evidence shows that this practice is indeed mostly concentrated on these stocks, our results strongly support the SEC's concerns. The popularity of sub-penny trading, substantiated by the broker-dealers' internalization activity that accounts for up to 17.5% of the US equity share volume, induced the Commission to outline (SEC, 2010) the concept of a "Trade-At" rule that "would prohibit any trading center from executing a trade at the price of the NBBO unless the trading center was displaying that price at the time it received the incoming contra-side order. Under this type of rule, for example, a trading center that was not displaying the NBBO at the time it received an incoming marketable order could either: (1) execute the order with significant price improvement (such as the minimum allowable quoting increment, generally one cent); or (2) route ISOs to fully displayed size of NBBO quotations and then execute the balance of the order at the NBBO price."

This rule would have the benefit of prohibiting broker-dealers to step in front of the NBBO by a fraction of a penny, thus avoiding the practice of pre-emptying the public limit order book;[19] however, it would have a detrimental effect on those public crossing networks -dark pools- that are designed to trade blocks and generally execute at the spread midpoint; similarly, such rule would negatively affect totally undisclosed orders that are allowed by several exchanges to be pegged at the spread midpoint. A possible solution that we suggest -but that our model does not tackle- is to introduce the Trade-At rule together with a waiver that allows sub-penny quoting and trading at the spread midpoint in dark venues. Future extensions of our model could aim at discussing this proposal.

Finally, our analysis suggests that future empirical work should focus on the effect of high frequency trading on market quality during the second half of the last decade. Existing preliminary evidence shows that, starting from 2005, sub-penny trading swamped in US markets and because this practice is tightly linked to algorithmic and fast trading programs, it would be interesting to verify whether Hendershott et al. (2010) positive results on the effects of high frequency trading on liquidity would still hold for the most recent time period.

[19] Notice that this proposal is consistent with the even more recent concerns raised after the turmoil of May 6, 2010 by market professionals who believe the idea of depth-of-book protection should be revisited (Chapman, 2010).

Appendix

A Proof of Lemma 1

Proof. At any period $t \neq T$, a trader selects his optimal strategy H_t^* by comparing the payoffs of all the strategies described in Table 2, $H_t = \{-1^B, +1^{A_k}, +1^{B_k}, -1^A, 0\}$. Assume that $p_t^*(A_k^{N_{-k},N_k}) > 0$ where in order to simplify the notation in this proof we omit to write that all the execution probabilities are conditional on S_t. We compute the threshold $\beta_{-1^B,0}$ between $H_t = -1^B$ and $H_t = 0$ by equalizing the profits from the two strategies: $B - \beta v = 0$, and we obtain $\beta_{-1^B,0} = \frac{B}{v}$. Similarly we compute the threshold between $H_t = -1^B$ and $H_t = +1^{A_k}$ and obtain $\beta_{-1^B,1^{A_k}} = \frac{B}{v} - \frac{p_t^*(A_k^{N_{-k},N_k})}{1-p_t^*(A_k^{N_{-k},N_k})} \cdot \frac{A_k - B}{v}$. Notice that under the assumption that $p_t^*(A_k^{N_{-k},N_k}) > 0$, then $\beta_{-1^B,1^{A_k}} < \beta_{-1^B,0}$ and hence there always exists a value for $\beta \in (\beta_{-1^B,1^{A_k}}, \beta_{-1^B,0})$ such that $H_t^*(\beta, S_{t-1}) = +1^{A_k}$. A similar result holds for the bid side. Clearly, traders' equilibrium β-thresholds and hence strategies crucially depend on the state of the book that traders face when arriving at the market, as these affect the execution probabilities of their limit orders. Hence we can have four possible scenarios. When there is room for limit orders on both sides of the market, equilibrium traders' strategies for $t \neq T$ are:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ +1^{A_k} & \text{if } \beta \in [\beta_1, \beta_3) \\ +1^{B_k} & \text{if } \beta \in [\beta_3, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \text{ if } p_t^*(A_k^{N_{-k},N_k}) \neq 0 \;\&\; p_t^*(B_k^{M_{-k},M_k}) \neq 0 \tag{7a}$$

where $\beta_1 = \frac{B}{v} - \frac{p_t^*(A_k^{N_{-k},N_k})}{1-p_t^*(A_k^{N_{-k},N_k})} \cdot \frac{A_k - B}{v}$, $\beta_3 = \frac{p_t^*(A_k^{N_{-k},N_k})A_k + p_t(B_k^{M_{-k},M_k})B_k}{p_t^*(A_k^{N_{-k},N_k}) + p_t(B_k^{M_{-k},M_k})} \cdot \frac{1}{v}$ and $\beta_5 = \frac{A}{v} + \frac{p_t^*(B_k^{M_{-k},M_k})}{1-p_t^*(B_k^{M_{-k},M_k})} \cdot \frac{A - B_k}{v}$. When instead the book open full either on the ask or on the bid side, the equilibrium strategies are respectively:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ 0 & \text{if } \beta \in [\beta_1, \beta_4) \\ +1^{B_k} & \text{if } \beta \in [\beta_4, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \text{ if } p_t^*(A_k^{N_{-k},N_k}) = 0 \;\&\; p_t^*(B_k^{M_{-k},M_k}) \neq 0 \tag{7b}$$

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ +1^{A_k} & \text{if } \beta \in [\beta_1, \beta_2) \\ 0 & \text{if } \beta \in [\beta_2, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \text{ if } p_t^*(A_k^{N_{-k},N_k}) \neq 0 \;\&\; p_t^*(B_k^{M_{-k},M_k}) = 0 \tag{7c}$$

where $\beta_2 = \frac{A_k}{v}$ and $\beta_4 = \frac{B_k}{v}$.Finally, when the book is full on both sides, equilibrium strategies are:

$$H_t^*(\beta, S_{t-1}) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_1) \\ 0 & \text{if } \beta \in [\overline{\beta}_1, \beta_5) \\ -1^A & \text{if } \beta \in [\beta_5, \overline{\beta}] \end{cases} \quad \text{if } p_t^*(A_k^{N_{-k},N_k}) = 0 \ \& \ p_t^*(B_k^{M_{-k},M_k}) = 0 \tag{7d}$$

Notice that if $p_t^*(A_k^{N_{-k},N_k}) = p_t^*(B_k^{M_{-k},M_k}) = 0$, $H_t^*(\beta, S_{t-1}) = H_T^*(\beta, S_{t-1})$. ■

B Proof of Lemma 2

Proof. As an example, we consider the ask side in period $T-1$; the other cases can be derived in a similar way. At $T-1$ limit orders have positive execution probability on both A_1 and A_2 only when the book opens empty on both sides, $S_{T-1} = [0, 0, 0, 0]$. In this case traders can optimally select their level of price aggressiveness. Profits from the two available limit order strategies are:

$$\boxed{\begin{array}{lcl} H_{T-1} = +1^{A_2} & : & (A_2 - \beta v) \cdot p_{T-1}^*(A_2 \,|[1000]\,) = (A_2 - \beta v) \cdot \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta}) v} \\ H_{T-1} = +1^{A_1} & : & (A_1 - \beta v) \cdot p_{T-1}^*(A_1 \,|[0100]\,) = (A_1 - \beta v) \cdot \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta}) v} \end{array}}$$

A limit order at A_1 is optimal if $\exists \beta$ such that $(A_1 - \beta v) \cdot p_{T-1}^*(A_1 \,|[0100]\,) > \max\{B_2 - \beta v, (A_1 - \beta v) \cdot p_{T-1}^*(A_2 \,|[1000]\,)\}$; in this case that the threshold between $H_{T-1} = -1^B$ and $H_{T-1} = +1^{A_1}$ is smaller than the threshold between $H_{T-1} = -1^B$ and $H_{T-1} = +1^{A_2}$. More precisely, as $\beta_{-1^B,1^{A_k}} = \beta_1 \,|_{B=B_2, A_k} = \frac{B_2}{v} - \frac{p_{T-1}^*(A_k|S_t)}{1 - p_{T-1}^*(A_k|S_t)} \cdot \frac{A_k - B_2}{v}$, in order for $\beta_{-1^B,1^{A_1}} < \beta_{-1^B,1^{A_2}}$ the lower selling price (by one tick, τ) must be compensated by a higher execution probability. As $p_{T-1}^*(A_1 \,|[0100]\,) - p_{T-1}^*(A_2 \,|[1000]\,)$ is an increasing function of the relative tick size, for $H_{T-1} = +1^{A_1}$ to be an optimal strategy, $\frac{\tau}{v}$ must be larger than $\widehat{\frac{\tau}{v}}$, where $\widehat{\frac{\tau}{v}}$ solves $(A_1 - \beta v) \cdot \frac{\overline{\beta} v - A_1}{(\overline{\beta} - \underline{\beta}) v} - (A_2 - \beta v) \cdot \frac{\overline{\beta} v - A_2}{(\overline{\beta} - \underline{\beta}) v} = 0$. ■

C Proof of Proposition 1

Proof. 1) Illiquid stocks: starting book at $T-2$ is [0000]

(1.1) Consider a LM where at $T-2$ traders' strategy space is $\{-1^B, +1^{A_2}, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$; each strategy corresponds to an opening book at $T-1$ equal to [0000], [1000], [0100], [0010], [0001] and [0000] respectively. Let's consider -as an example- the book that opens at $T-1$ with one share on A_2, [1000]. Traders' strategy space is $\{-1^B, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$ and the corresponding payoffs are:

$H_{T-1} = -1^B$	:	$B_2 - \beta v$
$H_{T-1} = +1^{A_1}$	:	$(A_1 - \beta v) \cdot p^*_{T-1}(A_1 \,\vert [1100]\,)$
$H_{T-1} = +1^{B_1}$	:	$(\beta v - B_1) \cdot p^*_{T-1}(B_1 \,\vert [1010]\,)$
$H_{T-1} = +1^{B_2}$	:	$(\beta v - B_2) \cdot p^*_{T-1}(B_2 \,\vert [1001]\,)$
$H_{T-1} = -1^A$	:	$\beta v - A_2$
$H_{T-1} = 0$	:	0

where the execution probabilities are given by (4) and (5), presented in Section 3.3. After comparing these payoffs, we obtain the equilibrium strategies at $T-1$ that depend on the relative tick size $\frac{\tau}{v}$, as shown in Lemma 2. For the time being we assume that the relative tick size is such (small enough) that traders are not aggressive and post limit orders at higher levels of the book when available. The equilibrium strategies at $T-1$ are given by formula (6):

$$H^{*LM}_{T-1}(\beta, [1000]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}\,\vert_{A_1,B=B_2}\,) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-1}\,\vert_{A_1,B=B_2}\,, \beta_{3,T-1}\,\vert_{A_1,B_2}\,) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-1}\,\vert_{A_1,B_2}\,, \beta_{5,T-1}\,\vert_{A=A_2,B_2}\,) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}\,\vert_{A=A_2,B_2}\,, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-1}\,\vert_{A_1,B=B_2} = \frac{B_2}{v} - \frac{p^*_{T-1}(A_1\vert[1100]\,)}{1-p^*_{T-1}(A_1\vert[1100]\,)} \cdot \frac{A_1-B_2}{v}$, $\beta_{3,T-1}\,\vert_{A_1,B_2} = \frac{p^*_{T-1}(A_1\vert[1100]\,)A_1 + p^*_{T-1}(B_2\vert[1001]\,)B_2}{p^*_{T-1}(A_1\vert[1100]\,)+p^*_{T-1}(B_2\vert[1001]\,)} \cdot \frac{1}{v}$ and $\beta_{5,T-1}\,\vert_{A=A_2,B_2} = \frac{A_2}{v} + \frac{p^*_{T-1}(B_2\vert[1001]\,)}{1-p^*_{T-1}(B_2\vert[1001]\,)} \cdot \frac{A_2-B_2}{v}$. This allows us to compute the execution probability of the strategy $H_{T-2} = +1^{A_2}$ that produces at $T-1$ the book that we are considering, i.e. [1000]:

$$\begin{aligned} p^*_{T-2}(A_2\,\vert[1000]\,) &= \frac{\overline{\beta} - \beta_{5,T-1}\big\vert_{A=A_2,B_2}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{5,T-1}\big\vert_{A=A_2,B_2} - \beta_{3,T-1}\big\vert_{A_1,B_2}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(A_2\,\vert[1001]\,) \\ &+ \frac{\beta_{1,T-1}\big\vert_{A_1,B=B_2} - \underline{\beta}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(A_2\,\vert[1000]\,) \end{aligned}$$

The first term on the RHS of this equation represents the probability of being executed at $T-1$, while the other two terms stand for the probability of being executed during the last period T. Similarly, we are able to get the execution probabilities at $T-2$ for the other order types, that produce respectively the opening books [0000],[0100],[0010] and [0001] at $T-1$. We can then replicate the procedure used at $T-1$ to compare all possible trader's payoffs at $T-2$:

$H_{T-2}=-1^{B}$	:	$B_2-\beta v$
$H_{T-2}=+1^{A_2}$	:	$(A_2-\beta v)\cdot p^*_{T-2}(A_2\,\vert[1000])$
$H_{T-2}=+1^{A_1}$	:	$(A_1-\beta v)\cdot p^*_{T-2}(A_1\,\vert[0100])$
$H_{T-2}=+1^{B_1}$	:	$(\beta v-B_1)\cdot p^*_{T-2}(B_1\,\vert[0010])$
$H_{T-2}=+1^{B_2}$	:	$(\beta v-B_2)\cdot p^*_{T-2}(B_2\,\vert[0001])$
$H_{T-2}=-1^{A}$	:	$\beta v-A_2$
$H_{T-2}=0$	:	0

Hence equilibrium strategies at $T-2$ are:

$$H^{*LM}_{T-2}(\beta,[0000])=\begin{cases}-1^{B} & \text{if } \beta\in[\underline{\beta},\beta_{1,T-2}\,|_{A_2,B=B_2})\\ +1^{A_2} & \text{if } \beta\in[\beta_{1,T-2}\,|_{A_2,B=B_2}\,,\beta_{3,T-2}\,|_{A_2,B_2})\\ +1^{B_2} & \text{if } \beta\in[\beta_{3,T-2}\,|_{A_2,B_2}\,,\beta_{5,T-2}\,|_{A=A_2,B_2})\\ -1^{A} & \text{if } \beta\in[\beta_{5,T-2}\,|_{A=A_2,B_2}\,,\overline{\beta}]\end{cases}$$

where $\beta_{1,T-2}\,|_{A_2,B=B_2}\;=\frac{B}{v}-\frac{p^*_{T-2}(A_2|[1000])}{1-p^*_{T-2}(A_2|[1000])}\cdot\frac{A_2-B}{v}$, $\beta_{3,T-2}\,|_{A_2,B_2}\;=\frac{p^*_{T-2}(A_2|[1000])A_2+p^*_{T-2}(B_2|[0001])B_2}{p^*_{T-2}(A_2|[1000])+p^*_{T-2}(B_2|[0001])}\cdot\frac{1}{v}$, $\beta_{5,T-2}\,|_{A=A_2,B_2}\;=\frac{A}{v}+\frac{p^*_{T-2}(B_2|[0001])}{1-p^*_{T-2}(B_2|[0001])}\cdot\frac{A-B_2}{v}$.

When instead $\frac{\tau}{v}$ is large, submitting limit orders on the first level of the book becomes also an optimal strategy. We present directly the equilibrium strategies at $T-2$, where formula (7a) is modified as follows:

$$H^{*LM}_{T-2}(\beta,[0000])=\begin{cases}-1^{B} & \text{if } \beta\in[\underline{\beta},\beta_{1,T-2}\,|_{A_1,B=B_2})\\ +1^{A_1} & \text{if } \beta\in[\beta_{1,T-2}\,|_{A_1,B=B_2}\,,\beta_{6,T-2}\,|_{A_1,A_2})\\ +1^{A_2} & \text{if } \beta\in[\beta_{6,T-2}\,|_{A_1,A_2}\,,\beta_{3,T-2}\,|_{A_2,B_2})\\ +1^{B_2} & \text{if } \beta\in[\beta_{3,T-2}\,|_{A_2,B_2}\,,\beta_{7,T-2}\,|_{B_1,B_2})\\ +1^{B_1} & \text{if } \beta\in[\beta_{7,T-2}\,|_{B_1,B_2}\,,\beta_{5,T-2}\,|_{A=A_2,B_1})\\ -1^{A} & \text{if } \beta\in[\beta_{5,T-2}\,|_{A=A_2,B_1}\,,\overline{\beta}]\end{cases}$$

where $\beta_{6,T-2}\,|_{A_1,A_2}\;=\frac{p^*_{T-2}(A_1|[0100])A_1-p^*_{T-2}(A_2|[1000])A_2}{p^*_{T-2}(A_1|[0100])-p^*_{T-2}(A_2|[1000])}\cdot\frac{1}{v}$, $\beta_{7,T-2}\,|_{B_1,B_2}\;=\frac{p^*_{T-2}(B_1|[0010])B_1-p^*_{T-2}(B_2|[0001])B_2}{p^*_{T-2}(B_1|[0010])-p^*_{T-2}(B_2|[0001])}\cdot\frac{1}{v}$.

(1.2) We solve the same problem in the small-tick market (SM) where traders can select among five price levels both on the ask and the bid side to post their limit orders, so that $a_l=v+\frac{\tau}{6}l$ and $b_l=v-\frac{\tau}{6}l$ for $l=1,..,5$, and $A_1=a_2$, $A_2=a_5$, $B_1=b_2$ and $B_2=b_5$. As the methodology is similar to the one for case 1.1, we directly present the equilibrium strategies at $T-2$ for the case with an empty book, that we indicate with $S_t=[0]$, and a small relative tick size value, $\frac{\tau}{v}$:

$$H^{*SM}_{T-2}(\beta,[0])=\begin{cases}-1^{b} & \text{if } \beta\in[\underline{\beta},\beta_{1,T-2}\,|_{a_5,b=b_5})\\ +1^{a_5} & \text{if } \beta\in[\beta_{1,T-2}\,|_{a_5,b=b_5}\,,\beta_{3,T-2}\,|_{a_5,b_5})\\ +1^{b_5} & \text{if } \beta\in[\beta_{3,T-2}\,|_{a_5,b_5}\,,\beta_{5,T-2}\,|_{a=a_5,b_5})\\ -1^{a} & \text{if } \beta\in[\beta_{5,T-2}\,|_{a=a_5,b_5}\,,\overline{\beta}]\end{cases}$$

When $\frac{\tau}{v}$ is large, equilibrium strategies at $T-2$ modify as follows:

$$H_{T-2}^{*SM}(\beta,[0]) = \begin{cases} -1^{b} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{a_1,b=b_5}) \\ +1^{a_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{a_1,b=b_5}, \beta_{6,T-2}\,|_{a_1,a_5}) \\ +1^{a_5} & \text{if } \beta \in [\beta_{6,T-2}\,|_{a_1,a_5}, \beta_{3,T-2}\,|_{a_5,b_5}) \\ +1^{b_5} & \text{if } \beta \in [\beta_{3,T-2}\,|_{a_5,b_5}, \beta_{7,T-2}\,|_{b_1,b_5}) \\ +1^{b_1} & \text{if } \beta \in [\beta_{7,T-2}\,|_{b_1,b_5}, \beta_{5,T-2}\,|_{a=a_5,b_1}) \\ -1^{a} & \text{if } \beta \in [\beta_{5,T-2}\,|_{a=a_5,b_1}, \overline{\beta}] \end{cases}$$

(1.3) As defined in Section 4, liquidity provision and executed volume at $T-2$ are determined by the probability that in this trading round the incoming trader submits a limit order or a market order respectively. To compare LM and SM in terms of these measures, it is sufficient to compare the thresholds that make the trader indifferent between submitting a market order and the most attractive among the available limit order strategies:

$$\widehat{\beta}_{5,T-2}^{LM}\,|_{A=A_2,B_k} = \max_{B_k} \beta_{5,T-2}\,|_{A=A_2,B_k} = \max_{B_k}\left\{\frac{A_2}{v} + \frac{p_{T-1}^{*}(B_k|S_{T-1})}{1-p_{T-1}^{*}(B_k|S_{T-1})}\cdot\frac{A_2-B_k}{v}\right\}$$

$$\widehat{\beta}_{5,T-2}^{SM}\,|_{a=a_5,b_l} = \max_{b_l} \beta_{5,T-2}\,|_{a=a_5,b_l} = \max_{b_l}\left\{\frac{a_5}{v} + \frac{p_{T-1}^{*}(b_l|S_{T-1})}{1-p_{T-1}^{*}(b_l|S_{T-1})}\cdot\frac{a_5-b_l}{v}\right\}$$

After substituting the execution probabilities computed in (1.1) and (1.2), we find that the probability of a market buy order is larger for the small tick market: $\widehat{\beta}_{5,T-2}^{SM}\,|_{a=a_5,b_l} < \widehat{\beta}_{5,T-2}^{LM}\,|_{A=A_2,B_k}$. A similar result holds for a market sell order: $\widehat{\beta}_{1,T-2}^{SM}\,|_{a_l,b=b_5} > \widehat{\beta}_{1,T-2}^{LM}\,|_{A_k,B=B_2}$. Consequently volume is higher in the small tick market:

$$VL_{T-2}^{SM} = \frac{\overline{\beta}-\widehat{\beta}_{5,T-2}^{SM}\big|_{a=a_5,b_l}}{\overline{\beta}-\underline{\beta}} + \frac{\widehat{\beta}_{1,T-2}^{SM}\big|_{a_l,b=b_5}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} > \frac{\overline{\beta}-\widehat{\beta}_{5,T-2}^{LM}\big|_{A=A_2,B_k}}{\overline{\beta}-\underline{\beta}} + \frac{\widehat{\beta}_{1,T-2}^{LM}\big|_{A_k,B=B_2}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} = VL_{T-2}^{LM}$$

Notice that when the book starts empty at $T-2$, no trading ($H_{T-2}=0$) is never optimal and hence in this single market model the submission probabilities of market and limit orders are complements. Thus, as $VL_{T-2}^{LM} < VL_{T-2}^{SM}$, $LP_{T-2}^{LM} > LP_{T-2}^{SM}$. Notice also that, as the book starts empty so that there is no depth already available, inside depth coincides with total depth and also with liquidity provision, $DPI_{T-2} = DPT_{T-2} = LP_{T-2}$, where:

$$LP_{T-2} = \frac{\widehat{\beta}_{5,T-2}-\widehat{\beta}_{1,T-2}}{\overline{\beta}-\underline{\beta}}$$

As a consequence, also total and internal depth are lower in a small tick market: $DPI_{T-2}^{LM} = DPT_{T-2}^{LM} > DPI_{T-2}^{SM} = DPT_{T-2}^{SM}$. Finally, in order to compute the spread, we need to differentiate two cases depending on the value of $\frac{\tau}{v}$. When $\frac{\tau}{v}$ is large, we obtain:

$$\begin{aligned} SP_{T-2}^{LM} &= E[A-B] = 3\tau - \left(\frac{\beta_{6,T-2}\big|_{A_1,A_2}-\beta_{1,T-2}\big|_{A_2,B=B_2}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{5,T-2}\big|_{A=A_2,B_2}-\beta_{7,T-2}\big|_{B_1,B_2}}{\overline{\beta}-\underline{\beta}}\right)\cdot\tau \\ &< 3\tau - \left(\frac{\beta_{6,T-2}\big|_{a_1,a_5}-\beta_{1.T-2}\big|_{a_1,b=b_5}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{5,T-2}\big|_{a=a_5,b_1}-\beta_{7,T-2}\big|_{b_1,b_5}}{\overline{\beta}-\underline{\beta}}\right)\cdot\frac{4\tau}{3} = SP_{T-2}^{SM} \end{aligned}$$

where, for example, $\frac{\beta_{6,T-2}\left|_{A_1,A_2} - \beta_{1,T-2}\right|_{A_2,B=B_2}}{\overline{\beta}-\underline{\beta}}$ is the probability of a limit sell order posted at A_1 and $\frac{\beta_{5,T-2}\left|_{A=A_2,B_2} - \beta_{7,T-2}\right|_{B_1,B_2}}{\overline{\beta}-\underline{\beta}}$ is the probability of a limit buy order posted at B_1.When instead $\frac{\tau}{v}$ is small and traders in equilibrium post limit orders only at $A_2 = a_5$ and $B_2 = b_5$, we obtain that $SP^{LM}_{T-2} = SP^{SM}_{T-2} = 3\tau$. So the spread never decreases after a tick size reduction, and it increases for stocks with large relative tick size.

2) Liquid stocks: starting book at $T-2$ is [0110]

(2.1) Consider again LM. The traders' strategy space at $T-2$ is the same as in case (1) and the possible opening books at $T-1$ are: [0100], [0010], [0210], [0120], [0110]. As an example, we consider the case where $H_{T-2} = +1^{A_1}$ so that the book opens at $T-1$ as [0210]; in this case the incoming trader never gets execution priority for limit orders and therefore his strategy space is $\{-1^B, -1^A, 0\}$; his corresponding payoffs are:

$$\boxed{\begin{array}{lcl} H_{T-1} = -1^B & : & B_2 - \beta v \\ H_{T-1} = -1^A & : & \beta v - A_2 \\ H_{T-1} = 0 & : & 0 \end{array}}$$

By comparing these payoffs, it is then straightforward to compute his equilibrium strategies, obtained by using (7d):

$$H^{*LM}_{T-1}(\beta, [0210]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-1}\,|_{B=B_1}) \\ 0 & \text{if } \beta \in [\beta_{1,T-1}\,|_{B=B_1}\,, \beta_{5,T-1}\,|_{A=A_1}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-1}\,|_{A=A_1}\,, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-1}\,|_{B=B_1} = \frac{B_1}{v}$ and $\beta_{5,T-1}\,|_{A=A_1} = \frac{A_1}{v}$. We can then compute the execution probabilities of the order submitted at $T-2$ as:

$$p^*_{T-2}(A\,|[0210]\,) = \frac{\overline{\beta}-\beta_{5,T-1}\big|_{A=A_1}}{\overline{\beta}-\underline{\beta}}$$

Following the same methodology, we are able to obtain the execution probabilities for all possible strategies at $T-2$. We can finally compute the strategies' payoffs in the same period:

$$\boxed{\begin{array}{lcl} H_{T-2} = -1^B & : & B_1 - \beta v \\ H_{T-2} = +1^{A_2} & : & (A_2 - \beta v) \cdot p^*_{T-2}(A_2^{1,1}\,|[1110]\,) \\ H_{T-2} = +1^{A_1} & : & (A_1 - \beta v) \cdot p^*_{T-2}(A_1^{2}\,|[0210]\,) \\ H_{T-2} = +1^{B_1} & : & (\beta v - B_1) \cdot p^*_{T-2}(B_1^{2}\,|[0120]\,) \\ H_{T-2} = +1^{B_2} & : & (\beta v - B_2) \cdot p^*_{T-2}(B_2^{1,1}\,|[0111]\,) \\ H_{T-2} = -1^A & : & \beta v - A_1 \\ H_{T-2} = 0 & : & 0 \end{array}}$$

where $p^*_{T-2}(A_2^{1,1}\,|[1110]\,) = \frac{\overline{\beta}-\beta_{5,T-1}\big|_{A=A_1,B_1}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(A)\,|_{A=A_2}$, $p^*_{T-2}(A_1^2\,|[0210] = \frac{\overline{\beta}-\beta_{5,T-1}\big|_{A=A_1,B_1}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(A)\,|_{A=A_1}$, $p^*_{T-2}(B_1^2\,|[0120]\,) = \frac{\beta_{1,T-1}\big|_{A=A_1,B_1}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(B)\,|_{B=B_1}$ and $p^*_{T-2}(B_2^{1,1}\,|[0111]\,) = \frac{\beta_{1,T-1}\big|_{A=A_1,B_1}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} \cdot p^*_{T-1}(B)\,|_{B=B_2}$. By comparing these payoffs we obtain the following equilibrium strategies:

$$H^{*LM}_{T-2}(\beta,[0110]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{A_2,B=B_1}) \\ +1^{A_2} & \text{if } \beta \in [\beta_{1,T-2}\,|_{A_2,B=B_1}, \beta_{3,T-2}\,|_{A_2,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\,|_{A_2,B_2}, \beta_{5,T-2}\,|_{A=A_1,B_2}) \\ -1^A & \text{if } \beta \in [\beta_{5,T-2}\,|_{A=A_1,B_2}, \overline{\beta}] \end{cases}$$

where $\beta_{1,T-2}\,|_{A_2,B=B_1} = \frac{B_1}{v} - \frac{p^*_{T-2}(A_2^{1,1}|[1110]\,)}{1-p^*_{T-2}(A_2^{1,1}|[1110]\,)} \cdot \frac{A_2-B_1}{v}$, $\beta_{3,T-2}\,|_{A_2,B_2} = \frac{p^*_{T-2}(A_2^{1,1}|[1110]\,)A_2+p^*_{T-2}(B_2^{1,1}|[0111]\,)B_2}{p^*_{T-2}(A_2^{1,1}|[1110]\,)-p^*_{T-2}(B_2^{1,1}|[0111]\,)} \cdot \frac{1}{v}$, and $\beta_{5,T-2}\,|_{A=A_1,B_2} = \frac{A_1}{v} + \frac{p^*_{T-2}(B_2^{1,1}|[0111]\,)}{1-p^*_{T-2}(B_2^{1,1}|[0111]\,)} \cdot \frac{A_1-B_2}{v}$.

(2.2) We solve the same problem in SM. Notice that in this case where at $T-2$ one share is available at a_2 and b_2, thanks to the finer price grid traders have room for undercutting the best quotes by submitting limit orders at a_1 and b_1. As the procedure to get to the equilibrium strategies is the same as in the previous case, we present directly the results. Notice that we indicate by $[Q^{a_2}=1, Q^{b_2}=1]$ the book that opens at $T-1$ with one unit on both a_2 and b_2.

$$H^{*SM}_{T-2}(\beta,[Q^{a_2}=1, Q^{b_2}=1]) = \begin{cases} -1^b & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{a_1,b=b_2}) \\ +1^{a_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{a_1,b=b_2}, \beta_{6,T-2}\,|_{a_1,a_5}) \\ +1^{a_5} & \text{if } \beta \in [\beta_{6,T-2}\,|_{a_1,a_5}, \beta_{3,T-2}\,|_{a_5,b_5}) \\ +1^{b_5} & \text{if } \beta \in [\beta_{3,T-2}\,|_{a_5,b_5}, \beta_{7,T-2}\,|_{b_1,b_5}) \\ +1^{b_1} & \text{if } \beta \in [\beta_{7,T-2}\,|_{b_1,b_5}, \beta_{5,T-2}\,|_{a=a_2,b_1}) \\ -1^a & \text{if } \beta \in [\beta_{5,T-2}\,|_{a=a_2,b_1}, \overline{\beta}] \end{cases}$$

(2.3) Similarly to (1.3), we compare the indicators of market quality by using the probabilities of observing different order types. Notice from the equilibrium strategies presented in (2.2) that here $\widehat{\beta}^{SM}_{5,T-2}\,|_{a=a_5,b_l} = \beta_{5,T-2}\,|_{a=a_2,b_1}$ and $\widehat{\beta}^{LM}_{5,T-2}\,|_{A=A_1,B_k} = \beta_{5,T-2}\,|_{A=A_1,B_2}$. After substituting for the optimal execution probabilities we find that $\beta_{5,T-2}\,|_{A=A_1,B_2} < \beta_{5,T-2}\,|_{a=a_2,b_1}$ and $\beta_{1,T-2}\,|_{A_2,B=B_1} > \beta_{1,T-2}\,|_{a_1,b=b_2}$, so that:

$$VL^{SM}_{T-2} = \frac{\overline{\beta}-\beta_{5,T-2}\big|_{a=a_2,b_1}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{1,T-2}\big|_{a_1,b=b_2}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} < \frac{\overline{\beta}-\beta_{5,T-2}\big|_{A=A_1,B_2}}{\overline{\beta}-\underline{\beta}} + \frac{\beta_{1,T-2}\big|_{A_2,B=B_1}-\underline{\beta}}{\overline{\beta}-\underline{\beta}} = VL^{LM}_{T-2}$$

Given that $H_{T-2}=0$ cannot occur in equilibrium and that as a consequence the probability of limit and market orders are complements, $VL^{LM}_{T-2} > VL^{SM}_{T-2}$ implies that $LP^{LM}_{T-2} < LP^{SM}_{T-2}$.

The other market indicators for LM are computed as follows:

$$\begin{aligned}
DPI_{T-2}^{LM} &= 2-1\cdot\left(\tfrac{\beta_{1,T-2}\left|_{A_2,B=B_1}-\underline{\beta}\right.}{\overline{\beta}-\underline{\beta}}+\tfrac{\overline{\beta}-\beta_{5,T-2}\left|_{A=A_1,B_2}\right.}{\overline{\beta}-\underline{\beta}}\right)=2-VL_{T-2}^{LM}=1+LP_{T-2}^{LM}\\
DPT_{T-2}^{LM} &= 2+LP_{T-2}^{LM}-VL_{T-2}^{LM}=1+2\cdot LP_{T-2}^{LM}\\
SP_{T-2}^{LM} &= LP_{T-2}^{LM}\cdot(\tau)+VL_{T-2}^{LM}\cdot(2\tau)=(2-LP_{T-2}^{LM})\tau
\end{aligned}$$

Following the same methodology, we compute market indicators for the small tick case, where $DPI_{T-2}^{SM}=1+LP_{T-2}^{SM}$, $DPT_{T-2}^{SM}=1+2\cdot LP_{T-2}^{SM}$ and

$$\begin{aligned}
SP_{T-2}^{SM} &= \Pr(H_{T-2}^{*SM}=+1^{a_2},+1^{b_2})\cdot(\tau)+\Pr(H_{T-2}^{*SM}=+1^{a_1},+1^{b_1})\cdot(\frac{2\tau}{3})+VL_{T-2}^{LM}\cdot(2\tau)\\
&= LP_{T-2}^{LM}\cdot(\tau)-\Pr(H_{T-2}^{*SM}=+1^{a_1},+1^{b_1})\cdot\frac{\tau}{3}+VL_{T-2}^{LM}\cdot(2\tau)\\
&< (2-LP_{T-2}^{SM})\tau
\end{aligned}$$

so that $DPI_{T-2}^{LM}<DPI_{T-2}^{SM}$, $DPT_{T-2}^{LM}<DPT_{T-2}^{SM}$ and $SP_{T-2}^{LM}>SP_{T-2}^{SM}$.

<u>3) Highly liquid stocks: starting book at $T-2$ is [0220]</u>

At $T-2$ traders' strategy space in a LM is now $\{-1^B,-1^A,0\}$ and the corresponding opening books at $T-1$ [0210], [0120], and [0220]. As the stock is very liquid, limit orders at $T-1$ never get executed except for the small tick market where traders can undercut existing liquidity by posting limit orders at A_1 and B_1.The equilibrium trading strategies at $T-2$ are:

$$H_{T-2}^{*LM}(\beta,[0220])=\begin{cases}-1^B & \text{if } \beta\in[\underline{\beta},\beta_{1,T-2}\left|_{A_1,B=B_1}\right.)\\ 0 & \text{if } \beta\in[\beta_{1,T-2}\left|_{A_1,B=B_1}\right.,\beta_{5,T-2}\left|_{A=A_1,B_1}\right.)\\ -1^A & \text{if } \beta\in[\beta_{5,T-2}\left|_{A=A_1,B_1}\right.,\overline{\beta}]\end{cases}$$

$$H_{T-2}^{*SM}(\beta,[Q^{a_2}=2,Q^{b_2}=2])=\begin{cases}-1^b & \text{if } \beta\in[\underline{\beta},\beta_{1,T-2}\left|_{a_1,b=b_2}\right.)\\ +1^{a_1} & \text{if } \beta\in[\beta_{1,T-2}\left|_{a_1,b=b_2}\right.,\beta_{3,T-2}\left|_{a_1,b_1}\right.)\\ +1^{b_1} & \text{if } \beta\in[\beta_{3,T-2}\left|_{a_1,b_1}\right.,\beta_{5,T-2}\left|_{a=a_2,b_1}\right.)\\ -1^a & \text{if } \beta\in[\beta_{5,T-2}\left|_{a=a_2,b_1}\right.,\overline{\beta}]\end{cases}$$

clearly, liquidity provision is positive only in the SM market so that $LP_{T-2}^{SM}>LP_{T-2}^{LM}=0$, thus $DPT_{T-2}^{SM}>DPT_{T-2}^{LM}$ and $SP_{T-2}^{SM}<SP_{T-2}^{LM}$. Comparing the thresholds for market orders we obtain:

$$\widehat{\beta}_{1,T-2}^{SM}\left|_{a_1,b=b_2}\right.=\min\{\tfrac{b_2}{v},\tfrac{b_2}{v}-\tfrac{p_{T-2}^*(\cdot)\left|_{a_1}\right.}{1-p_{T-2}^*(\cdot)\left|_{a_1}\right.}\cdot\tfrac{a_1-b_2}{v}\}<\tfrac{B_1}{v}=\widehat{\beta}_{1,T-2}^{LM}\left|_{A_1,B=B_1}\right.$$

so that $VL_{T-2}^{SM}<VL_{T-2}^{LM}$. Finally, $DPI_{T-2}^{SM}=3<3VL_{T-2}^{LM}+4(1-VL_{T-2}^{LM})=DPI_{T-2}^{LM}$. ■

D Proof of Lemma 3

Proof. Assume that the two markets have the same initial depth at the common price levels, so that $S_0^{A_2} = S_0^{a_5}$, $S_0^{A_1} = S_0^{a_2}$, $S_0^{B_2} = S_0^{b_5}$, and $S_0^{B_1} = S_0^{b_2}$. Consider an incoming seller who can submit either a market order or a limit order or refrain from trading. If he opts for a market order, he faces two cases: either the best ask price is different from the two common price levels ($A \neq A_2 \neq A_1$), or it is one of those two ($A = A_2$ or A_1). In the first case the market order will only change depth at the best ask price level and not at the common ones; in the second one the trader will randomize between the two trading venues as they both offer liquidity at the best ask and hence expected depth at the two common price levels will change equally. For this reason, we obtain that: $E[S_1^{A_2}] = E[S_1^{a_5}]$ and $E[S_1^{A_1}] = E[S_1^{a_2}]$. If instead the trader opts for a limit order, either he submits his order to a price level that is not common to the two markets, in which case the Lemma trivially holds, or if he decides to submit a limit order at one of the two common price levels (i.e. $A_2 = a_5$ and $A_1 = a_2$), he will optimally randomize between them so that expected depth will still be the same. For this case, consider as an example $A_1 = a_2$ and notice that profits from the two limit order strategies are indeed the same:

$$\left| \begin{array}{lcl} H_1 = +1^{A_1} & : & (A_1 - \beta v) \cdot p_1^*(A_1^{n_1, n_2+N_1+1}) \\ H_1 = +1^{a_2} & : & (a_2 - \beta v) \cdot p_1^*(a_2^{n_1, n_2+N_1+1}) \end{array} \right|$$

For this reason the incoming trader will randomize between the two trading venues:

$$E[h_1^{A_1}] = E[h_1^{a_2}] = \frac{1}{2} \int\limits_{\beta \in \{\beta : H_t(\beta) = 1^{A_1}, 1^{a_2}\}} H_t(\beta) d\beta$$

From equation (3), this implies that $E[S_1^{A_1}] = E[S_1^{a_2}]$. A similar result is obtained for $A_2 = a_5$. Finally, if the trader decides not to trade, then no change is observed on the depth associated to the two common price levels, and the Lemma will hold. As the same argument can be applied to the bid side and holds recursively for $t \geq 2$, we obtain that when $S_0^{A_2} = S_0^{a_5}$, $S_0^{A_1} = S_0^{a_2}$, $S_0^{B_2} = S_0^{b_5}$, and $S_0^{B_1} = S_0^{b_2}$, then $\forall t \in [1, T]$ $E[S_t^{A_2}] = E[S_t^{a_5}]$, $E[S_t^{A_1}] = E[S_t^{a_2}]$, $E[S_t^{B_2}] = E[S_t^{b_5}]$ and $E[S_t^{B_1}] = E[S_t^{b_2}]$. ■

E Proof of Proposition 2

Proof. Lemma 3 tells us that, if the two markets have the same initial depth at the common price levels, they should always have the same expected depth at those price levels. So in order to prove that liquidity provision concentrates on the SM, it is sufficient to show that traders optimally submit limit orders at price levels that are not common to the two trading venues. Consider hence a LM and a SM with the same initial depth at their common price levels and with $S_0^{a_j} = 0$ for $j = 1, 3, 4$. If, contrary to Proposition 2, depth does not concentrate on SM, it means that traders post their orders only to the common price levels, i.e. $S_t^{a_j} = 0$, $\forall t$. Assume that until period t it has never been indeed optimal to submit a limit order at any of the non-common price levels, so that $S_{t-1}^{a_j} = 0$ for $j = 1, 3, 4$. As an example, consider the payoffs of a seller who arrives at t and wants to submit a limit order. Define the payoff's difference between undercutting at a_1 and queuing at a_2 as:

$$f(n_2) = (a_1 - \beta v) \cdot p_t^*(a_1 \left| S_t^{LM}, S_t^{SM} \right.) - (a_2 - \beta v) \cdot p_t^*(a_2^{2n_2+1} \left| S_t^{LM}, S_t^{SM} \right.)$$

Notice that if the book is full at a_2(or A_1)(i.e. $2n_2 \geqslant T - t$), queuing at a_2 implies a zero execution probability. Hence we obtain: $f(n_2) = (a_1 - \beta v) \cdot p_t^*(a_1 \left| S_t^{LM}, S_t^{SM} \right.) > 0 \Rightarrow H_t = +1^{a_1} \succ H_t = +1^{a_2}$. If instead queuing has a non zero execution probability, $p_t^*(a_2^{2n_2+1} \left| S_{t-1}^{LM}, S_{t-1}^{SM} \right.) > 0$, then, as $\partial f(n_2)/\partial n_2 > 0$, $\exists \overline{n}_2$ such that $H_t = +1^{a_1} \succ H_t = +1^{a_2}$ for $n_2 > \overline{n}_2$. A similar result holds when comparing $H_t = +1^{a_1}$ and $H_t = +1^{a_5}$, where $\overline{n}_5 < \overline{n}_2$. Thus, as the book gets deeper at the common prices, there always exists a critical number of shares above which $H_t = +1^{a_1}$ is preferred to $H_t = +1^{a_2}, +1^{a_5}$. Moreover, keeping into account that in the case considered here the book opens empty (at t) at non common price levels, and hence $H_t = +1^{a_1}$ has always positive execution probability for $t \neq T$, it can be shown that for certain ranges of β, a limit order at $a_1$is preferable to a market order or not trade. That is, $\exists \beta \in (\beta_{-1^B, 1^{a_1}}, \beta_{-1^B, 0})$ $s.t$ $H_t^*(\beta, S_{t-1}^{LM}, S_{t-1}^{SM}) = +1^{a_1}$ (see proof of Lemma 2). Similar results can be obtained for the bid side. We can hence conclude that when the two markets compete, liquidity concentrates on the SM. ∎

F Proof of Proposition 3

Proof. Broker-dealers (BD) can choose the market where to submit their limit orders, while regular traders (RT) can only submit to the PLB. A smart router allows all market orders to spot the best price between the two markets. As shown in Proposition 2, when two markets with different tick size compete, liquidity provision concentrates in the market with the thinner priced grid (i.e. the IP). However, as only broker-dealers can access this market, the migration of liquidity provision will be stronger the larger α is.

1) Illiquid stocks: starting book at $T-2$ is [00|00]&[00000|00000]

At $T-2$ trader's strategy space, considering both BD and RT, is $\{-1^B, +1^i, +1^j, -1^A, 0\}$ with $i = A_{1:2}$ and $B_{1:2}$, $j = a_{1:5}$ and $b_{1:5}$. Therefore at the beginning of $T-1$ there are 15 possible states of the books: one share on the i-th level of the PLB and no shares on the others as well as on the IP ($Q^i = 1$, $Q^{-i} = 0$ and $Q^{-j} = 0$) (4 cases), one share on the j-th level of the IP and no shares on the other levels of the IP and on the PLB

($Q^j = 1$, $Q^{-j} = 0$ and $Q^{-i} = 0$) (10 cases), or both books are empty (1 case). Conditional on each case, the optimal equilibrium strategy for different types of traders can be obtained. Notice that at T, as all traders can observe the best available price, the equilibrium strategies of RT and BD coincide (market order only). As a result the orders' execution probabilities at $T-1$ are independent of traders' type incoming at T, so that for example: $p^{*RT}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1}) = p^{*BD}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1}) = p^{*}_{T-1}(A_k|S^{PLB}_{T-1}, S^{IP}_{T-1})$. An example for the case [10|00]&[00000|00000], i.e. $Q^{A_2} = 1$, is provided below.

If a RT arrives at $T-1$, his optimal strategies are the same as the one presented for case (1.1) in the proof of Proposition 1, where the opening book at $T-1$ was [10|00], and are hence omitted. By using the optimal β thresholds associated to these strategies, we can compute the execution probability of the limit order posted at $T-2$ at A_2 in case a RT arrives at $T-1$:

$$\begin{aligned} p^{*RT}_{T-2}(A_2\left|Q^{A_2} = 1\right.) \;=\; & \frac{\overline{\beta}-\beta^{RT}_{5,T-1}\left|_{A=A_2,B_2}\right.}{\overline{\beta}-\underline{\beta}} + \\ & \left(\frac{\beta^{RT}_{1,T-1}\left|_{A_1,B=B_2}\right.-\underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\beta^{RT}_{5,T-1}\left|_{A=A_2,B_2}\right.-\beta^{RT}_{3,T-1}\left|_{A_1,B_2}\right.}{\overline{\beta}-\underline{\beta}}\right)\cdot p^{*}_{T-1}(A)\left|_{A=A_2}\right. \end{aligned}$$

If instead a BD arrives at $T-1$, he will face the following payoffs:

$H_{T-1} = -1^{B}$	:	$B_2 - \beta v$
$H_{T-1} = +1^{A_1}$	:	$(A_1 - \beta v)\cdot p^{*}_{T-1}(A_1|S^{PLB}_{T-1}, S^{IP}_{T-1})$
$H_{T-1} = +1^{a_l}$	:	$(a_l - \beta v)\cdot p^{*}_{T-1}(a_l|S^{PLB}_{T-1}, S^{IP}_{T-1})$
$H_{T-1} = +1^{B_k}$	:	$(\beta v - B_k)\cdot p^{*}_{T-1}(B_k|S^{PLB}_{T-1}, S^{IP}_{T-1})$
$H_{T-1} = +1^{b_l}$	:	$(\beta v - b_l)\cdot p^{*}_{T-1}(b_l|S^{PLB}_{T-1}, S^{IP}_{T-1})$
$H_{T-1} = -1^{A}$	:	$\beta v - A_2$
$H_{T-1} = 0$	:	0

After substituting the execution probabilities at T, trader's equilibrium strategies at $T-1$ are obtained:

$$H^{*BD}_{T-1}(\beta, Q^{A_2} = 1) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-1}\left|_{a_4,B=B_2}\right.) \\ +1^{a_4} & \text{if } \beta \in [\beta^{BD}_{1,T-1}\left|_{a_4,B=B_2}\right., \beta^{BD}_{3,T-1}\left|_{a_4,B_2}\right.) \\ +1^{B_2} & \text{if } \beta \in [\beta^{BD}_{3,T-1}\left|_{a_4,B_2}\right., \beta^{BD}_{5,T-1}\left|_{A=A_2,B_2}\right.) \\ -1^{A} & \text{if } \beta \in [\beta^{BD}_{5,T-1}\left|_{A=A_2,B_2}\right., \overline{\beta}] \end{cases}$$

So, when a BD arrives at $T-1$, the execution probability of the limit order posted at $T-2$ at A_2 is:

$$\begin{aligned} p^{*BD}_{T-2}(A_2\left|Q^{A_2} = 1\right.) \;=\; & \frac{\overline{\beta}-\beta^{BD}_{5,T-1}\left|_{A=A_2,B_2}\right.}{\overline{\beta}-\underline{\beta}} + \\ & \left(\frac{\beta^{BD}_{1,T-1}\left|_{a_4,B=B_2}\right.-\underline{\beta}}{\overline{\beta}-\underline{\beta}} + \frac{\beta^{BD}_{5,T-1}\left|_{A=A_2,B_2}\right.-\beta^{BD}_{3,T-1}\left|_{a_4,B_2}\right.}{\overline{\beta}-\underline{\beta}}\right)\cdot p^{*}_{T-1}(A)\left|_{A=A_2}\right. \end{aligned}$$

We can hence compute the total execution probability of the limit order as:

$$p^*_{T-2}(A_2 \left| Q^{A_2} = 1\right.) = \alpha p^{*BD}_{T-2}(A_2 \left| Q^{A_2} = 1\right.) + (1-\alpha) p^{*RT}_{T-2}(A_2 \left| Q^{A_2} = 1\right.)$$

Similarly, we can compute the equilibrium strategies for all the other possible states of the book at $T-1$ and in this way obtain the execution probabilities of the different order types available at $T-2$ to a BD and a RT.

(1.1) At $T-2$, if a RT arrives, his strategy space is $\{-1^B, +1^{A_2}, +1^{A_1}, +1^{B_1}, +1^{B_2}, -1^A, 0\}$. His payoffs are:

$$\boxed{\begin{array}{lcl} H_{T-2} = -1^B & : & B_2 - \beta v \\ H_{T-2} = +1^{A_i} & : & (A_i - \beta v) \cdot p^*_{T-2}(A_i|[0]) \\ H_{T-2} = +1^{B_i} & : & (\beta v - B_i) \cdot p^*_{T-2}(B_i|[0]) \\ H_{T-2} = -1^A & : & \beta v - A_2 \\ H_{T-2} = 0 & : & 0 \end{array}}$$

where [0] indicates the case where both the PLB and the IP are empty and, for example, $p^*_{T-2}(A_i|[0]) = \alpha p^{*RT}_{T-2}(A_i \left| Q^{A_i} = 1\right.) + (1-\alpha) p^{*BD}_{T-2}(A_i \left| Q^{A_i} = 1\right.)$. Similar to (1.1) in the proof of Proposition 1, if $\frac{\tau}{v}$ is small:

$$H^{*RT}_{T-2}(\beta, [0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\left|_{A_2, B=B_2}\right.) \\ +1^{A_2} & \text{if } \beta \in [\beta_{1,T-2}\left|_{A_2, B=B_2}\right., \beta_{3,T-2}\left|_{A_2, B_2}\right.) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\left|_{A_2, B_2}\right., \beta_{5,T-2}\left|_{A=A_2, B_2}\right.) \\ -1^A & \text{if } \beta \in [\beta_{5,T-2}\left|_{A=A_2, B_2}\right., \overline{\beta}] \end{cases}$$

If instead $\frac{\tau}{v}$ is large:

$$H^{*RT}_{T-2}(\beta, [0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\left|_{A_2, B=B_2}\right.) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-2}\left|_{A_2, B=B_2}\right., \beta_{2,T-2}\left|_{A_2, B_2}\right.) \\ +1^{A_2} & \text{if } \beta \in [\beta_{2,T-2}\left|_{A_2, B=B_2}\right., \beta_{3,T-2}\left|_{A_2, B_2}\right.) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\left|_{A_2, B_2}\right., \beta_{4,T-2}\left|_{A=A_2, B_2}\right.) \\ +1^{B_1} & \text{if } \beta \in [\beta_{4,T-2}\left|_{A_2, B_2}\right., \beta_{5,T-2}\left|_{A=A_2, B_2}\right.) \\ -1^A & \text{if } \beta \in [\beta_{5,T-2}\left|_{A=A_2, B_2}\right., \overline{\beta}] \end{cases}$$

(1.2) At $T-2$, if a BD arrives, he has to decide where to trade. Thus his strategy space is $\{-1^B, +1^i, +1^j, -1^A, 0\}$ with $i = A_{1:2}$ and $B_{1:2}$, $j = a_{1:5}$ and $b_{1:5}$. His payoffs are:

$$\boxed{\begin{array}{lcl} H_{T-2} = -1^B & : & B_2 - \beta v \\ H_{T-2} = +1^{A_i} & : & (A_i - \beta v) \cdot p^*_{T-2}(A_i|[0]) \\ H_{T-2} = +1^{a_l} & : & (a_l - \beta v) \cdot p^*_{T-2}(a_l|[0]) \\ H_{T-2} = +1^{B_i} & : & (\beta v - B_i) \cdot p^*_{T-2}(B_i|[0]) \\ H_{T-2} = +1^{b_l} & : & (\beta v - b_l) \cdot p^*_{T-2}(b_l|[0]) \\ H_{T-2} = -1^A & : & \beta v - A_2 \\ H_{T-2} = 0 & : & 0 \end{array}}$$

where for example $p^*_{T-2}(a_l|[0]) = \alpha p^{*BD}_{T-2}(a_l\,|Q^{a_l}=1) + (1-\alpha)p^{*RT}_{T-2}(a_l\,|Q^{a_l}=1)$. Notice that if he submits a limit order to the PLB, his execution probabilities are the same as for a RT. If we consider again the case with $\frac{\tau}{v}$ small,[20] we obtain that $H^{*BD}_{T-2}(\beta,[0]) = H^{*RT}_{T-2}(\beta,[0]) = H^*_{T-2}(\beta,[0])$. The case with $\frac{\tau}{v}$ large will be discussed in (1.4).

(1.3) When $\frac{\tau}{v}$ is small, at $T-2$ expected volume on the PLB is:

$$\begin{aligned} VL^{PLB}_{T-2} &= \alpha[\Pr(H^{*BD}_{T-2}=-1^{A_2}) + \Pr(H^{*BD}_{T-2}=-1^{B_2})] + (1-\alpha)[\Pr(H^{*RT}_{T-2}=-1^{A_2}) + \Pr(H^{*RT}_{T-2}=-1^{B_2})] \\ &= VL^{PLB}_{T-2}(\alpha) \end{aligned}$$

As, being empty, there is no volume executed in the IP, we obtain that:

$$LP^{PLB}_{T-2} = 1 - VL^{PLB}_{T-2} - LP^{IP}_{T-2}$$

By taking into account that the book opens empty, we can compute the other market indicators for the PLB from the optimal order submission strategies at $T-2$:[21]

$$DPI^{PLB}_{T-2} = LP^{PLB}_{T-2} = DPT^{PLB}_{T-2}$$

$$SP^{PLB}_{T-2} = E[A-B] = 3\tau$$

To analyze the changes in the PLB after the introduction of an IP, we compare the obtained market indicators with those in (1.3) of proof of Proposition 1. Notice that the single market case is equivalent to the dual-market case with $\alpha = 0\%$. Thus the equilibrium strategies of a BD are the main determinants of changes in market quality. When $\frac{\tau}{v}$ is small, we obtained from Proposition 1: $VL^{LM}_{T-2} = \frac{\overline{\beta}-\widehat{\beta}^{LM}_{5,T-2}\big|_{A=A_2,B_k}}{\overline{\beta}-\underline{\beta}} + \frac{\widehat{\beta}^{LM}_{1,T-2}\big|_{A_k,B=B_2}-\underline{\beta}}{\overline{\beta}-\underline{\beta}}$. As $VL^{PLB}_{T-2} = VL^{PLB}_{T-2}(\alpha)$, which is increasing in α, we obtain that $VL^{PLB}_{T-2} > VL^{LM}_{T-2}$, $LP^{PLB}_{T-2} \leq 1 - VL^{PLB}_{T-2} < 1 - VL^{LM}_{T-2} = LP^{LM}_{T-2}$, $DPI^{PLB}_{T-2} < DPI^{LM}_{T-2}$, $DPT^{PLB}_{T-2} < DPT^{LM}_{T-2}$, and $SP^{PLB}_{T-2} > SP^{LM}_{T-2}$.

(1.4) When instead $\frac{\tau}{v}$is large, i.e. when given a fixed value for the tick size we consider a low priced stock, it can be shown that the equilibrium strategies are:

$$H^{*BD}_{T-2}(\beta,[0]) = \begin{cases} -1^B & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-2}) \\ +1^{a_1} & \text{if } \beta \in [\beta^{BD}_{1,T-2}, \beta^{BD}_{2,T-2}) \\ +1^{A_2} & \text{if } \beta \in [\beta^{BD}_{2,T-2}, \beta^{BD}_{3,T-2}) \\ +1^{B_2} & \text{if } \beta \in [\beta^{BD}_{3,T-2}, \beta^{BD}_{4,T-2}) \\ +1^{b_1} & \text{if } \beta \in [\beta^{BD}_{4,T-2}, \beta^{BD}_{5,T-2}) \\ -1^A & \text{if } \beta \in [\beta^{BD}_{5,T-2}, \overline{\beta}] \end{cases}$$

[20] If not so, the optimal strategy at $T-2$ can be $\{-1^B, +1^{A_1}, +1^{A_2}, +1^{B_2}, +1^{B_1}, -1^A\}$ or $\{-1^B, +1^{a_1}, +1^{A_1}, +1^{A_2}, +1^{B_2}, +1^{B_1}, +1^{b_1}, -1^A\}$ or $\{-1^B, +1^{a_1}, +1^{A_2}, +1^{B_2}, +1^{b_1}, -1^A\}$. To shorten this proof, we omit these calculations.

[21] When computing the spread, we do not consider those market orders that hit A_2 and B_2 as they are executed against the trading crowd and leave the spread unchanged.

$$
H^{*RT}_{T-2}(\beta,[0]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta_{1,T-2}\,|_{A_2,B=B_2}) \\ +1^{A_1} & \text{if } \beta \in [\beta_{1,T-2}\,|_{A_2,B=B_2}\,, \beta_{2,T-2}\,|_{A_2,B_2}) \\ +1^{A_2} & \text{if } \beta \in [\beta_{2,T-2}\,|_{A_2,B=B_2}\,, \beta_{3,T-2}\,|_{A_2,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta_{3,T-2}\,|_{A_2,B_2}\,, \beta_{4,T-2}\,|_{A=A_2,B_2}) \\ +1^{B_1} & \text{if } \beta \in [\beta_{4,T-2}\,|_{A_2,B_2}\,, \beta_{5,T-2}\,|_{A=A_2,B_2}) \\ -1^{A} & \text{if } \beta \in [\beta_{5,T-2}\,|_{A=A_2,B_2}\,, \overline{\beta}] \end{cases}
$$

After computing market quality indicators on the same lines as in (1.3), we observe the same effects on liquidity provision, depth and spread, but of larger size.

<u>2) Liquid stocks: starting book at $T-2$ is [01|10]&[00000|00000]</u>

At $T-2$ trader's strategy space is the same as that for illiquid stocks, and hence there are 17 possible states of the books. We provide directly equilibrium strategies at $T-2$.
(2.1) If at $T-2$ a RT arrives:

$$
H^{*RT}_{T-2}(\beta,[1]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta^{RT}_{1,T-2}\,|_{A_2,B=B_1}) \\ +1^{A_2} & \text{if } \beta \in [\beta^{RT}_{1,T-2}\,|_{A_2,B=B_1}\,, \beta^{RT}_{3,T-2}\,|_{A_2,B_2}) \\ +1^{B_2} & \text{if } \beta \in [\beta^{RT}_{3,T-2}\,|_{A_2,B_2}\,, \beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}) \\ -1^{A} & \text{if } \beta \in [\beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}\,, \overline{\beta}] \end{cases}
$$

where $\beta^{RT}_{1,T-2}\,|_{A_2,B=B_1} = \frac{B_1}{v} - \frac{p^*_{T-2}(A_2|S^{PLB}_{T-2},S^{IP}_{T-2})}{1-p^*_{T-2}(A_2|S^{PLB}_{T-2},S^{IP}_{T-2})}\cdot\frac{A_2-B_1}{v}$, $\beta^{RT}_{5,T-2}\,|_{A=A_1,B_2} = \frac{A_1}{v} + \frac{p^*_{T-2}(B_2|S^{PLB}_{T-2},S^{IP}_{T-2})}{1-p^*_{T-2}(B_2|S^{PLB}_{T-2},S^{IP}_{T-2})}\cdot \frac{A_1-B_2}{v}$, $\beta^{RT}_{3,T-2}\,|_{A_2,B_2} = \frac{p^*_{T-2}(A_2|S^{PLB}_{T-2},S^{IP}_{T-2})A_2+p^*_{T-2}(B_2|S^{PLB}_{T-2},S^{IP}_{T-2})B_2}{p^*_{T-2}(A_2|S^{PLB}_{T-2},S^{IP}_{T-2})+p^*_{T-2}(2|S^{PLB}_{T-2},S^{IP}_{T-2})}\cdot\frac{1}{v}$.
(2.2) If at $T-2$ a BD arrives:

$$
H^{*BD}_{T-2}(\beta,[1]) = \begin{cases} -1^{B} & \text{if } \beta \in [\underline{\beta}, \beta^{BD}_{1,T-2}) \\ +1^{a_1} & \text{if } \beta \in [\beta^{BD}_{1,T-2}, \beta^{BD}_{2,T-2}) \\ +1^{a_2} & \text{if } \beta \in [\beta^{BD}_{2,T-2}, \beta^{BD}_{3,T-2}) \\ +1^{b_2} & \text{if } \beta \in [\beta^{BD}_{3,T-2}, \beta^{BD}_{4,T-2}) \\ +1^{b_1} & \text{if } \beta \in [\beta^{BD}_{4,T-2}, \beta^{BD}_{5,T-2}) \\ -1^{A} & \text{if } \beta \in [\beta^{BD}_{5,T-2}, \overline{\beta}] \end{cases}
$$

(2.3) The PLB expected volume is:

$$
\begin{aligned} VL^{PLB}_{T-2} &= \alpha\cdot\Big(\frac{\overline{\beta}-\beta^{RT}_{5,T-2}\,|_{A=A_1,B_2}}{\overline{\beta}-\underline{\beta}} + \frac{\beta^{RT}_{1,T-2}\,|_{A_2,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}}\Big) \\ &\quad +(1-\alpha)\cdot\Big(\frac{\overline{\beta}-\beta^{BD}_{5,T-2}\,|_{A=A_1,B_2}}{\overline{\beta}-\underline{\beta}} + \frac{\beta^{BD}_{1,T-2}\,|_{A_2,B=B_1} - \underline{\beta}}{\overline{\beta}-\underline{\beta}}\Big) \end{aligned}
$$

As broker-dealers find it optimal to submit limit orders to the IP, we have $LP^{PLB}_{T-2} + LP^{IP}_{T-2} = 1 - VL^{PLB}_{T-2}$. The PLB evolves as follows:
$E[S^{PLB}_{T-2|T-3}] = S^{PLB}_{T-2} + E[h_{T-2}]$
$= [(1-\alpha)\Pr(H^{*RT}_{T-2} = +1^{A_2}), 1 - (1-\alpha)\Pr(H^{*RT}_{T-2} = -1^{A_1}) - \alpha\Pr(H^{*BD}_{T-2} = -1^{A_1}), 1 - (1-\alpha)\Pr(H^{*RT}_{T-2} = -1^{B_1}) - \alpha\Pr(H^{*BD}_{T-2} = -1^{B_1}), (1-\alpha)\Pr(H^{*RT}_{T-2} = +1^{B_2})]$
In addition, PLB has the following market quality indicators for inside depth, total depth and spread:

$$
\begin{aligned}
DPI^{PLB}_{T-2} &= 2 + LP^{PLB}_{T-2} - VL^{PLB}_{T-2} \\
DPT^{PLB}_{T-2} &= \sum_i E[Q^i_{T-2}] = 2 + LP^{PLB}_{T-2} - VL^{PLB}_{T-2} \\
SP^{PLB}_{T-1} &= E[A-B] = \tau + VL^{PLB}_{T-2}\tau = (1 + VL^{PLB}_{T-2})\tau
\end{aligned}
$$

To analyze how PLB market quality has changed after introducing IP, we compare these values with those in (2.3) of proof of Proposition 1, and obtain $LP^{PLB}_{T-2} < LP^{LM}_{T-2}$, $VL^{PLB}_{T-2} < VL^{LM}_{T-2}$ and thus $DPI^{PLB}_{T-1} > DPI^{LM}_{T-1}$, $SP^{PLB}_{T-1} < SP^{LM}_{T-1}$. ■

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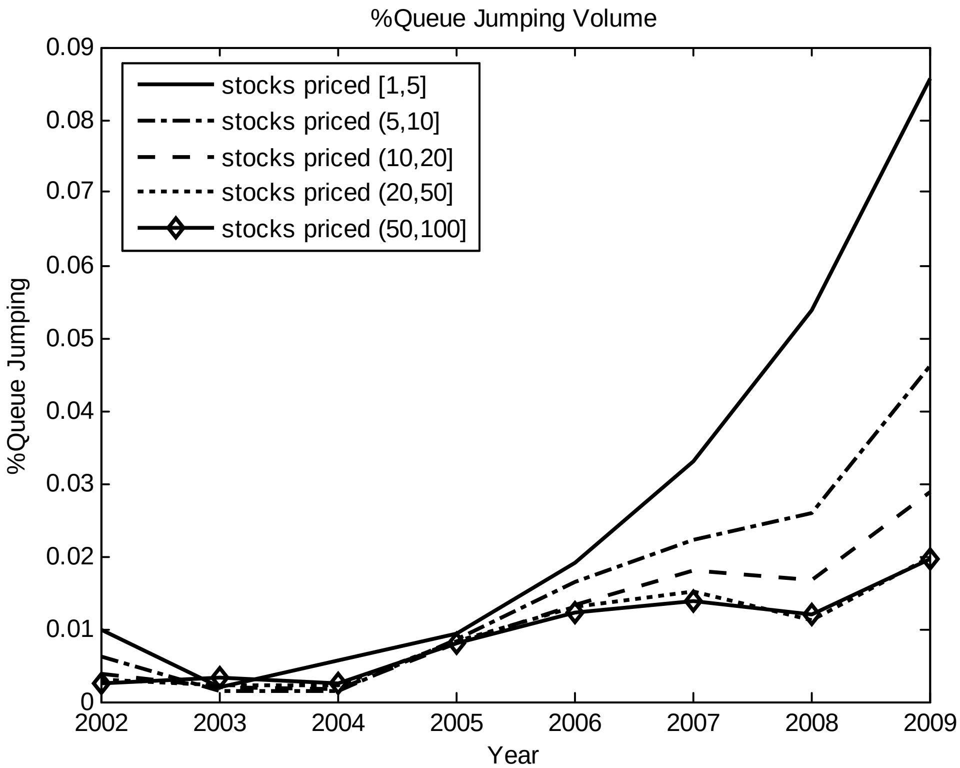

Figure 1: **NASDAQ Stocks: Queue Jumping.** This Figure shows the evolution of sub-penny trading over the last 10 years for different priced NASDAQ stocks. Daily data come from Thomson Reuters tick-by-tick hystorical and weekly statistics are from Delassus and Tyc (2010).

Table 1 Tick Size Reductions			
Country	**Market**		**Change time**
	Abbr.	**Full name**	
US	AMEX	American Stock Exchange	1992,1997
	NYSE	New York Stock Exchange	1992,1997,2001
	NASDAQ		1991,1997,2001
Canada	TSE	Toronto Stock Exchange	1996
France	EuP	Euronext Paris	1999
HK	HKSE	Hong Kong Stock Exchange	1994
Singapore	SES	Stock Exchange of Singapore	1994
Japan	TySE	Tokyo Stock Exchange	1998
Indonesia	ISE	Indonesia Stock Exchange	2001-2007
Thailand	SET	Stock Exchange of Thailand	2001

Table 1: Tick Size Reductions. This Table reports examples of tick size reductions which took place in some major exchages over the past two decades.

Table 2 Order Submission Strategy Space		
Strategy	H	$U(\cdot)$
Market Sell Order	-1^B	$B - \beta v$
Limit Sell Order	1^{A_k}	$p_t^*(A_k^{N_k}) \cdot (A_k - \beta v)$
No Trade	0	0
Limit Buy Order	1^{B_k}	$p_t^*(B_k^{M_k}) \cdot (\beta v - B_k)$
Market Buy Order	-1^A	$\beta v - A$

Table 2: Order Submission Strategy Space. This Table reports in column 3 the payoffs ($U(\cdot)$) of the order strategies (H_t) listed in column 2.

Table 3 Price Grid		
Large Tick	**Price**	**Small Tick**
A_2	$v + \frac{9}{6}\tau$	a_5
	$v + \frac{7}{6}\tau$	a_4
	$v + \frac{5}{6}\tau$	a_3
A_1	$v + \frac{3}{6}\tau$	a_2
	$v + \frac{1}{6}\tau$	a_1
	$v - \frac{1}{6}\tau$	b_1
B_1	$v - \frac{3}{6}\tau$	b_2
	$v - \frac{5}{6}\tau$	b_3
	$v - \frac{7}{6}\tau$	b_4
B_2	$v - \frac{9}{6}\tau$	b_5

Table 3: **Price Grid.** This Table shows price levels in different markets, where v indicates the asset value and τ the tick size. A large tick market has a coarse price grid while a small tick market has a fine price grid.

Table 4 Tick Size Change ($\tau = 0.1$, $S_{T-2} = [01|10] = [00010|01000]$)

		Large Tick [$Tick\ Size = \tau$]				Small Tick [$Tick\ Size = \frac{\tau}{3}$]				$\Delta(\times 100)$			
Asset Price = v		1	5	10	50	1	5	10	50	1	5	10	50
Limit Order = LP		.1006	.0226	.0114	.0024	.2334	.0492	.0248	.0050	13.28	2.66	1.34	.26
Market Order=VL		.8994	.9774	.9885	.9977	.7666	.9507	.9752	.9950	-13.28	-2.67	-1.33	-.27
No Trading		0	0	0	0	0	0	0	0	0	0	0	0
Spread		.1899	.1977	.1989	.1998	.1699	.1937	.1968	.1994	-2.00	-.40	-.21	-.04
Depth at BBO		1.1006	1.0226	1.0115	1.0023	1.2334	1.0493	1.0248	1.0050	13.28	2.67	1.33	.27
Agg. Depth		1.2012	1.0453	1.0230	1.0047	1.4671	1.0986	1.0496	1.0100	26.59	5.33	2.66	.53
Limit Order Sub Prob.	$A_2 = a_5$	.0503	.0113	.0057	.0012	.0147	.0045	.0024	.0005	-3.56	-.68	-.33	-.07
	a_4	-	-	-	-	0	0	0	0	-	-	-	-
	a_3	-	-	-	-	0	0	0	0	-	-	-	-
	$A_1 = a_2$	0	0	0	0	0	0	0	0	0	0	0	0
	a_1	-	-	-	-	.1020	.0201	.0100	.0020	-	-	-	-
	b_1	-	-	-	-	.1020	.0201	.0100	.0020	-	-	-	-
	$B_1 = b_2$	0	0	0	0	0	0	0	0	0	0	0	0
	b_3	-	-	-	-	0	0	0	0	-	-	-	-
	b_4	-	-	-	-	0	0	0	0	-	-	-	-
	$B_2 = b_5$	.0503	.0113	.0057	.0012	.0147	.0045	.0024	.0005	-3.56	-.68	-.33	-.07

Table 4: Tick Size Change - Liquid stock. This Table focuses on the case where the large tick (τ) and the small tick ($\frac{\tau}{3}$) market open at $T-2$ with one share on the first (A_1, B_1) and on the second (a_2, b_2) level of the book respectively. Results are reported for the large tick market (columns 2-5) and for the small tick market (columns 6-9) under the assumption of different asset values, $v = \{1, 5, 10, 50\}$. Columns (10-13) report the difference between the two markets (small-large) for each value of the asset considered. The following statistics are reported: liquidity provision, i.e. sum of limit order submission probabilities (row 4); trading volume, i.e. probability of market order submission (row 5); probability of "no trading" (row 6); spread, depth at the BBO and aggregate depth (rows 7-9), and limit order submission probabilities at different levels of the book (rows 10-19).

Table 5 Tick Size Change ($\tau = 0.1$, $S_{T-2} = [02|20] = [00020|02000]$)

		Large Tick [$Tick\ Size = \tau$]				Small Tick [$Tick\ Size = \frac{\tau}{3}$]				$\Delta(\times 100)$			
Asset Price = v		1	5	10	50	1	5	10	50	1	5	10	50
Limit Order = LP		0	0	0	0	.2334	.0492	.0248	.0050	23.34	4.92	2.48	.50
Market Order=VL		.9500	.9900	.9950	.9990	.7666	.9507	.9752	.9950	-18.34	-3.93	-1.98	-.40
No Trading		.0500	.0100	.0050	.0010	0	0	0	0	-5.00	-1.00	-0.50	-.10
Spread		.1000	.1000	.1000	.1000	.0922	.0984	.0992	.0998	-.78	-.16	-.08	-.02
Depth at BBO		3.0500	3.0100	3.005	3.0010	3.0000	3.0000	3.0000	3.0000	-5.00	-1.00	-.50	-.10
Agg. Depth		3.0500	3.0100	3.005	3.0010	3.4668	3.0986	3.0496	3.0100	41.68	8.86	4.46	.90
Limit Order Sub Prob.	$A_2 = a_5$	0	0	0	0	0	0	0	0	0	0	0	0
	a_4	-	-	-	-	0	0	0	0	-	-	-	-
	a_3	-	-	-	-	0	0	0	0	-	-	-	-
	$A_1 = a_2$	0	0	0	0	0	0	0	0	0	0	0	0
	a_1	-	-	-	-	.1167	.0246	.0124	.0025	-	-	-	-
	b_1	-	-	-	-	.1167	.0246	.0124	.0025	-	-	-	-
	$B_1 = b_2$	0	0	0	0	0	0	0	0	0	0	0	0
	b_3	-	-	-	-	0	0	0	0	-	-	-	-
	b_4	-	-	-	-	0	0	0	0	-	-	-	-
	$B_2 = b_5$	0	0	0	0	0	0	0	0	0	0	0	0

Table 5: Tick Size Change - Highly liquid stock. This Table focuses on the case where the large tick (τ) and the small tick ($\frac{\tau}{3}$) market open at $T-2$ with two shares on the first (A_1, B_1) and on the second (a_2, b_2) level of the book respectively. Results are reported for the large tick market (columns 2-5) and for the small tick market (columns 6-9) under the assumption of different asset values, $v = \{1, 5, 10, 50\}$. Columns (10-13) report the difference between the two markets (small-large) for each value of the asset considered. The following statistics are reported: liquidity provision, i.e. sum of limit order submission probabilities (row 4); trading volume, i.e. probability of market order submission (row 5); probability of "no trading" (row 6); spread, depth at the BBO and aggregate depth (rows 7-9), and limit order submission probabilities at different levels of the book (rows 10-19).

Table 6 Tick Size Change ($\tau = 0.1$, $S_{T-2} = [00|00] = [00000|00000]$)

		Large Tick [$Tick\ Size = \tau$]				Small Tick [$Tick\ Size = \frac{\tau}{3}$]				$\Delta(\times 100)$			
Asset Price = v		1	5	10	50	1	5	10	50	1	5	10	50
Limit Order = LP		.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
Market Order=VL		.4140	.8260	.9048	.9794	.4414	.8290	.9058	.9795	2.74	.30	.10	.01
No Trading		0	0	0	0	0	0	0	0	0	0	0	0
Bid-ask Spread		.2751	.3000	.3000	.3000	.2763	.3000	.3000	.3000	.12	0	0	0
Depth at BBO		.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
Agg. Depth		.5860	.1740	.0952	.0206	.5586	.1710	.0942	.0205	-2.74	-.30	-.10	-.01
Limit Order Sub Prob.	$A_2 = a_5$	.1684	.0870	.0476	.0103	.1905	.0855	.0471	.0103	2.21	-.15	-.05	0
	a_4	-	-	-	-	0	0	0	0	-	-	-	-
	a_3	-	-	-	-	0	0	0	0	-	-	-	-
	$A_1 = a_2$	.1246	0	0	0	0	0	0	0	-12.46	0	0	0
	a_1	-	-	-	-	.0888	0	0	0	-	-	-	-
	b_1	-	-	-	-	.0888	0	0	0	-	-	-	-
	$B_1 = b_2$	.1246	0	0	0	0	0	0	0	-12.46	0	0	0
	b_3	-	-	-	-	0	0	0	0	-	-	-	-
	b_4	-	-	-	-	0	0	0	0	-	-	-	-
	$B_2 = b_5$	.1684	.0870	.0476	.0103	.1905	.0855	.0471	.0103	2.21	-.15	-.05	0

Table 6: Tick Size Change - Illiquid stock. This table focuses on the case where the large tick (τ) and the small tick ($\frac{\tau}{3}$) market open empty at $T-2$. Results are reported for the large tick market (columns 2-5) and for the small tick market (columns 6-9) under the assumption of different asset values, $v = \{1, 5, 10, 50\}$. Columns (10-13) report the difference between the two markets (small-large) for each value of the asset considered. The following statistics are reported: liquidity provision, i.e. sum of limit order submission probabilities (row 4); trading volume, i.e. probability of market order submission (row 5); probability of "no trading" (row 6); spread, depth at the BBO and aggregate depth (rows 7-9), and limit order submission probabilities at different levels of the book (rows 10-19).

Table 7 Intermarket Competition: T-2 - LM: $[Tick\ Size = \tau = 0.1]$ SM: $[Tick\ Size = \frac{\tau}{3}]$													
		$S_{T-2} = [00\|00] \approx [00000\|00000]$						$S_{T-2} = [01\|10] \approx [00010\|01000]$					
		LM		SM		$\Delta(\times 100)$		LM		SM		$\Delta(\times 100)$	
Asset Price $= v$		1	10	1	10	1	10	1	10	1	10	1	10
Limit Order $= LP$		0	0	.5678	.0868	56.78	8.68	0	0	.2334	.0248	23.34	2.48
Market Order$=VL$		.2161	.4566	.2161	.4566	0	0	.3833	.4876	.3833	.4876	0	0
Spread		.3000	.3000	.2598	.2971	-4.02	-.29	.1383	.1488	.1306	.1479	-.77	-.09
Depth at BBO		0	0	.5678	.0868	56.78	8.68	1.6167	1.5124	1.6167	1.5124	0	0
Agg. Depth		0	0	.5678	.0868	56.78	8.68	1.6167	1.5124	1.8501	1.5372	23.34	2.48
	$A_2 = a_5$	0	0	0	0	0	0	0	0	0	0	0	0
	a_4	-	-	.1776	.0434	-	-	-	-	0	0	-	-
	a_3	-	-	0	0	-	-	-	-	0	0	-	-
Limit	$A_1 = a_2$	0	0	0	0	0	0	0	0	0	0	0	0
Order	a_1	-	-	.1063	0	-	-	-	-	.1167	.0124	-	-
Sub	b_1	-	-	.1063	0	-	-	-	-	.1167	.0124	-	-
Prob.	$B_1 = b_2$	0	0	0	0	0	0	0	0	0	0	0	0
	b_3	-	-	0	0	-	-	-	-	0	0	-	-
	b_4	-	-	.1776	.0434	-	-	-	-	0	0	-	-
	$B_2 = b_5$	0	0	0	0	0	0	0	0	0	0	0	0

Table 7: Intermarket Competition. This Table focuses on the case where the large tick (τ) and the small tick ($\frac{\tau}{3}$) market open *simultaneously* at $T-2$. Results are reported for the large tick market (columns 2-3 and 8-9) and for the small tick market (columns 4-5 and 10-11) under the assumption of different starting books ($S_{T-2} = [00|00] \approx [00000|00000]$ and $S_{T-2} = [01|10] \approx [00010|01000]$). Columns (6-7 and 12-13) report the difference between the two markets (small-large) for each case considered. The following statistics are reported: liquidity provision, i.e. sum of limit order submission probabilities (row 5); trading volume, i.e. probability of market order submission (row 6); spread, depth at the BBO and aggregate depth (rows 7-9), and limit order submission probabilities at different levels of the book (rows 10-19).

Table 8 - US Dark Pool Volume (% Consolidated US Equity Volume)

	Time	Jun.08	Aug. 08	May 09	Jun.09	Jul.09	Aug. 09	May 10	Jun.10	Jul. 10	Aug. 10
Internalized Pool (Broker-sponsored) (IP)	Credit Suisse Crossfinder	.81	.88	1.31	1.42	1.55	1.55	2.10	2.08	2.11	2.10
	Goldman Sachs Sigma X	1.37	3.00	1.09	1.19	1.29	1.44	1.57	1.47	1.56	1.55
	Knight Link	.98	1.17	1.23	1.05	1.08	1.23	1.62	1.71	1.44	1.38
	Getco Execution Service	.41	.64	1.17	1.06	1.08	1.11	1.02	1.03	1.16	1.28
	Morgan Stanley MS Pool	.26	.31	.44	.53	.50	.48	.52	.58	.61	.78
	Barclays LX	.47	-	.18	.20	.34	.25	.54	.57	.64	.67
	UBS Pin	.29	.58	.39	.42	.45	.43	.37	.37	.38	.41
	Citi Match	.43	.50	.33	.35	.38	.37	.35	.31	.38	.38
	%IP	5.02	7.08	6.14	6.22	6.67	6.86	8.09	8.12	8.28	8.55
Public Crossing Networks (PC)	ITG Posit	.32	.29	.20	.21	.24	.24	.25	.34	.39	.44
	Instinet CBX	.20	.27	.20	.19	.23	.24	.43	.41	.42	.41
	Liquidnet	.39	.41	.27	.26	.33	.28	.22	.21	.24	.25
	Convergex Millennium	.31	.30	.15	.16	.17	.16	.16	.21	.22	.18
	Convergex Vortex	.05	.06	.06	.08	.06	.05	.08	.07	.09	.10
	Pipeline Trading	.20	.15	.11	.09	.09	.12	.09	.08	.09	.09
	% PC	1.47	1.48	.99	.99	1.12	1.09	1.23	1.32	1.45	1.47
Consortium-based Pools (CBP)	Level	.40	.56	.48	.50	.58	.10	.73	.79	.86	.93
	Bid Trading	.13	.10	.10	.10	.14	.53	.21	.22	.27	.31
	%CBP	.53	.66	.58	.60	.72	.63	.94	1.01	1.13	1.24
	ALL	7.02	9.22	7.71	7.81	8.51	8.58	10.26	10.45	10.86	11.26

Table 8: US Dark Pool Volume. This Table shows the recent evolution of U.S. dark pool volume which breaks down into three parts: Internalization Pools, Public Crossing Networks and Consortium-based Pools. Statistics are kindly offered by Rosenblatt Securities Inc.

Table 9 Sub-penny Trading ($\tau = 0.1$, $S_{T-2} = [00\|00] = [00000\|00000]$)											
		$v = 1$					$v = 10$				
		PLB	PLB&IP: $\alpha = 10\%$		PLB&IP: $\alpha = 50\%$		PLB	PLB&IP: $\alpha = 10\%$		PLB&IP: $\alpha = 50\%$	
		$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$	$T-2$	$T-2$	$(\Delta \times 100)$	$T-2$	$(\Delta \times 100)$
Limit Order = LP		.5860	.5744	(-1.16)	.4658	(-12.02)	.0952	.0950	(-.02)	.0948	(-.04)
Market Order=VL		.4140	.4256	(1.16)	.4552	(4.12)	.9048	.9050	(.02)	.9052	(.04)
No Trading		0	0	(0)	0	(0)	0	0	(0)	0	(0)
Spread		.2751	.2775	(.24)	.2939	(1.88)	.3000	.3000	(0)	.3000	(0)
Depth BBO		.5860	.5744	(-1.16)	.4658	(-12.02)	.0952	.0950	(-.02)	.0947	(-.05)
Agg. Depth		.5860	.5744	(-1.16)	.4658	(-12.02)	.0952	.0950	(-.02)	.0947	(-.05)
			LOB	IP	LOB	IP		LOB	IP	LOB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.1684	.1746	0	.2022	0	.0476	.0475	0	.0474	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	.1246	.1126	0	.0307	0	0	0	0	0	0
Order	a_1	-	-	0	-	.0395	-	-	0	-	0
Sub	b_1	-	-	0	-	.0395	-	-	0	-	0
Prob.	$B_1 = b_2$	.1246	.1126	0	.0307	0	0	0	0	0	0
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.1684	.1746	0	.2022	0	.0476	.0475	0	.0474	0

Table 9: Sub-penny Trading - Illiquid stock. This Table focuses on the comparison between the single market case, where only the public limit order book (PLB) with tick size τ is active, and the dual market case, where the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ coexist. We report results for the case in which both the PLB and the IP open empty at $T-2$. We consider two different asset values, $v = \{1, 10\}$, and two different broker-dealer's arrival rates, $\alpha = \{10\%, 50\%\}$. Results for $v = 1$ are reported in column 2 (PLB), columns 3-4 (PLB&IP, $\alpha = 10\%$), and columns 5-6 (PLB&IP, $\alpha = 50\%$). Results for $v = 10$ are instead reported in column 7 (PLB), columns 8-9 (PLB&IP, $\alpha = 10\%$), and columns 10-11 (PLB&IP, $\alpha = 50\%$). The following statistics are reported: liquidity provision, i.e. sum of limit order submission probabilities (row 5); trading volume, i.e. probability of market order submission (row 6); probability of "no trading" (row 7); spread, depth at the BBO and aggregate depth (rows 8-10), and limit order submission probabilities at different levels of the book (rows 13-22).

Table 10 Sub-penny Trading ($\tau = 0.1$, $S_{T-2} = [01|10] = [00000|00000]$)

		$v=1$					$v=10$				
		PLB	PLB&IP: $\alpha=10\%$		PLB&IP: $\alpha=50\%$		PLB	PLB&IP: $\alpha=10\%$		PLB&IP: $\alpha=50\%$	
		$T-2$	$T-2$	$(\Delta\times 100)$	$T-2$	$(\Delta\times 100)$	$T-2$	$T-2$	$(\Delta\times 100)$	$T-2$	$(\Delta\times 100)$
Limit Order = LP		.1006	.0934	(-.72)	.0648	(-3.58)	.0114	.0108	(-.06)	.0082	(-.32)
Market Order=VL		.8994	.8858	(-1.36)	.8322	(-6.72)	.9886	.9872	(-.14)	.9818	(-.68)
No Trading		0	0	(0)	0	(0)	0	0	(0)	0	(0)
Spread		.1899	.1886	(-.13)	.1832	(-.67)	.1989	.1987	(-.02)	.1982	(-.07)
Depth BBO		1.1006	1.1142	(1.36)	1.1679	(6.73)	1.0115	1.0128	(.13)	1.0182	(.67)
Agg. Depth		1.2012	1.2076	(.64)	1.2327	(3.15)	1.0230	1.0236	(.06)	1.0263	(.33)
			LOB	IP	LOB	IP		LOB	IP	LOB	IP
Tick Size		τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$	τ	τ	$\frac{\tau}{3}$	τ	$\frac{\tau}{3}$
	$A_2 = a_5$	.0503	.0467	0	.0324	0	.0057	.0054	0	.0041	0
	a_4	-	-	0	-	0	-	-	0	-	0
	a_3	-	-	0	-	0	-	-	0	-	0
Limit	$A_1 = a_2$	0	0	0	0	0	0	0	0	0	0
Order	a_1	-	-	.0104	-	.0515	-	-	.0010	-	.0050
Sub	b_1	-	-	.0104	-	.0515	-	-	.0010	-	.0050
Prob.	$B_1 = b_2$	0	0	0	0	0	0	0	0	0	0
	b_3	-	-	0	-	0	-	-	0	-	0
	b_4	-	-	0	-	0	-	-	0	-	0
	$B_2 = b_5$	.0503	.0467	0	.0324	0	.0057	.0054	0	.0041	0

Table 10: Sub-penny Trading - Liquid stock. This Table focuses on the comparison between the single market case, where only the public limit order book (PLB) with tick size τ is active, and the dual market case, where the PLB and the internalization pool (IP) with tick size $\frac{\tau}{3}$ coexist. We report results for the case in which the PLB starts at $T-2$ with one share on the first level of the book (A_1, B_1). We consider two different asset values, $v = \{1, 10\}$, and two different broker-dealer's arrival rates, $\alpha = \{10\%, 50\%\}$. Results for $v = 1$ are reported in column 2 (PLB), columns 3-4 (PLB&IP, $\alpha = 10\%$), and columns 5-6 (PLB&IP, $\alpha = 50\%$). Results for $v = 10$ are instead reported in column 7 (PLB), columns 8-9 (PLB&IP, $\alpha = 10\%$), and columns 10-11 (PLB&IP, $\alpha = 50\%$). The following statistics are reported: liquidity provision, i.e. sum of limit order submission probabilities (row 5); trading volume, i.e. probability of market order submission (row 6); probability of "no trading" (row 7); spread, depth at the BBO and aggregate depth (rows 8-10), and limit order submission probabilities at different levels of the book (rows 13-22).